Geometries, non-geometries, and fluxes

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Abstract

Using F-theory/heterotic duality, we describe a framework for analyzing non-geometric T^2-fibered heterotic compactifications to six- and four dimensions. Our results suggest that among T^2-fibered heterotic string vacua, the non-geometric compactifications are just as typical as the geometric ones. We also construct four-dimensional solutions that have novel type-IIB and M-theory dual descriptions. These duals are non-geometric with three- and four-form fluxes not of $(2,1)$ or $(2,2)$ Hodge type, respectively, and yet preserve at least $N = 1$ supersymmetry.

CONTENTS

1   Introduction

1.1   The basics of non-geometries

The space of four-dimensional string compactifications is potentially vast. The degeneracy of these vacua comes about by the many choices of compactification metric and associated fluxes. When the size of the compactification space is large compared with the string scale, we can use supergravity to study the resulting low-energy four-dimensional physics. However, we expect generic stabilized vacua to involve string scale physics for which supergravity is inadequate.

One way in which a compactification space can become quantum is if the patching conditions involve symmetries present in string theory but not supergravity. The simplest example of this type is F-theory where the backgrounds involve seven-brane sources of type-IIB string theory [1]. Without knowing that S-duality is a good symmetry of type-IIB string theory, those
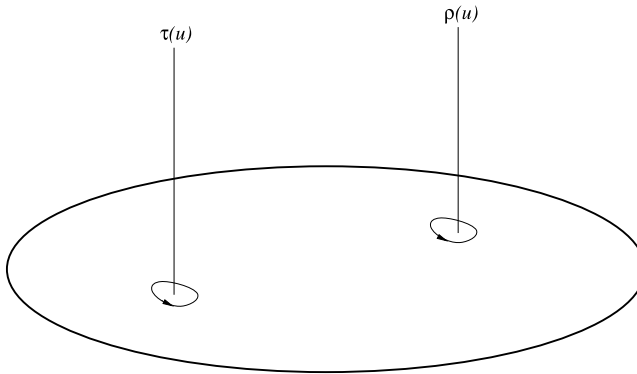
Figure 1: A schematic of the desired fibration data where u denotes coordinates on the base B. The loci of τ and ρ degenerations can be viewed as supporting 5-branes.

backgrounds would make no sense as solutions of type-IIB supergravity. A second example of quantum patching conditions are compactifications that involve T-duality, aspects of which we will explore here. This second case is an example of quantum geometry which arises in classical string theory, much like mirror symmetry.

Closed string theory on T^2 has two basic moduli: the complex structure parameter τ of T^2 and the Kähler modulus ρ, which determines the volume V of T^2 and the $\mathcal{B}$-field,

$$\rho = \rho_1 + \mathrm{i}\rho_2 = \mathcal{B} + \mathrm{i}V. \tag{1.1}$$

To build an elliptic compactification, one usually fibers τ over a base space allowing τ to undergo monodromies valued in $\mathrm{SL}(2, \mathbb{Z})$. These are large diffeomorphisms of the torus. In string theory, however, τ and ρ share the same symmetry group, appearing on equal footing and we should be able to describe quantum compactifications where both τ and ρ vary over a base space as depicted in figure 1. Since the action of $\mathrm{SL}(2, \mathbb{Z})$ on ρ includes $V \to 1/V$, these compactifications are typically inherently quantum. This is the class of compactifications we wish to explore.

In the purely geometric case where a large volume limit is possible, we can describe a torus fibration over a base B, depicted in figure 1, using a local semi-flat approximation for the metric

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}u^i\mathrm{d}u^j + \frac{\rho_2}{\tau_2}\left|\mathrm{d}w_1 + \tau(u)\mathrm{d}w_2\right|^2. \tag{1.2}$$

The base metric is g_{ij} and the torus has coordinates (w_1, w_2). The complex structure $\tau(u)$ varies over B while ρ is constant. This metric possesses $U(1) \times U(1)$ isometries acting on the torus fibers. For compact spaces, the semi-flat metric (1.2) is typically used as an approximation to a smooth Ricci flat metric with no isometries, with the approximation becoming exact as $V \to 0$. It is the smooth metric which is used to define the world-sheet sigma model, which flows to a conformal field theory defining the perturbative string background. However, with both τ and ρ varying, the existence of a smooth metric is no longer possible. Consequently, the condition analogous to the existence of a smooth metric should be the existence of a conformal field theory specified by τ, ρ, and B.

If we reduce ten-dimensional string theory on T^2 to eight-dimensions then we can view the resulting theory as possessing two families of (p, q) 5-branes in analogy with the (p, q) 7-branes of type-IIB string theory. One family is associated with τ degenerations while the other is associated with ρ degenerations. From this perspective, compactifications on B, like the one in figure 1, include 5-branes at the degeneration loci of τ and ρ. The standard NS5-brane corresponds to a purely perturbative ρ monodromy. If the moduli of the compactification can be tuned to make all the ρ monodromies perturbative then the model is likely to admit an asymmetric orbifold description. This is analogous to the orientifold limit of F-theory proposed by Sen [2].

The most desirable approach for studying stringy compactifications involving ingredients like T-duality is a world-sheet analysis where α' effects can be determined directly. In type-II string theory this kind of analysis can be further complicated by the presence of Ramond-Ramond (RR) fields, branes and orientifolds. These ingredients, needed for N = 1 compactifications with stabilized moduli [3], are difficult to analyze beyond the large volume supergravity limit, although it may be possible to understand their role in the Berkovits formalism [4]; see, for example [5].

In contrast, the heterotic string is a more desirable framework to use for two key reasons. Firstly, solutions are specified purely by the Neveu–Schwarz (NS) field content, which consists of the metric, torsion flux and bundle data. This avoids many of the complications of RR fluxes and, in principle, it is possible to construct world-sheet descriptions of heterotic vacua within the RNS formalism. Secondly, no orientifolds are needed. The Bianchi identity for the $\mathcal{H}_3$-flux of the heterotic string,

$$d\mathcal{H}_3 = \frac{\alpha'}{4} \left(\mathrm{Tr} \left(R \wedge R \right) - \mathrm{Tr} \left(F \wedge F \right) \right), \tag{1.3}$$

automatically includes a higher derivative curvature term that makes compact solutions possible. This removes the typically difficult task of

consistently patching orientifold actions together globally with T-dualities. This makes it much easier to construct non-geometric heterotic solutions than type-II or M-theory solutions. We will see how this simplifies the description of non-geometric vacua in the heterotic string versus type-IIB orientifolds in Sections 4 and 5.

One of the aims of this paper is to make use of heterotic — F-theory duality to provide a purely geometric description of a large class of non-geometric heterotic compactifications. The duality is typically stated as follows: F-theory compactified on a $K3$-fibered Calabi–Yau $(n + 1)$-fold, which is also elliptically fibered with section is equivalent to the heterotic string compactified on an elliptically fibered Calabi–Yau n-fold. Usually, one takes a particular limit in the moduli space of elliptic $K3$ surfaces to ensure that the heterotic solution is at large volume and well-described by supergravity.

From the point of view of F-theory, there is nothing special about this point in the moduli space, and one can ask what happens more generally. In this paper, refining some work of Clingher and Doran [6], we extend the heterotic — F-theory duality beyond the traditional limit, to all points in the moduli space where the heterotic gauge group remains unbroken.

As we will see, the generic heterotic solution with a dual F-theory description may not have a large volume limit but may instead involve patching by the T-duality group of the heterotic string on T^2. This provides a very nice way of determining fibration data for non-geometric compactifications. In fact, the F-theory fibration captures not only τ and ρ but also the Wilson line data for the heterotic gauge bundle on T^2.[1] So this approach should lead to the construction and description of quantum bundles. Exact conformal field theory descriptions of local heterotic models with abelian bundles have been found in [7,8]. It would be very interesting to see if that approach can be extended to accommodate non-geometric bundles.

It is important to stress that for compactifications with $N = 1$ supersymmetry, the F-theory/heterotic duality is not generally a quantum equivalence of string vacua. Rather, it is a means by which we can obtain classical data to describe a heterotic compactification. In the geometric case (without $\mathcal{H}_3$-flux), this data is an elliptic Calabi–Yau space over B and a holomorphic bundle which provides the defining data for a heterotic sigma model. In the non-geometric case, this data is replaced by a fibration of τ and ρ over B and a quantum bundle. However, aside from special BPS couplings, most

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[1]In geometric models where V can be made arbitrarily large, this Wilson line data describes a flat $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ connection on T^2.

space-time quantities such as Kähler potentials are going to be different in each theory.

1.2   Beyond T^2 and other approaches

It is natural to expect this geometrization of quantum heterotic compactifications to extend beyond T^2 fibrations. Indeed, if most Calabi–Yau spaces can be described as T^3-fibered spaces, as conjectured by Strominger–Yau–Zaslow [9], then we should expect "generic" heterotic compactifications to involve patching by the quantum symmetry group of the heterotic string on T^3 whose moduli space involves several distinct components [10]. The quantum patching conditions or monodromy data of the T^3-fibration should then be captured by M-theory compactified on a (potentially singular) $K3$-fibered manifold with G_2 holonomy. This is important to understand if we are to enumerate string vacua. Unfortunately, little is known about the construction of compact G_2 spaces let alone spaces admitting $K3$-fibrations so we will restrict our attention to heterotic compactifications with T^2-fibrations.

The final interesting case is a T^4-fibered heterotic compactification. In this case, we expect the quantum heterotic compactification to admit a dual description in terms of type-IIA on a $K3$-fibered space which also involves quantum patching conditions (namely, mirror transforms of the $K3$ fiber). In this case, both sides of the duality are generically quantum.

Some of the first attempts to construct quantum compactifications using U-duality appear in [11]. The type-II examples considered were compactifications to three dimensions or lower mainly because the solutions involved the full U-duality group rather than subgroups like the T-duality group. This work also pre-dates the discovery of flux vacua and the associated more general metric ansätze like the non-Kähler solutions of [3]. These more general metrics will be important in the examples we construct in Section 4.

More recently, a detailed discussion of non-geometric type-II solutions in six dimensions appeared in [12]. The type-II construction involves fibering $T^2 \times T^2$ which gives a double elliptic fibration over a base. The torus factors capture the τ and ρ monodromies. This doubled torus formalism has been further discussed in [13] where backgrounds using T-duality in the patching conditions have been termed "T-folds". The doubled torus approach has been extended to the heterotic string very recently in [14].

This doubled torus approach should be contrasted with the geometry of a $K3$-fibration that we use here. In principle, one should be able to understand

global properties like tadpole cancellation from the doubled torus formalism but it looks less intuitive for the heterotic string. This is partly because the definition of both sides of the Bianchi identity (1.3) are unclear, and partly because the bundle plays an important role in solving the tadpole condition (1.3); that bundle data is naturally encoded in the $K3$ fibration. For $N = 1$ compactifications, the tadpole conditions are really quite critical. For type-II non-geometric backgrounds, there are similar issues which remain to be understood [12].

The doubled torus approach might, however, be useful for constructing world-sheet descriptions; see, for example [15]. For example, it might be possible to extend the beta function computation of the doubled torus sigma model, developed in [16], to derive a complete version of the tadpole condition discussed in Section 3.4. That is a quite critical issue.

Our approach suggests a very different heterotic world-sheet description obtained naturally by studying an M5-brane wrapped on the $K3$-fiber of the dual geometry. Such an M5-brane sigma model can capture both torsional and torsion-free geometries along the lines discussed in [17]. We plan to explore this interesting wrapped brane configuration elsewhere. The last approach that leads naturally to non-geometric backgrounds is T-dualizing flux vacua. This approach was explored, for example, in [18]. For a review of past work on non-geometric backgrounds, see [19].

1.3 Some open issues and an outline

Some of the basic outstanding questions for non-geometric compactifications can be summarized as follows:

- What fibration data are needed to describe such compactifications?
- How do we construct and analyze world-sheet models, which involve quantum patching conditions?
- What new phenomenology or low-energy physics is possible in this wider class of compactifications?

We will set up a framework to answer the first two points. It would be very interesting to extend this framework beyond T^2 heterotic fibrations to T^3 fibrations. The third question is also extremely interesting. At least in type-II models, it appears that new low-energy couplings do emerge from non-geometric compactifications as described in [20]. It seems reasonable to suspect that new phenomenology might emerge in heterotic compactifications as well.

Most of the heterotic backgrounds we will describe are not left–right symmetric on the world-sheet. To describe a type-II compactification, we

would like to know if an analog of the standard embedding exists with varying ρ. It seems reasonable that such a generalization exists and will provide type-II solutions in a way quite different from the U-manifold geometrization discussed in [11].

Lastly, there should be nice methods of taking these solutions and generating non-geometric heterotic solutions *without* F-theory dual descriptions. For example, in the geometric setting, quotienting an ellipic Calabi–Yau with section by a free action can result in a torus-fibered Calabi–Yau without a section. The resulting space is still perfectly fine for the heterotic string but no longer fits into the heterotic/F-theory duality framework. We expect analogous constructions for these non-geometric models.

The outline for the paper is as follows: we first reconsider heterotic — F-theory duality in Section 2, focusing on the case of unbroken heterotic gauge group. Our analysis leads to a new construction of non-geometric heterotic compactifications in Section 3. The solutions we describe will be primarily phrased in terms of the heterotic string, although we later construct various type-IIB and M-theory duals. The vacua are typically non-geometric in the sense that they are locally geometric, satisfying the supergravity equations of motion, but globally well defined only in string theory. In particular, the complexified Kähler modulus will undergo non-trivial monodromies sourced by assorted heterotic 5-branes. We construct some simple examples and describe how to build general compactifications of this type.

In Section 4, we construct new non-geometric heterotic solutions with more general torsion. Such spaces have metrics which are locally non-Kähler. We do this by dualizing certain M-theory compactifications with flux which played a prominent role in constructing the first torsional (geometric) backgrounds [3]. The local supersymmetry constraints on the metrics and fluxes for these kinds of backgrounds were explored in [21].

These heterotic solutions, in turn, also have dual type-IIB and M-theory descriptions, obtained in Section 5, that exhibit novel characteristics. These are the compact U-folds sought in [11] but of a quite different local form. In particular, the space-time supersymmetry spinors have a more general structure than is usually considered. This allows us to construct, for example, four-dimensional type-IIB compactifications with three-form flux that is not necessarily of $(2, 1)$ Hodge type. We give an explicit example of such a construction and describe its M-theory lift.

Note added: We should mention that the solutions found in Sections 4 and 5 were obtained quite some time ago. During the completion of the project, several papers appeared with interesting related observations [14, 22–27]. It is also worth mentioning a very recent interesting conjecture that the

interpretation of black hole entropy might require the use of exotic branes associated to non-geometric monodromies [28].

2   F-theory and the heterotic string revisited

2.1   SL(2, $\mathbb{Z}$)-invariant scalar fields

Following our introductory comments, let us consider a physical theory which contains a scalar field τ invariant under an SL(2, $\mathbb{Z}$) action. It is natural to try to construct compactifications of this theory which exploit the SL(2, $\mathbb{Z}$)-invariance of the scalar. The general framework for doing so was laid out in [29] in the language of cosmic strings: the compactification space should have a multi-valued function τ on it, defined away from certain defects of codimension two, which will undergo SL(2, $\mathbb{Z}$) transformations around loops encircling the defects. These defects are depicted in figure 1.

The general problem of specifying such a multi-valued function arose in the work of Kodaira on elliptically fibered complex manifolds more than 45 years ago [30]. Any such elliptically fibered manifold gives rise to a multi-valued function τ defined on the base of the family, away from the subset of the base at which singular fibers are located. Conversely, given the multi-valued function τ, one can construct in a natural way an elliptically fibered manifold with fibers $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ over this subset of the base, which has the additional property that the family has a section (corresponding to $0 \in \mathbb{C}$).[2]

To close this circle of ideas, Kodaira showed that one can pass from an arbitrary elliptically fibered manifold to its associated "Jacobian fibration" (the one with the same τ function, and a section) in a natural way that does not involve finding τ explicitly.[3]   Moreover, Kodaira gave a way to characterize the set of all elliptically fibered manifolds with a fixed Jacobian fibration when the base has complex dimension one. This was later extended to bases of higher complex dimension by Nakayama [35, 36].

As Kodaira explained, two pieces of data are needed to specify τ: the natural SL(2, $\mathbb{Z}$)-invariant function $j = j(\tau)$ on the base (which Kodaira called the "functional invariant") and the precise SL(2, $\mathbb{Z}$) action on τ, which can

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[2]This result was obtained by Kodaira [30] when the base has complex dimension one, and subsequently generalized by Kawai [31] to dimension two and by Ueno [32] to arbitrary dimension.

[3]This is closely related to finding an equation in Weierstrass form, as described in an algebraic context by Deligne [33], and explored in this geometric context by Nakayama [34].

be equivalently thought of as the varying family of integer homology groups $H_1(\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau), \mathbb{Z})$ over the base (which Kodaira called the "homological invariant").

Given an elliptically fibered manifold $Z \to S$ with a section, there is a description of S as a *Weierstrass model* (cf. [33]). That is, there is a $\mathbb{P}^2$-bundle over S, and a birational map from Z to this $\mathbb{P}^2$-bundle, whose image has an (affine) equation of the form[4]

$$y^2 = x^3 + f(s)x + g(s), \tag{2.1}$$

where $f(s)$ and and $g(s)$ are sections of appropriate line bundles over S. To be precise, there is a line bundle $\mathcal{O}(L)$ on S such that $f(s) \in H^0(\mathcal{O}(4L))$, $g(s) \in H^0(\mathcal{O}(6L))$; we can regard x as a local section of $\mathcal{O}(2L)$ and y as a local section of $\mathcal{O}(3L)$ with the $\mathbb{P}^2$-bundle described as

$$\mathbb{P}\left(\mathcal{O} \oplus \mathcal{O}(2L) \oplus \mathcal{O}(3L)\right). \tag{2.2}$$

The total space may be singular, since certain subvarieties may be blown down in passing from the original elliptic fibration to the Weierstrass model.

The fibers of the Weierstrass model are singular[5] at the zeroes of the *discriminant*

$$\Delta(s) = 4f(s)^3 + 27g(s)^2, \tag{2.3}$$

and the functional invariant (the j-function) is given by the formula

$$j(s) = 1728\frac{4f(s)^3}{4f(s)^3 + 27g(s)^2}. \tag{2.4}$$

We will later make use of an equivalent formula for $j(s) - 1728$:

$$j(s) - 1728 = -1728\frac{27g(s)^2}{4f(s)^3 + 27g(s)^2}. \tag{2.5}$$

The homological invariant is determined by Kodaira's famous table, reproduced as Table 1. In that table, along any divisor D within S one calculates the orders of vanishing of $f(s)$, $g(s)$ and $\Delta(s)$ along D and learns about the

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[4]When comparing with [33], one should bear in mind that we are working over the complex numbers, so the exceptions to this form having to do with fields of characteristic 2 or 3 do not apply.

[5]Note that a singular point of a fiber is not necessarily a singular point of the total space, but for every singular point of the total space, the fiber passing through that point is singular.

Table 1: Kodaira's classification of singular fibers and monodromy.

	$\mathrm{ord}_D(f)$	$\mathrm{ord}_D(g)$	$\mathrm{ord}_D(\Delta)$	singularity	monodromy
I_0	≥ 0	≥ 0	0	none	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$I_n,\, n \geq 1$	0	0	n	A_{n-1}	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$
II	≥ 1	1	2	none	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$
III	1	≥ 2	3	A_1	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
IV	≥ 2	2	4	A_2	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$
I_0^*	≥ 2	≥ 3	6	D_4	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
$I_n^*,\, n \geq 1$	2	3	$n+6$	D_{n+4}	$\begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix}$
IV^*	≥ 3	4	8	E_6	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$
III^*	3	≥ 5	9	E_7	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
II^*	≥ 4	5	10	E_8	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$
non-minimal	≥ 4	≥ 6	≥ 12	non-canonical	$-$

singularity of the Weierstrass model over a general point of D, as well as the conjugacy class of the monodromy transformation about a loop encircling D. It is the latter which determines the homological invariant.

The last line of the table indicates a "non-minimal" Weierstrass equation: one whose singularities can be improved by making a birational transformation

$$(x, y) \mapsto \left(x/\psi(s)^2, y/\psi(s)^3 \right), \qquad (2.6)$$

(together with replacing $\mathcal{O}(L)$ by $\mathcal{O}(L + D)$), where $\psi(s)$ is a section of $\mathcal{O}(D)$ vanishing along D. This birational transformation does not affect the elliptic fibration away from the singular fibers in any way, and after a finite number of such improvements, a "minimal" Weierstrass model is obtained (i.e., one which fits into one of the earlier lines of the table). Because each non-minimal Weierstrass equation can be reduced to a minimal one by this process, it is customary to focus on the "minimal" case. We will comment

below on an additional reason that non-minimal Weierstrass equations would be unsuitable for the physical applications we have in mind.

Note that the Weierstrass equation is not uniquely specified by the τ function: we are free to rescale

$$(x, y, f, g) \mapsto \left(u(s)^2 x, u(s)^3 y, u(s)^4 f(s), u(s)^6 g(s)\right), \qquad (2.7)$$

using a nowhere vanishing function $u(s)$; this must be taken into account when describing the parameters of this construction.[6]

Kodaira also gave a formula for the canonical bundle of the total space of a minimal Weierstrass fibration when the base has complex dimension one (subsequently extended by others to higher dimension under certain hypotheses). The formula states that

$$\mathcal{O}(12K_{\overline{Z}}) = \pi^*(\mathcal{O}(12K_S + \Delta)), \qquad (2.8)$$

where $\pi : \overline{Z} \to S$ is the Weierstrass fibration.

To summarize: the data of a locally defined $\mathrm{SL}(2, \mathbb{Z})$-invariant scalar τ on some manifold S can be given in terms of an elliptic fibration $Z \to S$ with a section, and is effectively given by specifying a line bundle $\mathcal{O}(L)$ and describing Z as the desingularization of a hypersurface $\overline{Z}$ in the $\mathbb{P}^2$-bundle

$$\mathbb{P}\left(\mathcal{O} \oplus \mathcal{O}(2L) \oplus \mathcal{O}(3L)\right) \to S, \qquad (2.9)$$

defined by a Weierstrass equation

$$y^2 = x^3 + f(s)x + g(s), \qquad (2.10)$$

which does not fall into the last line of table 1 for any divisor D on S.

2.2 F-theory

The F-theory construction is a familiar application of the discussion in the previous section [1, 37, 38]. F-theory is a description of general type-IIB string backgrounds in which the complexified string coupling τ_F of the theory is allowed to be multi-valued and is defined away from defects of codimension two.

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[6]Note that allowing $u(s)$ to be a section of a line bundle would provide no greater generality, since a nowhere-vanishing section would trivialize the line bundle.

Kodaira's table allows a classification of the defects, using monodromy: a stack of n D7-branes corresponds to Kodaira's type I_n; a stack of n D7-branes on top of an orientifold O7-plane corresponds to Kodaira's type I_n^*; and various exotic 7-branes that are difficult to analyze from a perturbative string perspective correspond to the remaining Kodaira types II, III, IV, IV*, III*, II*.

There are special cases of the F-theory construction in which the τ function is constant [39, 40]. Firstly, for any constant value of the F-theory function τ_F we can choose data of the form

$$f(s) = \varphi h(s)^2, \quad g(s) = \gamma h(s)^3, \tag{2.11}$$

for some section $h(s)$ of the line bundle $\mathcal{O}(2L)$, and constants φ and γ. In this case,

$$j(s) = 1728 \cdot \frac{4\varphi^3}{4\varphi^3 + 27\gamma^2} = j(\tau_F), \tag{2.12}$$

is the constant value. The singular fibers occur at the zeros of $h(s)$, and are all of Kodaira type I_0^*, which corresponds to SO(8) enhanced gauge symmetry. (If the locus $h(s) = 0$ is reducible, there can be more than one SO(8) component.) This construction is equivalent to one made with orientifold planes and can be studied perturbatively (cf. [2,39]) by choosing τ_F near $i\infty$.

Secondly, if we take f to be identically zero, then we end up with $\tau_F = e^{2\pi i/3}$, while thirdly, if we take g to be identically zero, then we find $\tau_F = i$. Various Kodaira fibers and enhanced gauge symmetry groups are possible in these cases. Since τ_F is fixed away from $i\infty$ in these cases, a purely perturbative analysis is not possible.

Our confidence in F-theory is bolstered by F-theory/M-theory duality: after compactifying F-theory on an additional circle, one finds an equivalence with M-theory compactified on the elliptically fibered manifold Z, or more precisely, on the total space $\overline{Z}$ of the Weierstrass fibration.[7] Thus, to get a supersymmetric compactification of F-theory, we require $\overline{Z}$ to be Calabi–Yau, which — thanks to equation (2.8) — happens when $\mathcal{O}(12K_S + \Delta)$ is trivial. Since $\mathcal{O}(\Delta) = \mathcal{O}(12L)$, we should choose $\mathcal{O}(L) = \mathcal{O}(-K_S)$ (possibly up to torsion) to ensure that $\overline{Z}$ is Calabi–Yau (with at most canonical singularities).

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[7]This total space may have singularities, as indicated in table 1, and such singularities in an M-theory compactification give rise to non-abelian gauge symmetries of the compactified theory [41–44]. A non-minimal Weierstrass fibration will have a singularity which is non-canonical, that is, which does not preserve the holomorphic form of top degree on the fibration, and for this reason, such fibrations are not generally allowed when studying compactifications of M-theory or F-theory.

In Section 3.1, we will construct some new non-geometric compactifications of the heterotic strings, and will make use of a similar confidence-building duality: the corresponding F-theory/heterotic duality. In Section 3.1, we explain how those F-theory/heterotic dualities — in the absence of Wilson lines — are much more geometric than had originally been realized. The key insight about those dualities was found by Clingher and Doran [6], based in part on some old work of the second author of this paper [45]; our discussion is based on a refinement of these ideas.

First, though, we need to analyze F-theory models with certain large gauge groups. In anticipation of the duality with the heterotic string (to be reviewed in the next section), we construct F-theory models in dimension 8 and below with gauge groups $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$ or $G = \mathrm{Spin}(32)/\mathbb{Z}_2$. In eight dimensions, this amounts to giving an elliptic fibration $Z_G \to \mathbb{P}^1$ with gauge symmetry group G.

The Weierstrass model for $Z_{(E_8 \times E_8) \times \mathbb{Z}_2}$ was essentially given in [38] (see also [46]): there must be two fibers of Kodaira type II*. By choosing an appropriate coordinate σ on the base $\mathbb{P}^1$, we can assume that these fibers are located at $\sigma = 0$ and $\sigma = \infty$; the equation then takes the form

$$Y^2 = X^3 + a\sigma^4 X + b\sigma^5 + c\sigma^6 + d\sigma^7, \tag{2.13}$$

for some constants a, b, c and d. We review the argument for this in Appendix A. Note that the discriminant of equation (2.13) is

$$\Delta = \sigma^{10}\left(4a^3\sigma^2 + 27\left(b + c\sigma + d\sigma^2\right)^2\right), \tag{2.14}$$

since the (affine) degree of the discriminant in σ is 14, there is an implicit zero of order 10 at $\sigma = \infty$, the location of the second fiber of type II*. To prevent the zeros at $\sigma = 0$ and $\sigma = \infty$ from having order greater than 10 (which would lead to a non-minimal Weierstrass model), we should assume that neither b nor d is zero.

To obtain the Weierstrass model for $Z_{\mathrm{Spin}(32)/\mathbb{Z}_2}$, we need a fiber of type I^*_{12} and a Mordell–Weil group of $\mathbb{Z}_2$ (see [47, 48]). Note that by choosing an appropriate coordinate s on the base, we can assume that the fiber of type I^*_{12} is located at $s = \infty$. In this case, rather than using the traditional Weierstrass equation, we change coordinates so that the point of order 2 on the elliptic curves (which corresponds to the $\mathbb{Z}_2$ factor in the Mordell–Weil group) is at $x = 0$. Then, as we review in Appendix A, the equation takes

the form
$$y^2 = x^3 + (p_0 s^3 + p_1 s^2 + p_2 s + p_3)x^2 + \varepsilon x, \tag{2.15}$$

with discriminant
$$\Delta = -\varepsilon^2 (p(s)^2 - 4\varepsilon), \tag{2.16}$$

where
$$p(s) = p_0 s^3 + p_1 s^2 + p_2 s + p_3. \tag{2.17}$$

To ensure that the gauge group is precisely $\mathrm{Spin}(32)/\mathbb{Z}_2$, we must assume that neither ε nor p_0 is zero.

Remarkably, these two elliptically fibered $K3$ surfaces $Z_{(E_8 \times E_8) \rtimes \mathbb{Z}_2}$ and $Z_{\mathrm{Spin}(32)/\mathbb{Z}_2}$ are birational to each other if the coefficients are identified properly; we will make use of these birational equivalences in our constructions in the next section. If we start with the Weierstrass model $\overline{Z}_{(E_8 \times E_8) \rtimes \mathbb{Z}_2}$ given by equation (2.13) with $d \neq 0$, we can make a birational change to get to another $K3$ surface: let $X = x^2 s/d^2$, $Y = x^2 y/d^3$, $\sigma = x/d$, and multiply the equation by d^6/x^4, to obtain

$$y^2 = x^2 s^3 + a x^2 s + b d x + c x^2 + x^3. \tag{2.18}$$

This has the form of equation (2.15) with

$$p(s) = s^3 + as + c \quad \text{and} \quad \varepsilon = bd. \tag{2.19}$$

Conversely, if we start with the Weierstrass model $\overline{Z}_{\mathrm{Spin}(32)/\mathbb{Z}_2}$ described by equation (2.15) and assume $p_0 \neq 0$, setting $x = \sigma$, $y = Y/p_0 \sigma^2$, $s = \widehat{X}/p_0 \sigma^2$ and multiplying by $p_0^2 \sigma^4$ we find

$$Y^2 = p_0^2 \sigma^7 + \widehat{X}^3 + p_1 \sigma^2 \widehat{X}^2 + p_0 p_2 \sigma^4 \widehat{X} + p_0^2 p_3 \sigma^6 + p_0^2 \varepsilon \sigma^5. \tag{2.20}$$

To put this into Weierstrass form we need one more change of variables, completing the cube via $\widehat{X} = X - \frac{1}{3} p_1 \sigma^2$:

$$Y^2 = X^3 + \left(p_0 p_2 - \tfrac{1}{3} p_1^2\right) \sigma^4 X + p_0^2 \varepsilon \sigma^5 + \left(\tfrac{2}{27} p_1^3 - \tfrac{1}{3} p_0 p_1 p_2 + p_0^2 p_3\right) \sigma^6 + p_0^2 \sigma^7. \tag{2.21}$$

This has the form of equation (2.13) with

$$\begin{aligned} a &= p_0 p_2 - \tfrac{1}{3} p_1^2, \\ b &= p_0^2 \varepsilon, \\ c &= \tfrac{2}{27} p_1^3 - \tfrac{1}{3} p_0 p_1 p_2 + p_0^2 p_3, \\ d &= p_0^2. \end{aligned} \tag{2.22}$$

Table 2: The transformation properties of the coefficients in (2.13).

$a\sigma^4$	$\mathcal{O}(-4K_S) = \mathcal{O}\left(4\Sigma_0 + 4\Sigma_\infty + \varphi^*(-4K_B)\right)$
$b\sigma^5$	$\mathcal{O}(-6K_S) = \mathcal{O}\left(5\Sigma_0 + 7\Sigma_\infty + \varphi^*(-6K_B + \Lambda_{(E_8 \times E_8) \rtimes \mathbb{Z}_2})\right)$
$c\sigma^6$	$\mathcal{O}(-6K_S) = \mathcal{O}\left(6\Sigma_0 + 6\Sigma_\infty + \varphi^*(-6K_B)\right)$
$d\sigma^7$	$\mathcal{O}(-6K_S) = \mathcal{O}\left(7\Sigma_0 + 5\Sigma_\infty + \varphi^*(-6K_B - \Lambda_{(E_8 \times E_8) \rtimes \mathbb{Z}_2})\right)$

The existence of these birational isomorphims between the Weierstrass models $\overline{Z}_{(E_8 \times E_8) \rtimes \mathbb{Z}_2}$ and $\overline{Z}_{\mathrm{Spin}(32)/\mathbb{Z}_2}$ implies that the corresponding non-singular surfaces $Z_{(E_8 \times E_8) \rtimes \mathbb{Z}_2}$ and $Z_{\mathrm{Spin}(32)/\mathbb{Z}_2}$ are isomorphic; however, the isomorphism does not preserve the elliptic fibrations. Thus, if M-theory is compactified on either of these non-singular surfaces, the resulting seven-dimensional theory will have two distinct F-theory limits, corresponding to these two different elliptic fibrations (with section) on the surface.

For both gauge groups G, we can extend the above construction to a broader class of F-theory models by considering F-theory on a base S which is a $\mathbb{P}^1$-bundle over some space B. We can express S in the form $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(\Lambda_G))$ for some line bundle $\mathcal{O}(\Lambda_G)$ on B, with projection map $\varphi : S \to B$, and regard σ and s as sections of the appropriate $\mathcal{O}(\Lambda_G)$. If $\Sigma_0 \subset S$ is the divisor where $\sigma = 0$ in the first case (or $s = 0$ in the second case), and $\Sigma_\infty \subset S$ is the divisor where $\sigma = \infty$ in the first case (or $s = \infty$ in the second case), then $\mathcal{O}(\Sigma_\infty - \Sigma_0) = \varphi^*\mathcal{O}(\Lambda_G)$ and we can write

$$\begin{aligned} \mathcal{O}(-K_S) &= \mathcal{O}(\Sigma_0 + \Sigma_\infty + \varphi^*(-K_B)) \\ &= \mathcal{O}(2\Sigma_0 + \varphi^*(-K_B + \Lambda_G)). \end{aligned} \qquad (2.23)$$

This is the line bundle which we use to build an F-theory model whose Weierstrass fibration $\overline{Z}$ is Calabi–Yau.

In the case $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$, we get a Weierstrass equation of the form equation (2.13). To determine how the various coefficients in that equation transform, we illustrate in table 2 various forms of the appropriate line bundles. It follows that a, b, c, d are sections of

$$\begin{aligned} \mathcal{O}(-4K_B), \ \mathcal{O}(-6K_B + \Lambda_{(E_8 \times E_8) \rtimes \mathbb{Z}_2}), \ \mathcal{O}(-6K_B), \\ \mathcal{O}(-6K_B - \Lambda_{(E_8 \times E_8) \rtimes \mathbb{Z}_2}), \end{aligned} \qquad (2.24)$$

respectively.

Similarly, in the case of $G = \mathrm{Spin}(32)/\mathbb{Z}_2$, we get a Weierstrass equation of the form equation (2.15), whose coefficients are analyzed in table 3. It

Table 3: The transformation properties of the coefficients in (2.15).

$p_0 s^3$	$\mathcal{O}(-2K_S) = \mathcal{O}\left(3\Sigma_0 + \Sigma_\infty + \varphi^*(-2K_B - \Lambda_{\mathrm{Spin}(32)/\mathbb{Z}_2})\right)$
$p_1 s^2$	$\mathcal{O}(-2K_S) = \mathcal{O}\left(2\Sigma_0 + 2\Sigma_\infty + \varphi^*(-2K_B)\right)$
$p_2 s$	$\mathcal{O}(-2K_S) = \mathcal{O}\left(\Sigma_0 + 3\Sigma_\infty + \varphi^*(-2K_B + \Lambda_{\mathrm{Spin}(32)/\mathbb{Z}_2})\right)$
p_3	$\mathcal{O}(-2K_S) = \mathcal{O}\left(4\Sigma_\infty + \varphi^*(-2K_B + 2\Lambda_{\mathrm{Spin}(32)/\mathbb{Z}_2})\right)$
ε	$\mathcal{O}(-4K_S) = \mathcal{O}\left(8\Sigma_\infty + \varphi^*(-4K_B + 4\Lambda_{\mathrm{Spin}(32)/\mathbb{Z}_2})\right)$

follows that $(p_0, p_1, p_2, p_3, \varepsilon)$ are sections of

$$\mathcal{O}(-2K_B - \Lambda_{\mathrm{Spin}(32)/\mathbb{Z}_2}), \ \mathcal{O}(-2K_B), \ \mathcal{O}(-2K_B + \Lambda_{\mathrm{Spin}(32)/\mathbb{Z}_2}),$$
$$\mathcal{O}(-2K_B + 2\Lambda_{\mathrm{Spin}(32)/\mathbb{Z}_2}), \tag{2.25}$$

and $\mathcal{O}(-4K_B + 4\Lambda_{\mathrm{Spin}(32)/\mathbb{Z}_2})$, respectively.

Notice that the birational equivalence between the two models also extends to this higher-dimensional context, once we identify the line bundles correctly. Starting from $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$ using an arbitrary line bundle $\mathcal{O}(\Lambda_{(E_8 \times E_8) \rtimes \mathbb{Z}_2})$, we get a dual model with line bundle

$$\mathcal{O}(\Lambda_{\mathrm{Spin}\,32/\mathbb{Z}_2}) = \mathcal{O}(-2K_B), \tag{2.26}$$

compatible with equation (2.19). Conversely, starting from $G = \mathrm{Spin}\,32/\mathbb{Z}_2$ and an arbitrary line bundle $\mathcal{O}(\Lambda_{\mathrm{Spin}\,32/\mathbb{Z}_2})$, we get a dual model with line bundle

$$\mathcal{O}(\Lambda_{(E_8 \times E_8) \rtimes \mathbb{Z}_2}) = \mathcal{O}(-2K_B + 2\Lambda_{\mathrm{Spin}\,32/\mathbb{Z}_2}), \tag{2.27}$$

compatible with equation (2.22).

2.3   F-theory/heterotic dualities

The duality between F-theory and the heterotic string in dimension 8, originally proposed by Vafa [1], takes the following form when the heterotic gauge group is unbroken: for heterotic gauge group G, there is a family of elliptically fibered $K3$ surfaces $(X_G)_z$ (with section) parameterized by,

$$z \in \mathrm{SO}(2, 2; \mathbb{Z}) \backslash \mathrm{SO}(2, 2) / \mathrm{SO}(2) \times \mathrm{SO}(2), \tag{2.28}$$

and a family of heterotic string vacua $(Y_G)_z$ with gauge group G, such that F-theory on $(X_G)_z$ is dual to the heterotic string vacuum $(Y_G)_z$.

The data needed to specify the heterotic vacuum $(Y_G)_z$ consists of a flat metric and a $\mathcal{B}$-field on a two-torus.[8] There is a unique complex structure compatible with any given metric, so this data can be expressed as an elliptic curve E (i.e., a two-torus equipped with complex structure), as well as a Kähler class and $\mathcal{B}$-field on E. These latter two can be combined into the complex number ρ, defined in (1.1), which naturally lives in the upper half-plane and is invariant under the $\mathrm{SL}(2,\mathbb{Z})$ action. Similarly, the complex structure on E can be represented by a complex number τ in the upper half-plane, modulo $\mathrm{SL}(2,\mathbb{Z})$. The duality between F-theory and the heterotic string suggests that for each F-theory vacuum with gauge group G, τ and ρ should be expressible as functions of the coefficients of (2.13) or (2.15).

One should note that much of the discussion in the literature, including the analysis in [37,38], is limited to a particular limit, in which $(a^3/bd) \to \infty$ and $(c^2/bd) \to \infty$ while (c^2/a^3) remains finite.[9] As we will explain shortly, from the heterotic point of view this is equivalent to taking the large volume limit $\rho \to i\infty$, where the heterotic supergravity description is good. From the point of view of F-theory there is nothing special about this limit. One could consider generic values of (a^3/bd) and (c^2/bd) in $\mathbb{C}$, in which case the heterotic torus T^2 has some finite size and complex structure. As we will see, the fibered version of this case corresponds to non-geometric heterotic compactifications.

In fact, the explicit correspondence between F-theory and heterotic parameters in 8 dimensions was calculated in the case of $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$ in the early days of F-theory [50,51]. In the notation of the present paper,[10]

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[8]Since G is unbroken, all Wilson line expectation values must vanish.

[9]Of course, there are instances where two different limits of this kind are taken in order to study a duality. This was done for example in [37] which studied the duality of [49].

[10]To compare the two, one must make the substitution

$$X = b^{7/6}d^{-5/6}\widetilde{X},\, Y = b^{7/4}d^{-7/4}\widetilde{Y},\, \sigma = b^{1/2}d^{-1/2}\widetilde{\sigma}, \tag{2.29}$$

in equation (2.13) and multiply by $d^{5/2}b^{-7/2}$ to obtain

$$\widetilde{Y}^2 = \widetilde{X}^3 + ab^{-1/3}d^{-1/3}\widetilde{\sigma}^4\widetilde{X} + \widetilde{\sigma}^5 + cb^{-1/2}d^{-1/2}\widetilde{\sigma}^6 + \widetilde{\sigma}^7. \tag{2.30}$$

the authors of [50, 51] found:[11]

$$j(\tau)j(\rho) = -1728^2 \frac{a^3}{27bd}, \tag{2.31}$$

$$(j(\tau) - 1728)\,(j(\rho) - 1728) = 1728^2 \frac{c^2}{4bd}. \tag{2.32}$$

which implies that

$$\frac{c^2}{a^3} = -\left(1 - \frac{1728}{j(\tau)}\right)\left(1 - \frac{1728}{j(\rho)}\right); \tag{2.33}$$

the large volume heterotic limit $j(\rho) \to \infty$ thus corresponds to $(a^3/bd) \to \infty$ and $(c^2/bd) \to \infty$ while (c^2/a^3) remains finite.

Analogous formulas were found much more recently [6] for the case $G = $ Spin $32/\mathbb{Z}_2$. Our goal in this subsection is to refine these formulae in both cases, and to give a much more geometric explanation of them.

As stressed in Section 2.1, the heterotic elliptic curve E naturally encodes the information provided by the modular function τ. Similarly, since ρ is also an $\mathrm{SL}(2,\mathbb{Z})$ modular function, we can encode the information it provides in a second elliptic curve F. In dimension 8 this is not so crucial, but when we go to lower dimension, and want to use τ and ρ as fields which can vary in the compactification to lower dimension (exploiting the $\mathrm{SL}(2,\mathbb{Z})$ symmetry), this is an important step.

As Clingher and Doran [6] showed, the geometric connection between the heterotic and F-theory sides of this story is provided (in the absence of Wilson lines) by the notion of a Shioda–Inose structure for $K3$ surfaces. Following [45], we say that a $K3$ surface Z has a *Shioda–Inose structure* if there is an automorphism $\iota : Z \to Z$ of order two, preserving the holomorphic 2-form, and a complex torus A of complex dimension 2, such that Z/ι is birationally isomorphic to the Kummer surface $A/(-1)$. This definition was motivated by work of Shioda and Inose who considered such structures in special cases [53, 54].

The main theorem of [45] (combined with some known facts about the Néron–Severi group of a complex torus [55, 56]) implies that the $K3$ surfaces Z_G constructed in section 2.2 have Shioda–Inose structures with the complex torus taking the form $E \times F$ for two elliptic curves E and F. This is the

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[11]These same formulas were independently discovered in the mathematics literature in a slightly different context [52].

geometric form of F-theory/heterotic duality: the elliptic curves E and F associated to Z_G provide the data for the heterotic vacuum.

Clingher and Doran [6] have constructed the Shioda–Inose structure for $Z_{\text{Spin } 32/\mathbb{Z}_2}$ in a very explicit manner, and we refine their result in Appendix B. The result is stated in the opposite direction from the discussion above: starting with Weierstrass equations

$$v^2 = u^3 + \lambda_2 u + \lambda_3 \quad \text{and} \quad w^2 = z^3 + \mu_2 z + \mu_3, \qquad (2.34)$$

defining two elliptic curves E and F, respectively, the equation for the associated F-theory (Weierstrass) elliptic fibration $\overline{Z}_{\text{Spin } 32/\mathbb{Z}_2}$ is given by

$$y^2 = x^3 + (s^3 - 3\lambda_2\mu_2 s - \tfrac{27}{2}\lambda_3\mu_3)x^2 + \tfrac{1}{16}(4\lambda_2^3 + 27\lambda_3^2)(4\mu_2^3 + 27\mu_3^2)x. \tag{2.35}$$

In fact, letting $\iota_{\text{Spin } 32/\mathbb{Z}_2}$ be the automorphism of $\overline{Z}_{\text{Spin } 32/\mathbb{Z}_2}$ defined by translation by the point of order 2 in the Mordell–Weil group, the quotient $Z_{\text{Spin } 32/\mathbb{Z}_2}/\iota_{\text{Spin } 32/\mathbb{Z}_2}$ is birationally isomorphic to the Kummer surface $(E \times F)/(-1)$. (See Appendix B for the details of this.)

From this, and the birational equivalence we found between $\overline{Z}_{\text{Spin } 32/\mathbb{Z}_2}$ and $\overline{Z}_{(E_8 \times E_8) \rtimes \mathbb{Z}_2}$, we can find a model for the $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$ case as well. This time, we need to choose two factorizations

$$\begin{aligned} \frac{1}{4}(4\lambda_2^3 + 27\lambda_3^2) &= b(\lambda)d(\lambda), \\ \frac{1}{4}(4\mu_2^3 + 27\mu_3^2) &= b(\mu)d(\mu), \end{aligned} \tag{2.36}$$

and then the equation of $\overline{Z}_{(E_8 \times E_8) \rtimes \mathbb{Z}_2}$ takes the form

$$Y^2 = X^3 - 3\lambda_2\mu_2\sigma^4 X + b(\lambda)b(\mu)\sigma^5 - \frac{27}{2}\lambda_3\mu_3\sigma^6 + d(\lambda)d(\mu)\sigma^7. \tag{2.37}$$

In this case, the Shioda–Inose structure is induced by the automorphism $\iota_{(E_8 \times E_8) \rtimes \mathbb{Z}_2}$ which acts on the base of the elliptic fibration to exchange the two fibers of type II*, and acts on the fiber by multiplication by -1; it can be written as

$$\iota_{(E_8 \times E_8) \rtimes \mathbb{Z}_2} : (X, Y, \sigma) \mapsto \left(\frac{b^2 X}{d^2\sigma^4}, \frac{-b^3 Y}{d^3\sigma^6}, \frac{b}{d\sigma} \right), \tag{2.38}$$

where $b = b(\lambda)b(\mu)$ and $d = d(\lambda)d(\mu)$. Once again, the quotient $Z_{(E_8 \times E_8) \rtimes \mathbb{Z}_2}/\iota_{(E_8 \times E_8) \rtimes \mathbb{Z}_2}$ is birationally isomorphic to the Kummer surface $(E \times F)/(-1)$.

Let us verify that equations (2.31) and (2.32) are satisfied for this elliptic fibration. Since $bd = b(\lambda)d(\lambda)b(\mu)d(\mu)$ we have

$$-1728^2 \frac{a^3}{27bd} = -1728^2 \frac{(-3\lambda_2\mu_2)^3}{\frac{27}{16}(4\lambda_2^3 + 27\lambda_3^2)(4\mu_2^3 + 27\mu_3^2)}$$

$$= 1728^2 \frac{(4\lambda_2^3)(4\mu_2^3)}{(4\lambda_2^3 + 27\lambda_3^2)(4\mu_2^3 + 27\mu_3^2)} = j(\tau)j(\rho), \qquad (2.39)$$

using equation (2.4), and

$$1728^2 \frac{c^2}{4bd} = 1728^2 \frac{(-\frac{27}{2}\lambda_3\mu_3)^2}{\frac{4}{16}(4\lambda_2^3 + 27\lambda_3^2)(4\mu_2^3 + 27\mu_3^2)}$$

$$= (-1728)^2 \frac{(27\lambda_3^2)(27\mu_3^2)}{(4\lambda_2^3 + 27\lambda_3^2)(4\mu_2^3 + 27\mu_3^2)}$$

$$= (j(\tau) - 1728)(j(\rho) - 1728), \qquad (2.40)$$

using equation (2.5), verifying the formulas derived in [50].

3 Non-geometric heterotic models

3.1 Constructing non-geometric heterotic models

In this section, we wish to use the duality we have analyzed to construct F-theory duals to various heterotic models. We begin with an elliptically fibered space $\mathcal{E} \to B$, and consider the heterotic string on this space with unbroken gauge group. Maintaining unbroken gauge group requires two things: all Wilson lines must be trivial, and all instantons must be point-like.[12] The complex structure on the total space $\mathcal{E}$ determines complex structures on the elliptic fibers, but the complexified Kähler class on the fiber is left undetermined.

For simplicity, we assume that the elliptic fibration $\mathcal{E} \to B$ has a section, but in principle our construction can be made without that requirement. Under this assumption, $\mathcal{E}$ can be described by an equation

$$v^2 = u^3 + \lambda_2 u + \lambda_3, \qquad (3.1)$$

where λ_2 and λ_3 are sections of appropriate line bundles $\mathcal{O}(4L_\tau)$ and $\mathcal{O}(6L_\tau)$ on B. Note that to completely specify the geometry, we must also specify

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[12]There is also the possibility of "hidden obstructors" which do not break the gauge group [48], but we do not consider those here.

the locations of the point-like instantons on the space $\mathcal{E}$; we will return to this point later.

To build a (possibly) non-geometric model, we wish to allow the complexified Kähler parameter to be a non-constant function ρ on the base B. Strictly speaking, there will be some defect locus Δ_ρ at points of which either ρ is multiple-valued or ρ approaches infinity, so that ρ is only well-defined on $B - \Delta_\rho$. Moreover, there is an $SL(2, \mathbb{Z})$ ambiguity of ρ, so even on $B - \Delta_\rho$, ρ is only locally well-defined.

Hellerman *et al.* [12] took a "stringy cosmic string" point of view [29] in specifying the function ρ, but here we do something much closer in spirit to the construction of F-theory: we specify ρ via an auxiliary elliptic fibration $\pi_\rho : \mathcal{F} \to B$, so that the periods of the elliptic curve $\pi^{-1}(b)$ are $\mathbb{Z} \oplus \mathbb{Z}\rho(b)$. Just as in F-theory, in order to specify ρ in this way, we can assume that $\pi_\rho : \mathcal{F} \to B$ has a section. Thus, $\mathcal{F}$ will have a Weierstrass equation:

$$w^2 = z^3 + \mu_2 z + \mu_3, \tag{3.2}$$

where μ_2 and μ_3 are sections of appropriate line bundles $\mathcal{O}(4L_\rho)$ and $\mathcal{O}(6L_\rho)$ on the base B.

Because our construction does not necessarily have a large radius limit where supergravity techniques can be employed, we will derive certain restrictions on the families $\mathcal{E}$ and $\mathcal{F}$ indirectly via duality with F-theory. The restrictions to which we refer are the analogues of the restriction that the total space of $\mathcal{E}$ be Calabi–Yau if ρ is constant. In the $\mathrm{Spin}\,32/\mathbb{Z}_2$ case, the F-theory dual is given by equation (2.35), where now the coefficients $p_0, \ldots, p_3, s$ are considered as sections of appropriate line bundles. Comparing line bundles, we see that

$$
\begin{aligned}
\mathcal{O} &= \mathcal{O}(-2K_B - \Lambda_{\mathrm{Spin}\,32/\mathbb{Z}_2}), \\
\mathcal{O}(4L_\tau + 4L_\rho) &= \mathcal{O}(-2K_B + \Lambda_{\mathrm{Spin}\,32/\mathbb{Z}_2}), \\
\mathcal{O}(6L_\tau + 6L_\rho) &= \mathcal{O}(-2K_B + 2\Lambda_{\mathrm{Spin}\,32/\mathbb{Z}_2}), \\
\mathcal{O}(12L_\tau + 12L_\rho) &= \mathcal{O}(-4K_B + 4\Lambda_{\mathrm{Spin}\,32/\mathbb{Z}_2}),
\end{aligned}
\tag{3.3}
$$

where the first relation comes from the fact that p_0 is non-vanishing. Thus, $\mathcal{O}(\Lambda_{\mathrm{Spin}\,32/\mathbb{Z}_2}) = \mathcal{O}(-2K_B)$ and $\mathcal{O}(L_\tau + L_\rho) = \mathcal{O}(-K_B)$ (up to torsion).

It follows that for a given base B, we will be able to construct a non-geometric compactification for the $\mathrm{Spin}\,32/\mathbb{Z}_2$ heterotic string out of any two elliptic fibrations $\pi_\tau : \mathcal{E} \to B$ and $\pi_\rho : \mathcal{F} \to B$, provided that the associated

line bundles $\mathcal{O}(L_\tau)$ and $\mathcal{O}(L_\rho)$ satisfy

$$\mathcal{O}(L_\tau + L_\rho) = \mathcal{O}(-K_B), \tag{3.4}$$

up to torsion.

We can also find an F-theory dual in the case of the $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ heterotic string. For this, we need to specify a factorization of ε into bd, where b and d are sections of appropriate line bundles. Since ε is itself a product, this is accomplished by two factorizations:

$$\begin{aligned}
\tfrac{1}{4}(4\lambda_2^3 + 27\lambda_3^2) &= b(\lambda)d(\lambda), \\
\tfrac{1}{4}(4\mu_2^3 + 27\mu_3^2) &= b(\mu)d(\mu).
\end{aligned} \tag{3.5}$$

In other words (considering the vanishing loci), the discriminant locus Δ_τ of the first fibration is decomposed into two divisors $\Delta'_\tau = \{b(\lambda) = 0\}$ and $\Delta''_\tau = \{d(\lambda) = 0\}$, and similarly for Δ_ρ. It follows that $b(\lambda), d(\lambda), b(\mu), d(\mu)$ are sections of

$$\mathcal{O}(\Delta'_\tau), \mathcal{O}(\Delta''_\tau), \mathcal{O}(\Delta'_\rho), \mathcal{O}(\Delta''_\rho), \tag{3.6}$$

respectively. We can write the equation for the F-theory dual in the form[13]

$$Y^2 = X^3 - 3\lambda_2\mu_2\sigma^4 X + b(\lambda)b(\mu)\sigma^5 - \tfrac{27}{2}\lambda_3\mu_3\sigma^6 + d(\lambda)d(\mu)\sigma^7. \tag{3.9}$$

Again, we can determine line bundles from coefficients:

$$\begin{aligned}
\mathcal{O}(4L_\tau + 4L_\rho) &= \mathcal{O}(-4K_B), \\
\mathcal{O}(\Delta'_\tau + \Delta'_\rho) &= \mathcal{O}(-6K_B + \Lambda_{(E_8 \times E_8) \rtimes \mathbb{Z}_2}), \\
\mathcal{O}(6L_\tau + 6L_\rho) &= \mathcal{O}(-6K_B), \\
\mathcal{O}(\Delta''_\tau + \Delta''_\rho) &= \mathcal{O}(-6K_B - \Lambda_{(E_8 \times E_8) \rtimes \mathbb{Z}_2}).
\end{aligned} \tag{3.10}$$

Note that

$$\mathcal{O}(\Delta''_\tau + \Delta''_\rho) = \mathcal{O}(12L_\tau - \Delta'_\tau + 12L_\rho - \Delta'_\rho) = \mathcal{O}(-12K_B - \Delta'_\tau - \Delta'_\rho), \tag{3.11}$$

so the second and fourth equations above are equivalent.

---

[13]Here we are using the fact that

$$Y^2 = X^3 + a\sigma^4 X + b\sigma^5 + c\sigma^6 + d\sigma^7 \tag{3.7}$$

is birational to

$$y^2 = x^3 + (s^3 + as + c)x^2 + bdx. \tag{3.8}$$

It follows that (up to torsion):

$$
\begin{aligned}
\mathcal{O}(L_\tau + L_\rho) &= \mathcal{O}(-K_B), \\
\mathcal{O}(\Delta'_\tau + \Delta'_\rho) &= \mathcal{O}(-6K_B + \Lambda_{(E_8 \times E_8) \rtimes \mathbb{Z}_2}).
\end{aligned}
\tag{3.12}
$$

Thus, for a given base B, we will be able to construct a non-geometric compactification for the $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ heterotic string out of any two elliptic fibrations $\pi_\tau : \mathcal{E} \to B$ and $\pi_\rho : \mathcal{F} \to B$, together with decompositions of their discriminant divisors

$$
\Delta_\tau = \Delta'_\tau + \Delta''_\tau \quad \text{and} \quad \Delta_\rho = \Delta'_\rho + \Delta''_\rho,
\tag{3.13}
$$

provided that the associated line bundles $\mathcal{O}(L_\tau)$ and $\mathcal{O}(L_\rho)$ satisfy

$$
\mathcal{O}(L_\tau + L_\rho) = \mathcal{O}(-K_B),
\tag{3.14}
$$

up to torsion.

3.2 Compactifications to six dimensions

In six dimensions, it is possible to choose $B = T^2$ with $\mathcal{O}(\Lambda_{\mathrm{Spin}\,32/\mathbb{Z}_2})$, $\mathcal{O}(L_\tau)$ and $\mathcal{O}(L_\rho)$ all being torsion line bundles. This leads to the familiar compactification on $T^2 \times T^2$, or orbifolds thereof, and is not a case we will analyze in detail. In particular, both τ and ρ are constant in this case, and the heterotic model is geometric.

The other possibility in six dimensions is $B = \mathbb{P}^1$, and there are then three cases (bearing in mind that the Picard group has no torsion in this case), stemming from the formula $\mathcal{O}(L_\tau + L_\rho) = \mathcal{O}(-K_B)$, together with the fact that $\mathcal{O}(4L_\tau)$, $\mathcal{O}(6L_\tau)$, $\mathcal{O}(4L_\rho)$, and $\mathcal{O}(6L_\rho)$ all have sections:

1. $\mathcal{O}(L_\tau)$ has degree 2 which implies that $\mathcal{O}(L_\rho)$ is trivial and hence that ρ is constant. This is a geometric model in which $\mathcal{E}$ is an elliptically fibered $K3$ surface.
2. $\mathcal{O}(L_\tau)$ and $\mathcal{O}(L_\rho)$ each have degree one. This implies that both $\mathcal{E}$ and $\mathcal{F}$ are rational elliptic surfaces.[14] Both τ and ρ are non-constant; these are the Hellerman–McGreevy–Williams models.
3. $\mathcal{O}(L_\tau)$ is trivial, and $\mathcal{O}(L_\rho)$ has degree 2. In this case, τ is constant but ρ varies; this is (fiberwise) mirror symmetric to case (1).

---

[14]These are sometimes called "dP_9 surfaces."

In case (1), we recover the familiar geometric compactifications and their known F-theory duals. There is one additional feature of these models which we now spell out in detail: in order for the heterotic gauge group to remain unbroken, all instantons must be point-like, and as such, each must be located at a particular point on the heterotic side. As our basic construction shows, the complex structure of the F-theory model is determined by the ρ and τ data on the heterotic side, and appears to be independent of the location of the small instantons. However, each complex structure modulus on the F-theory side is part of a hypermultiplet which includes an additional complex scalar, and it is those scalars which dictate the locations of the small instantons. In a typical vacuum, the expectation values of those scalars vanish, so one would expect there to be a preferred location for small instantons.

In the Spin $32/\mathbb{Z}_2$ case, the physics of small instantons was described by Witten [57]; Aspinwall [48] used this analysis to identify the corresponding features of F-theory: small Spin $32/\mathbb{Z}_2$ instantons correspond to zeros of the coefficient ε in the basic equation (2.15). Aspinwall also gave an explanation of the zeros of p_0 in (2.15): they correspond to "hidden obstructors" [58] which occur at singular points of the heterotic $K3$ surface. As already mentioned, we do not consider hidden obstructors in our analysis and in fact we have set $p_0 = 1$.

Since we have

$$\varepsilon = \tfrac{1}{16}(4\lambda_2^3 + 27\lambda_3^2)(4\mu_2^3 + 27\mu_3^2), \qquad (3.15)$$

and since $4\mu_2^3 + 27\mu_3^2$ is constant in case (1), we see that the zeros of ε correspond to the singular fibers of the elliptic fibration $\mathcal{E} \to B$ (whose total space is the heterotic $K3$ surface). It is natural to suppose that the small instantons must be located along those singular fibers; in fact, the most natural place to locate these small instantons is at the singular points of the singular fibers. Similar remarks apply to case (3), using $\mathcal{F} \to B$ instead of $\mathcal{E} \to B$.

For the non-geometric compactifications, the zeros of ε correspond to places where *either* the fiber of $\mathcal{E} \to B$ is singular, *or* the fiber of $\mathcal{F} \to B$ is singular. The geometric part of this compactification is captured by $\mathcal{E} \to B$, whose total space is the heterotic rational elliptic surface: the bundle on this surface should have 12 small instantons, so we can again locate them at the singular points of fibers of $\mathcal{E} \to B$. The additional zeros of ε correspond to singular fibers of $\mathcal{F} \to B$ and don't have a straightforward geometric interpretation.

A similar analysis applies when the gauge group is $(E_8 \times E_8) \rtimes \mathbb{Z}_2$. This time, the small instantons involve tensionless strings [59,60], and in case (1)

we must choose how to distribute 24 small instantons between the two E_8 factors. In the F-theory interpretation [37], the zeros of the coefficients b and d in the basic equation (2.13) correspond to the small instantons in the two different E_8 factors. In our construction, the factorizations (3.5) show that the zeros of b and d, together, correspond to the singular fibers of the two elliptic fibrations $\mathcal{E} \to B$ and $\mathcal{F} \to B$. Thus, in case (1) the 24 singular fibers of $\mathcal{E} \to B$ get divided into two groups, corresponding to the two E_8 factors. As in the $\mathrm{Spin}\, 32/\mathbb{Z}_2$ case, we propose that the small instantons should be located at the singular points of those singular fibers. Case (3) is similar, with the roles of $\mathcal{E} \to B$ and $\mathcal{F} \to B$ reversed.

In case (2), the singular fibers of $\mathcal{E} \to B$ get divided into two groups, according to (3.5), and the singular fibers of $\mathcal{F} \to B$ likewise get divided. The total space of $\mathcal{E} \to B$ is the rational elliptic surface upon which we are compactifying the heterotic string, and we locate 12 small instantons at the singular fibers of $\mathcal{E} \to B$, divided into two groups as in (3.5). The additional zeros, corresponding to singular fibers of $\mathcal{F} \to B$, again do not have a straightforward geometric interpretation.

One interesting thing to note is that no new F-theory models were required in six dimensions to provide duals for non-geometric compactifications: all of the duals to non-geometric compactifications are in the same class of F-theory models as the duals to *geometric* compactifications, although presumably the dualities are occurring at different locations in the hyper-multiplet moduli space. In four dimensions, some of the semi-classical moduli are lifted by fluxes [3], so there may indeed be different F-theory models for geometric and non-geometric compactifications in that dimension.

3.3 Compactifications to four dimensions: an example

The general procedure described in Section 3.1 can also be used to construct examples in dimension four, which on the F-theory side will involve $K3$-fibered Calabi–Yau four-folds. These are much less constrained than was the corresponding set of $K3$-fibered Calabi–Yau 3-folds which we used in the previous section, so rather than attempting a general classification we will settle for examples. Our examples are easily generalizable to arbitrary $K3$-fibered Calabi–Yau four-fold and our results are characteristic of the general construction.

The class of examples we are interested in are Calabi–Yau four-folds, $\mathcal{M}_4$, with a $\mathbb{P}^2$ base which admit a $K3$-fibration. Schematically:

$$K3 \longrightarrow \mathcal{M}_4 \quad \text{with} \quad T^2 \longrightarrow K3$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\widetilde{\mathbb{P}}^2 \qquad\qquad\qquad\qquad \mathbb{P}^1$$

The elliptic curve is represented as a hypersurface in $\mathbb{P}^2$ via the vanishing of the Weierstrass equation (2.10). Let $[x, y, z]$ denote the homogeneous coordinates for this $\mathbb{P}^2$, $[t_1, t_2, t_3]$ the homogeneous coordinates for the base $B = \widetilde{\mathbb{P}}^2$, $[s_1, s_2]$ the coordinates for the $\mathbb{P}^1$. We construct a variety S which is fibered over $B = \widetilde{\mathbb{P}}^2$ with fiber $\mathbb{P}^1$, and a variety $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2L) \oplus \mathcal{O}(3L))$ which is fibered over S with fiber $\mathbb{P}^2$. The varieties $\mathbb{P}^1$ and $\mathbb{P}^2$ are fibered over the base $\widetilde{\mathbb{P}}^2$. We do this by lifting the torus action used to construct the base to act on the fiber: the coordinates $[x, y, z]$ and $[s_1, s_2]$ become sections of certain line bundles. Using λ and μ to denote the $\mathbb{C}^*$ actions of the base $\widetilde{\mathbb{P}}^2$ and $\mathbb{P}^1$ respectively we consider

$$[t_1, t_2, t_3] \sim \lambda[t_1, t_2, t_3],$$
$$[s_1, s_2] \sim [\lambda^{A_1}\mu s_1, \lambda^{A_2}\mu s_2],$$
$$[x, y, z] \sim [\lambda^{B_1}\mu^{C_1}x, \lambda^{B_2}\mu^{C_2}y, z], \tag{3.16}$$

with A_i, B_i, and C_i some real positive constants to be determined. Note we have made a basis choice such that the torus action on z is trivial. We require that (3.16) acts consistently on the Weierstrass polynomial (written here in the homogeneous coordinates of $\mathbb{P}^2$):

$$P = -y^2 z + x^3 + z^2 x f(s, t) + z^3 g(s, t), \tag{3.17}$$

and require that the variety defined by $P = 0$ have trivial canonical class; these conditions determine B_i, C_i in terms of A_i. Picking $A_1 = n$, $A_2 = 0$, it is convenient to write the exponents of the three $\mathbb{C}^*$ torus actions defining $\widetilde{\mathbb{P}}^2, \mathbb{P}^1, \mathbb{P}^2$ as a matrix:

$$\begin{pmatrix} t_1 & t_2 & t_3 & s_1 & s_2 & x & y & z \\ 1 & 1 & 1 & n & 0 & 2(3+n) & 3(3+n) & 0 \\ 0 & 0 & 0 & 1 & 1 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}. \tag{3.18}$$

This is precisely the charge matrix of a linear sigma model describing a toric variety [61]. Also $f(s, t)$ has charge $4(3 + n)$ and 8 under the first two $\mathbb{C}^*$ actions, while $g(s, t)$ has charge $6(3 + n)$ and 12. This gives a class of four-folds with a twist labeled by the integer n. We focus on F-theory duals of compactifications of the $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ heterotic string with unbroken

$(E_8 \times E_8) \rtimes \mathbb{Z}_2$; then the twist parameter n corresponds to choosing the line bundle $\mathcal{O}(\Lambda_{(E_8 \times E_8) \rtimes \mathbb{Z}_2})$ to be $\mathcal{O}_{\widetilde{\mathbb{P}}^2}(n)$. In order to get unbroken $(E_8 \times E_8) \rtimes \mathbb{Z}_2$, we need to restrict to $n \leq 3$.

Unbroken $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ implies that the Weierstrass equation takes the special form (2.13) with the coefficients interpreted as sections of appropriate bundles. In our case, this implies that

$$f(s,t) = a(t)s_1^4 s_2^4$$

and

$$g(s,t) = b(t)s_1^5 s_2^7 + c(t)s_1^6 s_2^6 + d(t)s_1^7 s_2^5,$$

where $a(t)$, $b(t)$, $c(t)$ and $d(t)$ are homogeneous of degrees 12, $18 + n$, 18, and $18 - n$, respectively.[15]

To find F-theory duals for geometric or non-geometric heterotic compactifications, following equation (3.14) we must choose line bundles $\mathcal{O}(L_\tau)$ and $\mathcal{O}(L_\rho)$ of degrees d and $3 - d$ (since $\mathcal{O}(-K_B)$ has degree 3), as well as a decomposition of the divisor Δ_τ (of degree $12d$) into two components $\Delta_\tau' + \Delta_\tau''$ of degrees k and $12d - k$, and a decomposition of the divisor Δ_ρ (of degree $36 - 12d$) into two components $\Delta_\rho' + \Delta_\rho''$ of degrees ℓ and $36 - 12d - \ell$, such that $k + \ell = 18 + n$.

The choice of $\mathcal{O}(L_\tau)$ and $\mathcal{O}(L_\rho)$ presents no particular problem for any value of $d \in \{0, 1, 2, 3\}$, so there are a variety of geometric and non-geometric heterotic compactifications with F-theory duals of this kind. In fact, there are equal numbers of geometric and non-geometric models (treating the constant τ models as geometric, even though strictly speaking they are the mirrors of geometric models), so neither type of heterotic compactification is favored. That is, not only are non-geometric models possible, they are just as typical as geometric models.

Note that the choice of decompositions $\Delta_\tau = \Delta_\tau' + \Delta_\tau''$ and $\Delta_\rho = \Delta_\rho' + \Delta_\rho''$, which affects the distribution of instantons (and their non-geometric counterparts) between the two E_8 factors of the gauge group, is trickier: the Weierstrass equations describing the bundles $\mathcal{E}$ and $\mathcal{F}$ must be carefully tuned to guarantee such a decomposition. This does not, however, affect our discussion of typicality for geometric versus non-geometric heterotic compactifications.

---

[15]Note that any model in dimension four with unbroken $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ has confusing aspects, such as an infinite tower of light solitonic states, if $b(t) = 0$ intersects $d(t) = 0$ [62]; such an intersection is unavoidable for our choice $B = \mathbb{P}^2$.

The description of moduli spaces which follows from the above analysis is a semi-classical one, and in general we expect a number of moduli to be lifted by fluxes [3]. As a consequence, we should expect that the F-theory duals of geometric and non-geometric compactifications live on different (quantum) moduli spaces in dimension four. It would be interesting to have a concrete example of this phenomenon.

3.4   Tadpoles and the Bianchi identity

One of the beautiful features of this class of heterotic models is that there is no need for extra ingredients like orientifold planes to construct compact models. Instead, the Bianchi identity for $\mathcal{H}_3$-flux,

$$d\mathcal{H}_3 = \frac{\alpha'}{4} \left(\text{Tr}\,(R \wedge R) - \text{Tr}\,(F \wedge F) \right), \qquad (3.19)$$

includes a higher derivative correction that in the presence of curvature induces a five-brane charge thereby allowing one to construct compact solutions. In the geometric setting, the five-brane charge tadpole must be satisfied by a combination of wrapped NS5-branes and finite-size gauge instantons.

We would like to understand this tadpole in the more general non-geometric setting. This is a subtle question for reasons that we will outline and is really best answered from a world-sheet approach.

Let us first think adiabatically from the perspective of a physicist who has reduced on the torus fiber and observes physics purely on the base B. From the perspective of such an observer, there are two scalar fields τ and ρ with monodromies around divisors of B. If Wilson line moduli were included, they would give additional scalars with the entire collection acted on by the full heterotic T^2 duality group. For simplicity, we will continue to restrict to unbroken maximal gauge symmetry.

Firstly, our usual intuition is that the total ρ monodromy measured around any divisor in B must be trivial for a compact solution. Said differently: the total NS5-brane charge must vanish. It is worth pointing out that this is not required in the heterotic string since the deficit can be made up by the gravitational contribution to the charge.

Secondly, there are really several distinct cases to this adiabatic tadpole analysis depending on whether $\mathcal{H}_3$ has support along the torus fiber. If $\mathcal{H}_3$ has one or two legs along the fiber then the left-hand side of (3.19) becomes intrinsically non-geometric, and would be sourced by some non-geometric

analog of a Pontryagin class. That is, the standard expression $\mathrm{Tr}\,(R \wedge R)$ is not invariant under the $\mathrm{SL}(2, \mathbb{Z})$ action on ρ and so is not well defined. To understand these components of the tadpole requires a world-sheet analysis. While this case does not seem to occur for models with F-theory duals, it would be very interesting to determine whether a non-trivial $d\mathcal{H}_3$ could be sourced this way since it would provide a new kind of non-Kähler solution which locally satisfies the quite restrictive supersymmetry constraints, while solving the Bianchi identity (3.19) via T-duality.

The only component of $d\mathcal{H}_3$ for which we might be able to use our adiabatic picture is when $d\mathcal{H}_3$ is supported completely on B, which requires a compactification to four dimensions or lower. This is the charge for NS5-branes which wrap the torus fiber. We note that ρ monodromies can never create NS5-branes which wrap the fiber. Those branes are always transverse to the torus fiber. So at least for this component, we might hope to treat τ and ρ in a similar fashion.

Let us try a direct attempt to understand this component of the charge tadpole. As a warm up case, let us take a geometric heterotic compactification on an elliptic space $\mathcal{M} \to B$. What we would like to do is express the Chern classes of $\mathcal{M}$ in terms of those of B together with the data defining the elliptic fibration. We can follow an approach used in [63–65]. For this geometric model we can present $\mathcal{M}$ in Weierstrass form as before. Let W be a $\mathbb{P}^2$ bundle over B with homogeneous coordinates $[u, v, w]$ which are sections of

$$\mathcal{O}(1) \otimes \mathcal{O}(2L_\tau), \quad \mathcal{O}(1) \otimes \mathcal{O}(3L_\tau), \quad \mathcal{O}(1), \qquad (3.20)$$

respectively. The line bundle $\mathcal{O}(1)$ is the degree one bundle over the $\mathbb{P}^2$ fiber. To describe $\mathcal{M}$, we consider

$$s = -wv^2 + u^3 + \lambda_2 uw^2 + \lambda_3 w^3 = 0, \qquad (3.21)$$

where λ_2 and λ_3 are sections of the line bundles $\mathcal{O}(4L_\tau)$ and $\mathcal{O}(6L_\tau)$, respectively while s is a section of $\mathcal{O}(3) \otimes \mathcal{O}(6L_\tau)$. In this purely geometric model, we set

$$\mathcal{O}(L_\tau) = \mathcal{O}(-K_B), \qquad (3.22)$$

to ensure that $\mathcal{M}$ is Calabi–Yau.

Let us set $\alpha = c_1(\mathcal{O}(1))$. The cohomology ring of W is then generated over the cohomology ring of B by the addition of α together with the relation,

$$\alpha(\alpha + 2c_1(B))(\alpha + 3c_1(B)) = 0. \qquad (3.23)$$

This relation states that (u, v, w) are not permitted to have any common zeroes. This relation holds in the cohomology ring of W. To restrict to $\mathcal{M}$,

we want to impose $s = 0$ but s is itself a section of a bundle with first Chern class $3(\alpha + 2c_1(B))$. Any class on $\mathcal{M}$ that can be extended to W can be integrated over $\mathcal{M}$ by multiplying by $3(\alpha + 2c_1(B))$.

Now we are ready to compute the Chern classes of $\mathcal{M}$ in terms of those of B. Let C_B denote the total Chern class of B. The total Chern class C_W of W is

$$C_W = C_B \cdot (1 + \alpha)(1 + \alpha + 2c_1(B))(1 + \alpha + 3c_1(B)). \tag{3.24}$$

To get the Chern class of $\mathcal{M}$, we use adjunction:

$$C_{\mathcal{M}} = C_W \cdot \frac{1}{1 + 3(\alpha + 2c_1(B))}. \tag{3.25}$$

To compute the 5-brane tadpole (3.19), we are really interested in $p_1(\mathcal{M})$ so we want to extract $c_1(\mathcal{M})$ and $c_2(\mathcal{M})$ from (3.25). It is easy to check that in this case, $c_1(\mathcal{M}) = 0$ as we expect. On expanding, we find

$$c_2(\mathcal{M}) = c_2(B) + 4\alpha c_1(B) + 11c_1(B)^2. \tag{3.26}$$

How might this computation generalize to include ρ monodromies? It is important to note that the choice of connection used to compute $\mathrm{Tr}\,(R \wedge R)$ is quite central. The connection required by duality is the torsional connection

$$\Omega_+ = \omega + \frac{1}{2}\mathcal{H}_3, \tag{3.27}$$

where ω is the spin connection; see [21,66] for a discussion about the role of the connection in constructing geometric torsional solutions.

At the level of cohomology classes, the choice of connection does not matter — at least for geometric backgrounds. For non-geometric backgrounds, the torsional connection will certainly depend on ρ, and this dependence might now involve non-trivial topology induced from ρ monodromies. In a patch where ρ and τ are single-valued, the metric itself depends on ρ_2 via the combination ρ_2/τ_2 in (1.2) while a dependence on ρ_1 emerges from $\mathcal{H}_3$ in the connection. From these arguments, it seems clear that ρ will contribute to the gravitational source for the tadpole although the precise form of the contribution is unknown. To proceed, let us treat τ and ρ symmetrically as an ansatz. This is somewhat suggested both by duality with F-theory and by a mirror transform on the fiber which exchanges ρ and τ but leaves wrapped NS5-branes invariant.

So rather than a single $\mathbb{P}^2$ bundle over B, let us consider a $\mathbb{P}^2 \times \mathbb{P}^2$ bundle over B. We will take a cubic surface in each $\mathbb{P}^2$ with one encoding the τ

variation and the other encoding the ρ variation, very much in the spirit of the doubled torus formalism. Let $\beta = c_1(\mathcal{O}(1))$ for the second $\mathbb{P}^2$. We will impose our earlier constraint (3.14) that

$$\mathcal{O}(L_\tau + L_\rho) = \mathcal{O}(-K_B)$$

and the relations

$$\begin{aligned}
\alpha(\alpha + 2c_1(L_\tau))(\alpha + 3c_1(L_\tau)) &= 0, \\
\beta(\beta + 2c_1(L_\rho))(\beta + 3c_1(L_\rho)) &= 0,
\end{aligned} \qquad (3.28)$$

where we have abbreviated $c_1(\mathcal{O}(L))$ by $c_1(L)$ to reduce notational clutter. Because there is really only one physical torus fiber, it only really makes sense to integrate out the fibers and discuss the anomaly on the base. Given this aim, we can simplify the relations (3.28) to

$$\begin{aligned}
\alpha(\alpha + 3c_1(L_\tau)) &= 0, \\
\beta(\beta + 3c_1(L_\rho)) &= 0.
\end{aligned} \qquad (3.29)$$

Now the analogue of (3.25) becomes

$$\begin{aligned}
C_{\mathcal{M}} = C_B \cdot (1+\alpha)(1+\alpha+2c_1(L_\tau))(1+\alpha+3c_1(L_\tau)) \\
\times \frac{(1+\beta)(1+\beta+2c_1(L_\rho))(1+\beta+3c_1(L_\rho))}{[1+3(\alpha+2c_1(L_\tau))][1+3(\beta+2c_1(L_\rho))]}.
\end{aligned} \qquad (3.30)$$

First, it is easy to check that $c_1(\mathcal{M}) = 0$ simply because of (3.14) and the linearity of the computation of c_1. This is completely natural. The more interesting structure is c_2 which is non-linear. We now find

$$\begin{aligned}
c_2(\mathcal{M}) = c_2(B) + 11c_1(B)^2 - 95c_1(L_\tau)c_1(L_\rho) - 9\alpha\beta \\
- 36(\beta c_1(L_\tau) + \alpha c_1(L_\rho)) + 4(\alpha c_1(L_\tau) + \beta c_1(L_\rho)).
\end{aligned} \qquad (3.31)$$

The first three terms of (3.31) can be directly compared with (3.26) since they are fully supported on the base. The interesting addition is the quadratic term $-95c_1(L_\tau)c_1(L_\rho)$ which is only present in the non-geometric case. While this computation suggests this coupling is present, it would be very interesting to understand whether this is true directly from a world-sheet computation.

It would also be nice to compare the NS5-brane anomaly to the D3-brane anomaly of F-theory [63], as was done in [64] for a class of dual pairs. The quantity to be determined in F-theory, namely the D3-brane charge, is unambiguous though the singularities of the F-theory four-fold, reflecting

the unbroken maximal gauge symmetry, make that computation potentially subtle. What is far less clear is what that number should be compared with in the heterotic string. In the geometric setting, U-duality related NS5-branes wrapping the elliptic fiber to D3-branes rather directly but that chain is certainly modified by the presence of ρ monodromies.

For models admitting an F-theory dual, an M5-brane wrapped on the $K3$-fiber of the F-theory geometry does naturally provide a realization of the world-sheet of the non-geometric heterotic string. This is similar to the proposal in [17] for studying sigma-models of (geometric) heterotic torsional backgrounds. We will not explore these interesting directions here, leaving them to future work.

4   Heterotic solutions with torsion

In the previous sections, we constructed non-geometric heterotic solutions by solving the Bianchi identity with point-like instantons — the only way the $\mathcal{H}_3$ flux appeared was via ρ monodromies. Yet the most physically interesting heterotic backgrounds involve more general torsion, or $\mathcal{H}_3$ flux, since they contain fewer moduli than conventional Calabi–Yau compactifications. From an F-theory perspective, the simplification we used in the preceding discussion is equivalent to setting any bulk filling G_4-flux to zero and looking for heterotic duals of the F-theory geometry. In this section, we wish to analyze the role of the bulk filling G_4-flux, and its various dual descriptions. Some notation and general relations of use in the following sections can be found in Appendix C.

Torsional solutions were first described by [67] in the context of supergravity, and the known compact examples are based on the solutions constructed in [3]. These geometries are constructed by dualizing M-theory compactified on four-folds,[16] $\mathcal{M}_4$, with bulk filling G_4, resulting in a four-dimensional heterotic compactification on a complex but non-Kähler geometric space with non-trivial $\mathcal{H}_3$. The solution of the Bianchi identity (1.3) is guaranteed by satisfying the tadpole condition in M-theory

$$n_{M_2} = -\int_{\mathcal{M}_4} X_8 - \frac{1}{2} \int_{\mathcal{M}_4} G_4 \wedge G_4, \qquad (4.1)$$

---

[16]In contrast to the prior discussion, the four-folds discussed in the next two sections are not necessarily Calabi–Yau. Hence, we will denote them by $\mathcal{M}_4$ and not $\overline{Z}$.

M–theory on $\mathcal{M}_4$ $\longrightarrow$ type I $\overset{\text{S–duality}}{\longrightarrow}$ Heterotic $\overset{\text{T–duality}}{\longrightarrow}$ Heterotic

Figure 2: Schematic of the duality chain that we use to generate non-geometric heterotic solutions with flux.

where n_{M2} is the number of space-time filling M2-branes. The integral of X_8 is given by the Euler character of the four-fold $\mathcal{M}_4$:

$$-\int_{\mathcal{M}_4} X_8 = \chi/24.$$

In the language of Section 3, those models involved τ monodromies but constant ρ. Using a duality chain shown in figure 2, we will show the presence of torsional flux gives us an additional way to generate non-geometric solutions. Dualizing flux to get "non-geometric fluxes" has been explored in past work like [18, 68].

These torsional non-geometric heterotic solutions, in turn, have novel type-IIB and M-theory duals, which we construct in Section 5. This basically completes a duality chain which starts with M-theory on a conformally Calabi–Yau four-fold and generates new compact solutions via U-duality.

We will not work in generality; rather we will focus on a simple example that will illustrate most of the germane features. The main simplification we use is an orbifold metric for a $K3$ surface. The advantage of this replacement is that the the orbifolded theory inherits part of the U-duality group of the covering toroidal compactification. Otherwise, we would need to worry about patching with mirror transforms of a $K3$ surface rather than the U-duality group of a torus.

M-theory on a Calabi–Yau four-fold $\mathcal{M}_4$

Let us briefly review the main differences between the M-theory compactifications discussed in Section 3 and compactifications with bulk filling G_4-flux. As described in [69], with flux the metric becomes warped so that the four-fold is now conformally Calabi–Yau,

$$ds^2 = \mathrm{e}^{-\phi}\eta_{\mu\nu}dx^\mu dx^\nu + \mathrm{e}^{\frac{1}{2}\phi}\widetilde{g}_{MN}dx^M dx^N, \qquad (4.2)$$

where $\widetilde{g}_{MN}$ is the metric on the four-fold. The flux must be a primitive $(2,2)$-form:

$$G_{abcd} = G_{abc\bar{d}} = 0, \ \ g^{c\bar{d}}G_{a\bar{b}c\bar{d}} = 0. \qquad (4.3)$$

There is also a space-time filling flux:

$$G_{\mu\nu\rho a} = \partial_a e^{-3\phi/2}. \tag{4.4}$$

The warp factor obeys a Poisson equation

$$\Box e^{3\phi/2} = \star_8 \left\{ 4\pi^2 X_8 - \frac{1}{2} G_4 \wedge G_4 \right\} - 4\pi^2 \sum_{i=1}^{n} \delta^8(x - x_i), \tag{4.5}$$

where the Laplacian and Hodge dual are taken with respect the unwarped internal metric g_{MN}, and we have allowed for the possibility of space-time filling M2-branes localized at points, x_i, in the four-fold.

There is an obstruction to solving the Poisson equation (4.5) unless the charge cancelation condition (4.1) is satisfied; the presence of the higher derivative coupling, X_8, is crucial for the existence of a solution. In our subsequent discussion, we will frequently arrive at expressions that depend on higher derivative terms which, although vanishing at the level of supergravity, are essential for the existence of the solution.

These M-theory solutions have three-dimensional heterotic duals when $\mathcal{M}_4$ admits a $K3$ fibration, and four-dimensional duals when that $K3$ fibration admits a compatible elliptic fibration. The duality chain sketched in figure 2 involves T-duality so we need a starting four-fold metric with suitable (approximate) isometries.

We could start with a semi-flat metric (1.2) for an elliptic four-fold. However, to make our life simpler and still illustrate the pertinent features of our solutions, we will further restrict to a particularly nice four-fold; namely, $K3 \times \widetilde{K3}$, where both $K3$ spaces admit elliptic fibrations. Further, we will take $\widetilde{K3}$ to be an orbifold space $\widetilde{K3} = T^4/\mathbb{Z}_2$. On shrinking the elliptic fiber of $\widetilde{K3}$, we will arrive at the orientifold limit of an F-theory compactification on $K3 \times \frac{T^2}{(-1)^{F_L}\,\Omega\,\mathbb{Z}_2}$.

Let us take a square complex structure on T^4, and choose the complex coordinates (w, v) to have canonical periodicity. The orbifold $\mathbb{Z}_2$ acts by sending

$$(w, v) \to (-w, -v). \tag{4.6}$$

We can choose v to coordinatize the elliptic fiber and w the base. The four-form flux takes the form:

$$G_4 = \alpha \wedge dw \wedge d\bar{v} + \beta \wedge d\bar{w} \wedge d\bar{v} + \text{c.c.}, \tag{4.7}$$

where $\alpha \in H^{1,1}(K3,\mathbb{Z})$, $\beta \in H^{2,0}(K3,\mathbb{Z})$ are primitive classes with respect to the Kähler form of $K3$. If $\beta = 0$ then this compactification preserves eight supersymmetries, otherwise it preserves four supersymmetries.

4.1 Type-IIB and Type-I torsional solutions

As a first step to constructing the new heterotic solutions, we take the F-theory limit by shrinking the elliptic fiber with coordinate v. This gives a type-IIB compactification on $K3 \times \frac{T^2}{(-1)^{F_L}\Omega\mathbb{Z}_2}$ with D7-branes and possibly D3-branes, depending on whether n_{M_2} is non-zero. The metric for this background is given by,

$$ds^2 = \mathrm{e}^{-3\phi/4}\eta_{\mu\nu}dx^\mu dx^\nu + \mathrm{e}^{3\phi/4}g_{mn}dx^m dx^n + \mathrm{e}^{3\phi/4}|dw_1 + \mathrm{i}dw_2|^2, \quad (4.8)$$

where $dw = dw_1 + idw_2$ is along the T^2 while the indices $m, n = 1, \ldots, 4$ parametrize the directions along the $K3$ surface, and g_{mn} is the Ricci-flat $K3$ metric. The M-theory four-form flux lifts to type-IIB three-form fluxes given by

$$H_3 = (\alpha + \overline{\beta}) \wedge dw + \text{c.c.}, \quad F_3 = \mathrm{i}(\overline{\beta} - \alpha) \wedge dw + \text{c.c.},$$

and a five-form flux that fills space-time

$$F_5 = dC_4 + H_3 \wedge C_2, \quad \text{where} \quad dC_4 = \varepsilon_4 \wedge d\,\mathrm{e}^{-3\phi/2}. \quad (4.9)$$

We will often find it convenient to write

$$F_3 = F_{w^1}dw^1 + F_{w^2}dw^2 = F_w dw + F_{\bar{w}}d\bar{w}, \quad (4.10)$$

and similarly for H_3. Writing $\alpha = \alpha_1 + \mathrm{i}\alpha_2$ and $\beta = \beta_1 + \mathrm{i}\beta_2$ we find

$$\begin{aligned} H_3 &= 2(\alpha_1 + \beta_1) \wedge dw^1 + 2(\beta_2 - \alpha_2) \wedge dw^2, \\ F_3 &= 2(\alpha_2 + \beta_2) \wedge dw^1 + 2(\alpha_1 - \beta_1) \wedge dw^2, \end{aligned} \quad (4.11)$$

The fluxes satisfy a constraint (corresponding to imaginary self-duality of G_3) given by

$$F_3 = \star_6(\mathrm{e}^{-\Phi_B}H_3), \quad (4.12)$$

where $\star_6$ is with respect to the unwarped metric. The type-IIB dilaton Φ_B is determined by the complex structure of the elliptic fiber with coordinate v. We set $g_s = \mathrm{e}^{\Phi_B}$. If $\beta = 0$ then $H_{w^1} = g_s F_{w^2}$, and $H_{w^2} = -g_s F_{w^1}$.

By T-dualizing along the w_1, w_2 coordinates, we arrive at a geometric type-I configuration with flux. This type-I solution consists of a six-dimensional manifold that is torus fibered with metric

$$ds^2 = \mathrm{e}^{-3\phi/4}\eta_{\mu\nu}dx^\mu dx^\nu + \mathrm{e}^{3\phi/4}g_{mn}dx^m dx^n + \mathrm{e}^{-3\phi/4}|dw + A_H|^2. \quad (4.13)$$

The one-form $A_H = B_{w^1} + iB_{w^2}$ is constructed out of a trivialization of the type-IIB field strength H_3. The trivialization is chosen such that the B_{w^i} are independent of the T^2 elliptic fiber in the $K3$ surface. This is a gauge choice which is convenient for the next step in the duality chain.

The only non-zero RR flux is

$$F'_3 = F_{w^1} \wedge dw^2 - F_{w^2} \wedge dw^1 + (F_{w^1} \wedge B_{w^2} - F_{w^2} \wedge B_{w^1}) + \star_{K3}d\mathrm{e}^{3\phi/2},$$
$$(4.14)$$

where in the last line we used $(F_5)_{w_1 w_2} = -\star_{K3} d\mathrm{e}^{3\phi/2}$. Note that $dF'_3 = 0$ at the level of supergravity, and $F'_1 = F'_5 = 0$ consistent with the type-I field content. These are the solutions of [3]; a similar chain starting with an elliptic Calabi–Yau three-fold in the semi-flat approximation gives more general metrics described in [21]. Fortunately, we can extract the physics we wish to see starting from this clean example.

4.2   New heterotic solutions with torsion

We follow the duality sequence illustrated in figure 2. Start with the type-I solution in Section 4.1 and S-dualize to the heterotic string. Then apply two T-dualities along the fiber of the $K3$ factor to generate the new non-geometric heterotic solution. This is the extra ingredient and the remaining unexplored duality direction in the possible dual realizations of F-theory on $K3 \times \widetilde{K3}$.

If we choose an orbifold metric $T^4/\mathbb{Z}_2$ for this $K3$ factor, we can write down explicit expressions for the metric and fluxes. Again, we could take a more general semi-flat metric but this should suffice.

So let us take a $K3$ surface realized as a Kummer surface $T^4/\mathbb{Z}_2$; further choose $T^4 = T^2 \times T^2$. Let $z_1 = x_1 + iy_1$ be coordinates for the T^2 fiber of the $K3$ surface, and let $z_2 = x_2 + iy_2$ be coordinates of the $T^2/\mathbb{Z}_2$ base. For simplicity, we choose square tori with canonical periodicities so that $dz_i = dx_i + idy_i$ with $i = 1, 2$ is a basis of holomorphic one forms.

We will construct an $N = 2$ solution, whose existence post-duality is more trust-worthy, by choosing $\beta = 0$ and α to be the following $(1,1)$-form:

$$\alpha = A dz_1 \wedge d\bar{z}_2. \tag{4.15}$$

The constant A is real. With this choice, the fluxes can be trivialized as follows (in real coordinates):

$$B_{w^1} = 2A(x_2 dx_1 + y_2 dy_1), \quad B_{w^2} = 2A(y_2 dx_1 - x_2 dy_1),$$
$$C_{w^1} = 2A(x_2 dy_1 - y_2 dx_1), \quad C_{w^2} = 2A(x_2 dx_1 + y_2 dy_1). \tag{4.16}$$

We pick this trivialization to ensure that there are isometries along the (x_1, y_1) directions. We can T-dualize along these directions to give a new heterotic solution (we denote the new field components by hats):

$$\widehat{ds}^2 = \eta_{\mu\nu} dx^\mu dx^\nu + e^{3\phi/2}[\varpi^2(dx_1^2 + dy_1^2) + (dx_2^2 + dy_2^2)] + (dw_1 + \varpi B_{w^1})^2$$
$$+ (dw_2 + \varpi B_{w^2})^2, \tag{4.17}$$

where ϖ is given by

$$\varpi = (e^{3\phi/2} + 4|A|^2(x_2^2 + y_2^2))^{-1}. \tag{4.18}$$

Note that at the level of supergravity, the solution to the warp factor equation (4.5) with this choice of fluxes is given by

$$e^{3\phi/2} = 1 - 4A^2(x_2^2 + y_2^2) + \O(\alpha') \tag{4.19}$$

implying $\varpi = 1 + \O(\alpha'^2)$. The remaining terms arise from higher derivative corrections to the warp factor equation (4.5) needed to ensure a solution exists. The $\mathcal{B}$-field is given by

$$\mathcal{B}_{w^1} = -2A\varpi(x_2 dx^1 + y_2 dy^1), \quad \mathcal{B}_{w^2} = -2A\varpi(y_2 dx^1 - x_2 dy^1), \tag{4.20}$$

and the heterotic dilaton is given by

$$e^{\Phi_h} = e^{3\phi/4} \varpi. \tag{4.21}$$

This solution is non-geometric in the following sense. Locally, the solution has a well-defined supergravity description and the above-field content solves the supersymmetry conditions and equations of motion (we show this explicitly in the type-IIB and M-theory duals below). On the other hand, the background is only globally well-defined when we include the $SO(4, 4, \mathbb{Z})$ transformations of the T^4 fiber. This is to be contrasted with the mechanism

for generating non-geometries described in Section 3. Since $\widetilde{K3}$ is trivially fibered over $K3$, that approach would give a geometric heterotic background.

It would be very interesting to explore compactifications in which this kind of torsion $\mathcal{H}_3$-flux and the ρ monodromies of Section 3 are combined.

5 Type-IIB and M-theory non-geometric duals

5.1 G-structures and local geometry

In this final section, we will describe the dual type-IIB and M-theory descriptions of the heterotic solutions derived in the previous section (see figure 3). These solutions are novel, having an interesting local and global geometry. We will characterize the local geometry in terms of G-structures, developed in [70, 71], in which the spinors classify the local geometry in terms of the fluxes; see, for example, [72, 73] for reviews.

In compactifications without flux to four dimensions, supersymmetry requires the existence of a covariantly constant spinor on the internal six-dimensional manifold. This implies the holonomy group is reduced to SU(3), which is a defining characteristic of a Calabi–Yau manifold. The supersymmetry spinors can be used to form spinor bilinears which correspond to forms on the internal space. Two forms play a distinguished role:

$$J_2 = -2i\eta^\dagger \gamma_{MN}\eta \, dx^M dx^N, \quad \Omega_3 = -2i\eta^T \gamma_{MNP}\eta dx^M dx^N dx^P. \qquad (5.1)$$

Since the spinor is covariantly constant, these forms are closed. It is not hard to show that J corresponds to the Kähler form and Ω the holomorphic three-form. The supersymmetry spinor therefore allows us to define forms which characterize the geometry of the internal space.

M–theory on $K3 \times T^4/\mathbb{Z}_2$ $\qquad\qquad\qquad\qquad$ **M/F–theory on $\widetilde{\mathcal{M}}_4$**

$\downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\uparrow$

type IIB on $K3 \times T^2/\mathbb{Z}_2$ $\xrightarrow{\text{T–duality}}$ type I $\xrightarrow{\text{T–duality}}$ **type IIB on $\mathcal{M}_3$**

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\downarrow$

$\qquad\qquad\qquad\qquad\qquad E_8 \times E_8 \text{ heterotic} \xrightarrow{\text{T–duality}} \mathbf{E_8 \times E_8} \text{ **Heterotic**}$
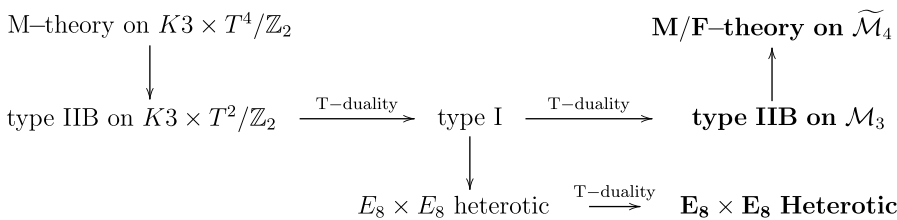
Figure 3: The duality chain used to generate the heterotic solutions in the previous section as well as their type-IIB and M-theory dual descriptions. The bold face indicates new solutions discussed in this paper.

How does this change when we include fluxes in the compactification? The first thing to observe is the fluxes enrich (and complicate) the supersymmetry variations, allowing more general backgrounds than just Calabi–Yau spaces. Secondly, the gravitino variation implies that the supersymmetry spinor is no longer covariantly constant with respect to the Levi–Civita connection, but is covariantly constant with respect to a connection that involves the flux. Schematically,

$$\nabla_M \eta = 0 \quad \longrightarrow \quad (\nabla + \text{flux})_M \eta = 0. \tag{5.2}$$

Although the manifold no longer has reduced holonomy, it still has a reduced structure group $G \subset \mathrm{SU}(4)$, and the deviation from special holonomy can be measured using instrinsic torsion. In particular, dJ and $d\Omega$ are no-longer zero, and are sourced by the fluxes. To be more specific, we will consider a general $N = 1$ type-II compactification. There are two supersymmetry spinors, which can be written (in string frame):

$$\epsilon^1 = \zeta_- \otimes \eta_+^1 + \text{c.c.},$$
$$\epsilon^2 = \zeta_- \otimes \eta_+^2 + \text{c.c.}, \tag{5.3}$$

where $\zeta_\pm$ are $d = 4$ Weyl spinors and $\eta_\pm^i$ are internal Weyl spinors, with the sign denoting chirality. In this notation, complex conjugation corresponds to a flip in chirality. In order to preserve $N = 1$ supersymmetry, the two spinors η^1 and η^2 need to be related. The type of relation characterizes the internal geometry in terms of the structure group of the manifold. In particular there are three obvious cases:

1. $\eta^1 \propto \eta^2$ everywhere. The structure group is at most $\mathrm{SU}(3) \subset \mathrm{SU}(4)$. This class of solutions typically come from large volume compactifications discussed in [3, 74], and are conformally Calabi–Yau.
2. $\eta^1 \perp \eta^2$ everywhere. The structure group of the internal manifold is reduced from $\mathrm{SU}(3)$ to $\mathrm{SU}(2)$ and this imposes strict topological conditions on the internal manifold; for example, $\chi = 0$. The geometry is labeled "static $\mathrm{SU}(2)$."
3. η^1 and η^2 interpolate between cases (1) and (2) at different points on the internal space: there may be points where they are parallel and other points where they are orthogonal. This is clearly the most general type of solution and is called "local $\mathrm{SU}(2)$."

In our case the geometry will have local $\mathrm{SU}(2)$ structure, with a structure group that includes the quantum $O(4, 4, \mathbb{Z})$ T-duality group.[17] The novelty

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[17]See Appendix E for a brief overview of the necessary $\mathrm{SU}(2)$ G-structure analysis.

arises in the kinds of flux one can write down without breaking supersymmetry. Our example is one that admits $(0,3)$ and $(3,0)$ G_3-flux along with non-trivial F_1 and F_5 fluxes.[18]

Although we derive the solution in the orbifold limit, the background has moduli which give rise to a family of type-IIB solutions. Our type-IIB solution can also be lifted to M-theory where the resulting flux is no longer necessarily $(2,2)$. This is in contrast to the solutions typically studied which are based on [69]. Our new solutions are therefore examples of the more general structures possible when a more general spinor ansätze is used in solving the supergravity equations of motion. We relegated some of the details required to demonstrate that the solutions preserve supersymmetry and obey the equations of motion to Appendix E.

5.2   A non-geometric type-IIB solution

Our starting point is again the type-I solution described in Section 4.1 with the choice of fluxes given by (4.16). Our parameterization of the flux and metric imply there are isometries along the (x_1, y_1) directions of $T^4/\mathbb{Z}_2$, so we can T-dualize these directions using the Buscher rules to construct a dual type-IIB solution. For convenience, these rules are summarized in Appendix D.

The $D9/O9$ system of type-I becomes $D7/O7$-branes localized in the fiber of $T^4/\mathbb{Z}_2$. Denoting the T-dualized fields by $\widetilde{G}$, $\widetilde{B}$, $\widetilde{\Phi}$ and $\widetilde{C}_n$ we find the NS–NS background:

$$\widetilde{ds}^2 = \mathrm{e}^{-3\phi/4}\eta_{\mu\nu}dx^\mu dx^\nu + \mathrm{e}^{3\phi/4}\left\{\varpi\left[dwd\bar{w} + dz_1 d\bar{z}_1\right] + dz_2 d\bar{z}_2\right\} \quad (5.4)$$

$$\mathrm{e}^{\widetilde{\Phi}_{IIB}} = \varpi, \quad (5.5)$$

$$\widetilde{B}_2 = -A\varpi\bar{z}_2 dz_1 \wedge dw + \text{c.c.}, \quad (5.6)$$

where $dz_i = dx_i + \mathrm{i}dy_i$, $dw = dw_1 + \mathrm{i}dw_2$ and ϖ is given in (4.18). Metrically the internal space $\mathcal{M}_3$ is a T^4 fibration over a T^2 base. We will explain below how to make sense of this globally. The RR field content is

$$\widetilde{F}_1 = -\star_{\mathbb{P}^1} d\varpi^{-1} = \varnothing(\alpha'^2), \quad (5.7)$$

---

[18]It is usually the case that the presence of $(0,3)$ or $(3,0)$ fluxes in compact string solutions breaks supersymmetry. That is true for models with a large volume limit. Here we relax that constraint.

$$\widetilde{F}_3 = -iAdw \wedge dz_1 \wedge d\bar{z}_2 + \text{c.c.} + \O(\alpha'^2),$$

$$\widetilde{F}_5 = -A^2\varpi[\star_{\mathbb{P}^1}d(|z_2|^2) \wedge dz^1 \wedge d\bar{z}^1 \wedge dw \wedge d\bar{w}] + \text{hodge dual} + \O(\alpha'^2),$$

$$(5.8)$$

where $\star_{T^2}$ denotes taking the Hodge dual with respect to the unwarped metric on the $\mathbb{P}^1$ base. In the last line, we have taken the ten-dimensional Hodge dual (so that $\widetilde{F}_5$ is self-dual). Note that $dF_5 = H_3 \wedge F_3$, which is a good consistency check. The spinors dualize as follows

$$\widetilde{\epsilon}_L = e^{-3\phi/16}\zeta_- \otimes \eta_+ + \text{c.c.},$$

$$\widetilde{\epsilon}_R = e^{-3\phi/16}\zeta_- \otimes [c\eta_+ + d\chi_+] + \text{c.c.},$$

$$(5.9)$$

where η_+ and χ_+ are two orthogonal spinors defined on the unwarped internal space. The coefficients are given by:

$$c = \varpi\left(4A^2|z_2|^2 - e^{3\phi/2}\right) \qquad d = 4z_2\varpi A\, e^{3\phi/4}. \qquad (5.10)$$

This is a compact type-IIB vacuum with local SU(2) structure: at generic points on the internal space, the spinors $\widetilde{\epsilon}_L$ and $\widetilde{\epsilon}_R$ are neither orthogonal nor parallel, even at the level of supergravity.

The solution is non-geometric in a fashion similar to the heterotic solution we described in Section 4.2. In this case, the internal space $\mathcal{M}_3$ looks like a T^4 fibration over a $T^2/\mathbb{Z}_2$ base. By including group elements from the $O(4,4,\mathbb{Z})$ T-duality group (which can be thought of as coming from compactifying type-IIB on the T^4 fiber), we find the metric is globally well defined.

The internal space is therefore a fibration (after including the non-geometric twists), with 7-branes localized in the T^4 fiber. Although we deduced the presence of the 7-branes via T-duality, the action of the non-geometric twists on the open string sector is quite complex. As alluded to in the introduction, this is one of the complications one must face when analyzing non-geometric compactifications in type-II string theory; M-theory and the heterotic string do not have such issues.

We have checked that the background given above locally satisfies the type-IIB supersymmetry constraints. The details are rather involved, and a summary is given in Appendix E.

As a further test, this solution can be lifted to M-theory (see the following section) and the equations of motion checked. Explicitly these are given by

$$R_{MN} - \tfrac{1}{2} G_{MN} R = \tfrac{1}{12} \left(G_{\mathrm{MPQR}} G_N^{\mathrm{PQR}} - \tfrac{1}{8} G_{MN} G_{\mathrm{PQRS}} G^{\mathrm{PQRS}} \right). \quad (5.11)$$

The fluxes must also satisfy the Bianchi identity,

$$d * G_4 + \tfrac{1}{2} G_4 \wedge G_4 = \varnothing(\ell_p^4). \quad (5.12)$$

After performing the lift to M-theory, one can see explicitly that our solution satisfies Einstein's equations.

It is interesting to note that the typical supersymmetry constraints used in the literature for studying type-IIB flux compactifications are that the G_3-flux must be imaginary self-dual, primitive and $(2,1)$ with respect to the complex structure [3]. Fluxes that do not obey these constraints are typically thought to break supersymmetry. We have constructed here a counter-example to this lore: a solution with G_3-flux that is not $(2,1)$, consistent with a non-holomorphic dilaton. Such solutions were first pointed out in [75] and here we have constructed an example. There are many ways to generalize this construction like starting with both α and β fluxes which would give $N = 1$ models.

5.3 Lift to M-theory

We now lift the type-IIB solution to M-theory in the usual way. This is useful because we avoid the difficulties in defining orientifold projections in non-geometric type-IIB. So assume $\mathcal{M}_3$ is the base of a torus-fibered four-fold $\widetilde{\mathcal{M}}_4$, with the complex structure of the torus determined by the type-IIB string coupling. This is depicted in figure 4. The three-fold $\mathcal{M}_3$, whose fibration structure is illustrated in figure 5, is itself T^4-fibered (including non-geometric twists). Therefore the M-theory solution itself only makes sense using the appropriate U-duality group. Using the standard relation between type-IIB and M-theory, we can read off the M-theory metric:

$$ds_{11}^2 = \mathrm{e}^{-\phi} \eta_{\mu\nu} dx^\mu dx^\nu + \mathrm{e}^{\phi/2} (\varpi dw d\bar{w} + \varpi dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2) + \mathrm{e}^{\phi/2} dv d\bar{v}. \quad (5.13)$$

The coordinate v parameterizes the torus fiber, and we absorb the volume into $v \sim v + 2\pi R \sim v + 2\pi R\tau$. The complex structure of the torus is given

$$T^2 \longrightarrow \widetilde{\mathcal{M}}_4$$
$$\downarrow$$
$$\mathcal{M}_3$$

Figure 4: Fibration structure of the $\widetilde{\mathcal{M}}_4$ solution.

$$T^4 \longrightarrow \mathcal{M}_3$$
$$\downarrow$$
$$T^2/\mathbb{Z}_2$$

Figure 5: Fibration structure of the type-IIB $\mathcal{M}_3$ solution.

by the axio-dilaton of type-IIB,

$$\tau = \widetilde{C}_0 + \mathrm{i}(\mathrm{e}^{3\phi/2} + 4A^2|z_2|^2), \tag{5.14}$$

where $d\widetilde{C}_0 = \widetilde{F}_1$ in (5.7).

There is also the M-theory three-form A_{MNP}, which has internal legs given by the type-IIB two-forms. Explicitly, the three form has one leg along the fiber and one along the base. It is determined in terms of type-IIB fluxes,

$$B_{\mu\nu} \longleftrightarrow A_{\mu\nu v^1}, \tag{5.15}$$
$$C_{\mu\nu} \longleftrightarrow A_{\mu\nu v^2}, \tag{5.16}$$

with $A_{MNP} = 0$, otherwise. There is also the type-IIB five-form field strength with four legs in space-time specified by (5.8). This lifts to a space-time filling four-form field strength G_{012a}, where $G = dA$.

Let us examine the global behavior. Under the periodicities $z_2 \sim z_2 + 2\pi \sim z_2 + 2\pi\tau$ the metric and complex structure are defined only up to U-duality transformations. We have arrived at solution of M-theory that is locally geometric, but globally non-geometric, requiring patching by U-duality. This is a U-fold as originally sought in [11] but of a quite different local form.

A short note on dualizing the spinor. It is possible to show that the type-IIB spinor lifts to a Majorana spinor in M-theory of the form,

$$\varepsilon = \mathrm{e}^{-5\phi/4}\psi \otimes (\xi_1 + \xi_2). \tag{5.17}$$

Here ξ_1 and ξ_2 are $d = 8$ Majorana spinors, which have chiral components:

$$\xi_i = \xi_i^+ + \xi_i^-, \tag{5.18}$$

with $\xi_i^\pm$ Majorana–Weyl spinors. Because of the zeroes in c and d defined in (E.10), these chiral components will also have zeroes. This is a background of the type discussed by Tsimpis [76], in which we preserve $N = 2$ in $d = 3$. This more general spinor ansatz is the reason one can have more general flux configurations (i.e., not necessarily $(2, 2)$ fluxes) without breaking supersymmetry.

Appendix A Weierstrass models for maximal gauge symmetry

In this appendix, we summarize the derivation of equations (2.13) and (2.15), following [38, 46–48]. To derive equation (2.13), we use Weierstrass form: in order to get fibers of type II* at both $\sigma = 0$ and $\sigma = \infty$, Kodaira's table implies that the coefficient of X must have zeros of order 4 at both 0 and ∞. Since this coefficient has degree 8, it therefore takes the form $a\sigma^4$ in affine coordinates. Similarly, the coefficient of X^0 must have zeros of order 5 at both 0 and ∞, with overall degree 12; thus, it must take the form $b\sigma^5 + c\sigma^6 + d\sigma^7$.

To derive equation (2.15), we start by imposing a $\mathbb{Z}_2$ subgroup of the Mordell–Weil group. (We do this because the desired gauge group $\overline{Z}_{\text{Spin32}/\mathbb{Z}_2}$ has fundamental group $\mathbb{Z}_2$, which implies that the torsion part of the Mordell–Weil group should be $\mathbb{Z}_2$ [77].) Having a $\mathbb{Z}_2$ subgroup of the Mordell–Weil group means that there should be a point of order 2 on the elliptic curve, and by a translation in the (x, y) plane we can move this point to $(0, 0)$. For $(0, 0)$ to be a point of order two, we need the tangent line of the elliptic curve at this point to be vertical; this implies that the equation takes the form

$$y^2 = x^3 + p_4(s)x^2 + \varepsilon_8(s)x, \tag{A.1}$$

where $p_4(s)$ and $\varepsilon_8(s)$ are polynomials of degree 4 and 8, respectively. Using the substitution $x = \bar{x} - p_4(s)/3$, we can restore this to Weierstrass form:

$$y^2 = \bar{x}^3 + \left(\varepsilon_8(s) - \tfrac{1}{3}p_4(s)^2\right)\bar{x} + p_4(s)\left(\tfrac{2}{27}p_4(s)^2 - \tfrac{1}{3}\varepsilon_8(s)\right), \tag{A.2}$$

which allows us to compute the discriminant:

$$
\begin{aligned}
\Delta &= 4\left(\varepsilon_8(s) - \tfrac{1}{3}p_4(s)^2\right)^3 + 27p_4(s)^2\left(\tfrac{2}{27}p_4(s)^2 - \tfrac{1}{3}\varepsilon_8(s)\right)^2 \\
&= \varepsilon_8(s)^2\left(4\varepsilon_8(s) - p_4(s)^2\right).
\end{aligned} \tag{A.3}
$$

To get a fiber of type I_{12}^* at $s = \infty$, f, g, and Δ must have zeros of order 2, 3, and 18 (respectively) at $s = \infty$. Thus, with respect to the affine coordinate s, we must have

$$
\begin{aligned}
\deg\left(\varepsilon_8(s) - \tfrac{1}{3}p_4(s)^2\right) &= 8 - 2 = 6, \\
\deg\left(p_4(s)\left(\tfrac{2}{27}p_4(s)^2 - \tfrac{1}{3}\varepsilon_8(s)\right)\right) &= 12 - 3 = 9
\end{aligned} \tag{A.4}
$$

and

$$
\deg\Delta = \deg\varepsilon_8(s)^2\left(4\varepsilon_8(s) - p_4(s)^2\right) = 24 - 18 = 6. \tag{A.5}
$$

From this data, we argue as follows. First, if $\deg p_4(s) = 4$, then there must be cancellation between leading terms of $p_4(s)^2$ and $\varepsilon_8(s)^2$ to get lower degree for both $\varepsilon_8(s) - \tfrac{1}{3}p_4(s)^2$ and $\tfrac{2}{27}p_4(s)^2 - \tfrac{1}{3}\varepsilon_8(s)$. But since those linear combinations are not proportional to each other, it is not possible to acheive both cancellations. Thus, $\deg p_4(s) \leq 3$. To get the correct reductions in degree, it is easy to see that also $\deg\varepsilon_8(s) \leq 6$. But now if $\deg p_4(s) < 3$, the second combination would have its degree reduced below 9. Thus $\deg p_4(s) = 3$. In equation (2.15) and also in Appendix B, we refer to this cubic polynomial as $p(s) = p_0 s^3 + p_1 s^2 + p_2 s + p_3$.

It then follows that $\deg(4\varepsilon_8(s) - p_4(s)^2) = 6$, and so to achieve the correct order of vanishing of the discriminant, the degree of $\varepsilon_8(s)$ must be ≤ 0, i.e., $\varepsilon_8(s)$ must be constant. In equation (2.15) and also in Appendix B, we simply refer to this constant as ε.

Appendix B   An explicit Shioda–Inose structure

In this appendix, we describe the explicit Shioda–Inose structure found by Clingher and Doran [6], and make it more precise. Our first step is to make explicit the involution on the $K3$ surface $\overline{Z}_{\mathrm{Spin32}/\mathbb{Z}_2}$, and to compute the quotient by that involution.

The involution on $\overline{Z}_{\mathrm{Spin32}/\mathbb{Z}_2}$ is induced by translation by the point of order 2. To work this out geometrically, we start with an arbitrary point

(x_0, y_0) on the elliptic curve and connect it by a line to $(0, 0)$; this line has equation $y = (y_0/x_0)x$. Substituting in, we find

$$(y_0^2/x_0^2)x^2 = x^3 + p(s)x^2 + \varepsilon x \tag{B.1}$$

or

$$\begin{aligned}
0 &= x^3 + \left(p(s) - y_0^2/x_0^2\right)x^2 + \varepsilon x \\
&= x(x^2 + \left(p(s) - y_0^2/x_0^2\right)x + \varepsilon) \\
&= x(x - x_0)(x - \varepsilon/x_0),
\end{aligned} \tag{B.2}$$

since

$$-x_0 - \varepsilon/x_0 = p(s)x - y_0^2/x_0^2. \tag{B.3}$$

The third point of intersection with the line is therefore

$$(\varepsilon/x_0, \varepsilon y_0/x_0^2). \tag{B.4}$$

The translation by $(0, 0)$ yields the point with the same x-value, but the negative of the y-value; that is, our automorphism is:

$$(x, y) \mapsto (\varepsilon/x, -\varepsilon y/x^2). \tag{B.5}$$

The quotient can be described by introducing invariants

$$\begin{aligned}
\xi &= x + \frac{\varepsilon}{x}, \\
\eta &= y - \frac{\varepsilon y}{x^2}
\end{aligned} \tag{B.6}$$

and observe that our equation can be written $y^2 = x^3 + p(s)x^2 + \varepsilon x = x^2 (\xi + p(s))$. Then

$$\begin{aligned}
\eta^2 &= y^2 \left(1 - \frac{\varepsilon y}{x^2}\right)^2 \\
&= x^2(\xi + p(s)) \left(1 - \frac{\varepsilon y}{x^2}\right)^2 \\
&= (\xi + p(s)) \left(x - \frac{\varepsilon y}{x}\right)^2 \\
&= (\xi + p(s))(\xi^2 - 4\varepsilon).
\end{aligned} \tag{B.7}$$

This is the equation of the quotiented surface. Its discriminant is

$$-16\varepsilon(p(s)^2 - 4\varepsilon)^2, \tag{B.8}$$

which has roots at $s = \infty$ and at the roots of $p(s) \pm 2\sqrt{\varepsilon}$ (the latter are all double roots).

Clingher and Doran [6] start with the Kummer surface of the product $E \times F$, where E is the double cover of $\mathbb{CP}^1$ with branch points $\{0, 1, \alpha, \infty\}$ and F is the double cover of $\mathbb{CP}^1$ with branch points $\{0, 1, \beta, \infty\}$. Clingher and Doran use the analysis of Oguiso [78] to locate the elliptic fibration on the Kummer surface $\mathrm{Km}(E \times F)$ which corresponds to unbroken Spin $32/\mathbb{Z}_2$, and find that the singular fibers of type I_2 of that fibration are located at 6 specific values of the parameter t of the fibration, divided into two groups of three [6, equation (59)]:

$$\left\{ 1, \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\alpha\beta}, \frac{\alpha\beta + 1}{\alpha\beta}, \frac{\alpha + \beta}{\alpha\beta} \right\} = \left\{ 1, \frac{1}{\alpha\beta}, \frac{\alpha + \beta}{\alpha\beta} \right\} \cup \left\{ \frac{1}{\alpha}, \frac{1}{\beta}, \frac{\alpha\beta + 1}{\alpha\beta} \right\}. \tag{B.9}$$

By rescaling the parameter t to $s = (\alpha\beta)t$, we can scale all of these singular values by $\alpha\beta$, giving the values

$$\{\alpha\beta, \beta, \alpha, 1, \alpha\beta + 1, \alpha + \beta\} = \{\alpha\beta, 1, \alpha + \beta\} \cup \{\beta, \alpha, \alpha\beta + 1\}. \tag{B.10}$$

The monic polynomials that vanish on these two sets

$$\begin{aligned} D_1(s) &= (s - \alpha\beta)(s - 1)(s - \alpha - \beta), \\ D_2(s) &= (s - \beta)(s - \alpha)(s - \alpha\beta - 1), \end{aligned} \tag{B.11}$$

then have the remarkable property that their difference

$$D_1(s) - D_2(s) = \alpha\beta(\alpha - 1)(\beta - 1) \tag{B.12}$$

is a constant and, in particular, is independent of s. Thus, for some monic polynomial $q(s)$, the polynomials $D_j(s)$ can be written as $q(s) \pm \frac{1}{2}\alpha\beta(\alpha - 1)(\beta - 1)$.

Since the only singular fibers of the elliptic fibration away from $s = \infty$ are the I_2 fibers which are located at the roots of $D_j(s)$, and at each of which the discriminant has a double zero, the discriminant must be (up to a multiplicative constant):

$$D_1(s)^2 D_2(s)^2 = (q(s) + \tfrac{1}{2}C)^2 (q(s) - \tfrac{1}{2}C)^2 = (q(s)^2 - \tfrac{1}{4}C^2)^2, \tag{B.13}$$

where $C = \alpha\beta(\alpha - 1)(\beta - 1)$. This is precisely the form which we derived in equation (B.8), if we identify $C^2/4$ with 4ε and $q(s)$ with $p(s)$.

In other words, in our quotient $\mathbb{Z}_{\text{Spin}(32)/\mathbb{Z}_2}/\iota_{\text{Spin}(32)/\mathbb{Z}_2}$, the roots of $p(s) \pm 2\sqrt{\varepsilon}$ are the two sets

$$\{\alpha\beta, 1, \alpha + \beta\} \text{ and } \{\beta, \alpha, \alpha\beta + 1\}. \tag{B.14}$$

It is then easy to derive the formulas:

$$p(s) = s^3 - (\alpha + 1)(\beta + 1)s^2 + ((\alpha + \beta)(1 + \alpha\beta) + \alpha\beta)s$$
$$- \tfrac{1}{2}\alpha\beta(\alpha + 1)(\beta + 1) \tag{B.15}$$
$$\varepsilon = \tfrac{1}{16}\alpha^2\beta^2(\alpha - 1)^2(\beta - 1)^2, \tag{B.16}$$

since

$$\frac{\alpha\beta(1 + \alpha\beta) + \alpha\beta(\alpha + \beta)}{2} = \frac{\alpha\beta(\alpha + 1)(\beta + 1)}{2} \tag{B.17}$$

and

$$\frac{\alpha\beta(1 + \alpha\beta) - \alpha\beta(\alpha + \beta)}{2} = \frac{\alpha\beta(\alpha - 1)(\beta - 1)}{2}. \tag{B.18}$$

We now wish to generalize this relation to a pair of elliptic curves for which the equations have been given but not the set of branch points. To this end, let

$$v^2 = (u - \alpha_1)(u - \alpha_2)(u - \alpha_3) = u^3 + \lambda_1 u^2 + \lambda_2 u + \lambda_3 \tag{B.19}$$

and

$$w^2 = (z - \beta_1)(z - \beta_2)(z - \beta_3) = z^3 + \mu_1 z^2 + \mu_2 z + \mu_3. \tag{B.20}$$

define E and F. We claim that in this case, the two triples of roots of $p(s) \pm 2\sqrt{\varepsilon}$ will be given by

$$\{\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3, \alpha_2\beta_1 + \alpha_3\beta_2 + \alpha_1\beta_3, \alpha_3\beta_1 + \alpha_1\beta_2 + \alpha_2\beta_3\} \tag{B.21}$$

and

$$\{\alpha_1\beta_1 + \alpha_3\beta_2 + \alpha_2\beta_3, \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_3\beta_3, \alpha_3\beta_1 + \alpha_2\beta_2 + \alpha_1\beta_3\}. \tag{B.22}$$

To verify that these are the same roots as before, we can set $\alpha_1 = \beta_1 = 0$, $\alpha_2 = \beta_2 = 1$, $\alpha_3 = \alpha$, and $\beta_3 = \beta$; then these two triples reduce to the previous case. Since the pair of triples is invariant under the action of the symmetric group on either set of roots, the sets are the same.

Now, however, the polynomials $p(s) \pm 2\sqrt{\varepsilon}$ can be determined by a computation with the elementary symmetric functions (λ's and μ's) of the roots (α's and β's). The result is:

$$p_1 = -\lambda_1 \mu_1,$$
$$p_2 = \lambda_1^2 \mu_2 + \lambda_2 \mu_1^2 - 3\lambda_2 \mu_2,$$
$$p_3 = -\lambda_1^3 \mu_3 - \lambda_3 \mu_1^3 - \tfrac{1}{2}\lambda_1 \lambda_2 \mu_1 \mu_2 + \tfrac{9}{2}\lambda_1 \lambda_2 \mu_3 + \tfrac{9}{2}\lambda_3 \mu_1 \mu_2 - \tfrac{27}{2}\lambda_3 \mu_3,$$
$$\varepsilon = \tfrac{1}{16}\mathrm{disc}_u(u^3 + \lambda_1 u^2 + \lambda_2 u + \lambda_3)\mathrm{disc}_z(z^3 + \mu_1 z^2 + \mu_2 z + \mu_3).$$

(B.23)

By completing the cube (in u, in z, and in s), we can set $\lambda_1 = \mu_1 = p_1 = 0$, leaving

$$p_2 = -3\lambda_2 \mu_2,$$
$$p_3 = -\tfrac{27}{2}\lambda_3 \mu_3,$$
$$\varepsilon = \tfrac{1}{16}(4\lambda_2^3 + 27\lambda_3^2)(4\mu_2^3 + 27\mu_3^2).$$

(B.24)

Appendix C   Some notation and useful relations

In this appendix, we summarize some notation and relations of use in the construction of the explicit solutions of Sections 4 and 5.

We take coordinates (x^1, y^1, x^2, y^2) for $K3$, while for $\widetilde{K3}$ we use (w^1, w^2, v^1, v^2). Both surfaces are assumed to be elliptically fibered. We use Roman indices to denote the following: $m, n = x^1, \ldots, y^2$ the coordinates for the total space $K3$; $p, q = x^2, y^2$ the $\mathbb{P}^1$ base of the $K3$; $i, j = x^1, y^1$ the elliptic fiber of $K3$. The indices a, b are tangent space indices for the internal space. We also use the complex combinations $dz^\alpha = dx^\alpha + \mathrm{i}dy^\alpha$, for $\alpha = 1, 2$ and $dw = dw^1 + \mathrm{i}dw^2$.

The function ϖ appears often:

$$\varpi = \frac{1}{e^{3\phi/2} + |B_{w^1}|^2 + |B_{w^2}|^2}.$$

(C.1)

where $|B_{w^i}|^2 = (B_{w^i x^1})^2 + (B_{w^i y^1})^2$ and ϖ always satisfies $\varpi = 1 + \mathcal{O}(\alpha'^2)$.

The relation between complex and real components for vectors and one-forms are summarized by:

$$C_w = \tfrac{1}{2}(C_{w^1} - \mathrm{i}C_{w^2}), \quad C_{w^1} = C_w + C_{\bar{w}}, \quad C_{w^2} = \mathrm{i}(C_w - C_{\bar{w}}). \quad \text{(C.2)}$$

Hodge duality on an m-dimensional manifold is defined as

$$\star F_p = \frac{\sqrt{G}}{p!(m-p)!} \varepsilon^{\nu_1 \ldots \nu_p}{}_{\mu_{p+1} \ldots \mu_m} F_{\nu_1 \ldots \nu_p} dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_m}. \qquad \text{(C.3)}$$

When taking Hodge dual it is useful to differentiate between warped and unwarped metrics. In particular, ε will be with respect to the warped metric, while ϵ is unwarped. The Hodge dual squares to

$$\star \star F_p = (-1)^{p(m-p)} F_p, \quad \text{for a Riemannian space,}$$
$$\star \star F_p = (-1)^{p(m-p)+1} F_p, \quad \text{for a Lorentzian space.} \qquad \text{(C.4)}$$

The adjoint differential operator is defined as

$$d^\dagger = (-1)^{(p+1)m+1} \star d\star, \quad \text{for a Riemannian space,}$$
$$d^\dagger = (-1)^{(p+1)m} \star d\star, \quad \text{for a Lorentzian space,} \qquad \text{(C.5)}$$

while the Laplacian $\square = dd^\dagger + d^\dagger d$.

The RR field strengths are defined by $F_{n+1} = dC_n + H_3 \wedge C_{n-3}$, obeying

$$\star_{10} F_n = (-1)^{\lfloor n/2 \rfloor} F_{10-n}.$$

Appendix D The T-duality rules

In this appendix, we summarize the Buscher rules which are extensively used in constructing the new metrics.

D.1 Metric and fluxes

The Buscher rules determine the value of the metric and B-field. They are given by [79],

$$\widetilde{G}_{99} = G_{99}^{-1},$$
$$\widetilde{G}_{i9} = G_{99}^{-1} B_{i9},$$
$$\widetilde{B}_{i9} = G_{99}^{-1} G_{i9},$$

$$\widetilde{G}_{ij} = G_{ij} - G_{99}^{-1}(G_{9i}G_{9j} - B_{9i}B_{9j}),$$
$$\widetilde{B}_{ij} = B_{ij} - G_{99}^{-1}(G_{9i}B_{9j} - B_{9i}G_{9j}),$$
$$2\widetilde{\phi} = 2\phi - \ln G_{99}, \tag{D.1}$$

where $i = 0, \ldots, 8$ and X^9 is the isometry direction along which we T-dualize. The dilaton becomes

$$e^{\Phi'} = e^{\Phi}\left(\frac{\det \widetilde{G}}{\det G}\right)^{1/4}. \tag{D.2}$$

The RR fluxes dualize as follows [79, 80]:

$$\widetilde{C}^{(n)}_{\mu\ldots\nu\alpha 9} = C^{(n-1)}_{\mu\ldots\nu\alpha} - (n-1)G_{99}^{-1}C^{(n-1)}_{[\mu\ldots\nu|9|}G_{\alpha]9},$$
$$\widetilde{C}^{(n)}_{\mu\ldots\nu\alpha\beta} = C^{(n+1)}_{\mu\ldots\nu\alpha\beta 9} + nC^{(n-1)}_{[\mu\ldots\nu\alpha}B_{\beta]9} + n(n-1)G_{99}^{-1}C^{(n-1)}_{[\mu\ldots\nu|9|}B_{\alpha|9|}G_{\beta]9}, \tag{D.3}$$

where C denotes the original fluxes and $\widetilde{C}$ the T-dualized fluxes. Here $\mu, \nu, \alpha \ldots \neq 9$.

D.2 Spinors under T-duality

The spinors dualize according to the rules written down by Hassan [81]. For T-duality of the supergravity spinors $\epsilon_{1,2}$ this is a simple generalization of the flat space T-duality rules:

$$\epsilon_{\mathrm{L}} \rightarrow \epsilon_{\mathrm{L}},$$
$$\epsilon_{\mathrm{R}} \rightarrow \beta_9 \epsilon_{\mathrm{R}}. \tag{D.4}$$

In flat space, the space-time indices coincide with tangent space indices and $\beta_9 = \Gamma\Gamma_9$ as usual. In a curved background, one simply generalizes β_9:

$$\beta_9 = \sqrt{G_{99}^{-1}}\Gamma\Gamma_9, \tag{D.5}$$

where G_{MN} the original metric, with M, N space-time indices. The gamma matrices satisfy $\{\Gamma^M, \Gamma^N\} = 2G^{MN}$, $\{\Gamma, \Gamma^M\} = 0$ and $\Gamma^2 = 1$. Further, $\Gamma_9 = G_{9M}\Gamma^M$, or in terms of Lorentz frame indices $a, b, \ldots$, we have $\Gamma_9 = e_{9a}\Gamma^a$, where $G_{MN} = e_M^a \eta_{ab} e_N^b$ and $e_{Ma} = G_{MN}e_a^N$. The normalization is determined by $\beta^2 = e^{i\pi F_R}$.

The vielbein transforms under T-duality as follows

$$\widetilde{e}_a^M = Q_N^M e_a^N \tag{D.6}$$

where

$$Q_N^M = \begin{pmatrix} G_{xx} & (G+B)_{xa} \\ 0 & 1 \end{pmatrix}. \tag{D.7}$$

The left-moving vielbein is invariant, while the right-moving Vielbein transforms in the following way

$$e_a^M \rightarrow Q_N^M \, e_a^N. \tag{D.8}$$

Appendix E   Type-IIB supergravity with SU(2) structure

In this section, the constraints supersymmetry imposes on the fluxes and geometry are reviewed for a general SU(2) structure spinor ansatz. Such an analysis was first performed in [75], and we review that work here because the solutions derived in Section 4 are examples of this type.

The spinor basis used in [75] is inconvenient for our purposes, and so we will rederive the pertinent results in a more convenient basis and notation.

E.1   Type-IIB supersymmetry

In this paper we are interested in $d = 4$ compactifications preserving $N = 1$ supersymmetry in space-time. The most general metric with Minkowski space-time takes the form

$$ds^2 = e^{-3\phi/4}\eta_{\mu\nu}dx^\mu dx^\nu + e^{3\phi/4}g_{ab}dy^a dy^b,$$

where the internal metric g_{ab} will in general have SU(2) structure. We formulate the type-IIB supergravity variations in Einstein frame where the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ symmetry of type-IIB supergravity is manifest. In Section E.3, we will switch to string frame which is convenient for performing string dualities.

The only non-trivial variations are those of the dilatino and gravitino:

$$\delta\lambda = \frac{i}{\kappa}\widetilde{\Gamma}^M P_M \varepsilon^* - \frac{i}{4}\widetilde{G}\varepsilon, \tag{E.1}$$

$$\delta\Psi_M = \frac{1}{\kappa}\widetilde{D}_M \varepsilon + \frac{i}{480}\widetilde{\Gamma}^{M^1...M^5} F_{M_1...M_5}\widetilde{\Gamma}_M \varepsilon - \frac{1}{16}\widetilde{\Gamma}_M \widetilde{G}\varepsilon^* - \frac{1}{8}\widetilde{G}\widetilde{\Gamma}_M \varepsilon^*. \tag{E.2}$$

The supersymmetry spinor ε is a complex $d = 10$ Weyl spinor, and the tilde denotes gamma matrices that are defined with respect to the warped metric. The field content of type-IIB consists of a three-form G_3, axio-dilaton τ and self-dual five-form F_5. Here $\widetilde{G} = \frac{1}{6}G_{MNP}\widetilde{\Gamma}^{MNP}$. The derivative of the dilaton and U(1) connection are given by

$$P_M = f^2 \partial_M B, \quad Q_M = f^2 \text{Im}\,(B\partial_M B^*), \quad \text{with}$$
$$B = \frac{1 + i\tau}{1 - i\tau}, \quad f^{-2} = 1 - BB^*, \quad \tau = C_0 + ie^{-\Phi}. \tag{E.3}$$

To preserve $d = 4$ Poincare invariance, we require all fields to depend only on internal coordinates, G_3 to have only internal legs, and F_5 to be space-time filling viz.

$$F_5 = (1 + \star)dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dh, \tag{E.4}$$

with $\star$ the $d = 10$ Hodge dual and $h = h(y)$ an arbitrary scalar defined on the internal manifold. The relation to the usual string frame quantities F_3, H_3 is given by

$$\kappa G_3 = ie^{i\theta}\frac{F_3^s - \tau H_3^s}{\tau_2^{1/2}}, \quad \text{with} \quad e^{i\theta} = \left(\frac{1 + i\tau^*}{1 - i\tau}\right)^{1/2}. \tag{E.5}$$

The five-form is rescaled

$$4\kappa F_5 = F_5^s, \tag{E.6}$$

where s denotes string frame quantities. Newton's constant is given by $2\kappa^2 = (2\pi)^7 g^2 \alpha'^2$. The metric and spinor are also rescaled

$$G_{MN} = e^{-\Phi/2}G_{MN}^s, \qquad \varepsilon = e^{-\Phi/8}(\epsilon_1 + i\epsilon_2). \tag{E.7}$$

E.2   SU(2) structure

Let us now turn to the spinor analysis of the type-IIB supersymmetry variations. The most general spinor ansatz preserving $N = 1$ space-time supersymmetry is given in (5.3), which in Einstein frame, takes the form

$$\varepsilon = \zeta_- \otimes \eta_+^1 + \zeta_+ \otimes \eta_-^2, \tag{E.8}$$

where $\zeta_\pm$ are the $d = 4$ space-time supersymmetry spinors, while $\eta_\pm^{1,2}$ are complex $d = 6$ Weyl spinors, with $\pm$ denoting chirality. The presence of SU(2) structure implies that there are two orthogonal well-defined spinors η_+ and χ_+. For a supersymmetric solution preserving local SU(2) structure, we can expand $\eta^{1,2}$ as

$$\eta_+^1 = a\eta_+ + b\chi_+, \tag{E.9}$$
$$\eta_+^2 = c\eta_+ + d\chi_+. \tag{E.10}$$

In SU(2) structure manifolds the spinors η_+ and χ_+ are used to form spinor bilinears. These bilinears give a unique characterization of the geometry. The spinors $\eta_\pm$ and $\chi_\pm$ can be normalized so that $\eta_\pm^\dagger \eta_\pm = \chi_\pm^\dagger \chi_\pm = 1$. They are related by

$$\chi_+ = \tfrac{1}{2} w_m \gamma^m \eta_-, \tag{E.11}$$

where $\{\gamma^m, \gamma^n\} = 2g^{mn}$ refer to the unwarped metric. We can now form the spinor bilinears:

$$w_m = \eta_+^{\mathrm{T}} \gamma_m \chi_+, \quad J_{mn} = -\mathrm{i}\eta_-^{\mathrm{T}} \gamma_{mn} \eta_+,$$
$$K_{mn} = \eta_+^{\mathrm{T}} \gamma_{mn} \chi_-, \quad \Omega_{npq} = -\eta_+^{\mathrm{T}} \gamma_{npq} \eta_+. \tag{E.12}$$

These bilinears encode information about the local geometry. By taking their exterior derivative one gets expressions for the intrinsic torsion modules. The intrinsic torsion allows one to read off various metric properties – for example, whether the metric is complex, Kähler, Calabi–Yau, etc. Note that we have defined the forms (E.12) with respect to the unwarped metric. Further, we have chosen a complex structure J defined by η_+ and corresponding $(3,0)$-form Ω. With SU(2) structure this is not the only choice: there is in fact, a U(1) of choices of complex structure given by

$$J = -\mathrm{i}\beta_-^{\mathrm{T}} \gamma_{mn} \beta_+,$$

where

$$\beta_+ = \eta_+ \cos\phi + \chi_+ \sin\phi.$$

The choice of ϕ will not affect whether the solution is supersymmetric or not, but does affect its interpretation. For example, the integrability of the complex structure depends on the choice of ϕ [75].

With respect to the choice complex structure in (E.12) above, the one-form w is holomorphic, as is the two-form K with $K = J^2 + iJ^3$. Further,

$$J = J^1 + \frac{i}{2}w \wedge \bar{w}, \quad \Omega = K \wedge w.$$

E.3   Solving type-IIB supergravity with SU(2) structure

We will now rewrite the type-IIB supersymmetry conditions (E.1) and (E.2) in terms of the forms w_m, J_{mn}, and K_{mn} defined in the previous section following [75]. As a first step we decompose the fluxes into SU(2) forms.

Dilaton:

$$P_m = p_1 w_m + p_2 \bar{w}_m + \Pi_m, \tag{E.13}$$

where Π_m is a real one-form with $w \lrcorner \Pi = 0$.

Three-form flux:

We decompose a general three-form flux in terms of the complex structure given above,

$$\begin{aligned}
G_3 = {} & g_{(3,0)} K \wedge w + g_{(2,1)} K \wedge \bar{w} + \widetilde{g}_{(2,1)} J^1 \wedge w + J^1 \wedge V^1 \\
& + w \wedge \bar{w} \wedge V^2 + w \wedge T^1 + \bar{w} \wedge T^2 + g_{(1,2)} \bar{K} \wedge w \\
& + \widetilde{g}_{(1,2)} J^1 \wedge \bar{w} + g_{(0,3)} \bar{K} \wedge \bar{w}.
\end{aligned} \tag{E.14}$$

This is the most general expansion of G_3 in terms of SU(2) modules. Here V^i and T^i are one-forms and two-forms, respectively, orthogonal to K and w:

$$V^i \lrcorner w = 0, \quad T^i \lrcorner w = 0.$$

The components $g_{(2,1)}$ and $g_{(1,2)}$ are primitive while $g_{(2,1)}$ and $g_{(1,2)}$ are non-primitive. Further, combining V^1 and V^2, the primitive and non-primitive

components can be made explicit viz.

$$J^1 \wedge V_1 + w \wedge \bar{w} \wedge V_2 = \tfrac{1}{2} \left(J^1 - \frac{\mathrm{i}}{2} w \wedge \bar{w} \right) \wedge (V_1 + 2\mathrm{i}V_2) + \tfrac{1}{2} J \wedge (V_1 - 2\mathrm{i}V_2).$$
(E.15)

The imaginary self-dual (ISD) limit corresponds to

$$g_{30} = g_{12} = \widetilde{g}_{21} = 0, \quad T^2 = 0,$$
$$(1 + \mathrm{i}J)(V^1 + 2\mathrm{i}V^2) = (1 - \mathrm{i}J)(V^1 - 2\mathrm{i}V^2) = 0. \qquad (\text{E.16})$$

Five-form flux and the warp factor:

Lastly, we may similarly expand the warp factor:

$$\partial_n (\log \mathrm{e}^{3\phi/2}) = \sigma w_n + \bar{\sigma}\bar{w}_n + \Sigma_n,$$
$$(\partial_n h) = \theta w_n + \bar{\theta}\bar{w}_n + H_n.$$

E.4   The supersymmetry variations

Dilatino:

First, we use the metric ansatz (E.1) together with the spinor ansatz (E.8) and plug it into the dilatino variation (E.1) giving:

$$\mathrm{e}^{-3\phi/8}\frac{\mathrm{i}}{\kappa}(\gamma^5 \otimes \gamma^n P_n) \left[\zeta_- \otimes (a^*\eta_+ + b^*\chi_+) + \zeta_+ \otimes (c^*\eta_- + d^*\chi_-] \right.$$
$$= \mathrm{e}^{-9\phi/8}\frac{1}{24}(\gamma^5 \otimes \gamma^{npq}) \left[\zeta_+ \otimes (a\eta_- + b\chi_-) + \zeta_- \otimes (c\eta_+ + d\chi_+) \right] G_{npq}.$$

The gamma matrices γ^a and γ^μ are defined with respect to the unwarped metric, which results in the warp factor appearing. Because ζ_+ and ζ_- are independent, the variations proportional to these spinors must vanish. This gives

$$\gamma^n P_n(a^*\eta_+ + b^*\chi_+) - \frac{\mathrm{e}^{-3\phi/4}\kappa}{24}\gamma^{npq}G_{npq}(c\eta_+ + d\chi_+) = 0,$$
$$\frac{\mathrm{e}^{-3\phi/4}\kappa}{24}\gamma^{npq}G_{npq}(a\eta_- + b\chi_-) - \gamma^n P_n(c^*\eta_- + d^*\chi_-) = 0.$$

A complete basis is specified by $\eta_\pm$ and $\gamma^n\eta_\pm$. We contract with these to rewrite the dilatino variation in terms of SU(2) invariants. This gives the

equations

$$2\mathrm{e}^{3\phi/4}p_1 d^* = -\kappa \left[2ag_{(3,0)} - \mathrm{i}b\widetilde{g}_{(2,1)}\right], \tag{E.17}$$

$$2\mathrm{e}^{3\phi/4}p_2 b^* = -\kappa \left(2c\,g_{(0,3)} + \mathrm{i}d\,\widetilde{g}_{(1,2)}\right), \tag{E.18}$$

$$2a^* c g_{(0,3)} = -\mathrm{i}(b^* c + a^* d)\widetilde{g}_{(1,2)} - 2b^* d g_{(2,1)}, \tag{E.19}$$

$$2ac^* g_{(3,0)} = \widetilde{g}_{(2,1)}\mathrm{i}(ad^* + bc^*) - 2bd^* g_{(1,2)}. \tag{E.20}$$

From contracting with $\gamma^k \eta_\pm$, we find

$$(g + \mathrm{i}J)^{kn}\Pi_n c^* + K^{kn}\Pi_n d^*$$
$$= -\mathrm{i}\frac{\kappa \mathrm{e}^{-3\phi/4}}{4}\left[(g - \mathrm{i}J)^{kn}(V_1 - 2\mathrm{i}V_2)_n a - bK^{kn}(V^1 + 2\mathrm{i}V^2)_n\right], \quad \text{(E.21)}$$
$$(g + \mathrm{i}J)^{kn}\Pi_n a^* + \bar{K}^{kn}\Pi_n b^*$$
$$= \mathrm{i}\frac{\kappa \mathrm{e}^{-3\phi/4}}{4}\left[c(g + \mathrm{i}J)^{kn}(V^1 - 2\mathrm{i}V^2)_n - d\bar{K}^{kn}(V^1 + 2\mathrm{i}V^2)_n\right]. \quad \text{(E.22)}$$

The terms $V^1 + 2\mathrm{i}V^2$ and $V^1 - 2\mathrm{i}V^2$ are part of the primitive and non-primitive components of the flux, respectively. These terms source the remaining part of the dilaton Π_n and the spinor via a, b, c and d.

Equation (E.17) shows that the appearance of $(3,0)$ and non-primitive $(2,1)$ flux source the holomorphic part of the dilaton, while (E.18) shows the analogous statement for the $(0,3)$ and $(1,2)$ components of G_3. The second pair of equations, (E.19) and (E.20), show that the appearance of $(0,3)$ flux is related to the primitive and non-primitive $(2,1)$-components of G_3, and vice versa for the $(3,0)$ component.

Thus, as noted in [75], we see that with SU(2) structure, the three-form flux no longer need be $(2,1)$ and primitive. Indeed, the non-primitive and $(3,0)$ parts result in a non-holomorphic dilaton. We see both of these features explicitly in our examples. Further, if we restrict to ISD fluxes, we are left with a strict relation between the warp factor and the five-form flux. Indeed, these equations determine a relation between specific components of the three-form flux and the five-form flux which sources the warp factor.

Gravitino:

First consider the space-time component $M = \mu$ of (E.2). We rewrite the gamma matrices in terms of the unwarped metric, and use the fact the

$\partial_\mu \varepsilon = 0$ to find

$$\delta \Psi_\mu = -\frac{1}{4\kappa} \left[\gamma_\mu \otimes \gamma^n \partial_n \log \mathrm{e}^{3\phi/2} \right] \varepsilon + \mathrm{e}^{3\phi/2} \left[\gamma_\mu \otimes \gamma^n \partial_n h \right] \varepsilon$$
$$+ \frac{\mathrm{e}^{-3\phi/2}}{48} \left[\gamma_\mu \otimes \gamma^{npq} G_{npq} \right] \varepsilon^* = 0.$$

We substitute the SU(2) spinor ansatz. For the dilatino variation, the spacetime component decouples giving two independent equations proportional to ζ_+ and ζ_-:

$$\left[-\frac{1}{4\kappa} \gamma^n \partial_n \log \mathrm{e}^{3\phi/2} + \mathrm{e}^{3\phi/4} \gamma^n \partial_n h \right] (a\eta_- + b\chi_-)$$
$$+ \frac{\mathrm{e}^{-3\phi/2}}{48} \gamma^{npq} G_{npq} (c^* \eta_- + d^* \chi_-) = 0,$$
$$\left[-\frac{1}{4\kappa} \gamma^n \partial_n \log \mathrm{e}^{3\phi/2} + \mathrm{e}^{3\phi/4} \gamma^n \partial_n h \right] (c\eta_+ + d\chi_+)$$
$$- \frac{\mathrm{e}^{-3\phi/2}}{48} \gamma^{npq} G_{npq} (a^* \eta_+ + b^* \chi_+) = 0.$$

We now contract with a complete basis of spinors, as above, to give constraints on the warp factor and the fluxes. As in the case of the dilaton, we may expand in terms of SU(2) invariants

$$\partial_n (\log \mathrm{e}^{3\phi/2}) = \sigma w_n + \bar{\sigma} \bar{w}_n + \Sigma_n,$$
$$(\partial_n h) = \theta w_n + \bar{\theta} \bar{w}_n + H_n.$$

We now get a series of equations

$$(\sigma - 4\kappa\theta \, \mathrm{e}^{3\phi/4}) a = \kappa \, \mathrm{e}^{-3\phi/2} (\mathrm{i}\widetilde{g}_{(2,1)} c^* - 2g_{(1,2)} d^*), \tag{E.23}$$
$$(\bar{\sigma} + 4\kappa\bar{\theta} \, \mathrm{e}^{3\phi/4}) d = -\kappa \, \mathrm{e}^{-3\phi/2} \left(2a^* g_{(0,3)} - \mathrm{i}b^* \widetilde{g}_{(1,2)} \right), \tag{E.24}$$

and

$$(g - \mathrm{i}J)^{kn}(\Sigma_n - 4\kappa \, \mathrm{e}^{3\phi/4} H_n) a + K^{kn}(\Sigma_n - 4\kappa \, \mathrm{e}^{3\phi/4} H_n) b$$
$$= -\mathrm{i}\frac{\kappa}{2} \mathrm{e}^{-3\phi/2} \left[(g - \mathrm{i}J)^{kn}(V^1 - 2\mathrm{i}V^2)_n c^* - K^{kn}(V^1 + 2\mathrm{i}V^2)_n d^* \right], \tag{E.25}$$

$$(g + \mathrm{i}J)^{kn}(\Sigma_n + 4\kappa \, \mathrm{e}^{3\phi/4} H_n) c + \bar{K}^{kn}(\Sigma_n + 4\kappa \, \mathrm{e}^{3\phi/4} H_n) d$$
$$= \mathrm{i}\frac{\kappa}{2} \mathrm{e}^{-3\phi/2} \left[(g + \mathrm{i}J)^{kn}(V^1 - 2\mathrm{i}V^2)_n a^* - \bar{K}^{kn}(V^1 + 2\mathrm{i}V^2)_n b^* \right]. \tag{E.26}$$

If we restrict to ISD fluxes, we are left with a strict relation between the warp factor and the five-form flux. This is to be expected from the type-B SUSY analysis.

In the specific examples we construct, this is not the case since the flux is not ISD. Indeed, these equations determine a relation between specific components of the three-form flux, and the five-form flux which source the warp factor.

We now solve the internal component of the gravitino with $M = m$ in (E.2). This is the most involved calculation, and will give a general equation determining the coefficients a, b, c and d in terms of the fluxes. Following the reasoning above, we find two independent equations from ζ_+ and ζ_-:

$$\nabla_m(a\eta_-) + \nabla_m(b\chi_-) + \left(\frac{1}{8}(\gamma_m^n - \delta_m^n)\partial_n \log \mathrm{e}^{3\phi/2} - \frac{i}{2}Q_m\right)(a\eta_- + b\chi_-)$$

$$= -\frac{\mathrm{e}^{3\phi/2}\kappa}{2}\gamma^n\gamma_m\partial_n h(a\eta_- + b\chi_-) + \frac{\mathrm{e}^{-3\phi/4}\kappa}{96}\left(\gamma_m^{npq} + 9\gamma^{[np}\delta_m^{q]}\right)$$
$$\times G_{npq}(c^*\eta_- + d^*\chi_-),$$

$$\nabla_m(c\eta_+) + \nabla_m(d\chi_+) + \left(\frac{1}{8}(\gamma_m^n - \delta_m^n)\partial_n \log \mathrm{e}^{3\phi/2} - \frac{i}{2}Q_m\right)(c\eta_+ + d\chi_+)$$

$$= \frac{\mathrm{e}^{3\phi/2}\kappa}{2}\gamma^n\gamma_m\partial_n h(c\eta_+ + d\chi_+) + \frac{\mathrm{e}^{-3\phi/4}\kappa}{96}\left(\gamma^{npq}_m + 9\gamma^{[np}\delta_m^{q]}\right)$$
$$\times G_{npq}(a^*\eta_+ + b^*\chi_+),$$

which give two independent equations for $\nabla_m\eta_-$ and $\nabla_m\chi_-$. Rewriting these equations gives

$$\nabla_m\eta_- = \frac{1}{\Delta}(d^*\partial_m a - b\partial_m c^*)\eta_- + \frac{1}{\Delta}(d^*\partial_m b - b\partial_m d^*)\chi_-$$

$$+ \frac{1}{8}(\gamma_m^n - \delta_m^n)\partial_n \log \mathrm{e}^{3\phi/2}\eta_- - \frac{i}{2}Q_m\eta_-$$

$$- \frac{\mathrm{e}^{3\phi/2}\kappa}{2\Delta}\gamma^n\gamma_m\partial_n h\left[(ad^* + bc^*)\eta_- + 2bd^*\chi_-\right]$$

$$+ \frac{\mathrm{e}^{-3\phi/4}\kappa}{96\Delta}\left(\gamma_m^{npq} + 9\gamma^{[np}\delta_m^{q]}\right)\left[(G_{npq}c^*d^* + \bar{G}_{npq}ab)\eta_-\right.$$
$$\left. + (G_{npq}d^{*\,2} + \bar{G}_{npq}b^2)\chi_-\right],$$

and

$$\nabla_m\chi_- = \frac{1}{\Delta}(c^*\partial_m a - a\partial_m c^*)\eta_- + \frac{1}{\Delta}(c^*\partial_m b - a\partial_m d^*)\chi_-$$

$$- \frac{1}{8}(\gamma_m^n - \delta_m^n)\partial_n \log \mathrm{e}^{3\phi/2}\chi_- + \frac{i}{2}Q_m\chi_-$$

$$
+ \frac{\mathrm{e}^{3\phi/2}\kappa}{2\Delta} \gamma^n \gamma_m \partial_n h \left[2ac^*\eta_- + (bc^* + ad^*)\chi_- \right]
$$

$$
- \frac{\mathrm{e}^{-3\phi/4}\kappa}{96\Delta} \left(\gamma^{npq}_m + 9\gamma^{[np}\delta^{q]}_m \right) [(G_{npq}c^{*\,2} - \bar{G}_{npq}a^2)\eta_-
$$

$$
+ (G_{npq}c^*d^* - \bar{G}_{npq}ab)\chi_-],
$$

where $\Delta = ad^* - bc^*$ is non-singular for SU(2) structure. Degenerate points are where the structure becomes SU(3). We now use $d(\eta^{\mathrm{T}}_+\eta_-) = 0$, $d(\chi^{\mathrm{T}}_+\chi_-) = 0$ and $d(\chi^{\mathrm{T}}_+\eta_-) = 0$ to give a series of equations determining a, b, c and d.

1. Using $d(\eta^{\mathrm{T}}_+\eta_-) = 0$ gives

$$
\frac{d^*\partial_m a - b\partial_m c^*}{\Delta\kappa} + \frac{d\partial_m a^* - b^*\partial_m c}{\Delta^*\kappa}
$$

$$
= -\frac{1}{8}\partial_m \log \mathrm{e}^{3\phi/2} - \frac{\mathrm{e}^{3\phi/2}}{2}(\delta^n_m - \mathrm{i}J^n_m)\partial_n h \frac{ad^* + bc^*}{\Delta} - \mathrm{e}^{3\phi/2}K^n_m H_n \left(\frac{bd^*}{\Delta} \right)
$$

$$
+ \frac{\mathrm{e}^{-3\phi/4}}{4\Delta} w_m \left[4g_{12}(d^*)^2 - \mathrm{i}c^*d^*\widetilde{g}_{21} - \mathrm{i}ab\,\widetilde{g}^*_{12} + 4b^2 g^*_{21} \right]
$$

$$
+ \frac{\mathrm{e}^{-3\phi/4}}{4\Delta^*} w_m \left[4g^*_{03}(d)^2 + 2\mathrm{i}cd\widetilde{g}^*_{12} + 2\mathrm{i}a^*b^*\,\widetilde{g}_{21} + 4(b^*)^2 g_{30} \right]
$$

$$
+ \frac{\mathrm{e}^{-3\phi/4}}{16\Delta} \left[(J^n_m - 3\mathrm{i}\delta^n_m)(V^1 + 2\mathrm{i}V^2)_n c^*d^* + \mathrm{i}K^n_m(3V^1 - 2\mathrm{i}V^2)_n(d^*)^2 \right]
$$

$$
+ \frac{\mathrm{e}^{-3\phi/4}}{16\Delta} \left[(J^n_m - 3\mathrm{i}\delta^n_m)(V^1 + 2\mathrm{i}V^2)_n ab + \mathrm{i}K^n_m(3V^1 - 2\mathrm{i}V^2)_n b^2 \right] + \text{c.c.}
$$

$$
\tag{E.27}
$$

2. Now we use $d(\chi^{\mathrm{T}}_+\chi_-) = 0$. This gives

$$
\frac{c^*\partial_m b - a\partial_m d^*}{\Delta\kappa} + \frac{c\partial_m b^* - a^*\partial_m d}{\Delta^*\kappa}
$$

$$
= -\frac{1}{8}\partial_m \log \mathrm{e}^{3\phi/2} + \frac{\mathrm{e}^{3\phi/2}}{2\Delta}[(bc^* + ad^*)[(\delta^n_m + \mathrm{i}J^n_m)\partial_n h - 2(w_m\bar{\theta} - \bar{w}_m\theta)]]
$$

$$
- \frac{ac^*\mathrm{e}^{3\phi/2}}{\Delta}\bar{K}^n_m H_n + \frac{\mathrm{e}^{-3\phi/4}}{16\Delta} \left\{ -(c^*)^2 \bar{K}^n_m(2V^2 + 3\mathrm{i}V^1)_n \right\}
$$

$$
+ \frac{\mathrm{e}^{-3\phi/4}}{16\Delta} \left\{ a^2 K^n_m(2V^2 + 3\mathrm{i}V^1)_n - (J^n_m + 3\mathrm{i}\delta^n_m)(V^1 + 2\mathrm{i}V^2)_n(c^*d^*) \right\}
$$

$$
+ \frac{\mathrm{e}^{-3\phi/4}}{16\Delta} \left\{ (ab)(J^n_m + 3\mathrm{i}\delta^n_m)(V^1 - 2\mathrm{i}V^2)_n + 12V^2_n(ab - c^*d^*) \right\}
$$

$$+ \frac{\mathrm{e}^{-3\phi/4}}{4\Delta} w \left[2g_{30}(c^*)^2 - 2g_{03}^* a^2 - 2\mathrm{i}\widetilde{g}_{21}^* c^* d^* + 2\mathrm{i}\widetilde{g}_{12}^* ab \right]$$

$$+ \frac{\mathrm{e}^{-3\phi/4}}{4\Delta^*} w \left[2g_{21}^* c^2 - 2g_{12}(a^*)^2 + \mathrm{i}\widetilde{g}_{12}^*(cd) - \mathrm{i}\widetilde{g}_{21}(ab)^* \right] + \text{c.c.} \qquad \text{(E.28)}$$

3. Finally, we obtain a complex equation using $d(\chi_+^{\mathrm{T}} \eta_-) = 0$. Using the above contractions, we get

$$\frac{b\partial_m d^* - d^* \partial_m b}{\Delta} + \frac{a^* \partial_m c - c\partial_m a^*}{\Delta^*} = -\frac{1}{8}(K + \bar{K})_m^n \Sigma_n$$

$$+ \frac{\mathrm{e}^{3\phi/2}\kappa}{2} \left[K_m^n H_n \frac{ad^* + bc^*}{\Delta} - \bar{K}_m^n H_n \frac{b^* c + a^* d}{\Delta^*} \right]$$

$$- \mathrm{e}^{3\phi/2}\kappa \left[\left(\frac{bd^*}{\Delta} + \frac{a^* c}{\Delta^*} \right) (\delta_m^n + \mathrm{i}J_m^n)\partial_n h + \frac{2bd^*}{\Delta} w_{[m} \bar{w}_{n]} \partial^n h \right]$$

$$+ \frac{\mathrm{e}^{-3\phi/4}\kappa}{16} \left[\mathrm{i}\bar{K}_m^n (3V^1 - 2\mathrm{i}V^2)_n \left(\frac{c^* d^* + ab}{\Delta} \right) + \mathrm{i}\bar{K}_m^n (3V^1 + 2\mathrm{i}V^2)_n \right.$$

$$\times \left(\frac{cd - a^* b^*}{\Delta^*} \right) + (J_m^n + 3\mathrm{i}\delta_m^n)(V^1 - 2\mathrm{i}V^2)_n \left(\frac{(d^*)^2}{\Delta} + \frac{(a^*)^2}{\Delta^*} \right)$$

$$+ (J_m^n + 3\mathrm{i}\delta_m^n)(V^1 + 2\mathrm{i}V^2)_n \left(\frac{b^2}{\Delta} - \frac{(c^*)^2}{\Delta^*} \right) \right]. \qquad \text{(E.29)}$$

As opposed to the previous two real equations, (E.27) and (E.28), this is a complex equation. We therefore find four independent real equations determining the coefficients a, b, c and d.

These results can be used to check that the type-IIB supersymmetry variations vanish for our explicit examples. As an additional check, we used Mathematica to check that the equations of motion are satisfied.

Acknowledgments

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