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On $\mathcal{N} = 1$ 4d effective couplings for F-theory and heterotic vacua

Hans Jockers[1], Peter Mayr[2], and Johannes Walcher[3]

[1]Department of Physics, Stanford University, Stanford, CA 94305-4060, USA
jockers@kitp.ucsb.edu
[2]Arnold Sommerfeld Center for Theoretical Physics, LMU, Theresienstrasse 37, D-80333 Munich, Germany
mayr@physik.uni-muenchen.de
[3]PH-TH Division, CERN, CH-1211 Geneva 23, Switzerland
johannes.walcher@cern.ch

Abstract

We show that certain superpotential and Kähler potential couplings of $\mathcal{N} = 1$ supersymmetric compactifications with branes or bundles can be computed from Hodge theory and mirror symmetry. This applies to F-theory on a Calabi–Yau four-fold and three-fold compactifications of type II and heterotic strings with branes. The heterotic case includes a class of bundles on elliptic manifolds constructed by Friedmann, Morgan and Witten. Mirror symmetry of the four-fold computes non-perturbative corrections to mirror symmetry on the three-folds, including D-instanton corrections. We also propose a physical interpretation for the observation by Warner that relates the deformation spaces of certain matrix factorizations and the periods of non-compact four-folds that are ALE fibrations.

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1 Introduction

Let Z_B be a Calabi–Yau (CY) three-fold and E a holomorphic bundle or sheaf on it. In a certain decoupling limit, where one neglects the backreaction of the full string theory to the degrees of freedom of the bundle, E can describe either a (sub-)bundle of a heterotic string compactification on Z_B, a heterotic five-brane or a B-type brane in a type II compactification on Z_B. In the latter case we will also be interested in the geometry (Z_A, L)

associated to (Z_B, E) by open string mirror symmetry, which consists of an A-type brane L on the mirror three-fold Z_A of Z_B. The contribution of the bundle to the space–time superpotential of a string compactification on Z_B is, in a certain approximation, given by the holomorphic Chern–Simons functional for both the heterotic bundle [1] and the B-type brane [2]

$$W_{\mathrm{CS}} = \int_{Z_B} \Omega \wedge \mathrm{tr}\left(\tfrac{1}{2} A \wedge \bar{\partial} A + \tfrac{1}{3} A \wedge A \wedge A\right). \tag{1.1}$$

Here Ω is the holomorphic (3,0) form on Z_B and A is the $(0,1)$ part of the connection on E. There is another superpotential proportional to the periods of Ω, which, again in a certain approximation, is of the form

$$W_G = \int_{Z_B} \Omega \wedge G = (N_\Sigma + S\hat{N}_\Sigma) \int_{\gamma_\Sigma} \Omega, \qquad \gamma_\Sigma \in H_3(Z_B, \mathbf{Z}). \tag{1.2}$$

In the type II compactification on Z_B, W_G is the superpotential induced by NS and RR three-form fluxes [3], and S the complex dilaton. In heterotic compactifications, W_G will be related below to the superpotential of a compactification on non-Kähler manifolds with H-flux [4]. Depending on the type of string theory and its compactification, the combined superpotential

$$W = W_{\mathrm{CS}} + W_G \tag{1.3}$$

may be exact or subject to various quantum corrections.

The purpose of this note is to show how the methods of mirror symmetry of [5–7] when combined with Hodge theory can be used to compute effective couplings of these heterotic/type II compactifications, including the superpotential and the Kähler potential. Hodge theory enters in two steps: A "classical" theory on the CY three-fold, which computes the integrals on the three-fold in (1.1), (1.2), and a "quantum" deformation of these three-fold data defined by the (classical) Hodge variation on a "dual" CY four-fold. Physicswise, the four-fold geometry represents the compactification manifold of a dual F-theory or type IIA compactification. We will argue that the four-fold result agrees with the three-fold result when it should, but gives more general results, including the case when the heterotic three-fold is not CY.

The first step on the three-fold can be realized by computing the Hodge variation on a *relative* cohomology group $H^3(Z_B, D)$, which captures the brane/bundle data in addition to the geometry of Z_B. This was shown previously in the context of B-type branes in [8–11] and we generalize this relation here to heterotic five-branes and general bundles, including the bundles on elliptically fibered three-fold Z_B constructed by Friedman *et al.* [12] (see also [13]). The "classical" Hodge theory on the three-fold gives an explicit evaluation of the three-fold integrals in (1.1), (1.2) and a preferred choice of physical coordinates, which leads to the prediction of world-sheet corrections from sphere and disc instantons of the appropriately defined mirror theories.

The second step involves Hodge theory and mirror symmetry on a mirror pair of dual CY four-folds. Four-folds enter the stage in two seemingly different ways, in remarkable parallel with the two appearances of (1.1) in heterotic and type II compactifications on Z_B. Firstly, through the duality of heterotic strings on elliptically fibered CY three-fold Z_B to F-theory on a CY four-fold X_B [14,15]. This duality motivated the systematic construction of "heterotic" bundles on elliptically fibered Z_B in [12, 13]. Secondly, four-folds appear in the computation of brane superpotentials of type II strings via an "open–closed string duality", which associates a non-compact four-fold geometry X_B^{nc} to a B-type brane on a three-fold Z_B [10, 16, 17]. In this approach, the superpotential (1.1) of the brane compactification on (Z_B, E) is computed from the periods of the holomorphic $(4, 0)$ form on the dual four-fold X_B^{nc}. Moreover, mirror symmetry of four-folds relates the sphere instanton corrected periods on the mirror four-fold X_A^{nc} of X_B^{nc} to the disc instanton corrected superpotential of the compactification with A-type brane L on the mirror manifold Z_A of Z_B. This surprising relation between mirror symmetry of the four-folds X_A^{nc} and X_B^{nc} and open string mirror symmetry of the brane geometries (Z_B, E) and (Z_A, L) has been tested in various different contexts, see, e.g., [11, 18–20].

As we will argue below, these two four-fold strands are in fact connected by a certain physical and geometrical limit, that relates open–closed duality to heterotic/F-theory duality.[1] In this limit part of the bundle degrees of freedoms decouple (in a physical sense) from the remaining compactification and the type II brane and the heterotic bundle are equalized. Geometrically, this can be viewed as a local mirror limit in the open string sector of type II strings or a local mirror limit for bundles considered in [22, 23], respectively. In this limit, the F-theory/type IIA superpotential on the dual four-fold X_B reduces to the "classical" type II/heterotic superpotential (1.3) on the three-fold Z_B, as has been observed previously in [11].

The result obtained from an F-theory/type IIA compactification on the dual four-fold differs from the three-fold result away from the decoupling limit. We assert that these deviations represent physical corrections to

[1] A related explanation of type II open–closed duality based on T-duality of five-branes [21] has been recently given in [17].

the dual type II/heterotic compactification from perturbative and instanton effects and describe how Hodge theory and mirror symmetry on the four-fold provides a powerful computational tool to determine these perturbative and non-perturbative contributions. Depending on the point of view, the corrections computed by mirror symmetry of four-folds describe world-sheet, D-brane, or space–time instanton effects in the dual type II and heterotic compactifications.

Finally, we discuss the type II/heterotic duality in the context of non-compact four-folds that arise as two-dimensional ALE fibrations. For a particular choice of background fluxes these models admit a description in terms of certain Kazama–Suzuki coset models [24, 25], whose deformation spaces coincide with the deformation spaces of matrix factorizations of $\mathcal{N} = 2$ minimal models [26]. We give a physical interpretation of this relation via type II/heterotic duality and we propose that this correspondence holds even more generally.

The organization of this note is as follows. In Section 2 we discuss the application of Hogde theory to the evaluation of the Chern–Simons functional (1.1) with a focus on bundles on elliptic CY three-fold constructed by Friedman *et al.* [12]. For a perturbative bundle with structure group $SU(N)$ the superpotential captures obstructions to the deformation of the spectral cover Σ imposed by a certain choice of line bundle. We discuss also the case of a general structure group G and heterotic five-branes. In Section 3 we describe the decoupling limit in the type II and heterotic compactifications and use it to relate open–closed string duality to F-theory/heterotic duality, giving an explicit map between type II and heterotic compactifications. We discuss the relevant string dualities and the meaning of the quantum corrections in the dual theories. In Section 4, we argue, that the F-theory superpotential on the four-fold captures more generally the heterotic superpotential for a bundle compactification on a generalized CY manifold and describe the map from the F-theory superpotential to the superpotential for heterotic bundles and heterotic five-branes. In Section 5 we extend the previous discussion to the Kähler potential and the twisted superpotential by studying the effective supergravity for the two-dimensional compactification of type IIA on the four-fold and heterotic strings on $T^2 \times Z_B$. In Section 6 we start to demonstrate our techniques for an example of an $\mathcal{N} = 1$ supersymmetric bundle compactification on the quintic. We discuss the perturbative heterotic theory, the general structure of the quantum corrections and give explicit results for the example. In Section 7 we consider other interesting examples, including heterotic five-branes wrapping a curve in the base of the heterotic CY manifold and bundles with non-trivial Jacobians. In Section 8 we connect via heterotic/type II duality the deformation spaces of certain matrix factorizations to the deformation spaces of type II on non-compact four-folds that are ALE fibrations with fluxes. Section 9 contains our conclusions. In the appendix we present further technical details on the computations for the toric hypersurface examples analyzed in the main text.

2 Hodge theoretic data and $\mathcal{N} = 1$ superpotentials

2.1 Hodge variations in open–closed duality

In the approach of [8, 9, 11], the superpotential of B-type brane compactifications with five-brane charge on a CY Z_B is computed from the mixed Hodge variation on a certain relative cohomology group $H^3(Z_B, D)$. The superpotential is a linear combination of the period integrals of the relative (3,0) form $\underline{\Omega} \in H^{3,0}(Z_B, D)$

$$W_{\mathrm{II}}(Z_B, D) = \sum_{\gamma_\Sigma \in H_3(Z_B)} N_\Sigma \int_{\gamma_\Sigma} \underline{\Omega}^{(3,0)} + \sum_{\substack{\gamma_\Sigma \in H_3(Z_B, D) \\ D \supset\ \partial\gamma_\Sigma \neq 0}} \hat{N}_\Sigma \int_{\gamma_\Sigma} \underline{\Omega}^{(3,0)}. \quad (2.1)$$

The first term is the RR "flux" superpotential [3, 24] on three-cycles $\gamma_\Sigma \in H_3(Z_B)$ and the second term an off-shell version of the brane superpotential [7, 27, 28] defined on three-chains γ_Σ with non-empty boundary. Note that the superpotential $W_{\mathrm{II}}(Z_B, D)$ associated with the Hodge bundle does not include the NS part of the type II flux potential.

The boundary $\partial\gamma_\Sigma$ is required to lie in a hypersurface $D \subset Z_B$, $\partial\gamma_\Sigma \in H_2(D)$. The moduli of the hypersurface D parametrize certain deformations of the brane configuration (Z_B, E). Infinitesimally, the accessible deformations are described by elements in $H^{2,1}(Z_B, D)$ and come in two varieties,

$$\phi_a \in H^{2,1}(Z_B), \qquad \hat{\phi}_\alpha \in H^{2,0}(D). \quad (2.2)$$

Here $H^{2,1}(Z_B)$ captures the deformations of the complex structure of the three-fold Z_B and $H^{2,0}(D)$ the deformations of the holomorphic hypersurface $i : D \hookrightarrow Z_B$.

Mirror symmetry maps the B-type brane configuration (Z_B, E) to an A-type brane configuration (Z_A, L) on the mirror three-fold Z_A. The flat Gauss–Manin connection on $H^3(Z_B, D)$ determines the mirror map $z(t)$ between the complex structure moduli z of (Z_B, E) and the Kähler moduli t of (Z_A, L). Inserting the mirror map into (2.1) then gives the disc instanton corrected superpotential of the A-type geometry near a suitable large volume point of (Z_A, L) [11].

The relative cohomology problem and open string mirror symmetry is related to absolute cohomology and mirror symmetry of CY four-folds by a certain open–closed string duality [10, 16, 17]. The constructions of these papers associate to a B-type brane compactification (Z_B, E) and its mirror (Z_A, L) a pair of non-compact mirror four-folds $(X_A^{\rm nc}, X_B^{\rm nc})$, such that the "flux" superpotential of [24] agrees with the combined "flux" and brane superpotential (2.1) of the three-fold compactification,

$$W(X_B^{\rm nc}) = \sum_{\gamma_\Sigma \in H_4(X_B^{\rm nc})} \underline{N}_\Sigma \int_{\gamma_\Sigma} \Omega^{(4,0)} = W_{\rm II}(Z_B, D), \tag{2.3}$$

for appropriate choice of coefficients $N_\Sigma, \hat{N}_\Sigma, \underline{N}_\Sigma$. Open-closed string duality thus links the pure Hodge variation on $H^4_{\rm hor}(X_B^{\rm nc})$ to the mixed Hodge variation on the relative cohomology space $H^3(Z_B, D) \simeq H^3(Z_B) \oplus H^2_{\rm var}(D)$. The relation between the pure Hodge spaces appearing in this relation is schematically

$$\begin{array}{ccccccccc}
H^{3,0}(Z_B) & \xrightarrow{\delta} & H^{2,1}(Z_B) & \xrightarrow{\delta} & H^{1,2}(Z_B) & \xrightarrow{\delta} & H^{0,3}(Z_B) & & \\
\uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\alpha} & & \\
H^{4,0}(X_B^{\rm nc}) & \xrightarrow{\delta} & H^{3,1}(X_B^{\rm nc}) & \xrightarrow{\delta} & H^{2,2}_{\rm hor}(X_B^{\rm nc}) & \xrightarrow{\delta} & H^{1,3}(X_B^{\rm nc}) & \xrightarrow{\delta} & H^{0,4}(X_B^{\rm nc}) \\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & \\
 & & H^{2,0}(D) & \xrightarrow{\delta} & H^{1,1}_{\rm var}(D) & \xrightarrow{\delta} & H^{0,2}(D) & &
\end{array} \tag{2.4}$$

Here δ denotes universally a variation in the complex structure of the respective geometries, represented by the Gauss–Manin derivative and projecting onto pure pieces.

The two maps $\alpha, \beta:\ H^4_{\rm hor}(X_B^{\rm nc}) \to H^3(Z_B, D)$ identify an element of $H^4_{\rm hor}(X_B^{\rm nc})$ either with an element in $H^3(Z_B)$ of the closed string state space or an element in $H^2(D)$ associated with the brane geometry $i: D \hookrightarrow Z_B$. These maps can be explicitly realized on the level of four-fold period integrals by integrating out certain directions of the four-cycles $\Gamma_\Sigma \in H_4(X_B^{\rm nc})$ [16,17]. The map $\alpha:\ H^4_{\rm hor}(X_B^{\rm nc}) \to H^3(Z_B)$ can be represented as an integration over a particular S^1 in $X_B^{\rm nc}$ and shifts the Hodge degree by $(-1, 0)$. The other class of contours produces a delta function on the hypersurface D as in [5], and leads to the map $\beta:\ H^4_{\rm hor}(X_B^{\rm nc}) \to H^2(D)$ that shifts by $(-1, -1)$.

Specifically, the infinitesimal deformations of the complex structure of X_B^{nc} split into the closed and open string deformations (2.2) as

$$H^{3,1}(X_B^{\mathrm{nc}}) \simeq H^{2,1}(Z_B) \oplus H^{2,0}(D).$$

The above deformation problem is a priori *unobstructed*, but becomes obstructed by the superpotential (2.3) upon adding the appropriate "flux." In the brane geometry (Z_B, E) this can be realized by a brane flux, adding a D5-charge $\tilde{\gamma} \in H^2(D)$ [11, 17, 19]. A non-trivial obstruction in the open string direction arises for the choice

$$\tilde{\gamma} \in H^2_{\mathrm{var}}(D) = \mathrm{coker}\big(H^2(Z_B) \xrightarrow{i^*} H^2(D)\big). \tag{2.5}$$

Restricting the open string moduli to the subspace where the class $\tilde{\gamma}$ remains of type (1,1) leads to a superpotential for the closed string moduli as in [29,30]. Note also that a class $\tilde{\gamma}$ in the image of i^* is always of type (1,1) and thus does not impose a restriction on the moduli of D, as the variation δW_{II} of equation (2.1) is automatically zero for a holomorphic boundary $\partial\Gamma_\Sigma$.

2.2 Hodge variations for heterotic superpotentials

In the following we consider a similar Hodge theoretic approach to superpotentials of "heterotic" bundles on elliptically fibered CY manifolds constructed in [12, 13].

In the framework Friedmann, Morgan and Witten, an $SU(n)$ bundle E on an elliptically fibered CY three-fold $\pi_{Z_B} : Z_B \to B$ with section $\sigma : B \to Z_B$ is described in terms of a spectral cover Σ, which is an n-fold cover $\pi_\Sigma : \Sigma \to B$, and certain twisting data specifying a line bundle on Σ. Fixing the projection of the second Chern class of E to the base B, the latter comprise a continuous part related to the Jacobian of Σ and a discrete part from elements

$$\gamma \in \ker\big(H^{1,1}(\Sigma) \xrightarrow{\pi_{\Sigma *}} H^{1,1}(B)\big). \tag{2.6}$$

In the duality to F-theory on a four-fold X_B, the elements of the Hodge spaces of the spectral cover are related to those on X_B schematically as [12, 13, 31]:

Σ	X_B
$H^{2,0}$	$H^{3,1}$
$H^{1,1}$	$H^{2,2}$
$H^{1,0}$	$H^{2,1}$

The first line identifies the infinitesimal deformations of Σ with infinitesimal deformations of the four-fold. The second relation relates the discrete data described by the class γ with four-form flux in the F-theory compactification on X_B. The last relation reflects the isomorphism of the Jacobian of Σ and the corresponding Jacobian in X_B related to it by duality (see also [32]). Note that the heterotic/F-theory relation between $H^4(X_B)$ and $H^2(\Sigma)$ is formally given by the same $(-1,-1)$ shift in Hodge degree as in the map β in the open–closed duality relation (2.4). As argued below, this similarity is not accidental, but a reflection of the fact, that the heterotic and type II data can be related by the afore mentioned decoupling limit.

Again the deformations of the spectral cover Σ in $H^{2,0}(\Sigma)$ are unobstructed if γ is the "generic" $(1,1)$ class discussed in [12].[2] Consider instead a class γ that is of type $(1,1)$ only on a subspace $\hat{z}=0$ of the deformation space. Twisting by γ then should obstruct the deformations of Σ in the direction $\hat{z}\neq 0$, which destroy the property $\gamma\in H^{1,1}(\Sigma)$.

We propose that the heterotic superpotential describing this obstruction is captured by the chain integral

$$W_{\mathrm{het}}(Z_B,\Sigma,\gamma)=\int_\Gamma \underline{\Omega}^{3,0}, \qquad (2.7)$$

for $\Gamma\in H_3(Z_B,\Sigma)$ a three-chain with non-zero boundary on Σ. The dual space $H^3(Z_B,\Sigma)\simeq H^3(Z_B)\oplus H^2_{\mathrm{var}}(\Sigma)$ is the relative cohomology group defined by the spectral cover Σ with $H^2_{\mathrm{var}}(\Sigma)$ the mid-dimensional horizontal Hodge cohomology of Σ. Moreover, the boundary two-cycle $C=\partial\Gamma\subset\Sigma$ is the cycle Poincaré dual to γ. The chain integral can then be computed from the Hodge variation on the relative cohomology group, as has been used in [8, 9, 11] to compute brane superpotentials in type II strings. As a first check on the relevance of the mixed Hodge variation on $H^3(Z_B,\Sigma)$ for the heterotic theory, note that the deformation space $H^{2,0}(\Sigma)$ is indeed captured by the Hodge space $H^{2,1}(Z_B,\Sigma)$, as in the type II case.

In the type II context, the mixed Hodge variation gives more physical information than just the superpotential, specifically appropriate coordinates on the deformation space, which lead to the interpretation of the superpotential as a disc instanton sum in the mirror A model. The physical interpretation of the corrections in the heterotic theory will be discussed below.

[2]However, the existence of this class is a consequence of insisting on a section for $\pi_\Sigma:\Sigma\to B$.

Expression (2.7) of the heterotic string can be argued for by relating it to the holomorphic Chern–Simons functional (1.1), which is the holomorphic superpotential for the bundle moduli in the heterotic string [1]. Before turning to the derivation for a genuine CY three-fold of holonomy $SU(3)$, it is instructive to reflect on the argument at the hand of the simpler $\mathcal{N} = 2$ supersymmetric case of dual compactifications of F-theory on $K3 \times K3$ and heterotic string on $T^2 \times K3$. The perturbative F-term superpotential associated with a heterotic flux on K3 in the ith $U(1)$ factor is [33, 34]

$$W_{\text{het}}^{\mathcal{N}=2} = A_i \int_C \omega^{2,0}, \qquad (2.8)$$

where A_i is the Wilson line on T^2, C the cycle Poincaré dual to the flux and $\omega^{2,0}$ the holomorphic $(2, 0)$ form on the heterotic K3. In this simple case, the spectral cover is just points on the dual T^2 times K3, and the chain integral (2.7) over the holomorphic $(3, 0)$ form $dz \wedge \omega^{2,0}$ becomes

$$W_{\text{het}} = \int_\Gamma \Omega = \int_0^{p_i} dz \int_C \omega^{2,0} = A_i \int_C \omega^{2,0}, \qquad (2.9)$$

reproducing (2.8). Here we used that the holomorphic Wilson lines with periods $A_i \sim A_i + 1 \sim A_i + \tau$ appearing in (2.8) are defined by the Abel–Jacobi map on T^2. Furthermore, p_i denotes the associated point in the Jacobian. In the $\mathcal{N} = 1$ case, the points p_i vary over the base and the bounding two-cycles are not of the simple form $(0, p_i) \times C$. An important consequence is that holomorphy of C gets linked to the deformations A_i.[3]

There is also a simple generalization of this $\mathcal{N} = 2$ superpotential to the case, where the heterotic vacuum contains heterotic five-branes [36], and this is also true for the $\mathcal{N} = 1$ supersymmetric case studied below. The five-brane superpotential is in fact the most straightforward part starting from the results on type II brane superpotentials of [8, 9, 11], as the brane deformations of the type II brane map to the brane deformations of the heterotic five-brane in a simple way. The type II/heterotic map providing this identification and explicit examples will be discussed later on.

2.3 Holomorphic Chern–Simons functional for heterotic bundles

The holomorphic Chern–Simons functional is (a projection of) the transgression of the Chern–Weil representation of the algebraic second Chern class

[3]See [35] for a similar discussion.

for a supersymmetric vector bundle configuration. Thus, in order to establish for a supersymmetric heterotic bundle configuration that (1.1) agrees with equation (2.7) on-shell, we need to show that the boundary two-cycle $C = \partial\Gamma$ of the three-chain Γ in equation (2.7) is given by a curve representing the algebraic second Chern class of the holomorphic heterotic vector bundle. The latter is encoded in the zero and pole structure of a global meromorphic section $s_E : Z \to E$ of the supersymmetric holomorphic heterotic bundle E [37]. This is described in [38] for a general $SU(2)$ bundle and in [30] for a bundle associated with a matrix factorization.

To apply this reasoning to the $SU(N)$ bundles of [12], we need to construct an explicit representative for the algebraic Chern class.[4] As explained in [12], the spectral cover Σ together with the class γ of equation (2.6) defines the $SU(n)$ bundle E over the elliptically fibered three-fold $\pi_Z : Z \to B$ by

$$E = \pi_{2*}\mathcal{R}, \qquad \mathcal{R} = \mathcal{P}_B \otimes \mathcal{S}, \qquad \mathcal{R} \to \Sigma \times_B Z.$$

Here π_2 is the projection to the second factor of the fiberwise product $\Sigma \times_B Z$ of the three-fold Z and of the spectral cover Σ over the common base B. $\mathcal{P}_B$ is the restriction of the Poincaré bundle of the product $Z \times_B Z$ to $\Sigma \times_B Z$, while $\mathcal{S} \to \Sigma$ denotes the line bundle over the spectral cover Σ, which is given by[5]

$$\mathcal{S} = \mathcal{N} \otimes \mathcal{L}_\gamma.$$

The bundle $\mathcal{N}$ ensures that the first Chern class $c_1(E)$ of the $SU(n)$ bundle vanishes and its explicit form is thoroughly analyzed in [12]. The holomorphic line bundle $\mathcal{L}_\gamma$ with $c_1(\mathcal{L}_\gamma) = \gamma$ governs the twisting associated to the class γ in (2.6), and it is responsible for the discussed obstructions to the deformations of the spectral cover Σ. Note that, due to the property (2.6), the line bundle $\mathcal{L}_\gamma$ does not further modify the first Chern class $c_1(E)$ [12].

In order to construct a section s_E of the $SU(n)$-bundle, we need to push-forward a global (meromorphic) section $s_\mathcal{R} = s_\mathcal{P} \cdot s_\mathcal{S}$ of the line bundle $\mathcal{R}$, which in turn is the product of a section of the Poincaré bundle $\mathcal{P}_B$ and the line bundle $\mathcal{S}$. The Poincaré bundle is given by $\mathcal{P}_B = \mathcal{O}(\Delta - \Sigma \times \sigma) \otimes K_B$, where Δ is (the restriction of) the diagonal divisor in $Z \times_B Z$, K_B is the canonical bundle of the base (pulled back to $\Sigma \times_B Z$) and $\sigma : B \to Z$ the section of the elliptic fibration Z. Therefore the section $s_\mathcal{P} = s_K \cdot s_F$ can be chosen to be the product of the section s_K of the canonical bundle of the

[4]To avoid cluttering of notation, the heterotic manifold Z_B is denoted simply by Z in the following argument.

[5]For ease of notation its pull-back to $\Sigma \times_B Z$ is also denoted by the same symbol $\mathcal{S}$.

base B and the section s_F, which has a (simple) zero set along the diagonal divisor Δ and a (simple) pole set along the divisor $\Sigma \times_B \sigma$. Finally, the zero set/pole set of the section $s_{\mathcal{S}}$ is induced from the (algebraic) first Chern class $c_1(\mathcal{S})$ of the line bundle $\mathcal{S}$ over the spectral cover Σ. Here we are in particular interested in the contribution from the line bundle $\mathcal{L}_\gamma$, whose global (meromorphic) section extended to the fiber-product space $\Sigma \times_B Z$ is denoted by s_γ.

For an $SU(n)$-bundle the projection map π_2 is an n-fold branched cover of the three-fold Z, and therefore in a open neighborhood $U \subset B$ of the base the push-forward of the section $s_{\mathcal{R}}$ yields

$$s_E = \pi_{2*} s_{\mathcal{R}} = s_K \cdot (s_F^1 \cdot s_{\mathcal{S}}^1, s_F^2 \cdot s_{\mathcal{S}}^2, \ldots, s_F^n \cdot s_{\mathcal{S}}^n). \tag{2.10}$$

As the section s_K originates from the canonical bundle over the base, it appears as an overall pre-factor of the bundle section s_E, while the entries s_F^i and $s_{\mathcal{S}}^i$ arise from the n sheets of the n-fold branched cover. The entries s_F^i restrict on the elliptic fiber to a section of $\oplus_{i=1}^n (\mathcal{O}(p_i) \otimes \mathcal{O}(0)^{-1})$ that have a simple zero at p_i and a simple pole at 0. Here 0 denotes the distinguished point corresponding to the section $\sigma : B \to Z$ and $\sum_i p_i = 0$ for $SU(n)$.[6] The n entries $s_{\mathcal{S}}^i$ arise again from the section $s_{\mathcal{S}}$ on the n different sheets. Since the section $s_{\mathcal{S}}$ is induced from a line bundle over the spectral cover, the zeros/poles of the sections $s_{\mathcal{S}}^i$ correspond to co-dimension one sub-spaces on the base.

Now we are ready to determine the algebraic Chern classes of the $SU(n)$-bundle E from the global section (2.10). By construction the first topological Chern class is trivial, which implies that also the first algebraic Chern class vanishes since the Abel–Jacobi map is trivial for the simply-connected CY three-folds discussed here. The second algebraic Chern class is determined by the "transverse zero/pole sets" of the section s_E, which correspond to the co-dimension two cycles of the mutual zero/pole sets of distinct entries s_E^i and s_E^j, $i \neq j$.

Since $s_E^i = s_F^i \cdot s_{\mathcal{S}}^i$, this computation exhibits $c_2(E)$ as a sum of three contributions: The joint vanishing of s_F^i and s_F^j is empty since $p_i \neq p_j$ generically. The joint vanishing of $s_{\mathcal{S}}^i$ and $s_{\mathcal{S}}^j$ is a sum of fibers, which we may

[6] At branch points of the spectral cover (at least) two points p_i and p_j, $i \neq j$, coincide, and the restriction of the bundle E to the elliptic fiber becomes a sum of $n-2$ line bundles plus a rank two bundle, which is a non-trivial extension of two line bundles [12]. However, due to the splitting principle the second algebraic Chern class is insensitive to these non-trivial extension, and we can simply work with the direct sum of n line bundles.

neglect since, moving in a rational family, they do not contribute to the superpotential.[7]

Equivalently, we may use the relation $\mathrm{ch}_2(E) = \frac{1}{2}c_1(E)^2 - c_2(E)$ between the second Chern class and the second Chern character $\mathrm{ch}_2(E)$, which thanks to the vanishing of c_1 reduces to $\mathrm{ch}_2(E) = -c_2(E)$, to compute $c_2(E)$ from the transverse zero/pole sets of the local sections s^k_F and $s^k_{\mathcal{S}}$ of the *same* entry k. This will more directly lead to the desired boundary two-cycle $C = \partial\Gamma$. (Again, we may neglect the self-intersections of s^k_F and $s^k_{\mathcal{S}}$.)

We focus now on the contribution $c_2(E_\gamma)$ to the second algebraic Chern class, which is associated to the intersection of the zero/pole sets of the local sections s^k_γ and the local sections s^k_F for $k = 1, \ldots, n$. As argued the obtained divisor is rational equivalent to the (negative) boundary two-cycle C arising form the Poincaré dual of the two-form γ on the spectral cover Σ, and we obtain for the second algebraic Chern class

$$c_2(E) = c_2(E_\gamma) + c_2(V), \qquad c_2(E_\gamma) = -[C], \tag{2.11}$$

where we denote by $[C]$ the cycle class, which arises from embedding the two-cycle C of the spectral cover Σ into the CY three-fold Z. Due to the property (2.6) the curve associated to $c_2(E_\gamma)$ is (up to a minus sign) rational equivalent to the boundary of the same three-chain Γ appearing in equation (2.7). The other piece $c_2(V)$, which is (locally) independent of the analyzed deformations of the spectral cover, is discussed in detail in [12]. In general, it gives rise to a non-trivial second topological Chern class. In a globally consistent heterotic string compactification this contribution is compensated by the second topological Chern class of the tangent bundle as dictated by the anomaly equations of the heterotic string.[8]

Thus, by reproducing the three-chain Γ from the second algebraic Chern class of the holomorphic $SU(n)$ bundles, the holomorphic Chern–Simons functional is demonstrated to be agreement with the holomorphic superpotential (2.7). Analogously to the non-supersymmetric off-shell deformations of branes in type II compactifications [11, 17], we propose that the correspondence between the superpotential (2.7) and the Chern–Simons functional even persists along deformations of the spectral cover, which yield non-supersymmetric $SU(n)$ bundle configurations.

[7]An equivalent way to see this is to note that five-branes wrapped on the fiber on the elliptic threefold map under heterotic/F-theory duality to mobile D3-branes which clearly have no superpotential.

[8]In generalized CY compactifications of the heterotic string additional contributions enter into the anomaly equation due to non-trivial background fluxes and the modified generalized geometry [4].

To illustrate the presented construction, we briefly return to the $\mathcal{N} = 2$ compactification of the heterotic string on $T^2 \times K3$. For this example the spectral cover of an $SU(n)$ bundle is a disjoint union of n K3 surfaces $\coprod_{i=1}^{n} \{p_i\} \times K3$ embedded into $T^2 \times K3$. A class γ fulfilling the property (2.6) can be thought of as a non-trivial (1,1)-form ω_γ, which appears in the component $p_i \times K3$ and $p_j \times K3$, $i \neq j$, with opposite signs. Then the Poincaré dual curve C of γ embedded into $T^2 \times K3$ is the boundary of the three-chain $\Gamma = (p_i, p_j) \times C$, where (p_i, p_j) denotes the one-chain on the torus bounded by the points p_i and p_j. The resulting chain integral over $dz \wedge \omega^{2,0}$ exhibits the same structure as the naive equation (2.8).

2.4 Chern–Simons versus F-theory/heterotic duality

In the next section we will consider a dual F-theory compactification on a four-fold and argue that mirror symmetry of the four-fold computes interesting quantum corrections to the Chern–Simons functional. Here we want to motivate the following "classical" relation between the four-fold periods and the Chern–Simons functional (1.1)

$$\int_{X_B} \Omega^{4,0} \wedge G_{\mathbf{A}} = \int_Z \Omega^{3,0} \wedge \mathrm{tr}\left(\frac{1}{2} A \wedge \bar{\partial} A + \frac{1}{3} A \wedge A \wedge A\right) + \mathcal{O}(S^{-1}, \mathrm{e}^{2\pi \mathrm{i} S}). \tag{2.12}$$

In the above, X_B is a CY four-fold which will support the F-theory compactification dual to the heterotic compactification on the three-fold Z and $G_{\mathbf{A}}$ is a four-form "flux" related to the connection A of a bundle $E \to Z$ as described below. Moreover S is a distinguished complex structure modulus of the four-fold X_B such that Im $S \to \infty$ imposes a so-called stable degeneration (s.d.) limit in the complex structure of X_B. In this limit the four-fold X degenerates into two components

$$X \overset{\mathrm{Im}\ S \to \infty}{\longrightarrow} X^\sharp = X_1 \cup_Z X_2,$$

intersecting over the elliptically fibered heterotic three-fold $Z \to B_2$ [12, 13, 15, 39]. The two four-fold components X_i are also fibered over the same base B_2 and capture (part of) the bundle data of the two E_8 factors of the heterotic string, respectively.

The idea is now to view Z as a complex boundary within one of the components X_i and to apply a theorem of [38], which relates the holomorphic Chern–Simons functional on a three-fold Z to an integral of the Pontryagin class of a connection $\mathbf{A}$ on an extension $\mathbf{E} \to X'$ of the bundle $E \to Z$ defined over a Fano four-fold X':

$$\int_{X'} \mathrm{tr}(F_{\mathbf{A}}^{0,2} \wedge F_{\mathbf{A}}^{0,2}) \wedge s_1^{-1} = \mathrm{CS}(Z, A). \tag{2.13}$$

Here $\mathrm{CS}(Z, A)$ is short for the Chern–Simons functional on the r.h.s. of (2.12) without the finite S corrections. Moreover $s \in H^0(K_{X'}^{-1})$ is a section of the anti-canonical bundle of X' whose zero set defines the three-fold Z as a 'boundary' of X'.

Now it is straightforward to show, that the components X_i of the degenerate F-theory four-fold $X^\sharp$ are Fano in the sense required by the theorem and moreover that the heterotic CY three-fold Z can be defined as the zero set of appropriate sections s_i of the anti-canonical bundles $K_{X_i}^{-1}$, as required by the theorem. This will be discussed in more detail in Section 4.2, where we explicitly discuss hypersurface representations for $X^\sharp$ to match the F-theory/heterotic deformation spaces.

The above line of argument then leads to a relation of the form (2.12), provided one identifies the four-form flux $G_{\mathbf{A}}$ with the Pontryagin class of a gauge connection $\mathbf{A}$ on an extension $\mathbf{E}$ of the bundle over the component X_1. Up to terms of lower Hodge type, we shall have

$$G_{\mathbf{A}}|_{X_1} \sim \mathrm{tr}(F_{\mathbf{A}}^{0,2} \wedge F_{\mathbf{A}}^{0,2}). \tag{2.14}$$

Note that this identification of the four-form flux is a non-trivial prediction of the outlined duality.

The real challenge posed by relations (2.14), (2.12) is not the on-shell relation, which has been argued for in a special case in the previous section, but a proper off-shell extension of both sides. On the four-fold side, the standard lore of string compactifications is to not fix the Hodge type of G, but rather to view the flux superpotential as a potential on the moduli space of the four-fold X, which fixes the moduli to the critical locus. The idea is, that the periods $\int_X \Omega^{4,0}$ on the l.h.s. of (2.12) have a well-defined meaning as the section of a bundle over the unobstructed complex structure moduli space $\mathcal{M}_{\mathrm{CS}}(X)$ of the four-fold *before* turning on a the flux; in particular they define the Kähler metric on $\mathcal{M}_{\mathrm{CS}}(X_B)$. In this way, viewing non-zero G as a "perturbation" on top of an unobstructed moduli space, the section $W(X_B)$ is considered as an off-shell potential for fields parametrizing $\mathcal{M}_{\mathrm{CS}}(X)$. Although it is not clear, in general, under which conditions it is valid to restrict the effective field theory to the fields parametrizing $\mathcal{M}_{\mathrm{CS}}(X)$ and to interprete $W(X_B)$ as the relevant low-energy potential for the light fields, this working definition for an off-shell deformation space seems to make sense in many situations.[9]

Relation (2.12) suggests that it should be possible to give a sensible notion of a distinguished, finite-dimensional "off-shell moduli space" for non-holomorphic bundles and to treat the obstruction induced by the Chern–Simons superpotential as some sort of "perturbation" to an unobstructed problem. This is also suggested by the recent success to compute off-shell superpotentials for brane compactifications from open string mirror symmetry. We plan to circle around these questions in the future.

3 Quantum corrected superpotentials in F-theory from mirror symmetry of four-folds

In this section we show, that the various Hodge theoretic computations of superpotentials in CY three-fold and four-fold compactifications discussed above are in some cases linked together by a chain of dualities. The unifying framework is the type IIA compactification on a pair (X_A, X_B) of compact mirror CY four-folds and its F-theory limits. As will be argued below, mirror symmetry of the four-folds computes interesting quantum corrections, most notably D-instanton corrections to type II orientifolds and world-sheet corrections to heterotic (0,2) compactifications, which are hard to compute by other means at present. Another interesting connection is that to the heterotic superpotential for generalized CY manifold. The purpose of this section is to study the general framework, which involves a somewhat involved chain of dualities, while explicit examples are given in Sections 6 and 7.

3.1 Four-fold superpotentials: a first look at the quantum corrections

For orientation it is useful to keep in mind the concrete structure of the superpotential on compact four-folds that we want to study, as it links the different dual theories discussed below at the level of effective supergravity. The compact four-fold X_B for F-theory compactification is obtained from the non-compact four-fold X_B^{nc} of open–closed in equation (2.3) by a simple compactification [10, 11, 19], discussed in more detail later on. In a

[9] There is a considerable literature on this subject. We suggest [40] for a justification in the context of type IIA flux compactifications on three-folds, [41] in the type IIB context, [42] in non-geometric phases and [43] for a recent general discussion.

certain decoupling limit defined in [11], the F-theory superpotential on X_B reproduces the type II superpotential (2.1) plus further terms:

$$\begin{aligned} W_{\mathrm{F}}(X_B) &= \sum_{\gamma_\Sigma \in H_4(X_B)} \underline{N}_\Sigma \int_{\gamma_\Sigma} \Omega^{(4,0)} \\ &\overset{\mathrm{Im}\ S\to\infty}{=} \sum_{\gamma_\Sigma \in H_3(Z_B)} (N_\Sigma + S\ M_\Sigma) \int_{\gamma_\Sigma} \underline{\Omega}^{(3,0)} \\ &\quad + \sum_{\substack{\gamma_\Sigma \in H_3(Z_B, D) \\ \partial\gamma_\Sigma \neq 0}} \hat{N}_\Sigma \int_{\gamma_\Sigma} \underline{\Omega}^{(3,0)} + \cdots . \end{aligned} \tag{3.1}$$

The essential novelty in the superpotential of the compact four-fold, as compared to the previous result (2.1), is the additional dependence on the new, distinguished complex structure modulus S of the compactification X_B of X_B^{nc}. This modulus is identified in [11] with the decoupling limit

$$\mathrm{Im}\ S \sim 1/g_s \to \infty. \tag{3.2}$$

A similar weak coupling expansion of the four-fold Kähler potential leads to a conjectural Kähler potential for the open–closed deformation space, as will be discussed in more detail in Section 5.

Note that the flux terms $\sim S\, M_\Lambda$ in the four-fold superpotential $W_{\mathrm{F}}(X_B)$ correspond to NS fluxes in the type II string on Z_B, which were missing in (2.1).[10] In addition there are subleading corrections for finite S, denoted by the dots in (3.1), which include an infinite sum of exponentials with the characteristic weight e^{-1/g_s} of D-instantons. Before studying these corrections in detail, it is instructive to consider the dualities involved in the picture, which leads to a somewhat surprising reinterpretation of the open–closed duality of [10, 16].

3.2 $\mathcal{N} = 1$ duality chain

The relevant duality chain for understanding the quantum corrections in (3.1), and the relation to open–closed duality, relates the following $\mathcal{N} = 1$

[10] This has been observed already earlier in a related context in [44], see also the discussion in Section 5 below.

supersymmetric compactifications:[11]

$$\frac{\text{type II OF}}{T^2 \times Z_B} \sim \frac{\text{F-theory}}{K3 \times Z_B} \sim \frac{\text{heterotic}}{T^2 \times Z_B} \sim \frac{\text{type IIA}}{X_B/X_A} \sim \frac{\text{F-theory}}{X_B \times T^2} \qquad (3.3)$$

where Z_B is a CY three-fold and (X_A, X_B) a mirror pair of four-folds which is related to the heterotic compactification on Z_B by type IIA/heterotic duality. Here and in the following it is assumed that the three-fold Z_B and the four-fold X_B have suitable elliptic fibrations, in addition to the K3 fibration of X_B required by heterotic/type IIA duality [45]. This guarantees the existence of the F-theory dual in the last step. For an appropriate choice of bundle one can take the large volume of the T^2 factor to obtain the four-dimensional duality between heterotic on Z_B and F-theory on X_B [15].

The remaining section will center around the identification of the limit (3.2) in the various dual theories. Note that there are two different F-theory compactifications involved in the duality chain (3.3), namely on the manifolds $K3 \times Z_B$ and $X_B \times T^2$, respectively. Identification (3.2) is associated with the F-theory compactification on $K3 \times Z_B$, or the type II orientifold on $T^2 \times Z_B$, in the orientifold limit [46]. The decoupling limit describes also a certain limit of the heterotic compactification *on the same three-fold* Z_B, which will be identified as a large fiber limit of the elliptic fibration Z_B below.

In order to make contact with the brane configuration (Z_B, E) discussed in Section 2.1, we combine the orientifold limit of F-theory with a particular Fourier–Mukai transformation [47, 48]

$$\frac{\text{type II OF}}{\check{T}^2 \times \check{Z}_B} \sim \frac{\text{type II OF}}{T^2 \times Z_B} \sim \frac{\text{F-theory}}{K3 \times Z_B}.$$

The relevant Fourier–Mukai transformation is discussed in detail in [48]. Heuristically, it implements T duality in both directions of the torus T^2 to the dual torus $\check{T}^2$ together with a fiberwise T duality in both directions of the elliptic fibers of the three-fold Z_B to the three-fold $\check{Z}_B$ with dual elliptic fibers. This operation does not change the complex structure of the bulk geometry, but instead it transforms the brane configuration to the open–closed geometry (Z_B, E). These orientifold limits of F-theory, the type II

[11] In this note, for ease of notation and to emphasize the relation to four-dimensional theories, $\mathcal{N} = 1$ compactifications to two space–time dimensions also refer to low-energy effective theories with four supercharges.

and heterotic compactifications on Z_B can be also connected as

$$\begin{array}{ccccc} \dfrac{\text{type II OF}}{\check{T}^2 \times \check{Z}_B} & \sim & \dfrac{\text{type I}}{\check{T}^2 \times Z_B} & \sim & \dfrac{\text{heterotic}}{T^2 \times Z_B} \\ & & \wr & & \\ & & \dfrac{\text{type II OF}}{T^2 \times Z_B}. & & \end{array} \tag{3.4}$$

Here S duality associates the type I to the heterotic string, T duality on $\check{T}^2$ relates the type I compactification to the type II orientifold on $T^2 \times Z_B$, while the afore mentioned Fourier–Mukai transformation, which realizes fiberwise T duality, applied to the three-fold Z_B of the type I theory maps to the type II orientifold on $\check{T}^2 \times \check{Z}_B$ [46–48].

3.3 The decoupling limit as a stable degeneration

The meaning of the decoupling limit in the mirror pair (X_A, X_B) of four-folds and the dual heterotic string on $Z_B(\times T^2)$ can be understood with the help of the following two propositions obtained in the study of F-theory/heterotic duality and mirror symmetry on toric four-folds in [23]. It is shown there that[12]

(C1) If F-theory on the four-fold X_B is dual to a heterotic compactification on a three-fold Z_B then the mirror four-fold X_A is a fibration $Z_A \to X_A \to \mathbf{P}^1$, where the generic fiber Z_A is the three-fold mirror of Z_B.

(C2) In the above situation, the large base limit in the Kähler moduli of the fibration $X_A \to \mathbf{P}^1$ maps under mirror symmetry to a "stable degeneration" limit in the complex structure moduli of the mirror X_B.

The first part applies, since the four-fold duals constructed in the context of open–closed string duality have precisely the fibration structure required by (C1); indeed the mirror pair $(X_A^{\mathrm{nc}}, X_B^{\mathrm{nc}})$ of open–closed dual four-folds, dual to an A-brane geometry (Z_A, L) and its mirror B-brane geometry (Z_B, E), is constructed in [10, 16] as a fibration over the complex plane, where the generic fiber is the CY three-fold Z_A:

$$\begin{array}{ccccc} Z_A & \longrightarrow & X_A^{\mathrm{nc}} & \xleftrightarrow[\text{mirror symmetry}]{\text{four-fold}} & X_B^{\mathrm{nc}} \\ & & \Big\downarrow{\scriptstyle \pi(L)} & & \\ & & \mathbf{C} & & \end{array} \tag{3.5}$$

[12]For concreteness, we quote the result for F-theory on a four-fold, although it applies more generally to n-folds, as will be also used below.

The notation $\pi(L)$ for the fiber projection is a reminder of the fact that the data of the bundle L are encoded in the singularity of the central fiber as described in detail in [10, 11, 16, 18]. The manifold X_B^{nc} may be defined as the four-fold mirror of the fibration X_A^{nc}. Since the pair of compact four-folds (X_A, X_B) is obtained by a simple compactification of the base to a $\mathbf{P}^1$ [11,19], it follows that the F-theory four-fold X_B has a mirror X_A, which is a three-fold fibration $\pi : X_A \to \mathbf{P}^1$ with generic fiber Z_A. The multiple fibration structures are summarized below:

	$\frac{\text{F-theory}}{X_B} \sim \frac{\text{heterotic}}{Z_B}$	$\frac{\text{closed}}{X_A} \sim \frac{\text{open}}{(Z_A,L)}$
Elliptic fib.	$\begin{array}{ccc} T^2 \to & X_B \\ & \downarrow \\ & B_3 \end{array}$ $\begin{array}{ccc} T^2 \to & Z_B \\ & \downarrow \\ & B_2 \end{array}$	–
K3 fib.	$\begin{array}{ccc} K3 \to & X_B \\ & \downarrow \\ & B_2 \end{array}$	–
three-fold fib.	$\begin{array}{ccc} Z_A \to & X_A \\ & \downarrow \\ & \mathbf{P}^1 \end{array}$	$\begin{array}{ccc} Z_A \to & X_A \\ & \downarrow \\ & \mathbf{P}^1 \end{array}$

(3.6)

Here B_3 and B_2 denote the corresponding three- and two-dimensional base spaces, where B_2 is common to the F-theory manifold and the heterotic dual. The crucial link is the three-fold fibration of X_A, which is required by both, F-theory/heterotic *and* open–closed duality. Claim (C1) then implies that F-theory on X_B has an open–closed dual interpretation as a B-type brane on a three-fold Z_B and an A-type brane on the mirror Z_A. The reverse conclusion, namely that an open–closed dual pair (X_A, X_B) also has an F-theory/heterotic interpretation, requires the additional condition, that X_B is elliptic and K3 fibered. This leaves the possibility, that open–closed duality holds for more general four-fold geometries than F-theory/heterotic duality. For simplicity we impose in the following, that X_B is elliptically and K3 fibered, which implies that (C1) holds also in the reverse direction.

Part two of the proposition applies, since the decoupling limit Im $S \to \infty$ in the complex structure of X_B was defined in [11] as the mirror of the large base volume in the Kähler moduli of the fibration $\pi : X_A \to \mathbf{P}^1$. The image of this limit under the mirror map in the complex structure of X_B is a local mirror limit in the sense of [22] and effectively imposes the s.d. limit of X_B studied in [12, 15, 39]. Under F-theory/heterotic duality, the s.d. limit maps to a large fiber limit of the heterotic string compactification on the elliptic fibration Z_B and this is the sought for identification of limit (3.2) in the heterotic string. The meaning as a physical decoupling limit of a sector of the heterotic string can be understood from both, the world-sheet and the effective supergravity point of view, as will be discussed in Section 5. Explicit examples for the relation between the hypersurface geometries X_B and Z_B in the s.d. limit will be considered in Sections 6 and 7.

3.4 Open-closed duality as a limit of F-theory/heterotic duality

The relation in (3.3) between the type II orientifold on Z_B and type IIA on the four-folds (X_B, X_A) is similar as in the open–closed duality of [10,16,17]. These papers claim to compute the type II superpotential for a B-type brane compactification on Z_B with a given five-brane charge from the periods of a dual (non-compact) four-fold $X_B^{\rm nc}$. As explained in [11, 17, 19], this five-brane charge can be generated by non-trivial fluxes on higher dimensional branes. The only difference to the type II orientifold on $T^2 \times Z_B$ appearing in (3.3) is the extra T^2 compactification and the presence of seven-branes wrapping Z_B, which does not change the superpotential associated with the five-brane charge.

In the decoupling limit Im $S \to \infty$, which sends X_B to the non-compact manifold $X_B^{\rm nc}$, the "local" B-type brane with five-brane charge decouples from the global orientifold compactification and we recover the type II result $W_{\rm II}(Z_B)$ in equation (1.2).[13] Note that in this limit there are two different paths connecting the B-type orientifold to the non-compact open–closed string dual $X_B^{\rm nc}$. The first one goes via the open–closed string duality of [10, 16, 17], while the second goes via F-theory/heterotic/type IIA duality of equation (3.3).

$$\begin{array}{ccc} \begin{matrix}\text{type II OF}\\ T^2\times Z_B\end{matrix} & \xrightarrow[\text{duality}]{\text{F/het/IIA}} & \begin{matrix}\text{type IIA}\\ X_B\end{matrix} \\ \Big\downarrow{\scriptstyle g_s\to 0} & & \Big\downarrow{\scriptstyle \text{Im } S\to\infty} \\ \begin{matrix}\text{local B-brane}\\ (Z_B,E)\end{matrix} & \xrightarrow[\text{duality}]{\text{open-closed}} & \begin{matrix}\text{type IIA}\\ X_B^{\rm nc}\end{matrix} \end{array} \qquad (3.7)$$

Commutativity of the diagram implies that for this special case, open–closed duality of [10, 16, 17] coincides with heterotic/F-theory duality in the decoupling limit.

Note that the duality (3.4) maps a D3-brane wrapping a curve C in Z_B in the orientifold to a heterotic five-brane wrapping the same curve C in the heterotic dual Z_B. The heterotic five-brane can be locally viewed as an

[13] In the type II string without branes/orientifold, $\hat{N}_\Sigma = 0$ and the subleading corrections to the superpotential would be absent [3].

M-theory five-brane [49], which is in turn related to the type IIA five-brane used in [17] to derive open–closed string duality from T-duality.

The original observation of open–closed string duality of [16] is that it maps the disc instanton generated superpotential of the brane geometry (Z_A, L) (mirror to (Z_B, E)) to the sphere instanton generated superpotential for the dual four-fold $X_A^{\rm nc}$ (mirror to $X_B^{\rm nc}$). At tree-level, this map is *term by term*, that is it maps an individual Ooguri–Vafa invariant for a given class $\beta \in H^2(Z_A, L)$ to a Gromov–Witten invariant for a related class $\beta' \in H^2(X_A^{\rm nc})$. This genus zero correspondence left the important question, whether there is a full string duality, that extends this relation between the three-fold and the four-fold data beyond the superpotential. From the above diagram we see that there is at least one true string duality that reduces to open–closed string duality of [10, 16, 17] at $g_s = 0$ and extends it to a true string duality: F-theory/heterotic duality!

3.5 Instanton corrections and mirror symmetry in F-theory

The above discussion has lead to the qualitative identification of the dual interpretations of the expansion in (3.1) in terms of a weak coupling limit of the type II orientifold, a large fiber volume of the heterotic string on the elliptic fibration Z_B, a stable degeneration limit of the F-theory four-fold X_B and a large base limit of the three-fold fibration $X_A \to \mathbf{P}^1$. We will now argue that the quantum corrections computed by four-fold mirror symmetry can be tentatively assigned to the two four-fold superpotentials in [24, 50] as

$$\begin{aligned} W(X_B) &= \int_{X_B} \Omega \wedge F_{\rm hor} \leftrightarrow \text{D-1,D1/finite-fiber corrections} \\ &\qquad\qquad\qquad\qquad \text{in type II OF/Het,} \\ \widetilde{W}(X_B) &= \int_{X_B} \mathrm{e}^{B+\mathrm{i}J} \wedge F_{\rm ver} \leftrightarrow \text{D3/space–time instantons} \\ &\qquad\qquad\qquad\qquad \text{in type II OF/Het.} \end{aligned} \tag{3.8}$$

Here $W(X_B)$ is the four-fold superpotential of equation (3.1), while $\widetilde{W}(X_B)$ is the twisted super-potential associated with the type IIA compactification on X_B.[14] The latter computes also world-sheet instanton corrections to the large volume limit of the type II/heterotic compactification.

The details of the argument are somewhat involved and may be skipped on a first reading. It is again instructive to first consider the simpler case of

[14]See the discussion in Section 5 below.

a closely related duality chain with $\mathcal{N} = 2$ supersymmetry:

$$\frac{\text{type II OF}}{T^2 \times Z_H} \sim \frac{\text{F-theory}}{\tilde{Z}_V \times Z_H} \sim \frac{\text{heterotic}}{T^2 \times Z_H} \sim \frac{\text{type IIA/IIB}}{X_B/X_A} \sim \frac{\text{F-theory}}{X_B \times T^2} \tag{3.9}$$

where $\tilde{Z}_V$, Z_H are two K3 manifolds and (X_A, X_B) denotes a mirror pair of CY three-folds; differently then in (3.3), mirror symmetry of the three-folds exchanges the IIA compactification on X_B with a type IIB compactification on X_A. As before, we assume that the three-fold X_B is elliptically fibered, such that one can decompactify the T^2 of the heterotic string to obtain F-theory in six dimensions. Note that the $\mathcal{N} = 1$ duality chain (3.3) can be heuristically thought of as a chain of dualities obtained by "fibering" (3.9) over $\mathbf{P}^1$, so that some observations from the $\mathcal{N} = 2$ supersymmetric case will carry over to $\mathcal{N} = 1$.

The two basic questions that we want to study in this simpler setup are the meaning of mirror symmetry in F-theory and the identification of quantum corrections computed by it. It will turn out that, under favorable conditions, the distinguished modulus S has a mirror partner ρ and mirror symmetry of the CY manifolds X_A and X_B exchanges the two weak coupling expansions in Im S and Im ρ.

The quantum corrections to the $\mathcal{N} = 2$ supersymmetric duality chain (3.9) have a rich structure studied previously in [36, 51]. The F-theory superpotential for the $K3 \times K3$ compactification, which arises in the effective $\mathcal{N} = 2$ supergravity theory from certain gaugings in the hypermultiplet sector, can be written as a bilinear in the period integrals on the two K3 factors [33, 52]

$$W_{F,\text{pert}} = \sum_{I,\Lambda} \left(\int_{Z_H} \omega^{2,0} \wedge \mu^I \right) \underline{G}_{I\Lambda} \left(\int_{\tilde{Z}_V} \omega^{2,0} \wedge \tilde{\mu}^\Lambda \right). \tag{3.10}$$

Here $\underline{G}_{I\Lambda}$ labels the four-form flux in F-theory, decomposed on a basis $\{\tilde{\mu}^\Lambda\}$ for $H^2_{\text{prim}}(\tilde{Z}_V)$ and $\{\mu^I\}$ for $H^2_{\text{prim}}(Z_H)$ as $G = \sum_{I,\Lambda} \underline{G}_{I\Lambda} \mu^I \wedge \tilde{\mu}^\Lambda$.

The periods on Z_H depend on $\mathcal{N} = 2$ hyper multiplets and are mapped under duality to the type IIA/F-theory compactification on X_B to the three-fold periods, by a similar relation as (3.1):

$$\int_{X_B} \omega^{3,0} \wedge \gamma^I = \int_{Z_H} \omega^{2,0} \wedge \mu^I + \mathcal{O}(\mathrm{e}^{2\pi \mathrm{i} S}, S^{-1}). \tag{3.11}$$

This equation describes, how the periods on the F-theory three-fold X_B defined on the basis $\gamma^I \in H^3(X_B, \mathbf{Z})$ compute finite S corrections to the periods on the two-fold Z_H of the dual type II compactification. As explained in the four-fold case, (C2) says that these are corrections to the s.d. limit in the complex structure of X_B.

Note that (3.10) is apparently symmetric in the periods of the two K3 factors. This is somewhat misleading, as the periods on $\tilde{Z}_V$ depend on $\mathcal{N}=2$ vector multiplets.[15] It was argued in [36], that there is also a similar relation as (3.11) for the second period vector on $\tilde{Z}_V$ (3.11),

$$\int_{X_A} \omega^{3,0} \wedge \tilde{\gamma}^\Lambda = \int_{\tilde{Z}_V} \omega^{2,0} \wedge \tilde{\mu}^\Lambda + \mathcal{O}(\mathrm{e}^{2\pi \mathrm{i}\rho}, \rho^{-1}), \tag{3.12}$$

where ρ is a distinguished vector multiplet related to the heterotic string coupling as discussed below. This relation describes corrections to the result (3.10) computed by the periods *of the mirror manifold* X_A. Here it is understood, that one uses mirror symmetry to map the periods of the holomorphic (3,0) form on $H^3(X_A, \mathbf{Z})$ defined on the basis $\tilde{\gamma}^\Lambda \in H^3(X_A, \mathbf{Z})$ to the periods of the Kähler form on a dual basis $\gamma^\Lambda \in \oplus_k H^{2k}(X_B, \mathbf{Z})$,

$$\int_{X_A} \omega^{3,0} \wedge \tilde{\gamma}^\Lambda \;\longrightarrow\; \int_{X_B} \frac{1}{k!} J^k \wedge \tilde{\gamma}^\Lambda. \tag{3.13}$$

Note that these "Kähler periods" of X_B are the three-fold equivalent of the integrals appearing in the twisted superpotential $\widetilde{W}(X_B)$ in (3.8). However, replacing the K3 periods in (3.10) by the quantum corrected expressions (3.11) and (3.12), we obtain a superpotential that is proportional to both, the periods of the manifold X_B *and* of its mirror X_A. It was argued in [36], that this "quadratic" superpotential in the three-fold periods is in agreement with the S-duality of topological strings predicted in [53]. Similar expressions have been obtained in [54,55] from the study of type II compactification on generalized CY manifolds.

The similarity of the two expansions (3.11) and (3.12) is no accident. By (C2), the s.d. limit Im $S \to \infty$ is mirror to the large base limit of the fibration $X_A \to \mathbf{P}^1$, which is a K3 fibration by (C1) in the three-fold case. By type IIA/heterotic duality, X_B is also a K3 fibration $X_B \to \mathbf{P}^1$ and equation (3.12) represents the large base limit Im $\rho \to \infty$ of X_B, where ρ is the Kähler volume of the base $\mathbf{P}^1$. By heterotic/type IIA duality, the Kähler volume of the base of X_B is identified with the four-dimensional heterotic string coupling [56]. Adding the identification of S provided by (C2), we obtain the following heterotic interpretation of the volumes $V_{A/B}$ of the base

[15] See [52] for a discussion of the effective supergravity theory for the orientifold limit of $K3 \times K3$.

$\mathbf{P}^1$'s of the fibrations $X_{A/B} \to \mathbf{P}^1$:

$$V_B = \lambda_{4,\text{het}}^{-2} = \text{Im } \rho, \qquad V_A = V_{E_\text{het}} = \text{Im } S. \tag{3.14}$$

Here V_{E_het} denotes the volume of the elliptic fiber of Z_H in the heterotic compactification in (3.9). Clearly, mirror symmetry exchanges the two expansions (3.11) and (3.12) associated with a compactification on X_A or on X_B, respectively

$$\begin{matrix} S \\ (3.11) \end{matrix} \quad \overset{\text{mirror}}{\underset{\text{symmetry}}{\longleftrightarrow}} \quad \begin{matrix} \rho \\ (3.12) \end{matrix}. \tag{3.15}$$

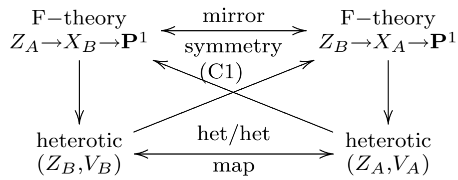
In the dual F-theory compactification on $K3 \times K3$, mirror symmetry represents the exchange of the two K3 factors [51, 57], which gives rise to two dual heterotic $T^2 \times K3$ compactifications. Starting from the duality relation between M-theory on $K3 \times K3$ and heterotic string on $T^2 \times S^1 \times K3$ [58], it is shown in [51], that the exchange of the two K3 factors in M-theory generates the following $\mathbf{Z}_2$ transformation on the moduli of the two heterotic duals:

$$V_{E'_\text{het}} = \lambda_4^{-2}, \qquad \lambda_4'^{-2} = V_{E_\text{het}}.$$

Comparing with relation (3.14) between the four-dimensional heterotic coupling and the volumes of the bases of the fibrations (X_A, X_B), one concludes that the result of [51] is in accord with the claim (C2) of [23] and its consequence (3.15) in this case. It is reassuring to observe that these conclusions, reached by rather different arguments in [23, 36, 51], agree so nicely.

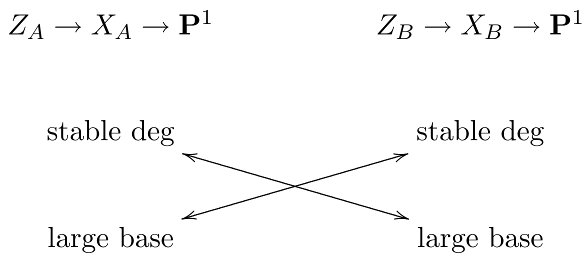
As further argued in [36], expansion (3.12) computed from mirror symmetry of the three-folds X_B and X_A computes D3 instanton corrections to the orientifold on $K3 \times T^2$ (or F-theory on $K3 \times K3$). The basic instanton is a D3-brane wrapping $K3$, which is mapped under the duality (3.4) to a five-brane instanton of the heterotic brane wrapping $T^2 \times K3$. In the type II orientifold, ρ is the K3 volume.

Compactifying the $\mathcal{N} = 2$ chain on a further $\mathbf{P}^1$, the previous arguments leads to the assignments (3.8). In particular, the identification of D3 instantons in [36] continues to hold with the appropriate replacement of K3 with four-cycles in Z_B. The above argument based on (C2) is, in fact, independent of the dimension and can be phrased more generally as the following statement on mirror symmetry in F-theory. Let X_B be an F-theory n-fold with heterotic dual (Z_B, V_B), where V_B denotes the gauge bundle. If the mirror X_A of X_B is also elliptically and K3 fibered, we have the following relations between the F-theory compactifications on (X_A, X_B) and heterotic compactifications on (Z_A, Z_B):



(3.16)

Under mirror symmetry, the s.d. limit and the large base limit are exchanged:



(3.17)

Note that the two theories on the left and on the right are in general *not* dual but become dual after further circle compactifications.

The simplest example is F-theory on a K3 X_B dual to heterotic on $(Z_B = T^2, V_G)$, where V_G denotes a flat gauge bundle on T^2 with structure group G. The eight-dimensional heterotic compactification has an unbroken gauge group H, where H is the centralizer of G in the 10-dimensional heterotic gauge group. In a further compactification on T^2 one has to choose a flat H bundle on the second T^2. Assuming that the bundles factorize, one can exchange the two T^2 factors and thus H and G. In F-theory this exchange corresponds to mirror symmetry of K3 and this was used in [22,23] to construct local mirrors of bundles on T^2 from local ADE singularities.

The next simple example is the above $\mathcal{N} = 2$ supersymmetric case, where X_B is the three-fold in (3.9), with a heterotic dual compactified on $K3 \times T^2$. Assuming a suitable factorization of the heterotic bundle, the action of three-fold mirror symmetry maps to the exchange of the two K3 factors $(\tilde{Z}_V, Z_H)$ in the dual F-theory compactification in 3.16. In the heterotic string this symmetry relates two *different* K3 compactifications (Z_H, V) and $(\tilde{Z}_V, V')$ which become dual after compactification on $T^2 \times S^1$ [23, 59].[16]

[16]One needs the T^2 compactification to get two type IIA compactifications on the mirror pair (X_A, X_B), which become T-dual after a further circle compactification.

In the four-fold case, the fibrations required by the above arguments are not granted, since (C1) now implies that the four-fold X_A is a three-fold fibration $X_A \to \mathbf{P}^1$ (as opposed to the K3 fibration in the three-fold case). *If* X_A is K3 fibered, the $\mathcal{N}=1$ chain can be viewed as a $\mathcal{N}=2$ chain fibered over $\mathbf{P}^1$ and the above arguments apply, leading to the assignment (3.8). In the other case, the large Im S expansion of $W(X_B)$ always exists, but there is no corresponding large ρ expansion of the twisted superpotential $\widetilde{W}(X_B)$.

4 Heterotic superpotential from F-theory/heterotic duality

Having identified the limit $S \to \mathrm{i}\infty$ as a large fiber limit in the heterotic interpretation, the next elementary question is to identify the "flux quanta" of the four-fold superpotential (3.1) in the context of the heterotic string. This task can be divided into identifying the origin of the terms $\sim N_\Sigma$, M_Σ captured by the bulk periods and the terms $\sim \hat{N}_\Sigma$ proportional to chain integrals.

4.1 Generalized CY contribution to $W_{\mathrm{F}}(X_B)$

The back-reaction of the bulk background fluxes in the heterotic string requires the compactification space to be a generalized CY space [4, 60–65]. Using dimensional reduction techniques of the heterotic string on such generalized CY geometries $\tilde{Z_B}$ reveals that the flux-induced superpotential reads [64–68]

$$W_{\mathrm{het}} = \int_{\tilde{Z_B}} \tilde{\Omega} \wedge \left(H - i\, d\tilde{J}\right), \tag{4.1}$$

where H is the non-trivial NS three-form flux and $d\tilde{J}$ is often called the geometric flux of the generalized three-fold $\tilde{Z_B}$. The three-forms $\tilde{\Omega}$ and the two-form $\tilde{J}$ are the generalized counterparts of the holomorphic three-form Ω and the (complexified) Kähler form J of the associated CY three-fold Z_B.[17] In general, the direct evaluation of the heterotic superpotential (4.1) of the three-fold $\tilde{Z_B}$ is rather complicated, therefore we argue here that under certain circumstances the heterotic fluxes can be computed from the periods of the original three-fold Z_B.

It is instructive to examine first the fluxes of the heterotic string compactified on the $\mathcal{N}=2$ background $T^2 \times K3$. For this particular geometry the

[17] In the context of generalized CY spaces $\tilde{J}$ and $\tilde{\Omega}$ are in general not closed with respect to the de Rahm differential d.

analyzed fluxes induce a deformation to the non-Kähler geometry $\tilde{K}$, which is a non-trivial toroidal bundle $\pi : T^2 \to \tilde{K} \to K3$ over the $K3$ base [69–71].

In order to show the relation to the superpotential (4.1) we first construct the cohomology classes, which capture the twisting to the toroidal bundle $\tilde{K}$. Choosing a good open covering $\mathcal{U} = \{U_\alpha\}$ of the $K3$ base together with a trivialization of the toroidal bundle, the non-trivial bundle structure is captured by transition funcions $\varphi^{(k)}_{\alpha\beta} : U_{\alpha\beta} \to \mathbf{R}, k = 1, 2$, in the open sets $U_{\alpha\beta} = U_\alpha \cap U_\beta$. These transition functions patch together the angular coordinates of the two circles $S^1 \times S^1$ in the torodial fibers. Due to the periodicity of the angular variables the transition functions fulfill on triple overlaps $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$ the condition

$$\varepsilon^{(k)}_{\alpha\beta\gamma} = \frac{1}{2\pi}\left(\varphi^{(k)}_{\alpha\beta} - \varphi^{(k)}_{\alpha\gamma} + \varphi^{(k)}_{\beta\gamma}\right) \in \mathbf{Z}\ , \qquad k = 1, 2.$$

The constructed functions $\varepsilon^{(k)} : U_{\alpha\beta\gamma} \to \mathbf{Z}$ specify two-cocycles in the Čech cohomology group $\check{H}^2(K3, \mathbf{Z})$. The classes $\varepsilon^{(k)}$ correspond to the Euler classes $e^{(k)}$ of the two circular bundles in the integral de Rham cohomology $H^2(K3, \mathbf{Z})$.[18]

The non-Kähler manifold $\tilde{K}$ is equipped with the hermitian form $\tilde{J}$ and the holomorphic three-form $\tilde{\Omega}$ [70, 71][19]

$$\tilde{J} = \pi^* J_{K3} - S\, i\, \theta^{(1)} \wedge \theta^{(2)}, \qquad \tilde{\Omega} = \omega^{2,0} \wedge (\theta^{(1)} + i\theta^{(2)}).$$

Here $\theta^{(k)}, k = 1, 2$, are the two one-forms of the toroidal fibers, while J_{K3} is the (complexified) Kähler form and $\omega^{2,0}$ is the holomorphic two-form of the $K3$ base. S is the (complexified) Volume modulus of the toroidal fiber. On-shell the value of the volume modulus S becomes stabilized at $S = i$ [71], since the equations of motions impose the torsional constraint [4, 60, 61][20]

$$H = (\partial - \bar{\partial})\tilde{J}. \tag{4.2}$$

[18] For details and background material on Čech cohomology and on the construction of the Euler classes we refer the interested reader, for instance, to [72].

[19] For simplicity, we ignore a warp factor in front of the Kähler form J_{K3}, as it is not relevant for the analysis of the superpotential. Also note that in our conventions the imaginary part of $\tilde{J}$ corresponds to the hermitian volume form.

[20] The stabilization of volume moduli in the context of heterotic string compactifications with fluxes is also discussed in [62, 67].

As the two-forms $d\theta^{(k)}$ restrict to the Euler classes $e^{(k)}$ on the $K3$ base, the non-Kähler three-fold $\tilde{K}$ encodes the background fluxes

$$d\tilde{J} = -\mathrm{i}\, S(\pi^* e^{(1)} \wedge \theta^{(2)} - \pi^* e^{(2)} \wedge \theta^{(1)}), \qquad H = \pi^* e^{(1)} \wedge \theta^{(1)} + \pi^* e^{(2)} \wedge \theta^{(2)},$$

where the H-flux is determined by imposing the torsional constraint (4.2) for the on-shell value $S = i$ of the fiber volume. Then evaluating the superpotential (4.1) with these fluxes yields

$$W_{\mathrm{het}} = \int_{\tilde{K}} \tilde{\Omega} \wedge (H - \mathrm{i}\, d\tilde{J}) = \int_{C_H} dz \wedge \omega^{2,0} - \mathrm{i}S \int_{C_J} dz \wedge \omega^{2,0}. \tag{4.3}$$

In the last equality the toroidal fibers of the twisted manifold $\tilde{K}$ are integrated out, and in a second step the resulting period integrals of the $K3$ base are transformed into periods of the holomorphic 3-form $dz \wedge \omega^{2,0}$ on the original three-fold $T^2 \times K3$ with respect to the three-cycles C_H and C_J, which are Poincaré dual to the integral three-forms $e^{(1)} \wedge dy - e^{(2)} \wedge dx$ and $e^{(1)} \wedge dx + e^{(2)} \wedge dy$.

Note that the structure of the derived superpotential is in agreement with the superpotential periods obtained in [36].

The idea is now to generalize the construction by "twisting" the fibers of the elliptically fibered three-fold $\pi : Z_B \to B$ with a section $\sigma : B \to Z_B$, such that we arrive at the generalized CY three-fold $\tilde{Z}_B$. In order to eventually relate the periods of the two manifolds Z_B and $\tilde{Z}_B$, we first translate the three-form cohomology of the three-fold Z_B to appropriate cohomology groups on the common base B. This is achieved with the Leray–Serre spectral sequence, which associates the cohomology of a fiber bundle to cohomology groups on the base.

Let $\mathcal{U} = \{U_\alpha\}$ be a good open covering of the base B. Then the cohomology group $H^k(Z_B, \mathbf{Z})$ is iteratively approximated by the Leray–Serre spectral sequence. The leading order E_2 terms of the spectral sequence read [72]

$$E_2^{p,q} = \check{H}^p(B, \mathcal{H}^q) \simeq H^p(B, \mathcal{H}^q).$$

Here the (pre-)sheaf $\mathcal{H}^q$ of the base B is defined by assigning to each open set U the group $\mathcal{H}^q(U) = H^q(\pi^{-1}U, \mathbf{Z})$, and the inclusion of open sets $\iota_U^V : V \hookrightarrow U$ induces the homomorphism $\iota_U^{V*} : \mathcal{H}^q(U) \to \mathcal{H}^q(V)$ via pullback of forms. Then the spectral sequence abuts to $H^3(Z_B,\mathbf{Z})$, and we obtain[21]

$$H^3(Z_B,\mathbf{Z}) \simeq \bigoplus_{n=0}^{3} E_2^{n,3-n} = \bigoplus_{n=0}^{3} H^n(B,\mathcal{H}^{3-n}).$$

Due to the simple connectedness of the examined CY three-fold Z_B we arrive at the simplified relation

$$H^3(Z_B,\mathbf{Z}) \simeq E_2^{2,1} = \check{H}^2(B,\mathcal{H}^1) \simeq H^2(B,\mathcal{H}^1). \tag{4.4}$$

Note that the (pre)sheaf $\mathcal{H}^1$ is not locally constant, because the dimension of the sheaf $\mathcal{H}^1$ differs at a singular fiber from the dimension of the sheaf $\mathcal{H}^1$ at a generic regular fiber.

In terms of the open covering $\mathcal{U}$ a Čech cohomology element ε in $\check{H}^2(B,\mathcal{H}^1)$ is a map that assigns to each triple intersection set $U_{\alpha\beta\gamma}$ an element in $\mathcal{H}^1(U_{\alpha\beta\gamma})$ and fulfills the cocycle condition on quartic intersections $U_{\alpha\beta\gamma\delta}$

$$0 = (\rho_\delta\circ\varepsilon)(U_{\alpha\beta\gamma}) - (\rho_\gamma\circ\varepsilon)(U_{\alpha\beta\delta}) + (\rho_\gamma\circ\varepsilon)(U_{\alpha\beta\delta}) - (\rho_\alpha\circ\varepsilon)(U_{\beta\gamma\delta}).$$

The map ρ_δ, for instance, is the pull-back induced from the inclusion $\iota_\delta : U_{\alpha\beta\gamma} \hookrightarrow U_{\alpha\beta\gamma\delta}$. Then the cohomology element ε is called a two-cocycle with coefficients in the (pre-)sheaf $\mathcal{H}^1$, and it is non-trivial if it does not arise form a one-cochain on double intersections $U_{\alpha\beta}$.

To proceed we assume that the generalized CY manifold $\tilde{Z}_B$ is also fibered $\tilde{\pi} : \tilde{Z}_B \to B$ over the same base B and that it arises from "twisting" the elliptic fibers of the three-fold Z_B. This "twist" is measured by the one-cochain φ, which assigns to each double intersection $U_{\alpha\beta}$ an element in $\mathcal{H}^1(U_{\alpha\beta})\otimes_{\mathbf{Z}}\mathbf{R}$ and which captures the distortion of the angular variable of the one-cycles in the elliptic fibers of the original three-fold Z_B.

In general, the one-chain φ does not fulfill the cocycle condition due to the periodicity of the angular variables of the one-cycles. Instead we find on triple intersections $U_{\alpha\beta\gamma}$

$$\varepsilon : U_{\alpha\beta\gamma} \mapsto \frac{1}{2\pi}\left[(\rho_\gamma\circ\varphi)(U_{\alpha\beta}) - (\rho_\beta\circ\varphi)(U_{\alpha\gamma}) + (\rho_\alpha\circ\varphi)(U_{\beta\gamma})\right] \in \mathcal{H}^1(U_{\alpha\beta\gamma}),$$

which defines a two-cocycle in $\check{H}^2(B,\mathcal{H}^1)$ characterizing the "twist" of the three-fold $\tilde{Z}_B$.

[21]Strictly speaking the first relation is not an equality "$\simeq$" but an inclusion "$\subseteq$", because we ignore the "higher order corrections" from the spectral sequence. This implies that some of the elements on the right-hand side might actually be trivial in $H^3(Z_B,\mathbf{Z})$.

Analogously the element e in $H^3(Z_B, \mathbf{Z})$, which corresponds to the Čech cohomology element ε in $\check{H}^2(B, \mathcal{H}^1)$, is explicitly constructed. Namely, there are one-forms ξ_α defined on the open sets U_α, which are exact on double overlaps $U_{\alpha\beta}$ [72]

$$\frac{1}{2\pi} d\varphi(U_{\alpha\beta}) = \rho_\beta(\xi_\alpha) - \rho_\alpha(\xi_\beta). \tag{4.5}$$

Therefore the two-forms $d\xi_\alpha$ patch together to a global two-form s_e in $H^2(B, \mathcal{H}^1)$, which in turn can be identified with the three-form e in H^3 $(Z_B, \mathbf{Z})$ according to (4.4).

In order to extract the geometric flux from the three-fold $\tilde{Z}_B$, we need to get a handle on the 2-form $\tilde{J}$ in the superpotential (4.1). Due to the fibered structure of the three-fold Z_B the Kähler form J splits into two pieces

$$J = \pi^* J_B + J_F,$$

where $J_B = \sigma^* J$, $J_F = J - \pi^* J_B$ and $J_F = S\,\omega_F$ in terms of the integral generator ω_F and the (complexified) Kähler volume of the generic elliptic fiber. Then upon the "twist" to the three-fold $\tilde{Z}_B$ the Kähler form J is transformed into the two-form

$$\tilde{J} = \tilde{\pi}^* J_B + \tilde{J}_F = \tilde{\pi}^* J_B + S\,\tilde{\omega}_F.$$

The two-form $\tilde{\omega}_F$ is defined on each open-set $\tilde{\pi}^{-1}U_\alpha$ by

$$\tilde{\omega}_F|_{\tilde{\pi}^{-1}U_\alpha} = \omega_F|_{\tilde{\pi}^{-1}U_\alpha} + \xi_\alpha,$$

where we now view ξ_α as a two form in the open set $\tilde{\pi}^{-1}U_\alpha$. Due to the "twist" the two-forms $\tilde{\omega}_F$, which are defined on open sets, patch together to a global two-form on the three-fold $\tilde{Z}_B$. Furthermore, as a consequence of equation (4.5) we observe that

$$d\tilde{J} = S\,d\tilde{\omega}_F = S\,s_e, \tag{4.6}$$

in terms of the element s_e in $H^2(B, \mathcal{H}^1)$.

In order to evaluate the heterotic superpotential (4.1) we express the three-forms of $\tilde{\Omega}$, H and $d\tilde{J}$ of the "twisted" three-fold $\tilde{Z}_B$ as elements s_Ω, s_H and s_e of the sheaf cohomology $H^2(B, \mathcal{H}^1 \otimes \mathbf{C})$. Using equation (4.4) we induce s_Ω from the holomorphic three-form Ω in $H^{3,0}(Z_B)$ and the NS flux s_H from an integral three-form in $H^3(Z_B, \mathbf{Z})$. Furthermore, we also inherit the pairing $\langle\cdot,\cdot\rangle$ on $H^2(B, \mathcal{H}^1 \otimes \mathbf{C})$ from the three-form pairing $\int_{Z_B} \cdot \wedge \cdot$

on the CY three-fold Z_B. Then the superpotential (4.1) for the "twisted" manifold $\tilde{Z}_B$ becomes

$$W_{\text{het}} = \langle s_\Omega, s_H \rangle - \mathrm{i}\, S \,\langle s_\Omega, s_e \rangle = \int_{C_H} \Omega - \mathrm{i} S \int_{C_J} \Omega. \tag{4.7}$$

In the last step we have again related the integral elements s_H and s_e to their Poincaré dual three-cycles C_H and C_J in the original CY manifold Z_B.

In the context of heterotic string compactifications on the three-fold Z_B the presented arguments provide further evidence for the encountered structure of the closed-string periods in equation (3.1). In particular, we find that the complex modulus S should be identified with the complexified volume of the generic elliptic fiber.

There is, however, a cautious remark overdue. We tacitly assumed that the manifold $\tilde{Z}_B$ can be constructed by simply "twisting" the elliptic fibers of Z_B. In general, however, we expect that such a construction is obstructed and additional modifications are necessary to arrive at a "true" generalized CY manifold. A detailed analysis of such obstructions is beyond the scope of this note. However, we believe that the outlined construction is still suitable to anticipate the (geometric) flux quanta, which are responsible for the transition to the generalized CY manifold $\tilde{Z}_B$ to leading order. From the duality perspective of the previous section we actually expect further corrections to the superpotential (4.7). These corrections should be suppressed in the large fiber limit Im $S \to \infty$. It is in this limit, in which we expect the "twisting" construction to become accurate.

4.2 Chern–Simons contribution to $W_{\mathbf{F}}(X_B)$

The F-theory prediction from the last term in (3.1) is the equality, up to finite S corrections, of certain four-fold period integrals on X_B and the Chern–Simons superpotential on Z_B, for appropriate choice of $G \in H^4(X_B)$ and a connection on $E \to Z_B$. The general relation of this type has already been described in Section 2.4 where we used that the three-fold Z_B may be viewed as a "boundary" within the F-theory four-fold X_B in the s.d. limit. Here we complete the argument and discuss the map of the deformation spaces by using hypersurface representations for $X^\sharp$ and Z_B. This will also lead to a direct identification of the open–closed dual four-fold geometries for type II branes and the local mirror geometries for (heterotic) bundles of [22, 23].

To this end, we represent the s.d. limit $X_B^\sharp$ of the F-theory four-fold X_B as a reducible fiber of a CY five-fold W obtained by fibering X_B over $\mathbf{C}$

as in [23, 39]. Let μ be the local coordinate on the base $\mathbf{C}$ which serves as a deformation space for the four-fold fiber X_B. We start from the Weierstrass form

$$p_{\mathrm{W}} = y^2 + x^3 + x \sum_{\alpha,\beta} s^{4-\alpha} \tilde{s}^{4+\alpha} \mu^{4-\beta} a_{\alpha,\beta} f_\alpha(x_k) + \sum_{\alpha,\beta} s^{6-\alpha} \tilde{s}^{6+\alpha} \mu^{6-\beta} b_{\alpha,\beta} g_\alpha(x_k),$$

where $f_\alpha(x_k)$ and $g_\alpha(x_k)$ are functions of the coordinates on the two-dimensional base B_2 of the K3 fibration of the four-fold X_B. Moreover (y, x) and $(s, \tilde{s})$ can be thought of as (homogeneous) coordinates on the elliptic fiber and the base $\mathbf{P}^1$ of the K3 fiber, respectively. Finally $a_{\alpha,\beta}$, $b_{\alpha,\beta}$ are some complex constants entering the complex structure of W. The fiber of $W \to \mathbf{C}$ over a point $p \in \mathbf{C}$ represents a smooth F-theory four-fold X_B with a complex structure determined by the values of the constants $a_{\alpha,\beta}$, $b_{\alpha,\beta}$ and of the coordinate μ at p.

Tuning the complex structure of W by choosing $a_{\alpha,\beta} = 0$ for $\alpha + \beta > 4$ and $b_{\alpha,\beta} = 0$ for $\alpha + \beta > 6$, the central fiber of W at $\mu = 0$ acquires a non-minimal singularity at $y = x = s = 0$, which can be blown up by

$$y = \rho^3 y, \;\; x = \rho^2 x, \;\; s = \rho s, \;\; \mu = \rho\mu,$$

to obtain the hypersurface[22]

$$p_{W^\sharp} = y^2 + x^3 + x \sum_{\alpha,\beta} s^{4-\alpha} \tilde{s}^{4+\alpha} \mu^{4-\beta} \rho^{4-\alpha-\beta} f_\alpha(x_k) + \sum_{\alpha,\beta} s^{6-\alpha} \tilde{s}^{6+\alpha} \mu^{6-\beta} \rho^{6-\alpha-\beta} g_\alpha(x_k), \tag{4.8}$$

The singular central fiber has been replaced by a fiber $X^\sharp = X_1 \cup X_2$ with two components X_i defined by $\rho = 0$ and $\mu = 0$, respectively. The component $\rho = 0$ is described by

$$\begin{aligned} p_{X_1} &= p_0 + p_+, \\ p_0 &= y^2 + x^3 + x f_0(x_k) + g_0(x_k), \\ p_+ &= x \sum_{\alpha>0} s^{4-\alpha} \mu^\alpha f_\alpha(x_k) + \sum_{\alpha>0} s^{6-\alpha} \mu^\alpha g_\alpha(x_k), \end{aligned} \tag{4.9}$$

we have collected the terms with zero and positive powers in μ into the two polynomials p_0 and p_+ for later use. The hypersurface X_1 is a fibration

[22] The non-zero constants $a_{\alpha,\beta}$, $b_{\alpha,\beta}$ are set to one in the following.

$X_1 \to B_2$ with fiber a rational elliptic surface S_1. The expressions in (4.9) are sections of line bundles, specifically the anti-canonical bundle $\mathcal{L} = K_{B_2}^{-1}$, a line bundle $\mathcal{M}$ over B_2 that enters the definition of the fibration $X_B \to B_2$ and a bundle $\mathcal{N}$ associated with a $\mathbf{C}^*$ symmetry acting on the homogeneous coordinates (y, x, s, μ). The powers of the line bundles appearing in these sections are

	p_{X_1}	y	x	s	μ	$f_\alpha(x_k)$	$g_\alpha(x_k)$
$\mathcal{L}$	6	3	2	0	0	4	6
$\mathcal{M}$	6	3	2	1	0	α	α
$\mathcal{N}$	6	3	2	1	1	0	0

(4.10)

e.g. $p_{X_1} \in \Gamma(\mathcal{L}^6 \otimes \mathcal{M}^6 \otimes \mathcal{N}^6)$.

The hypersurface X_1 has a positive first Chern class $c_1(X_1) = c_1(\mathcal{N})$ and the CY three-fold Z_B is embedded in X_1 as the divisor $\mu = 0$,

$$p_{Z_B} = p_{X_1} \cap \{\mu = 0\} = p_0,$$

verifying a claim that was needed in the argument of Section 2.4. According to the picture of F-theory/heterotic duality developed in [12, 15], the polynomial p_+ containing the positive powers in s describes part of the bundle data in a single E_8 factor of the heterotic string compactified on Z_B. Using a different argument, based on the type IIA string compactified on fibrations of ADE singularities, more general n-fold geometries $\hat{X}$ of the general form (4.9) have been obtained in [22, 23] as local mirror geometries of bundles with arbitrary structure group on elliptic fibrations. Mirror symmetry gives an entirely explicit map between the moduli of a given toric n-fold and the geometric data of a G bundle on a toric $n-1$-fold Z_B, which applies to any geometry $\hat{X}$ of the form (4.9) [23]. The application of these methods will be illustrated at the hand of selected examples in Sections 6 and 7.

A special case of the above discussion is the one, where the heterotic gauge sector is not a smooth bundle, but includes also non-perturbative small instantons [49]. The F-theory interpretation of these heterotic five-branes as a blow up of the base of elliptic fibration $X_B \to B_3$ has been studied in detail in [13, 15, 39]; see also [23, 73] for details in the case of toric hypersurfaces and [74] for an elegant discussion of the moduli space in M-theory.

From the point of Hodge variations and brane superpotentials this is in fact the most simple case, starting from the approach of [8, 9, 11], as the brane moduli of the type II side map to moduli of the heterotic five-brane. An explicit example from [10] will be discussed in Section 7

4.3 Type II / heterotic map

The above argument also provides a means to describe an explicit map between a type II brane compactification on Z_B and a heterotic bundle compactification on Z_B. The key point is again the afore mentioned relation (C2) between the large volume limit of the fibration $\pi : X_A \to \mathbf{P}^1$ and the s.d. limit of the F-theory four-fold X_B. The relation between the F-theory four-fold geometry, the heterotic bundle on Z_B and the type II branes on Z_B is concisely summarized by the following diagram:

$$\begin{array}{ccc} Z_A \to X_A \to \mathbf{P}^1 & \xrightarrow[+\ \textit{local limit}]{\text{large base}} & Z_A \to X_A^{\mathrm{nc}}(L) \to \mathbf{C}^1 \\ {\scriptstyle \text{mirror symmetry}}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{local} \mid \text{mirror symmetry}} \\ X_B & \xrightarrow[+\ \text{local limit}]{\text{stable deg}} & \hat{X}(E) \end{array} \tag{4.11}$$

The upper line indicates how the open–closed string dual $X_A^{\mathrm{nc}}(L)$ of an A-type bundle L on the three-fold Z_A sits in the compact four-fold X_A mirror to X_B. The details of the bundle L are encoded in the toric resolution of the central fiber Z_A^0 at the origin $0 \in \mathbf{C}^1$, as described in terms of toric polyhedra in [10, 16, 18]. The limit consists of concentrating on a local neighbourhood of the point $0 \in \mathbf{P}^1$ and taking the large volume limit of $\mathbf{P}^1$ base.

The lower row describes how the heterotic bundle E on the elliptic manifold Z_B dual to F-theory on X_B is captured by a local mirror geometry of the form (4.9). Assuming that the large base/local limit commutes with mirror symmetry, the diagram is completed to the right by another vertical arrow, which represents local mirror symmetry of the non-compact manifolds. The mirror of the open–closed dual $X_A^{\mathrm{nc}}(L)$ has been previously called $X_B^{\mathrm{nc}}(E)$, and we see that commutativity of the diagram requires that the open–closed dual $X_B^{\mathrm{nc}}(E)$ *is the same* as the heterotic dual $\hat{X}(E)$. Indeed, the hypersurface equations for $G = SU(N)$ given in [23] for the heterotic four-fold $\hat{X}$ and in [10] for the open–closed four-fold X_B^{nc} can be both written in the form

$$\begin{aligned} p(\hat{X}) &= p_0(Z_B) + v\, p_+(\Sigma) \qquad && \text{(heterotic/F–theory duality)}, \\ p(X_B^{\mathrm{nc}}) &= P(Z_B) + v\, Q(D) \qquad && \text{(open–closed duality)}, \end{aligned} \tag{4.12}$$

where v is a local coordinate defined on the cylinder related to s in (4.9). In both cases, the v^0 term specifies the three-fold Z_B on which the type II/heterotic string is compactified. In the type II context, $Q(D) = 0$ is the hypersurface $D \subset Z_B$, which is part of the definition of the B-type brane [10, 16, 18]. In the heterotic dual of [23], $p_+(\Sigma) = 0$ specifies the $SU(n)$ spectral cover [12].

The agreement of the local geometries dual to the type II/heterotic compactification on Z_B predicted by the commutativity of (4.11) is now obvious with the identification

$$\text{type II/heterotic map}: \qquad P(Z_B) = p_0(Z_B), \qquad Q(D) = p_+(\Sigma). \quad (4.13)$$

This map between the dual four-folds in (4.12) can be interpreted as a geometric reflection of the physical fact that the decoupling limit conforms the heterotic and type II bundles.

Note that, with the identification (4.13), the proofs of [11, 16–18], which relate the relative periods $H^3(Z_B, D)$ to the periods of the four-fold X_B^{nc} in the context of open–closed duality, carry also over to the heterotic string setting for $G = SU(N)$. More ambitiously, one would like to have an explicit relation between the four-fold periods and the holomorphic Chern–Simons integral also for a heterotic bundle with general structure group G. The approach of [22, 23] gives an explicit map from the moduli of a G bundle on Z_B to a local mirror geometry $\hat{X}$ for any G and evaluation of the periods of $\hat{X}$ gives the four-fold side. A computation on the heterotic side could proceed by a generalization of the arguments of Section 2.3, e.g., by constructing the sections of the bundle from the more general approaches to G bundles described in [12, 31]. In Section 8 we outline a possible alternative route, using a conjectural relation between two two-dimensional theories associated with the three-fold and the four-fold compactification.

5 Type II/heterotic duality in two space–time dimensions

In the previous sections we demonstrated the chain of dualities in equation (3.3) by matching the holomorphic superpotentials of the various dual theories. In this section we further supplement this analysis by relating the two-dimensional low energy effective theories of the type IIA compactificatons on the four-folds X_A and X_B with the dual heterotic compactification on $T^2 \times Z_B$. Many aspects of the type II/heterotic duality on the level of the low-energy effective action are already examined in [44]. We further extend this discussion here.

For the afore mentioned string compactifications the low-energy effective theory is described by two-dimensional $\mathcal{N} = (2, 2)$ supergravity.[23] Chiral multiplets φ and twisted chiral multiplets $\tilde{\varphi}$ comprise the dynamical degrees

[23]Note that these two-dimensional theories describe the effective space–time theory and not the two-dimensional field theory of the underlying microscopic string worldsheet.

of freedom of these supergravity theories [75,76]. In a dimensional reduction of four-dimensional $\mathcal{N}=1$ theories the two-dimensional chiral multiplets/twisted chiral multiplets arise from four-dimensional chiral multiplets/vector multiplets, respectively.

The scalar potential of the two-dimensional $\mathcal{N}=(2,2)$ Lagrangian arises from the holomorphic chiral and twisted chiral superpotentials $W(\varphi)$ and $\widetilde{W}(\tilde{\varphi})$, and the kinetic terms are specified by the two-dimensional Kähler potential[24]

$$K^{(2)}(\varphi,\bar{\varphi},\tilde{\varphi},\bar{\tilde{\varphi}}) = K^{(2)}(\varphi,\bar{\varphi}) + \widetilde{K}^{(2)}(\tilde{\varphi},\bar{\tilde{\varphi}}). \tag{5.1}$$

Here $K^{(2)}$ and $\widetilde{K}^{(2)}$ can be thought of individual Kähler potentials for the chiral and twisted chiral sectors. In this section we mainly focus on the Kähler potential (5.1) to further establish the type II/heterotic string duality of equation (3.3).

5.1 Type IIA on CY four-folds

The low-energy degrees of freedom of type IIA compactifications on the CY four-fold X are the twisted chiral multiplets $T^A, A=1,\dots,h^{1,1}(X)$ and the chiral multiplets $z^I, I=1,\dots,h^{3,1}(X)$.[25] They arise from the Kähler and the complex structure moduli of the four-fold X.[26] Then the tree-level Kähler potential is given by [44]

$$K^{(2)}_{\mathrm{IIA}} = K^{(2)}_{\mathrm{CS}}(z,\bar{z}) + \widetilde{K}^{(2)}_{\mathrm{K}}(T,\bar{T}) = -\ln Y^{\mathrm{IIA}}_{\mathrm{CS}}(z,\bar{z}) - \ln \widetilde{Y}^{\mathrm{IIA}}_{\mathrm{K}}(T,\bar{T}), \tag{5.2}$$

where the exponential of the potential $K^{(2)}_{\mathrm{CS}}$ for the complex structure moduli is determined by

$$Y^{IIA}_{\mathrm{CS}}(z,\bar{z}) = \int_X \Omega(z)\wedge\bar{\Omega}(\bar{z}), \tag{5.3}$$

[24]This splitting of the Kähler potential does not represent the most general form. In fact, in general the target space metric need not even be Kähler [75]. The given ansatz, however, suffices for our purposes.

[25]In two dimensions the graviton and the dilaton are not dynamical [77].

[26]For $h^{2,1}(X)\neq 0$ there are additional $h^{2,1}$ chiral multiplets, which we do not take into account here. With these multiplets the simple splitting ansatz (5.1) ceases to be sufficient [44].

in terms of the holomorphic $(4,0)$ form Ω of the CY X. In the large radius regime the twisted potential $\widetilde{K}_{\rm K}^{(2)}$ for the Kähler moduli reads

$$\begin{aligned}\widetilde{Y}_{\rm K}^{\rm IIA} &= \frac{1}{4!}\int_X J^4\\ &= \frac{1}{4!}\sum_{A,B,C,D}\mathcal{K}_{ABCD}(T^A-\bar{T}^A)(T^B-\bar{T}^B)(T^C-\bar{T}^C)(T^D-\bar{T}^D),\end{aligned} \tag{5.4}$$

with $\mathcal{K}_{ABCD}$ the topological intersection numbers of the four-fold X. The Kähler moduli T^A appear in the expansion of the complexified Kähler form $B+iJ=T^A\omega_A$, $\omega_A\in H^2(X,\mathbf{Z})$, where B and J are the NS two-form and the real Kähler form, respectively. Finally, in the presence of background fluxes, we obtain the holomorphic superpotentials [24, 50]

$$W(z)=\int_X \Omega\wedge F_{\rm hor},\qquad \widetilde{W}(t)=\int_X e^{B+iJ}\wedge F_{ver}. \tag{5.5}$$

Here $F_{\rm hor}\in H^4_{\rm hor}(X)$ is a non-trivial horizontal RR four-form flux, whereas $F_{\rm ver}\in H^{\rm ev}_{\rm ver}(X)$ is a non-trivial even-dimensional vertical RR flux.[27] The twisted chiral superpotential $\widetilde{W}$ receives non-perturbative world-sheet corrections away from the large radius point [78, 79].

5.2 Type IIA on the CY four-folds X_A and X_B

We now turn to the type IIA compactification on the special CY four-fold X_A. As discussed in Section 4.1. the four-fold geometry X_A is a fibration over the $\mathbf{P}^1$ base, where the generic fiber is the CY three-fold Z_A. Geometries of this type have been studied previously in [44, 79] and we extend the discussion here to fibrations with singular fibers, which support the brane/bundle degrees of freedom in the context of open–closed/heterotic duality.

For the divisor D_S dual to the base this implies

$$\int_{D_S} c_3(X_A)=\chi(Z_A). \tag{5.6}$$

Here $c_3(X_A)$ is the third Chern class of the four-fold X_A and $\chi(Z_B)$ is the Euler characteristic of three-fold Z_A. Hence the divisor D_S is homologous to the generic (non-singular) fiber Z_A.

[27] The six- and eight-forms are the magnetic dual fluxes to the RR four- and two-form fluxes in type IIA.

For type IIA compactified on the four-fold X_A we are interested in the twisted chiral sector, and hence in the twisted Kähler potential (5.4). This means we need to obtain the intersection numbers of the fibered four-fold X_A. We use similar arguments as in [56], where the intersection numbers of $K3$-fibered CY threefold are determined.

We denote by S the (complexified) Kähler modulus that measures the volume of the $\mathbf{P}^1$ base, which is dual to the divisor D_S representing the generic fiber Z_A. Consider now a divisor H_a of the generic fiber Z_A. As we move this divisor about the base by mapping it to equivalent divisors in the neighboring generic fibers, we define a divisor D_a in the CY four-fold X_A.[28] The remaining (inequivalent) divisors of the four-fold X_A are associated to singular fibers, and we denote them by $\hat{D}_{\hat{a}}$.

The two-forms ω_S, ω_a and $\hat{\omega}_{\hat{a}}$, which are dual to the divisors D_S, D_a and $\hat{D}_{\hat{a}}$, furnish now a basis of the cohomology group $H^2(X_A, \mathbf{Z})$, and we denote the corresponding (complexified) Kähler moduli by S, t_a and $\hat{t}_{\hat{a}}$. They measure the volume of the $\mathbf{P}^1$-base, the volume of the two-cycles in the generic three-fold fiber Z_A, and the volume of the remaining two-cycles arising from the degenerate fibers.

From this analysis we can extract the structure of intersection numbers. Since D_S is a homology representative of the generic fiber it intersects only with the CY divisors D_a according to the triple intersection numbers κ_{abc} of the three-fold Z_A. The intersection numbers for divisors, which do not involve D_S, cannot be further specified by these general considerations. Therefore we find

$$\frac{1}{4!}\mathcal{K}_{ABCD}T^A\,T^B\,T^C\,T^D = \frac{1}{3!}\kappa_{abc}\,S\,t_a\,t_b\,t_c + \frac{1}{4!}\mathcal{K}'_{\alpha\beta\gamma\delta}t'_\alpha\,t'_\beta\,t'_\gamma\,t'_\delta, \tag{5.7}$$

where t'_α are the Kähler moduli $(t_a, \hat{t}_{\hat{a}})$ with their quartic intersection numbers $\mathcal{K}'_{\alpha\beta\gamma\delta}$. The twisted Kähler potential for the four-fold X_A then reads

$$\begin{aligned}\widetilde{Y}_{\mathrm{K}}(X_A) = {}&\frac{1}{3!}(S-\bar{S})\sum\kappa_{abc}(t_a-\bar{t}_a)(t_b-\bar{t}_b)(t_c-\bar{t}_c)\\ &+\frac{1}{4!}\sum\mathcal{K}'_{\alpha\beta\gamma\delta}(t'_\alpha-\bar{t}'_\alpha)(t'_\beta-\bar{t}'_\beta)(t'_\gamma-\bar{t}'_\gamma)(t'_\delta-\bar{t}'_\delta).\end{aligned} \tag{5.8}$$

[28]Due to monodromies with respect to the degenerate fibers, it may happen that two inequivalent divisors H_a and H_b are identified globally, and hence yield the same divisor $D_a = D_b$. Then we work on the three-fold Z_A with monodromy-invariant (linear combinations of) divisors such that only inequivalent divisors D_a are generated on the four-fold X_A.

The essential point here is that the leading term for large S involves only the moduli t_a associated with the bulk fields in the dual compactifications, whereas the brane/bundle degrees of freedom appear in the subleading term. In the decoupling limit Im $S \to \infty$, the kinetic terms derived from (5.8) factorize into the bulk and bundle sector of the dual theories as

$$G_{A\bar{B}}(T^C)\,\partial_\mu T^A \partial^\mu \bar{T}^{\bar{B}} \to G^{\text{bulk}}_{a\bar{b}}(t^c)\,\partial_\mu t^a \partial^\mu \bar{t}^{\bar{b}} + \frac{1}{\text{Im }S}\,G^{\text{bundle}}_{\alpha\bar{\beta}}(t^c,t^\gamma)\,\partial_\mu t^\alpha \partial^\mu \bar{t}^{\bar{\beta}},$$

illustrating the separation of the physical scales at which the fields in the two sectors fluctuate. In this limit, the backreaction of the (dual) bulk fields to the (dual) bundle fields vanishes and the latter fluctuate in the fixed background determined by the bulk fields. A more detailed treatment of the heterotic dual will be given below.

Analogously to the three contributions to $H^2(X_A,\mathbf{Z})$ distinguished above, we can decompose the even-dimensional fluxes F_{V} into three distinct classes

$$F_{\text{V}} = f^{(1)} + f^{(2)} \wedge \omega_S + f^{(3)}, \tag{5.9}$$

where the components $f^{(1)}$ and $f^{(2)}$ pull back to even-forms in $H^{\text{ev}}(Z_B)$, while the fluxes $f^{(3)}$ vanish upon pullback to the regular three-fold fiber Z_A. With the vertical fluxes (5.9) the (semi-classical) twisted chiral superpotential $\widetilde{W}(X_A)$ simplifies to

$$\widetilde{W}(X_A) = \int_{Z_B} e^{\sum_a t_a\omega_a} \wedge (Sf^{(1)} + f^{(2)}) + \int_{X_A} e^{\sum_\alpha t'_\alpha\omega'_\alpha} \wedge (f^{(1)} + f^{(3)}), \tag{5.10}$$

with the generators $(\omega_a, \hat{\omega}_{\hat{a}})$ collectively denoted by ω'_α.

Next, we turn to the chiral sector of type IIA strings compactified on the mirror four-fold X_B. The Kähler potential (5.3) is then expressed in terms of the periods $\Pi^\Sigma = \int_{\gamma_\Sigma}\Omega^{(4,0)}$ of the CY four-fold X_B

$$Y_{\text{CS}}(X_B) = \sum_{\gamma_\Sigma,\gamma_\Lambda \in H_4(X_B)} \Pi^\Sigma(z)\eta_{\Sigma\Lambda}\bar{\Pi}^\Lambda(\bar{z}), \tag{5.11}$$

where $\eta_{\Sigma\Lambda}$ is the topological intersection paring on $H_4(X_B)$. The horizontal background fluxes F_{H} induce the chiral superpotential $W(X_B)$ given in equation (3.1), where the quanta $\underline{N}_\Sigma$ correspond to the integral flux quanta of four-form flux F_{H}.

By four-fold mirror symmetry the superpotential $\widetilde{W}(X_A)$ and $W(X_B)$ are equal on the quantum level. In comparing the semi-classical expression

(5.10) for the twisted superpotential to the structure of the chiral superpotential (3.1) in the stable degeneration limit (3.1), we observe that the vertical fluxes $f^{(1)}$, $f^{(2)}$ and $f^{(3)}$ give rise to the flux quanta M_Σ, N_Σ and $\hat{N}_\Sigma$, respectively.

5.3 Heterotic string on $T^2 \times Z_B$

The low-energy effective action of the heterotic string compactified on the four-fold $T^2 \times Z_B$ together with a (non-trivial) gauge bundle V has in the large radius regime the structure [44]

$$K^{(2)}_{\text{het}} = K^{(4)}_{\text{het}}(\Phi, \bar{\Phi}) + \widetilde{K}^{(2)}_{\text{het}}(\tilde{\Phi}, \bar{\tilde{\Phi}}). \tag{5.12}$$

The chiral Kähler potential $K^{(4)}_{\text{het}}$ coincides with the four-dimensional Kähler potential of the heterotic string compactified on the CY three-fold Z_B with the gauge bundle V. Apart from the heterotic dilaton, which is not a dynamic field in two dimensions [77], it comprises all the kinetic terms for both the chiral multiplets of the Kähler/complex structure moduli of the three-fold Z_B and the chiral multiplets from the gauge bundle V. The Kähler potential $\widetilde{K}^{(2)}_{\text{het}}$ of the twisted chiral multiplet consists of the modes arising from the torus T^2 and the gauge fields, which correspond to the vector multiplets in higher dimensions.

For heterotic CY compactifications with the standard embedding of the spin connection the Kähler potential $K^{(4)}_{\text{het}}$ splits further according to

$$K^{(4)}_{\text{het}} = K^{(4)}_{\text{CS}}(z, \bar{z}) + K^{(4)}_{\text{K}}(t, \bar{t}) + \ldots,$$

where $K^{(4)}_{\text{CS}}$ and $K^{(4)}_{\text{K}}$ are the Kähler potentials for the chiral complex structure and Kähler moduli z and t of the CY Z_B. For a general heterotic string compactification, we do not know of any generic model independent properties of the Kähler potential. However, in the context of type IIA/heterotic duality (3.3), we expect a special subsector associated with the kinetic terms of the complex structure moduli z^a of the three-fold together with the specific moduli fields $\hat{z}^{\hat{a}}$ of the bundle captured by the dual four-fold.

In order to infer some qualitative information about the relevant kinetic terms of the moduli z^a and $\hat{z}^{\hat{a}}$, we briefly discuss the general structure of the bosonic part of the four-dimensional low-energy effective heterotic action in the four-dimensional Einstein frame

$$S^{(4)}_{\text{het}} = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{g_4} \left(R^{(4)} - \frac{1}{2} \left(C_{a\bar{b}} \partial_\mu z^a \partial^\mu \bar{z}^{\bar{b}} \right) - \frac{1}{2} \left(B_{\hat{a}\bar{\hat{b}}} \partial_\mu \hat{z}^{\hat{a}} \partial^\mu \bar{\hat{z}}^{\bar{\hat{b}}} \right) + \cdots \right). \tag{5.13}$$

Here $R^{(4)}$ is the Einstein–Hilbert term, κ_4 is the four-dimensional gravitational coupling constant. $C_{a\bar{b}}$ and $B_{\hat{a}\bar{\hat{b}}}$ denote the Kähler metrics of the chiral fields z^a and $\hat{z}^{\hat{a}}$. For simplicity cross terms among bulk and bundle moduli and the kinetic terms of other moduli fields are omitted. Note that the α' dependence of the bundle moduli is absorbed into the Kähler metric $B_{\hat{a}\bar{\hat{b}}}$.

From a dimensional reduction point of view the bundle moduli $\hat{z}^{\hat{a}}$ arise from a Kaluza–Klein reduction of the 10-dimensional vector field $A^{(10)}$, which in terms of four-dimensional coordinates x and internal coordinates y enjoys the expansion

$$A^{(10)}(x, y) = A^{(4)}_\mu(x)\, dx^\mu + \sum_{\hat{a}} (\hat{z}^{\hat{a}}(x)\, v_{\hat{a}}(y) + \text{c.c.}) + \cdots .$$

The four-dimensional vector $A^{(4)}$ gives rise to the Yang–Mills kinetic term, while the internal vectors fields $v_{\hat{a}}$ are integrated out in the dimensional reduction process and yield the metric $B_{\hat{a}\bar{\hat{b}}}$

$$B_{\hat{a}\bar{\hat{b}}} = \frac{1}{V(Z_B)} \int_{Z_B} d^6y \sqrt{g_6}\, \alpha' g^{i\bar{\jmath}}\, \text{Tr} \left(v_{\hat{a},i}\, \bar{v}_{\bar{\hat{b}},\bar{\jmath}} \right). \tag{5.14}$$

The volume factor $V(Z_B)$ arises due to the Weyl rescaling to the four-dimensional Einstein frame, and it compensates the scaling of the (internal) measure $d^6y\sqrt{g_6}$. Thus the dimensionless quantity $\frac{\alpha'}{\ell^2}$, where ℓ is the length scale of the internal CY manifold Z_B, governs the magnitude of the kinetic terms $B_{\hat{a}\bar{\hat{b}}}$.

As discussed in Section 4.1, the decoupling limit Im $S \to \infty$ defined in [11] is mapped on the heterotic side to the large fiber limit of the elliptically fibered CY three-fold $Z_B \to B$. In order to work in at semi-classical regime, the volume $V(B)$ of the base B, common to the K3 fibration $X_B \to B$ and the elliptic fibration $Z_B \to B$, has to be taken of large volume as well, due to the relations [44]

$$\lambda^{-2}_{\text{II},2\text{d}} = \lambda^{-2}_{\text{het},2\text{d}}, \qquad V_{\text{het}}(B) \cdot V_{\text{II}}(B) = \lambda^{-4}_{\text{II},2\text{d}},$$

which follow from the relations $\lambda_{\text{II},6\text{d}} = \lambda^{-1}_{\text{het},6\text{d}}$, $g_{\text{het}} = \lambda^{-2}_{\text{II},6\text{d}} g_{\text{II}}$ in six dimensions [58]. As we move away from the stable degeneration point in the dual type IIA description, the volume of the elliptic fiber in the three-fold Z_B becomes finite while we keep the volume of the base large

$$0 \ll \ell_F \ll \ell_B. \tag{5.15}$$

Here ℓ_F is the length scale for the generic elliptic fiber and ℓ_B is the length scale for the base.

As a consequence, as we move away from the stable degeneration point, the bundle components, which scale with the dimensionless quantity

$$g_F \equiv \frac{\alpha'}{\ell_F^2},$$

are the dominant contributions to the metric (5.14). The moduli of the spectral cover correspond on the (dual) elliptic fiber to vector fields $v_{\hat{a}}$, which are contracted with the metric component scaling as g_F. Therefore the bundle moduli $\hat{z}^{\hat{a}}$ associated to the subbundle E of the spectral cover becomes relevant.

Thus for the heterotic string compactification on the three-fold Z_B with gauge bundle the complex structure/bundle moduli space of the pair (Z_B, E) is governed by the deformation problem of a family of CY three-folds Z_B together with a family of spectral covers Σ_+. As proposed in (3.1), this moduli dependence is encoded in the relative periods $\underline{\Pi}^\Sigma(z, \hat{z})$ of the relative three forms $H^3(Z_B, \Sigma_+)$, and therefore in the semi-classical regime the Kähler potential of the complex structure/bundle moduli space (Z_B, E) is expressed explicitly by [11, 80]

$$\begin{aligned} K^{(4)}_{\mathrm{CS},E} &= -\ln Y_{\mathrm{CS},E}(Z_B, \Sigma_+), \\ Y_{\mathrm{CS},E}(Z_B, \Sigma_+) &= \sum_{\gamma_\Sigma, \gamma_\Lambda \in H_3(Z_B, \Sigma_+)} \underline{\Pi}^\Sigma(z, \hat{z}) \underline{\eta}_{\Sigma\Lambda} \bar{\underline{\Pi}}^\Lambda(\bar{z}, \bar{\hat{z}}). \end{aligned} \tag{5.16}$$

The topological metric $\eta_{\Sigma\Lambda}$ arises from the intersection matrix of the relative cycles γ_Σ. This intersection matrix has the form [11]

$$(\underline{\eta}) = \begin{pmatrix} \eta_{Z_B} & 0 \\ 0 & \mathrm{i}\, g_F\, \hat{\eta}_{\Sigma_+} \end{pmatrix},$$

where η_{Z_B} is the topological metric of the absolute cohomology $H^3(Z_B)$ and $\hat{\eta}_{\Sigma_+}$ is the topological metric of the variable cohomology sector $H^2_{\mathrm{var}}(\Sigma_+)$ of the relative cohomology group $H^3(Z_B, \Sigma_+)$.

Note that the structure of the Kähler potential (5.16) is also in agreement with the mirror Kähler potential of type IIA compactified on the four-fold X_A. By the arguments of Section 4, the Kähler modulus S of the $\mathbf{P}^1$ base of the four-fold X_A is related to the heterotic volume modulus of the elliptic fiber of the fibration $Z_B \to B$. In the large base limit of X_A/bundle decoupling limit of (Z_B, V) the leading order terms are the Kähler moduli of the three-fold fiber Z_A/complex structure moduli of the three-fold Z_B. These moduli spaces are identified by mirror symmetry of the three-fold mirror pair (Z_A, Z_B). The subleading terms for type IIA on X_A in equation (5.8) should be compared to the subleading bundle moduli terms in equation (5.16) on the heterotic side.

Finally, we remark that since the chiral sector of the heterotic string compactification on $T^2 \times Z_B$ and on Z_B are equivalent (cf. equation (5.12)), the identification of the chiral Kähler potentials in the type IIA/heterotic duality in two space–time dimensions carries over to the analog identification of Kähler potentials in the F-theory/heterotic dual theories in four space–time dimensions discussed in Section 4.

6 A heterotic bundle on the mirror of the quintic

Our first example will be an $\mathcal{N}=1$ supersymmetric compactification on the quintic in $\mathbf{P}^4$ and its mirror. This was the first compact manifold for which disc instanton corrected brane superpotentials have been computed from open string mirror symmetry in [29, 30]. This computation was confirmed by an A model computation in [81]. An off-shell version of the superpotential was later obtained in [9–11, 17], both in the relative cohomology approach, equation (2.1), as well as from open–closed duality, equation (2.3).

6.1 Heterotic string on the three-fold in the decoupling limit

Here we follow the treatment in [10, 11], In the framework of [82], the mirror pair (X_A, X_B) of toric hypersurfaces can be defined by a pair (Δ, Δ^*) of toric polyhedra, given in Appendix B.1 for the concrete example. The $h^{1,1}=3$ Kähler moduli t_a, $a=1,2,3$, of the fibration $Z_A \to X_A \to \mathbf{P}^1$ describe the volume $t=t_1+t_2$ of the generic quintic fiber of the type Z_A, the volume $S=t_3$ of the base $\mathbf{P}^1$ and one additional Kähler volume $\hat{t}=t_2$ measuring the volume of an exceptional divisor intersecting the singular fiber Z_A^0. This divisor is associated with the vertex $\nu_6 \subset \Delta$ in equation (B.1) and its Kähler modulus represents an open string deformation of a toric A brane geometry (Z_A, L) of the class considered in [7].

The hypersurface equation for the mirror four-fold X_B is given by the general expression

$$P(X_B) = \sum_{i=0}^{N} a_i \prod_{j=0}^{M} x_j^{\langle \nu_i, \nu_j^\star \rangle + 1}. \tag{6.1}$$

Here the sums for i and j run over the relevant integral points of the polyhedra Δ and Δ^*, respectively, and a_i are complex coefficients that determine the complex structure of X_B. A similar expression holds for the hypersurface equation of the mirror manifold X_A, with the roles of Δ and Δ^* exchanged.

Instead of writing the full expression, which would be too complicated due to the large number of relevant points of Δ^*, we first write a simplified expression in local coordinates that displays the quintic fibration of the mirror:

$$P(X_B) = p_0 + v^1 p_+ + v^{-1} p_-, \tag{6.2}$$

with

$$\begin{aligned} p_0 &= x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - (z_1 z_2)^{-1/5}\, x_1 x_2 x_3 x_4 x_5, \\ p_+ &= x_1^5 + z_2\, (z_1 z_2)^{-1/5}\, x_1 x_2 x_3 x_4 x_5, \qquad p_- = z_3\, x_1^5. \end{aligned} \tag{6.3}$$

Here v is a local coordinate on $\mathbf{C}^*$ and z_a the three complex structure moduli of X_B related to the afore mentioned Kähler moduli of X_A by the mirror map, $t_i = t_i(z_.)$. In the large volume limit the leading behavior is $t_i(z_.) = \frac{1}{2\pi \mathrm{i}} \ln(z_i) + \mathcal{O}(z_.)$. The special combination $z_1 z_2$ appearing above is mirror to the volume of the quintic fiber of $\pi : X_A \to \mathbf{P}^1$, We refer to Appendix B.1 for further details of the parametrization used here and in the following.

Although the above expression for $P(X_B)$ is oversimplified (most of the coordinates x_j in (6.1) have been set to one), it suffices to illustrate the general structure and to sketch the effect of the decoupling limit, which, again simplifying, corresponds to setting $z_3 = 0$, removing the term $\sim p_-$ in (6.3).[29] This produces a hypersurface equation of the promised form (4.12). In particular, $p_0(Z_B) = 0$ defines the mirror of the quintic, which has a single complex structure deformation parametrized by $z = z_1 z_2$. The hypersurface D for the relative cohomology space $H^3(Z_B, D)$, which specifies the Hodge variation problem, is defined by $p_+ = 0$, that is

$$Z_B \supset D: \; x_1^5 + z_2\, (z_1 z_2)^{-1/5}\, x_1 x_2 x_3 x_4 x_5 = 0. \tag{6.4}$$

More precisely the component of (6.4) relevant to the brane superpotential of [10, 29] is in a patch with $x_i \neq 0 \; \forall i$ and passing to appropriate local

[29] A more precise description of this process as a local mirror limit is given in [23].

coordinates for this patch, the Hodge variation on D is equivalent to that on a quartic K3 surface in $\mathbf{P}^3$ [10].

The F-theory content of the toric hypersurface X_B and its heterotic dual are exposed in different local coordinates on the ambient space, which put the hypersurface equation into the form studied in the context of F-theory/heterotic duality in [23]:

$$\begin{aligned} p_0 &= Y^3 + X^3 + YXZ(stu + s^3 + t^3) - z_1 z_2\, Z^3(s^2t^2u^5), \\ p_+ &= X^3 - z_2\, YXZ(stu)\ , \qquad p_- = z_3\, X^3. \end{aligned} \tag{6.5}$$

Here (Y, X, Z) are the coordinates on the elliptic fiber, a cubic in $\mathbf{P}^2$. Again the zero set $p_0 = 0$ defines the three-fold geometry Z_B, while the polynomials $p_\pm$ specify the components $\Sigma_\pm$ of the spectral cover of the heterotic bundle in the two E_8 factors. While p_- corresponds to the trivial spectral cover, p_+ describes a non-trivial component

$$\Sigma_+ :\ X^2 - z_2\, YZ(stu) = 0. \tag{6.6}$$

This equation can be seen to correspond to a bundle with structure group $SU(2)$ as follows. The intersection of the equation Σ_+ with the cubic elliptic equation gives six zeros. However these zeros are identified by the Greene–Plesser orbifold group $\mathbf{Z}_3$, acting on the coordinates $\{Z, Y, X\}$ according to

$$\{Z, Y, X\} \to \{\rho^2\, Z, \rho\, Y, X\}\ , \qquad \rho^3 = 1, \tag{6.7}$$

where ρ is a third root of unity. Note that the spectral cover Σ_+ represents the most general polynomial of degree two invariant with respect to the orbifold group (6.7). As a consequence, the six zeros become just two distinct zeros in the elliptic fiber E, adding up to zero. Therefore the spectral cover describes a $SU(2)$ bundle on the heterotic manifold Z_B.

Alternatively, one may study the perturbative gauge symmetry of the heterotic compactification from studying the singularities of the elliptic fibration X_B. The result of this procedure, described in detail in the appendix, is that the bundle leads to the gauge symmetry breaking pattern

$$E_6 \times E_6 \longrightarrow SU(6) \times E_6 \tag{6.8}$$

in agreement with a new component of the bundle of structure group $SU(2)$.

6.1.1 Flux superpotential in the decoupling limit

To be more precise, the above discussion describes only the data of the bundle geometrized by F-theory and ignores the "non-geometric" part of the bundle arising from fluxes on the seven-branes, which may lead to a larger structure group of the bundle, and thus smaller gauge group of the compactification then the one described above [13].

In particular, to compute the heterotic superpotential (2.7), we have to specify the class γ of Section 2.2, which determines the flux number $\hat{N}_\Sigma$ in (3.1), and thus the superpotential as a linear combination of the four-fold periods. This is the heterotic analog of choosing the five-brane flux on the type II brane (6.4). Since equation (4.13) identifies the type II open string brane modulus z_2 literally with the heterotic bundle modulus in the decoupling limit Im $S \to \infty$, the relative cohomology space and the associated Hodge variation problem are identical to the one studied in the context of type II branes in [11]. Using the identification $\gamma = \tilde{\gamma}$ between the classes defined in (2.5) and (2.6), the heterotic superpotential in the decoupling limit is identical to that for the type II brane computed in Section 5 of [11], see equation (5.3). We now discuss the corrections to this result for finite Im S.

6.2 F-theory superpotential on the four-fold X_B

According to the arguments of Section 3, Hodge theory on the F-theory four-fold X_B computes further corrections to the superpotential of the type II/heterotic compactification for finite S. We will now perform a detailed study of the periods of X_B using mirror symmetry of the four-folds (X_A, X_B).

Mirror symmetry is vital in two ways. Firstly, it allows to determine the geometric periods on $H_4(X_B, \mathbf{Z})$, appearing as the coefficients of the flux numbers N_Σ in (3.1), from an intersection computation on the mirror X_A. Secondly, the mirror map $t(z)$ can be used to define preferred local coordinates on the complex structure moduli space $\mathcal{M}_{\mathrm{CS}}(X_B)$ near a large complex structure point. In the context of open–closed string duality, these two steps are central to extracting the large volume world-sheet instanton expansion of the periods for the mirror A-model geometry X_A, as they yield the disc instanton expansion of the superpotential for A-type brane geometry (Z_A, L) by open–closed duality [10,11]. In the present context, we use this A model expansion to describe the superpotential $W_F(X_B)$ near a large complex structure limit of X_B, which by the previous arguments describes the decoupling limit Im $S \to \infty$ of the dual heterotic compactification (Z_B, E) near large complex structure of Z_B.[30]

The methods of mirror symmetry for toric four-fold hypersurfaces used in the following have been described in detail in [79, 83, 84] and we refer to these papers to avoid excessive repetitions. We work at the large complex structure point of X_B defined by the values $z_a = 0,\ a = 1, 2, 3$ for the moduli in the hypersurface equation (6.2). This corresponds to a large volume phase $t_a \sim \frac{1}{2\pi i} \ln(z_a) \to \mathrm{i}\infty$ in the Kähler moduli of the mirror manifold X_A generated by the charge vectors

$$\begin{aligned} l^1 &= (-4 \quad\ 0\ \ 1\ \ 1\ \ 1\ \ 1\ \ -1 \quad\ 1\ \ 0), \\ l^2 &= (-1 \quad\ 1\ \ 0\ \ 0\ \ 0\ \ 0 \quad\ 1\ \ -1\ \ 0), \\ l^3 &= (\ \ 0\ \ -2\ \ 0\ \ 0\ \ 0\ \ 0 \quad\ 0 \quad\ 1\ \ 1). \end{aligned} \tag{6.9}$$

The topological intersection data for this phase can be determined from toric geometry in the standard way, see [19, 23, 79, 84] for examples. We refer to the appendix of [11] for details on this particular example and restrict here to quote the quartic intersections

$$\begin{aligned} \mathcal{F}_4 &= \frac{1}{4!} \int_{X_c} J^4 = \frac{1}{4!} \sum_{a,b,c,d} K_{\alpha\beta\gamma\delta} t^\alpha t^\beta t^\gamma t^\delta \\ &= \frac{5}{6}(t_1 + t_2)^3 t_3 + \frac{5}{12}(t_1 + t_2)^4 - \frac{1}{6} t_1^4 = \frac{5}{6} \check{t}_1^3 \check{t}_3 + \left(\frac{5}{12} \check{t}_1^4 - \frac{1}{6} \check{t}_2^4 \right). \end{aligned} \tag{6.10}$$

Here $J = \sum_a t_a J_a = \sum_a \check{t}_a \check{J}_a$ denotes the Kähler form on X_A, with J_a, $a = 1, 2, 3$ a basis of $H^{1,1}(X_A)$ dual to the Mori cone defined by (6.9). In the above, we have introduced the linear combinations

$$\check{t}_1 = t = t_1 + t_2, \qquad \check{t}_2 = \hat{t} - t = -t_1, \qquad \check{t}_3 = S = t_3, \tag{6.11}$$

and the corresponding basis $\{\check{J}_a\}$ of $H^{1,1}(X_A)$ to expose the simple dependence on the Kähler modulus $\check{t}_1 = \mathrm{Vol}(Z_A)$ of the generic quintic fiber of $\pi :\ Z_A \to X_A \to \mathbf{P}^1$.

The leading terms of the period vector $\Pi_\Sigma = \int_{\gamma_\Sigma} \Omega$ for X_B in the limit $z_a \to 0$ can be computed from the classical volumes of even-dimensional

[30]The fact that the large complex structure limit of the four-fold X_B implies a large structure limit of the dual heterotic three-fold Z_B follows already from the hypersurface equation, equation (6.5), and is explicit in the monodromy weight filtration of the four-fold periods discussed below.

algebraic cycles in X_A

$$\Pi_\Sigma(X_B) = \int_{\gamma_\Sigma} \Omega(z) \sim \frac{1}{q!} \int_{\tilde{\gamma}_\Sigma} J^q,$$

where $\gamma_\Sigma \in H_4(X_B, \mathbf{Z})$ refers to a basis of primitive four-cycles in X_B and $\tilde{\gamma}_\Sigma$ a basis for the $2q$ dimensional algebraic cycles in $H_{2q}(X_A)$, $q = 0, ..., 4$, related to the former by mirror symmetry. Except for $q = 2$, there are canonical basis elements for $H_{2q}(X_A, \mathbf{Z})$, given by the class of a point, the class of X_A, the divisors dual to the generators $\check{J}_a$ and the curves dual to these divisors, respectively. On the subspace $q = 2$ we choose as in [11] the basis $\gamma_1 = D_1 \cap D_2, \gamma_2 = D_2 \cap D_8, \gamma_3 = D_2 \cap D_6$. Here the $D_i = \{x_i = 0\}$, $i = 0, \ldots, 8$ are the toric divisors defined by the coordinates x_i on the ambient space for X_A (cpw. equation (6.1)), which correspond to the vertices of the polyhedron Δ in (B.1). The classical volumes of these basis elements computed from the intersections (6.10) are

$$\begin{gathered} \Pi_0 = 1, \quad \Pi_{1,i} = \check{t}_i, \quad \Pi_{2,1} = 5\check{t}_1\check{t}_3, \quad \Pi_{2,2} = \tfrac{5}{2}\check{t}_1^2, \quad \Pi_{2,3} = 2\check{t}_2^2, \\ \Pi_{3,1} = \tfrac{5}{2}\check{t}_1^2\check{t}_3 + \tfrac{5}{3}\check{t}_1^3, \quad \Pi_{3,2} = -\tfrac{2}{3}\check{t}_2^3, \quad \Pi_{3,3} = \tfrac{5}{6}\check{t}_1^3, \Pi_4 = \mathcal{F}_4, \end{gathered} \tag{6.12}$$

where the first index q on $\Pi_{q,}$ denotes the complex dimension of the cycle.

The entries of the period vector $\Pi(X_B)$ are solutions of the Picard–Fuchs system for the mirror manifold X_B with the appropriate leading behavior (6.12) for $z_a \to 0$. The Picard–Fuchs operators can be derived from the toric GKZ system [79, 84] and are given in equation (A.6) in the Appendix.

The Gauss–Manin system for the period matrix imposes certain integrability conditions on the moduli dependence of the periods of a CY n-fold. For $n = 2$ these conditions imply that there are no instanton corrections on K3 and for $n = 3$ they imply the existence of a prepotential $\mathcal{F}$ for the periods. For $n = 4$ the periods can no longer be integrated to a prepotential, but still satisfy a set of integrability conditions discussed in [11].

Applying the integrability condition to the example the leading behavior of Π near $\check{t}_3 = i\infty$, is captured by only seven functions denoted by $(1, \check{t}_1, \check{t}_2, \tilde{F}_t, \tilde{W}, \tilde{F}_0, \tilde{T})$. The 11 solutions can be arranged into a period vector of the form

$$\begin{array}{lll} \Pi_0 = 1, & & \\ \Pi_{1,1} = \check{t}_1, & \Pi_{1,2} = \check{t}_2, & \Pi_{1,3} = \check{t}_3, \\ \Pi_{2,1} = 5\check{t}_1\check{t}_3 + \pi_{2,1}, & \Pi_{2,2} = -\tilde{F}_t, & \Pi_{2,3} = -\tilde{W}, \\ \Pi_{3,1} = \check{t}_3\,\tilde{F}_t + \pi_{3,1}, & \Pi_{3,2} = \tilde{T}, & \Pi_{3,3} = -\tilde{F}_0, \\ \Pi_4 = \check{t}_3\,\tilde{F}_0 + \pi_4, & & \end{array} \tag{6.13}$$

where the index q on Π_q, now labels the monodromy weight filtration w.r.t. to the large volume monodromy $\check{t}_a \to \check{t}_a + 1$.

Since the decoupling limit sends the compact four-fold X_B to its non-compact open–closed dual X_B^{nc}, these functions should reproduce the relative three-fold periods on $H^3(Z_B, D)$ in virtue of equation (2.3). Indeed the four functions $(1, \check{t}_1, \tilde{F}_t, \tilde{F}_0)$ converge to the four periods on $H^3(Z_B)$

$$\lim_{\check{t}_3 \to i\infty} (1, \check{t}_1, \tilde{F}_t, -\tilde{F}_0) = (1, t, \partial_t \mathcal{F}(t), -2\mathcal{F}(t) + t\partial_t \mathcal{F}(t)), \tag{6.14}$$

where $\mathcal{F}(t) = \frac{5}{6}t^3 + \mathcal{O}(\mathrm{e}^{2\pi i t})$ is the closed string prepotential on the mirror quintic.[31] The remaining three functions reproduce the three chain integrals on $H_3(Z_B, D)$ with non-trivial $\partial\gamma \in H_2(D)$:

$$\lim_{\check{t}_3 \to i\infty} (\check{t}_2, \tilde{W}, \tilde{T}) = (\hat{t} - t, W(t, \hat{t}), T(t, \hat{t})), \tag{6.15}$$

with classical terms $W(t, \hat{t}) = -2\check{t}_2^2 + \mathcal{O}(\mathrm{e}^{2\pi \mathrm{i} \check{t}_k})$, $T(t, \hat{t}) = \frac{2}{3}\check{t}_2^3 + \mathcal{O}(\mathrm{e}^{2\pi \mathrm{i} \check{t}_k})$, $k = 1, 2$. In the context of open–closed duality, the double logarithmic solution $W(t, \hat{t})$ of the four-fold is conjectured [16] to be the generating function of disc instantons in the type II mirror configuration (Z_A, L),

$$W(t, \hat{t}) = -2\check{t}_2^2 + \sum_{\beta} \sum_{k=1}^{\infty} N_\beta \frac{q^{k\beta}}{k^2},$$

similarly as $\mathcal{F}(t)$ is the generating function of closed string sphere instantons [85]. In the above formula, β denotes the homology class of the disc and the N_β are the integral Ooguri–Vafa disc invariants [86].

Since the closed string period vector (6.11) appears twice in (6.13), with coefficients 1 and $\check{t}_3 = S$, respectively, the leading terms of the 11 periods on X_B are proportional to the seven relative periods on $H^3(Z_B, D)$

$$\lim_{\mathrm{Im}\ S \to \infty} \Pi_{q,} \sim \begin{cases} (1, S) \times (1, t, \partial_t \mathcal{F}, -2\mathcal{F} + t\partial_t \mathcal{F}), \\ (\hat{t} - t, W(t, \hat{t}), T(t, \hat{t})). \end{cases}$$

A linear combination of these leading terms gives a large S expansion for the superpotential of the form (3.1).

[31] Here and in the following we neglect terms in the geometric periods from polynomials of lower degree in $\check{t}_i$.

6.3 Finite S corrections: perturbative contributions

There are two types of finite S contributions in the four-fold periods, which correct the three-fold result: linear corrections $\sim S^{-1}$ and exponential corrections $\sim e^{2\pi i S}$. In the type II orientifold where Im $S \sim 1/g_s$, the first should correspond to perturbative corrections.

These linear corrections are described by the three additional functions $\pi_{2,1}, \pi_{3,1}, \pi_4$ in (6.13) with leading behavior

$$\begin{aligned}\lim_{\check{t}_3\to i\infty} \pi_{2,1} &= f_{2,1}(\check{q}_1, \check{q}_2),\\ \lim_{\check{t}_3\to i\infty} \pi_{3,1} &= -\tfrac{5}{3}\check{t}_1^3 + f_{3,1}(\check{t}_1, \check{t}_2, \check{q}_1, \check{q}_2),\\ \lim_{\check{t}_3\to i\infty} \pi_4 &= \tfrac{5}{12}\check{t}_1^4 - \tfrac{1}{6}\check{t}_2^4 + f_4(\check{t}_1, \check{t}_2, \check{q}_1, \check{q}_2),\end{aligned} \tag{6.16}$$

An immediate observation is that these terms seem to break the naive S-duality symmetry of the type II string (and the T-duality of the heterotic string) even in the large S limit where one ignores the D-instanton corrections $\sim e^{2\pi i S}$. The above functions $f_{q,}$ vanish exponentially in the $\check{q}_i = e^{2\pi i \check{t}_i}$ for $i = 1, 2$ near the large complex structure limit of Z_B, but contribute in the interior of the complex structure moduli space of Z_B.

For example, the ratio of two periods corresponding to the central charges of an "S-dual" pair of BPS domain walls with classical tension $\sim \tilde{F}_t$ is

$$Z_2/Z_1 = \frac{S\tilde{F}_t + \pi_{3,1}}{\tilde{F}_t} = S + \frac{2}{3}t + \tilde{f}(\check{t}_k, \check{q}_k) + \mathcal{O}(e^{-2\pi/g_s}).$$

In principle, there are various possibilities regarding the fate of S duality. Firstly, there could be a complicated field redefinition which corrects the relation Im $S = \frac{1}{g_s}$ away from the decoupling limit such that there is an S duality for a redefined field $\tilde{S}$ including these corrections. Such a redefinition is known to be relevant in four-dimensional $\mathcal{N} = 2$ compactifications of the heterotic string, where one may define a perturbatively modular invariant dilaton [87]. On the other hand, duality transformations often originate from monodromies of the periods in the CY moduli space, which generate simple transformations at a boundary of the moduli space, such as Im $S = \infty$, but correspond to complicated field transformations away from this boundary. Again, such a "deformation" of a duality transformation is known to happen in the heterotic string [88]. At this point we cannot decide between these options, or a simple breaking of S-duality, without a detailed study of the monodromy transformations in the three-dimensional moduli space of the four-fold, which beyond the scope of this work.

6.4 D-instanton corrections and Gromov–Witten invariants on the four-fold

There are further exponential corrections $\sim e^{2\pi i S}$ to the moduli dependent functions in equations (6.13). Recall that we are considering here the classical periods of X_B, which describe the complex structure moduli space of the four-fold X_B and complex deformations of the dual heterotic bundle compactification on Z_B. From the point the type IIA compactification on X_B, obtained by compactifying F-theory on $X_B \times T^2$, these are B model data and do not have an immediate instanton interpretation.

However, according to the identification of the decoupling limit in Section 2, we expect these B model data to describe D-instanton corrections $\sim e^{-2\pi/g_s}$ to the type II orientifold on the three-fold, see (3.3). Lacking a sufficient understanding of the afore mentioned issue of field redefinitions, we will express the expansion in exponentials $\sim e^{2\pi i S}$ in terms of Gromov–Witten invariants, or rather in terms of integral invariants of Gopakumar–Vafa type, using the multi-cover formula for four-folds given in [79,83]. These invariants capture the world-sheet instanton expansion of the A-model on the mirror X_A of X_B. Note that if mirror pair (X_A, X_B) supports a duality of the type (3.16), then this expansion captures world-sheet and D-instanton corrections computed by the twisted superpotential $\widetilde{W}(X_A)$, according to the arguments in Section 3.5. However, according to equation (3.6) such a duality can only exist if the mirror four-fold X_A is given in terms of a suitable fibration structure, which is not true for the quintic example of this section (since X_A is neither elliptically nor K3 fibered), but for other examples considered in Section 7.

The integral A model expansion of the four-fold is defined by [79,83][32]

$$\Pi_{2,\gamma} = p_2^{\gamma}(t_a) + \sum_{\beta} \sum_{k>0} N_{\beta}^{\gamma} \frac{q^{\beta \cdot k}}{k^2}, \tag{6.17}$$

where $\Pi_{2,\gamma}$ is one of the periods in the $q = 2$ sector, double logarithmic near the large complex structure limit $z_a = 0$, and p_2 a degree two polynomial

[32] The fact that this multi-cover formula for spheres in a four-fold is formally the same as the multi-cover formula for discs in a three-fold [86] is at the heart of the open–closed duality of [10,16,17].

in the coordinates t_a defined by (6.9). Moreover β is a label, which in the A model on the mirror X_A specifies a homology class $\beta \in H_2(X_A, \mathbf{Z})$ with exponentiated Kähler volume $q^\beta = \prod_a q_a^{n_a}$, $q_a = \mathrm{e}^{2\pi \mathrm{i} t_a}$. As discussed above, these Kähler moduli of X_A map under mirror symmetry to coordinates on the complex structure moduli space of the F-theory compactification on X_B,[33] and we use these coordinates to write an expansion for the B model on X_B.

We restrict here to discuss only the few leading coefficients N_β^γ for the three linearly independent $q = 2$ periods of X_B. We label the "class" β by tree integers (m, n, k), such that N_β^γ is the coefficient of the exponential $\exp(2\pi i(mt_1 + nt_2 + kt_3)$ in the basis (6.9). Thus k is the exponent of $\mathrm{e}^{2\pi \mathrm{i} S}$ in the expansion.

6.4.1 Deformation of the closed string prepotential $\mathcal{F}_t$

The leading term of the period $\Pi_{2,2}$ is the closed string prepotential (6.13). This period is mirror to a four-cycle in the quintic fiber of X_A and depends only on the closed string variable $t = t_1 + t_2$ in the limit Im $S \to \infty$. The leading terms in the expansion (6.17) of the four-fold period are

$k=0$	0	1	2	3
0	0	0	0	0
1	0	2,875	0	0
2	0	0	1,218,500	0
3	0	0	0	951,619,125

$k=1$	0	1	2	3
0	5	20	0	0
1	0	8,895	33,700	600
2	0	19,440	16,721,375	63,071,800
3	0	−1,438,720	49,575,600	32,305,559,000

$k=2$	0	1	2	3
0	0	0	0	0
1	0	0	3,060	3,750
2	0	0	5,03,8070	98,649,500
3	0	0	19,074,160	47,957,485,000

$k=3$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	−2,010	−1,300
2	0	0	0	1,710,620	13,806,200
3	0	0	0	4,610,786,345	243,610,412,900

(6.18)

where the vertical (horizontal) directions corresponds to m (n). The $k = 0$ expansion is a power series in the closed string exponential, which displays the independence of the closed string prepotential on the open string sector. This independence is lost taking into account $\mathrm{e}^{2\pi \mathrm{i} S}$ corrections, as is expected

[33] The t_a are the distinguished flat coordinates of the Gauss–Manin connection.

from the backreaction of the closed string to the open string degrees of freedom at finite g_s.

The mixture between the closed and open string sector at finite S is already visible in the definition of mirror map. In [8,89] it had been observed, that the definition of the flat closed string coordinate does *not* depend on the open string moduli in the non-compact case, in other words, the mirror map $t = t(z)$ for the closed string modulus $t = t_1 + t_2$ is the same as in the theory without branes, with $z = z_1 z_2$. This is no longer the case for finite S, as there are corrections to the mirror map of the form $t(z_a) = t(z) + \mathrm{e}^{2\pi \mathrm{i} S} f(z, \hat{z})$.

6.4.2 Deformation of disc superpotential $W(t, \hat{t})$

The leading term of the period $\Pi_{2,3}$ is the brane superpotential of [11], which conjecturally computes the disc instanton expansion of an A type brane on the quintic. The leading terms in the expansion (6.17) of the four-fold period with respect to the corrections $\mathrm{e}^{2\pi \mathrm{i} kS}$ are

$k=0$	0	1	2	3	4	5
0	0	20	0	0	0	0
1	−320	1,600	2,040	−1,460	520	−80
2	13,280	−116,560	679,600	1,064,180	−1,497,840	1,561,100
3	−1,088,960	12,805,120	−85,115,360	530,848,000	887,761,280	−1,582,620,980

$k=1$	0	1	2	3	4	5
0	0	20	0	0	0	0
1	0	1,600	30,640	3,180	−1,160	160
2	0	−116,560	3,772,320	55,277,220	10,018,200	−6,906,880
3	0	12,805,120	−351,282,880	7,862,229,440	104,899,190,560	23,999,809,580

$k=2$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	2,040	3,180	480	−40	0
2	0	0	679,600	55,277,220	151,559,040	10,282,300	−4,775,320
3	0	0	−85,115,360	78,62,229,440	333,857,152,320	974,522,062,840	92,723,257,200

(6.19)

6.4.3 Deformation of $\Pi_{2,1}$

As discussed in the previous subsections, the corrections to the third period $\Pi_{2,1}$ contain S^{-1} corrections and are in this sense the most relevant. The leading terms of the expansion (6.17) are

$k=0$	0	1	2	3
0	0	20	0	0
1	0	6,020	3,060	−2,010
2	0	19,440	3,819,570	1,710,620
3	0	−1,438,720	19,074,160	3,659,167,220
4	0	148,132,440	−2,365,073,280	20,826,366,840

$k=1$	0	1	2	3
0	−10	−20	0	0
1	0	−6,020	0	3,150
2	0	−19,440	0	35,577,700
3	0	1,438,720	0	15,651,926,000
4	0	−148,132,440	0	79,135,362,000

(6.20)

The $k = 0$ corrections capture the linear corrections discussed in Section 6.3. These should arise from a one-loop effect on the brane; it would be interesting to verify this by an independent computation.

7 Heterotic five-branes and non-trivial Jacobians

In this section we discuss a number of further examples to illustrate the duality relations and the application of the method. The geometries are mostly taken from [10], where the brane superpotential for B-type branes has been already computed. Since the superpotential (2.7) for the heterotic compactification on Z_B with the appropriate bundle E agrees with the brane superpotential in the decoupling limit, the explicit heterotic superpotential in this limit can be read off from the results of [10]. We have performed also a computation of the finite S corrections to the heterotic superpotential for the examples below, by the methods described in detail in the previous section. The results are of a similar general structure as in the quintic case. Detailed expressions for the examples are available upon request.

The main focus of this section will be to describe some additional aspects arising from the point of F-theory and the heterotic compactification on Z_B. Let us recall the following basic result on F-theory/heterotic duality which will help to understand the different outcomes in the following examples. The elements of the Hodge group $H_{1,1}(X_B)$ of the four-fold can be roughly divided into the following sets w.r.t. their meaning in the dual heterotic compactification on the CY three-fold Z_B with bundle E (see [13, 15, 90]):

Generic classes:

The first set arises from the two generic classes from the K3 fiber Y of the K3 fibration $X_B \to B_2$:

1. The class E of the fiber of the elliptic fibration $Y \to \mathbf{P}^1$, which is also the elliptic fiber of X_B. This curve shrinks in the 4D F-theory limit and does not lead to a field in four dimensions;
2. The class F of the section of the elliptic fibration $Y \to \mathbf{P}^1$, which provides the universal tensor multiplet associated with the heterotic dilaton.

Geometry of Z_B:

3. $h^{1,1}(B_2)$ classes of the base of the K3 fibration $X_B \to B_2$ with K3 fiber Y.
4. $h^{1,1}(Z_B) - h^{1,1}(B_2) - 1$ classes associated with singular fibers of the elliptic fibration $Z_B \to B_2$.

Gauge fields and five-branes:

5. $h^{1,1}(Y) - 2 = \text{rank } G_{\text{pert}}$ classes from singular fibers of the elliptic fibration $Y \to \mathbf{P}^1$, corresponding to the Cartan subgroup of the perturbative gauge group G_{pert}.
6. $h^{1,1}(B_3) - h^{1,1}(B_2) - 1$ classes arising from blow ups of the $\mathbf{P}^1$ bundle $B_3 \to B_2$ with fiber of class F. These blow ups correspond to heterotic five-branes wrapping a curve $C \in B_2$.
7. The remaining rank $G_{\text{non-pert}}$ classes of X_B arise from extra singularities of the elliptic fibration, which correspond to the Cartan subgroup of a non-perturbative gauge group $G_{\text{non-pert}}$.

Fixing the heterotic three-fold Z_B, one can still vary the four-fold data in the last group, to choose a bundle E. In the framework of toric geometry, this step can be made very explicit by using local mirror symmetry of bundles [22]. Starting from the toric three-fold polyhedron for Z_B one may to "geometrically engineer" the bundle in terms of a four-fold polyhedron, by appropriately adding or removing exceptional divisors, as described in great detail in [23, 73]. By the type II/heterotic map (4.13), this is the complement of adding singular fibers to the mirror fibration $X_A \to \mathbf{P}^1$ in (3.5), to define a toric A type brane on the three-fold mirror Z_A [10].

Items 5–7 in the above describe, how an element of $H^{1,1}(X_B)$ added in the engineering of the bundle falls into one of the three classes in the last set, depending on the relative location of the exceptional divisor w.r.t.

the fibration structure. It follows that the B-type branes in the type II compactification may map to quite different heterotic degrees of freedom under the type II/heterotic map (4.13): perturbative gauge fields, heterotic five-branes and non-perturbative gauge fields. This variety can be seen already in the examples of [10], as discussed below.

7.1 Structure group $SU(1)$: heterotic five-branes

As seen in the previous section, the quintic example of [9, 10, 29] corresponds to a perturbative heterotic bundle with structure group $SU(2)$. Another example of a brane compactification taken from [10] turns out to have a quite different interpretation. In this case, the brane deformation of the type II string does not translate to a bundle modulus on the heterotic side under the type II/heterotic map (4.13), but rather to a brane modulus. On the heterotic side, this is a five-brane representing a small instanton [49].

Let us first recall the brane geometry on the type II side, which is defined in [10] as a compactification of a non-compact brane in the non-compact CY $\mathcal{O}(-3)_{\mathbf{P}^2}$, i.e., the anti-canonical bundle of $\mathbf{P}^2$. This example has been very well studied in the context of open string mirror symmetry in [18, 89, 91]. The non-compact CY can be thought of as the large fiber limit of an elliptic fibration $Z_A \to \mathbf{P}^2$ which gives the interesting possibility to check the result obtained from the compact four-fold against the disc instanton corrected three-fold superpotential computed by different methods in [18,89,91]. Indeed it was shown in [10] that four-fold mirror symmetry reproduces the known results for the non-compact brane in the large fiber limit, including the normalization computed from the intersections of the four-fold X_A. The result for the local result is corrected by instanton corrections for finite fiber volume.[34]

Two different three-fold compactifications of $\mathcal{O}(-3)_{\mathbf{P}^2}$ were considered in [10], with a different model for the elliptic fiber.[35] As the two examples produce very similar results, we discuss here the degree 18 case of [10] in some detail and only briefly comment on the difference for the degree 9 hypersurface, below.

The B-type brane is defined in [10] by adding a new vertex

$$\nu_8 = (-1, 0, 2, 3, -1) \tag{7.1}$$

[34] Note that this is a large fiber limit in the type IIA theory compactified on Z_A, not the previously discussed large fiber limit of the heterotic string compactified on Z_B.

[35] A cubic in $\mathbf{P}^2$ for the degree 9 and a sixtic in $\mathbf{P}^2(1, 2, 3)$ for the degree 18 hypersurface.

in the base of the "enhanced" toric polyhedron Δ. The Hodge numbers of the space X_B obtained in this way are $X_B: \ h^{1,3}=4, \ h^{1,2}=0, \ h^{1,1}=2796$, $\chi = 16,848\,(=0 \bmod 24)$. We refer the interested reader again to Appendix B for the details on the toric geometry and the parametrizations used in the following and continue with a non-technical discussion of the geometry. The addition of the vertex ν_8 corresponds to the blow up of a divisor in the singular central fiber of the four-fold fibration $X_A \to \mathbf{P}^1$. The new element in $H^{1,1}(X_A)$ is identified as the deformation parameter of the A-brane on the three-fold Z_A, via open–closed duality.

On the mirror side, the blow up modulus corresponds to a new complex structure deformation parametrizing a holomorphic divisor in Z_B. As will be explained now, this deformation maps in the heterotic compactification to a modulus moving a heterotic five-brane that wraps a curve C in the base B_2 of the three-fold Z_B.

In appropriate local coordinates, the form (6.2) of the hypersurface equation, exposing the elliptic fibration of both, Z_B and X_B, is

$$\begin{aligned} p_0 &= Y^2 + X^3 + (z_1^3 z_2 z_3)^{-1/18}\, YXZ\, stu \\ &\quad + Z^6\, \left((z_2 z_3)^{-1/3}\,(stu)^6 + s^{18} + t^{18} + u^{18}\right), \\ p_+ &= Z^6\,((stu)^6 + \hat{z} s^{18}), \qquad p_- = Z^6\,(stu)^6. \end{aligned} \tag{7.2}$$

The brane geometry in Z_B, reducing to the mirror of the non-compact brane in $\mathcal{O}(-3)_{\mathbf{P}^2}$ of [89], is defined by the hypersurface $D: \, p_+ = 0$ within Z_B defined by $p_0 = 0$ [10].

The hypersurface constraint (7.2) is already in the form to which the methods of [23] can be applied. The relevant component of p_+ deforming with the modulus $\hat{z}$ lies in a patch with $s, t, u \neq 0$ and is given by

$$\Sigma_+ : \ Z^6\,(t^6 u^6 + \hat{z} s^{12}) = 0. \tag{7.3}$$

Here the deformation $\hat{z}$ does not involve the coordinates of the elliptic fiber, and therefore it does *not* correspond to a bundle modulus. Instead this F-theory geometry describes heterotic five-branes wrapping a curve C in the base B_2 of the heterotic compactification. As described in detail in [13,15,39] (see also [74]), F-theory describes these heterotic five-branes by a blow ups of the $\mathbf{P}^1$ bundle $B_3 \to B_2$.

The toric four-fold singularities associated with heterotic five-branes of type (7.2) were also studied in great detail in [23,73]. In the present case, the five-branes wrap a set of curves C in the elliptic fibration $Z_B \to B_2$, defined by the zero of the function $f(s,t,u) = s^6(t^6u^6 + \hat{z}s^{12})$. The deformation $\hat{z}$ moves the branes on the second component, similarly as it moves the type II brane in the dual type II compactification on Z_B.

By the F-theory/heterotic dictionary developed in [13, 15, 39], the above singularity describes a small E_8 instanton, which can be viewed as an M-theory/type IIA 5-brane [49]. Note that there are also exceptional blow up divisors in X_B associated with the 5-brane wrapping, which support the elements in $H^{1,1}(X_B)$ dual to the world-volume tensor fields on the five-branes [13, 15, 39]. However, these Kähler blow ups are not relevant for the purpose of computing the superpotential $W(X_B)$.

The above conclusions may again be cross-checked by analyzing the perturbative gauge symmetry of the heterotic compactification, which does not changes in this case for $\hat{z} \neq 0$

$$E_8 \times E_8 \longrightarrow E_8 \times E_8 \ , \tag{7.4}$$

as is expected from the trivial structure group of the bundle, with the anomaly cancelled entirely by five-branes.

The compactification of the non-compact brane in $\mathcal{O}(-3)_{\mathbf{P}^2}$ in the degree 9 hypersurface leads to similar results. The four-fold considered in [10] has the Hodge numbers

$$X_B: \ h^{1,3} = 6(2),\ h^{1,2} = 0,\ h^{1,1} = 586,\ \chi = 3600\,(= 0 \bmod 24)$$

and describes a heterotic compactification with five-branes wrapping a curve given by the equation

$$\Sigma_+: \ Z^3\, s^3\, (t^3u^3 + \hat{z}s^6) = 0. \tag{7.5}$$

The further discussion is as above, except for the gauge symmetry breaking pattern, which is in this case $E_6 \times E_6 \to E_6 \times E_6$.

In the decoupling limit Im $S \to \infty$ limit, the heterotic superpotential for the five-branes in these two cases agrees with the type II brane superpotential computed in Section 3.2 and Appendix B of [10], respectively. See also Section 5 of [19] for a reconsideration of the first case, with an identical result (Table 3a/5.2).

7.2 Non-trivial Jacobians: $SU(2)$ bundle on a degree 9 hypersurface

A new aspect of another example of [10] is the appearance of a non-trivial Jacobian $J(\Sigma)$ of the spectral surface, corresponding to non-zero $h^{1,2}$ [12]. In this case there are additional massless fields associated with the Jacobian $J(X_B) = H^3(X_B, \mathbf{R})/H^3(X_B, \mathbf{Z})$ in the F-theory compactification, and the non-trivial Jacobian of Σ in the heterotic dual [12, 31, 32].

The present example has been considered in Section 3.3 of [10] and describes a brane compactification on the same degree 9 hypersurface Z_A as in the previous section, but with a different gauge background. Z_A is defined as a hypersurface in the weighted projective space $\mathbf{P}^4(1, 1, 1, 3, 3)$ with hodge numbers and Euler number

$$Z_A : \; h^{1,1} = 4(2), \; h^{1,2} = 112, \; \chi = -216, \tag{7.6}$$

The numbers in brackets denote the non-toric deformations of Z_A, which are unavailable in the given hypersurface representation.

As familiar by now, the technical details on toric geometry are relegated to Appendix B. The Hodge numbers of the dual F-theory four-fold X_B are

$$X_B : \; h^{1,3} = 4, \; h^{1,2} = 3, \; h^{1,1} = 246(11), \; \chi = 1530 = 18 \text{ mod } 24.$$

The local form (6.2) of the hypersurface equation for X_B, exposing the elliptic fibration and the hypersurface Z_B is

$$\begin{aligned} p_0 &= a_1 Y^3 + a_2 X^3 + Z^3 \; (a_3 (stu)^3 + a_4 s^9 + a_5 t^9 + a_6 u^9) + a_0 Y X Z \, stu, \\ p_+ &= Y \, (a_8 Y^2 + a_7 X Z \, stu), \qquad p_- = a_9 Y^3. \end{aligned} \tag{7.7}$$

Again the zero set $p_0 = 0$ defines the three-fold geometry Z_B for the compactification of the type II/heterotic string, while the brane geometry considered in [10] is defined by the hypersurface $D : p_+ = 0$. By the type II/heterotic map (4.13), we reinterprete these equations in terms of a heterotic bundle on Z_B. While p_- corresponds to the trivial spectral cover, p_+ describes a component with non-trivial dependence on a single modulus $\hat{z}$:

$$\Sigma_+ : \; Y^2 + \hat{z} X Z stu = 0, \tag{7.8}$$

where $\hat{z}$ is the brane/bundle deformation. As in the quintic case, Σ_+ may be identified with a component with structure group $SU(2)$. This is confirmed by a study of the perturbative gauge symmetry of the heterotic compactification, which changes for $\hat{z} \neq 0$ as

$$E_6 \times E_6 \longrightarrow SU(6) \times E_6 \ . \tag{7.9}$$

The Im $S \to \infty$ limit of the heterotic superpotential for this bundle coincides with the type II result computed in [10].

8 ADE singularities, Kazama–Suzuki models and matrix factorizations

In the above we have described how four-fold mirror symmetry computes quantum corrections to the superpotential and the Kähler potential of supersymmetric compactifications to four and lower dimensions with four supercharges. Specifically, these corrections are expected to correspond to D(-1), D1 and D3 instanton contributions in the type II orientifold compactification on Z_B and to world-sheet and space–time instanton corrections to a $(0,2)$ heterotic string compactification on the same manifold. At present, it is hard to concretely verify these predictions by an independent computation. A particularly neat way to find further evidence for our proposal (in the $\mathcal{N}=2$ supersymmetric situation) would be to establish a connection with [92]. In these works, considerable progress has been made in understanding corrections to the hyper-multiplet moduli, especially the interaction with mirror symmetry. It would be very interesting to study the overlap with the non-perturbative corrections discussed in the present paper. In this section, we discuss a different application of heterotic/F-theory duality which might be viewed as an interesting corroboration of our main statements, and is also of independent interest.

8.1 $\mathcal{N}=2$ supersymmetry

It is best again to begin with eight supercharges. Consider a heterotic string compactification on a K3 manifold near an ADE singularity with a trivial gauge bundle on the blown up two-spheres. The hypermultiplet moduli space of this heterotic compactification is corrected by α' corrections from perturbative and world-sheet instanton effects. It has been shown in [93] that for an A_1 singularity, the heterotic moduli space in the hyperkähler limit is given by the Atiyah–Hitchin manifold, which is also the moduli space of three-dimensional $\mathcal{N}=4$ $SU(2)$ Yang–Mills theory. This relation between the moduli space of the heterotic string on a singular K3 and the moduli space of a three-dimensional gauge theory can be derived and generalized by studying the stable degeneration limit of the dual type IIA/F-theory three-fold. Specifically it is shown in [94, 95] that the three-fold X_B dual to the heterotic string on an ADE singularity of type G and with a certain local behavior of the gauge bundle V develops a singularity, which 'geometrically engineers' a three-dimensional gauge theory of gauge group and matter content depending on G and V, see [96]. In connection with the $\mathcal{N} = 2$ version of the decoupling limit Im $S \to \infty$, equation (3.11), this leads to a very concrete relation between the three-fold period and the world-sheet instanton corrections to the heterotic hypermultiplet space in the hyperkähler limit. This could be explicitly checked against the known result, at least in the case dual to 3D $SU(2)$ SYM theory.

8.2 $\mathcal{N} = 1$ supersymmetry

The above situation has an interesting $\mathcal{N} = 1$ counter part. Namely, it has been conjectured in [95] that one may use the heterotic string on a certain three-fold singularity to geometrically engineer (the moduli space of) interesting two-dimensional field theories. The three-fold singularities are of the type

$$y^2 + H(x_k) = 0, \tag{8.1}$$

where $H(x_k)$ describes an ADE surface singularity. The idea is the obvious generalization of the above, by first applying heterotic/F-theory duality and then exploiting the relation of [24] between similar four-fold singularities and Kazama–Suzuki models. We here make this correspondence more precise.

Recall that the identification of [24] proceeded through the comparison of the vacuum and soliton structure of a type IIA compactification on CY four-fold with its superpotential from four-form flux, and the Landau–Ginzburg description [97] of the deformed Kazama–Suzuki coset models [98]. The four-folds relevant for this connection are local manifolds that are fibered by singular two-dimensional ALE spaces and their deformations. The ADE type of the singularity in the fiber determines the numerator G of the $\mathcal{N} = 2$ coset G/H, while the flux determines the denominator H and the level. More precisely, the fluxes studied in [24] are the minimal fluxes corresponding to a minuscule weight of G. These give rise to the so-called SLOHSS models (simply-laced, level one, Hermitian symmetric space), which is the subset of Kazama–Suzuki models admitting a Landau–Ginzburg description. This identification was checked for the A-series in [24] and worked out in detail for D and E in [25]. It has remained an interesting question to identify the theories for non-minimal flux, see e.g., the conclusions of [25].

An important clue to address this question has come from the study of matrix factorizations and their deformation theory. In particular, it was observed in [26], see also [99], that the superpotential resulting from the deformation theory of certain matrix factorization in $\mathcal{N} = 2$ minimal models coincides with the Landau–Ginzburg potential of a corresponding SLOHSS model. More precisely, the matrix factorizations are associated with the fundamental weights of ADE simple Lie algebras via the standard McKay correspondence, and the relevant subset are those matrix factorizations corresponding to the minuscule weights. We argue that this coincidence of superpotentials can be explained via heterotic/F-theory/type II duality.

The missing link is provided by Curto and Morrison [100]. Among the results of this work is that the matrix factorizations of ADE minimal models can be used to describe bundles on partial resolutions (Grassmann blowups) of the three-fold singularities of ADE type (8.1) that appear in the above-mentioned conjecture of [95]. The bundles have support only on the smooth part of the partial blowup, which is important to apply the arguments of [93].

The combination of the last three paragraphs suggests that we should couple the heterotic worldsheet to the matrix factorizations of [100]! This can be implemented by using the $(0, 2)$ linear sigma model [76] resp. $(0, 2)$ Landau–Ginzburg models [101], along the lines of [102], in combination with an appropriate non-compact Landau–Ginzburg model to describe the fibration structure. The resulting strongly coupled heterotic world-sheet theories are conjectured to be dual to those 2D field theories that are engineered on the four-fold side. The ADE type of the minimal model is that of the fiber of the four-fold, while the fundamental weight specifies the choice of four-form flux.

As formulated, the above conjecture makes sense for all, fundamental weights. The main testable prediction is thus the coincidence of the deformation superpotentials of the higher rank matrix factorizations corresponding to non-minuscule fundamental weights with the appropriate periods of the four-folds of [24, 25]. Note that the Kazama–Suzuki models only appear for the minuscule weights, and that we have not covered the case of fluxes corresponding to non-fundamental weights. We plan to return to these questions in the near future.

9 Conclusions

In this note we study the variation of Hodge structure of the complex structure moduli space of certain CY four-folds. These moduli spaces capture certain effective couplings of the $\mathcal{N} = 1$ supergravity theory arising from the associated F-theory four-fold compactification. Furthermore, through a chain of dualities we relate such F-theory scenarios to heterotic compactifications with non-trivial gauge bundle and small instanton five-branes and to type II compactifications with branes.

The connection to the heterotic string is made through the stable degeneration limit of the F-theory four-fold [12,15,39]. Taking this limit specifies the corresponding heterotic geometry. Due to the employed F-theory/heterotic duality the resulting heterotic geometry is given in terms of elliptically fibered CY three-folds. Furthermore, in the simplest cases, the geometric bundle moduli are described in terms of the spectral cover, which is also encoded in the four-fold geometry in the stable degeneration limit [12]. Alternatively, depending on the details of the F-theory four-fold, we describe the moduli space of heterotic five-branes instead of bundle moduli. On the other hand, the link to the open–closed type II string theories is achieved through the weak coupling limit [11], and it realizes the open–closed duality introduced in [16–18].

We argue that the two distinct limits to the heterotic string and to the open–closed string map the variation of Hodge structure of the F-theory CY four-fold to the variation of mixed Hodge structure of the corresponding CY three-fold relative to a certain divisor. For the heterotic string this divisor is either identified with the spectral cover of the heterotic bundle or with the embedding of small instantons. In the context of open–closed type II geometries the divisor encodes a certain class of brane deformations as studied in [8–11, 17, 19, 20, 103, 104].

Starting from the F-theory four-fold geometry we discuss in detail non-trivial background fluxes and compute the $\mathcal{N} = 1$ superpotential, which couples to the moduli fields described by the variation of Hodge structure. We trace these F-terms along the chain of dualities to the open–closed and heterotic string compactifications. For the heterotic string we find that, depending on the characteristics of the four-fold flux quanta, these fluxes either deform the bulk geometry of the heterotic string to generalized CY manifolds [69–71], or they give rise to superpotential terms for the bundle/five-brane moduli fields. The superpotentials associated to the flux quanta encode obstructions to deformations of the spectral cover. Furthermore, we show that in the stable degeneration limit the holomorphic Chern–Simons functional of the heterotic gauge bundle gives rise to these F-terms for the geometric bundle moduli.

The underlying four-fold description of the heterotic and the type II strings allows us to extract (non-perturbative) corrections to the stable degeneration limit and the weak coupling limit, respectively. We discuss the nature of these corrections, and we find that they encode world-sheet instanton, D-instanton and space–time instanton corrections depending on the specific theory in the analyzed web of dualities. In order to exhibit the origin of these corrections we compare our analysis with the analog $\mathcal{N} = 2$ scenarios, which have been studied in detail in [23, 36].

Apart from these F-term couplings we demonstrate that our techniques are also suitable to extract the Kähler potentials for the metrics of the studied moduli spaces in appropriate semi-classical regimes. In [11] the connection to the open–closed Kähler potential for three-fold compactifications with seven-branes has been developed. Here, starting from the Kähler potential of the complex structure moduli space of the CY four-fold, we also extract the corresponding Kähler potential associated to the combined moduli space of the complex structure and certain moduli of the heterotic gauge bundle. In leading order these Kähler potentials are in agreement with the results obtained by dimensional reduction of higher dimensional supergravity theories [44, 80]. In addition our calculation predicts subleading corrections.

Thus, the used duality relations together with the presented computational techniques offer novel tools to extract (non-perturbative) corrections to $\mathcal{N} = 1$ string compactifications arising from F-theory, from heterotic strings or from type II strings in the presence of branes. It would be interesting to confirm the anticipated quantum corrections by independent computations and to understand in greater detail the physics of various (non-perturbative) corrections discussed here. In particular, our analysis suggests a connection to the quantum corrections in the hypermultiplet sector of $\mathcal{N} = 2$ compactifications analyzed in [92].

Our techniques should also be useful to address phenomenological interesting questions in the context of F-theory, type II or heterotic string compactifications. As discussed in Sections 5 and 6, the finite S corrections to the superpotential capture the backreaction of the geometric moduli to the bundle moduli. Such corrections are a new and important ingredient in fixing the bundle moduli in phenomenological applications, as emphasized, e.g., in [35]. Thus the calculated (quantum corrected) superpotentials provide a starting point to investigate moduli stabilization and/or supersymmetry breaking for the class of models discussed here. In the context of the heterotic string it seems plausible that our approach can be extended to more general heterotic bundle configurations, which can be described in terms of monad constructions [101,105]. Such an extension is not only interesting from a conceptual point of view, but in addition it also gives a handle on analyzing the effective theory of phenomenologically appealing heterotic bundle configurations as discussed, for instance, in [106].

In Section 8, we propose an explanation, and conjecture an extension of, an observation originally made by Warner, which relates the deformation superpotential of matrix factorizations of minimal models to the flux superpotential of local four-folds near an ADE singularity. One of the results of this connection is the suggestion that (higher rank) matrix factorizations should also play a role in constructing the $(0, 2)$ world-sheet theories of heterotic strings.

The presented approach to calculate deformation superpotentials by studying adequate Hodge problems is ultimately linked to the derivation of effective obstruction superpotentials with matrix factorization or, more generally, world-sheet techniques [107–112]. While the latter approach leads to effective superpotentials up to field redefinitions, our computations give rise to effective superpotentials in terms of flat coordinates due to the underlying integrability of the associated Hodge problem. It would be interesting to explore the physical origin and the necessary conditions for the emergence of such a flat structure in the context of the deformation spaces studied in this note.

Appendix A Some toric data for the examples

A.1 The quintic in $\boldsymbol{P}^4(1,1,1,1,1)$

A.1.1 Parametrization of the hypersurface constraints

The toric polyhedra for the example considered in Section 6 are defined as the convex hull of the vertices

$$\begin{array}{llc} \Delta & \nu_0 = & (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0) \\ & \nu_1 = & (-1,\ \ 0,\ \ 0,\ \ 0,\ \ 0) \\ & \nu_2 = & (\ \ 0,-1,\ \ 0,\ \ 0,\ \ 0) \\ & \nu_3 = & (\ \ 0,\ \ 0,-1,\ \ 0,\ \ 0) \\ & \nu_4 = & (\ \ 0,\ \ 0,\ \ 0,-1,\ \ 0) \\ & \nu_5 = & (\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 0) \\ & \nu_6 = & (\ \ 0,\ \ 0,\ \ 0,\ \ 0,-1) \\ & \nu_7 = & (-1,\ \ 0,\ \ 0,\ \ 0,-1) \\ & \nu_8 = & (-1,\ \ 0,\ \ 0,\ \ 0,\ \ 1) \end{array} \qquad \begin{array}{llc} \Delta^* & \nu^\star_0 = & (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0) \\ & \nu^\star_1 = & (\ \ 1,-4,\ \ 1,\ \ 1,\ \ 0) \\ & \nu^\star_2 = & (\ \ 1,\ \ 1,-4,\ \ 1,\ \ 0) \\ & \nu^\star_3 = & (\ \ 1,\ \ 1,\ \ 1,-4,\ \ 0) \\ & \nu^\star_4 = & (\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 0) \\ & \nu^\star_5 = & (-4,\ \ 1,\ \ 1,\ \ 1,\ \ 1) \\ & \nu^\star_6 = & (-4,\ \ 1,\ \ 1,\ \ 1,-5) \\ & \nu^\star_7 = & (\ \ 0,-3,\ \ 1,\ \ 1,\ \ 1) \\ & \nu^\star_8 = & (\ \ 0,\ \ 1,-3,\ \ 1,\ \ 1) \\ & \nu^\star_9 = & (\ \ 0,\ \ 1,\ \ 1,-3,\ \ 1) \\ & \nu^\star_{10} = & (\ \ 0,\ \ 1,\ \ 1,\ \ 1,\ \ 1) \end{array} \tag{A.1}$$

The local coordinates in expressions (6.3) and (6.5) are defined by the following selections Ξ_1 and Ξ_2 of points of Δ^*, respectively:

$$\begin{array}{c|rrrrr} \Xi_1 & & & & & \\ x_1' & -4 & 1 & 1 & 1 & 0 \\ x_2 & 1 & -4 & 1 & 1 & 0 \\ x_3 & 1 & 1 & -4 & 1 & 0 \\ x_4 & 1 & 1 & 1 & -4 & 0 \\ x_5 & 1 & 1 & 1 & 1 & 0 \\ a & -4 & 1 & 1 & 1 & -1 \\ b & -4 & 1 & 1 & 1 & 1 \end{array} \qquad \begin{array}{c|rrrrr} \Xi_2 & & & & & \\ Z & 1 & 1 & 0 & 0 & 0 \\ Y & 1 & -2 & 0 & 0 & 0 \\ X' & -2 & 1 & 0 & 0 & 0 \\ s & 1 & 1 & -2 & 1 & 0 \\ t & 1 & 1 & 1 & -2 & 0 \\ u & 1 & 1 & 1 & 1 & 0 \\ a & -2 & 1 & 0 & 0 & 1 \\ b & -2 & 1 & 0 & 0 & -1 \end{array} \tag{A.2}$$

As described in Section 6, the local coordinates $\{x_i\}$ and $\{Z, Y, X', s, t, u\}$ may be associated with the "heterotic" manifold Z_B encoded in the F-theory four-fold X_B. In the example, Z_B is the mirror quintic, which is embedded in a toric ambient space with a large number $h^{1,1} = 101$ of Kähler classes, resulting in 101 coordinates x_k in the hypersurface constraint (6.1). $\{x_i\}$ and $\{Z, Y, X', s, t, u\}$ are special selections of these 101 coordinates, where the latter display (one of) the elliptic fibration(s) of Z_B.

On the other hand (a, b) are coordinates inherent to the four-fold X_B, parametrizing a special $\mathbf{P}^1$, F, which plays the central role in the stable degeneration limit of [12, 39] and the local mirror limit of [22, 23]. F is the base of the elliptic fibration of a K3 Y, which in turn is the fiber of the K3 fibration of X_B:

$$Y \to F, \qquad Y \to X_B \to B_2.$$

In the above example, B_2 can be thought of as a blow up of $\mathbf{P}^2$. The stable degeneration limit of the toric hypersurface can be defined as a local mirror limit in the complex structure moduli of X_B, where one passes to new coordinates [23]

$$(6.3)\ :\ x_1 = x_1' ab,\ v = a/b\ , \qquad (6.5)\ :\ X = X'ab,\ v = a/b.$$

The distinguished local coordinate $v = a/b$ on $\mathbf{C}^*$ parametrizes a patch near the local singularity associated with the bundle/brane data for a Lie group G [22]. For $G = SU(n)$, v appears linearly, which leads to a substantial simplification of the Hodge variation problem, as described in the appendices of [16, 17].

A.1.2 Perturbative gauge symmetry of the heterotic string

The perturbative gauge symmetry of the dual heterotic string is determined by the singularities in the elliptic fibration of the K3 fiber Y [15]. There is a simple technique to read off fibration structures for the CY four-fold X_B from the toric polyhedra described in [113]. Namely a fibration of $X_B \to B_{4-n}$ with fibers a CY n-fold Y_n corresponds to the existence of a hypersurface H of codimension $4 - n$, such that the integral points in the set $H \cap \Delta^*$ define the toric polyhedron of Y_n.

In the present case, the toric polyhedron Δ^*_{K3} for the K3 fiber Y is given as the convex hull of the points in Δ^* lying on the hypersurface $H : \{x_3 = x_4 = 0\}$:

$$\begin{array}{llllll} \Delta_{K3} & \mu_0 = & (\ 0,\ \ 0,\ \ 0) & \Delta^*_{K3} & \mu^\star_0 = & (\ 0,\ \ 0,\ \ 0) \\ & \mu_1 = & (\ 0, -1,\ \ 0) & & \mu^\star_1 = & (-2,\ \ 1, -3) \\ & \mu_2 = & (\ 1,\ \ 1,\ \ 0) & & \mu^\star_2 = & (-2,\ \ 1,\ \ 1) \\ & \mu_3 = & (\ 0,\ \ 0, -1) & & \mu^\star_3 = & (\ 0, -1,\ \ 1) \\ & \mu_4 = & (-1,\ \ 0, -1) & & \mu^\star_4 = & (\ 0,\ \ 1,\ \ 1) \\ & \mu_5 = & (-1,\ \ 0,\ \ 1) & & \mu^\star_5 = & (\ 1, -2,\ \ 0) \\ & & & & \mu^\star_6 = & (\ 1,\ \ 1,\ \ 0) \end{array} \qquad \text{(A.3)}$$

where the zero entries at the third and fourth position have been deleted and Δ_{K3} is the dual polyhedron of Δ^*_{K3}. The elliptic fibration of Y is visible as the polyhedron $\Delta^*_E = \Delta^*_{K3} \cap \{x_5 = 0\}$ of the elliptic curve

$$\Delta_E = \mathrm{Conv}\,\{(-1,0),(0,-1),(1,1)\}\,, \quad \Delta^*_E = \mathrm{Conv}\,\{(-2,1),(1,-2),(1,1)\}\,.$$

Since the model for the elliptic fiber is not of the standard form, but the cubic in $\mathbf{P}^2$ orbifolded by the action (6.7), the application of the standard methods to determine the singularity of the elliptic fibration and thus the perturbative heterotic gauge group should be reconsidered carefully. The singularity of the elliptic fibration can be determined directly from the hypersurface equation of X of the elliptically fibered K3 polynomial

$$p(K3) \ = \ Z^3 + Y^3 + X'^3(a^2b^4 + a^4b^2 + a^3b^3) + ZYX'(ab + b^2), \qquad \text{(A.4)}$$

which is associated to the toric data (A.3). The $\mathbf{Z}_3$ orbifold singularity is captured by $r^3 = p\,q$ in terms of the invariant monomials $p = \frac{Y^3}{X'^3}$, $q = \frac{Z^3}{X'^3}$ and $r = \frac{ZY}{X'^2}$. Then, to leading order, the singularities of the elliptic fiber E in the vicinity $a = 0$ and in the vicinity $b = 0$ are, respectively, given by

$$p_{a\to 0}(K3) \ = \ a^2q + q^2 + qr + r^3, \qquad p_{b\to 0}(K3) \ = \ b^2q + q^2 + bqr + r^3.$$

Using a computer algebra system, such as [114], it is straightforward to check that the polynomials $p_{a\to 0}(K3)$ and $p_{b\to 0}(K3)$ correspond to the ADE singularities $SU(6)$ and E_6.

In fact it turns out that the same answer is obtained by naively applying the method developed in [115, 116] for the standard model of the elliptic fiber, which implements the Kodaira classification of singular elliptic fibers in the language of toric polyhedra, such that the orbifold group is taken into account automatically. The polyhedron Δ^*_{K3} splits into a top and bottom piece Ξ_+ and Ξ_- with the points

$$
\begin{array}{rrr}
 & \Xi_+ & \\
-2 & 1 & 1\\
-1 & 0 & 1\\
-1 & 1 & 1\\
0 & -1 & 1\\
0 & 0 & 1\\
0 & 1 & 1
\end{array}
\qquad\qquad
\begin{array}{rrr}
 & \Xi_- & \\
-2 & 1 & -3\\
-2 & 1 & -2\\
-2 & 1 & -1\\
-1 & 0 & -2\\
-1 & 1 & -2\\
0 & -1 & -1\\
0 & 1 & -1
\end{array}
$$

which build up the affine Dynkin diagrams of $SU(6)$ and E_6, respectively. As asserted in [90, 115, 116], these toric vertices corresponds to two ADE singularities of the same type, in agreement with the direct computation. Moreover, deleting the vertex $\nu_7 \in \Delta$ which is associated with the exceptional toric divisor that described the brane/bundle modulus $\hat{z}$, the same analysis produces a K3 fiber with two ADE singularities of type E_6, leading to the pattern (6.8).

A.1.3 Moduli and Picard–Fuchs system

The moduli z_a are related to the parameters a_i in (6.1) by

$$z_a = (-)^{l_0^a} \prod_i a_i^{l_i^a}, \tag{A.5}$$

where l_i^a are the charge vectors that define the phase of the linear sigma model for the mirror X_A. For the phase considered in [10,11], these are given in (6.9). The complex structure modulus $z \sim e^{2\pi i t}$ mirror to the volume of the generic quintic fiber, the brane/bundle modulus $\hat{z} \sim e^{2\pi i \hat{t}}$ and the distinguished modulus $z_S \sim e^{2\pi i S}$ capturing the decoupling limit are given by

$$z = z_1 z_2 = -\frac{a_1 a_2 a_3 a_4 a_5}{a_0^5}, \qquad \hat{z} = z_2 = -\frac{a_1 a_6}{a_0 a_7}, \qquad z_S = z_3 = \frac{a_7 a_8}{a_1^2}.$$

The GKZ system for CY four-folds has been discussed in the context of mirror symmetry e.g., in [10, 79, 84]. A straightforward manipulation of it leads to the following system of Picard–Fuchs operators for the above example:

$$\begin{aligned}
\mathcal{L}_1 &= \theta_1^4(\theta_1+\theta_3-\theta_2) - z_1(-\theta_1+\theta_2)(4\theta_1+1+\theta_2)(4\theta_1+2+\theta_2)\\
&\quad \times (4\theta_1+3+\theta_2)(4\theta_1+4+\theta_2),\\
\mathcal{L}_2 &= (\theta_1+\theta_3-\theta_2)\theta_3 - z_3(2\theta_3-\theta_2)(2\theta_3+1-\theta_2),\\
\mathcal{L}_3 &= -(2\theta_3-\theta_2)(-\theta_1+\theta_2) - z_2(\theta_1+\theta_3-\theta_2)(4\theta_1+1+\theta_2),\\
\mathcal{L}_4 &= (-\theta_1+\theta_2)\theta_3 + z_2 z_3(2\theta_3-\theta_2)(4\theta_1+1+\theta_2),\\
\mathcal{L}_5 &= -(2\theta_3-\theta_2)\theta_1^4 - z_1 z_2(4\theta_1+1+\theta_2)(4\theta_1+2+\theta_2)(4\theta_1+3+\theta_2)\\
&\quad \times (4\theta_1+4+\theta_2)(4\theta_1+5+\theta_2),\\
\mathcal{L}_6 &= -(2\theta_3-\theta_2)\theta_1^3 - 5z_1 z_2(4\theta_1+1+\theta_2)(4\theta_1+2+\theta_2)(4\theta_1+3+\theta_2)\\
&\quad \times (4\theta_1+4+\theta_2) - z_2\theta_1^3(\theta_1+\theta_3-\theta_2). \qquad \text{(A.6)}
\end{aligned}$$

Here $\theta_a = z_a \frac{\partial}{z_a}$ are the logarithmic derivatives in the coordinates z_a, $a = 1,2,3$.

A.2 Heterotic five-branes

A.2.1 Degree 18 hypersurface in $\mathbf{P}^4(1,1,1,6,9)$

The polyhedra for the mirror pair (X_A, X_B) of four-folds dual to the three-fold compactifications on (Z_A, Z_B) are defined as the convex hull of the points:

	Δ					Δ^*					x_i	Ξ				
ν_0	0	0	0	0	0	6	6	1	1	0	Y	0	0	1	−1	0
ν_1	0	0	0	−1	0	6	6	1	1	−6	X	0	0	−2	1	0
ν_2	0	0	−1	0	0	6	−12	1	1	0	Z'	0	0	1	1	0
ν_3	0	0	2	3	0	6	−12	1	1	−6	s	−12	6	1	1	0
ν_4	−1	0	2	3	0	0	6	1	1	6	t	6	−12	1	1	0
ν_5	0	−1	2	3	0	0	0	1	−1	0	u	6	6	1	1	0
ν_6	1	1	2	3	0	0	0	−2	1	0	a	0	0	1	1	−1
ν_7	0	0	2	3	−1	0	−6	1	1	6	b	0	0	1	1	1
ν_8	−1	0	2	3	−1	−12	6	1	1	6						
ν_9	0	0	2	3	1	−12	6	1	1	−6						

(A.7)

Δ is the enhanced polyhedron for X_A^{nc} in Table 2 of [10], with the point ν_9 added in the compactification X_A of X_A^{nc}. The polyhedron Δ_3 for the three-fold Z_A defined as a degree 18 hypersurface in $\mathbf{P}^4(1,1,1,6,9)$ is given by the points on the hypersurface $\nu_{i,5}=0$, with the last entry deleted. The vertices of the dual polyhedron Δ_3^* of Δ_3 are given by the points of Δ^* with

$\nu^\star_{i,5} = 0$ and on extra vertex $(-12, 6, 1, 1)$. On the r.h.s. we have given the selection Ξ of points in Δ^* used to define local coordinates in (7.2). The relation to the coordinates used there is $Z = Z'ab,\ v = a/b$.

The relevant phase of the Kähler cone considered in [10, 19] is

$$\begin{aligned} l^1 &= (\;\; -6 \;\; 3 \;\; 2 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0), \\ l^2 &= (\;\; 0 \;\; 0 \;\; 0 \;\; -2 \;\; 0 \;\; 1 \;\; 1 \;\; -1 \;\; 1 \;\; 0), \\ l^3 &= (\;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; 1 \;\; 0 \;\; 0 \;\; 1 \;\; -1 \;\; 0), \\ l^4 &= (\;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 1). \end{aligned} \tag{A.8}$$

In the coordinates (A.5), the brane modulus in (6.6) is given by $\hat{z} = z_2^{1/3} z_3^{-2/3}$.

A.2.2 Degree 9 hypersurface in $\mathbf{P}^4(1,1,1,3,3)$

The polyhedra for the mirror pair (X_A, X_B) of four-folds dual to the three-fold compactifications on (Z_A, Z_B) are defined as the convex hull of the points:

	Δ					Δ^*					x_i	Ξ				
ν_0	0	0	0	0	0	−6	3	1	1	3	Y	0	0	1	−2	0
ν_1	0	0	0	−1	0	0	3	1	1	3	X	0	0	−2	1	0
ν_2	0	0	−1	0	0	0	−3	1	1	3	Z'	0	0	1	1	0
ν_3	0	0	1	1	0	3	3	1	1	−3	s	−6	3	1	1	0
ν_4	−1	0	1	1	0	−6	3	1	1	−3	t	3	−6	1	1	0
ν_5	0	−1	1	1	0	3	−6	1	1	−3	u	3	3	1	1	0
ν_6	1	1	1	1	0	3	3	1	1	0	a	0	0	1	1	−1
ν_7	0	0	1	1	−1	3	−6	1	1	0	b	0	0	1	1	1
ν_8	−1	0	1	1	−1	0	0	−2	1	0						
ν_9	0	0	1	1	1	0	0	1	−2	0						

(A.9)

The polyhedron Δ_3 for the three-fold Z_A defined as a degree 9 hypersurface in $\mathbf{P}^4(1,1,1,3,3)$ is again given by the points on the hypersurface $\nu_{i,5} = 0$. On the r.h.s. we have given the selection Ξ of points in Δ^* used in (7.5), with the redefinitions $Z = Z'ab,\ v = a/b$. The phase of the Kähler cone considered in [10] is

$$\begin{aligned} l^1 &= (-3 \;\; 1 \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0), \\ l^2 &= (\;\; 0 \;\; 0 \;\; 0 \;\; -2 \;\; 0 \;\; 1 \;\; 1 \;\; -1 \;\; 1 \;\; 0), \\ l^3 &= (\;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; 1 \;\; 0 \;\; 0 \;\; 1 \;\; -1 \;\; 0), \\ l^4 &= (\;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 1). \end{aligned} \tag{A.10}$$

In the coordinates (A.5), the brane modulus in (7.5) is given by $\hat{z} = z_2^{1/3} z_3^{-2/3}$.

A.3 $SU(2)$ bundle of the degree 9 hypersurface in $\mathbf{P}^4(1,1,1,3,3)$

The polyhedra for the mirror pair (X_A, X_B) of four-folds dual to the three-fold compactifications on (Z_A, Z_B) are defined as the convex hull of the points:

$$
\begin{array}{c|rrrrr|rrrrr|c|rrrrr}
 & & & \Delta & & & & & \Delta^* & & & x_i & & & \Xi & & \\
\nu_0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 1 & 1 & 0 & Y' & 0 & 0 & 1 & -2 & 0 \\
\nu_1 & 0 & 0 & 0 & -1 & 0 & 3 & -6 & 1 & 1 & 0 & X & 0 & 0 & -2 & 1 & 0 \\
\nu_2 & 0 & 0 & -1 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & Z & 0 & 0 & 1 & 1 & 0 \\
\nu_3 & 0 & 0 & 1 & 1 & 0 & 2 & -4 & 1 & 0 & 1 & s & -6 & 3 & 1 & 1 & 0 \\
\nu_4 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & -2 & 1 & t & 3 & -6 & 1 & 1 & 0 \\
\nu_5 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 1 & -2 & -3 & u & 3 & 3 & 1 & 1 & 0 \\
\nu_6 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & a & 0 & 0 & 1 & -2 & -1 \\
\nu_7 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2 & 1 & 0 & b & 0 & 0 & 1 & -2 & 1 \\
\nu_8 & 0 & 0 & 0 & -1 & -1 & -4 & 2 & 1 & 0 & 1 & & & & & & \\
\nu_9 & 0 & 0 & 0 & -1 & 1 & -6 & 3 & 1 & 1 & 0 & & & & & &
\end{array}
\qquad \text{(A.11)}
$$

Δ is the enhanced polyhedron for X_A^{nc} in Table 4 of [10], with the point ν_9 added in the compactification X_A of X_A^{nc}. The polyhedron Δ_3 for the three-fold fiber Z_A of the fibration $X_A \to \mathbf{P}^1$ is given by the points on the hypersurface $\nu_{i,5} = 0$, with the last entry deleted [10]. The vertices of the dual polyhedron Δ_3^* of Δ_3 are given by the points of Δ^* with $\nu^\star_{i,5} = 0$ and one extra vertex $(0, 0, 1, -2)$ (which is a point, but no vertex, in Δ^*). On the r.h.s. we have given the selection Ξ of points in Δ^* used to define local coordinates in (7.7). The relation to the coordinates used there is $Y = Y'ab,\ v = a/b$. The charge vectors for the phase of the linear sigma model considered in [10] is

$$
\begin{array}{rl}
l^1 = (-2 & 0\ \ 1\ \ \ 1\ \ 0\ \ 0\ \ 0\ \ -1\ \ \ 1\ \ 0), \\
l^2 = (\ \ 0 & 0\ \ 0\ \ -3\ \ 1\ \ 1\ \ 1\ \ \ \ 0\ \ \ 0\ \ 0), \\
l^3 = (-1 & 1\ \ 0\ \ \ 0\ \ 0\ \ 0\ \ 0\ \ \ \ 1\ \ -1\ \ 0), \\
l^4 = (\ \ 0 & -2\ \ 0\ \ \ 0\ \ 0\ \ 0\ \ 0\ \ \ \ 0\ \ \ 1\ \ 1).
\end{array}
\qquad \text{(A.12)}
$$

In the coordinates (A.5), the brane modulus in (7.8) is given by $\hat{z} = z_3\ (z_1^3 z_2 z_3^3)^{-1/9}$. The combination $z_1^3 z_2 z_3^3$ of complex structure parameters is mirror to the overall volume of Z_A.

Explicit expressions for the superpotential in the decoupling limit can be found in Section 3.3 and Appendix B. of [10].

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