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Early universe models from noncommutative geometry

Matilde Marcolli[1] **and Elena Pierpaoli**[2]

[1]Department of Mathematics, California Institute of Technology,
Pasadena, CA 91125, USA
matilde@caltech.edu
[2]Department of Physics and Astronomy, University of Southern California,
Los Angeles, CA 90089, USA
pierpaol@usc.edu

Abstract

We investigate cosmological predictions on the early universe based on the noncommutative geometry (NCG) models of gravity coupled to matter. Using the renormalization group analysis for the standard model with right-handed neutrinos and Majorana mass terms, which is the particle physics content of the most recent NCG models, we analyze the behavior of the coefficients of the gravitational and cosmological terms in the Lagrangian derived from the asymptotic expansion of the spectral action functional of NCG. We find emergent Hoyle–Narlikar and conformal gravity at the see-saw scales and a running effective gravitational constant, which affects the propagation of gravitational waves and the evaporation law of primordial black holes and provides Linde models of negative gravity in the early universe. The same renormalization group analysis also governs the running of the effective cosmological constant of the model. The model also provides a Higgs-based slow-roll inflationary mechanism, for which one can explicitly compute the slow-roll parameters. The particle physics content allows for dark matter models based on sterile neutrinos with Majorana mass terms.

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1 Introduction

The idea of using noncommutative geometry (NCG) to give a conceptual mathematical formulation of the standard model of elementary particle physics dates back to the work of Connes [19]. It was shown more recently in [17] (see also [4], [20], and Chapter 1 of [22]) that the NCG model of particle physics can be made compatible with right-handed neutrinos and neutrino masses and that the full Lagrangian of the standard model with Majorana mass terms for the right-handed neutrinos can be *derived* by a computation from a very simple input of an almost commutative space, the product of an ordinary spacetime manifold and a finite noncommutative space. The resulting physical Lagrangian in this NCG model is obtained from the asymptotic expansion in the energy scale Λ of a natural action functionals defined on noncommutative spaces, the *spectral action*, [14]. Among the most interesting features of these models of particle physics based on NCG is the fact that the physical Lagrangian of the model is completely *computed* from a simple geometric input (the choice of a finite-dimensional algebra), so that the physics is very tightly constrained by the underlying geometry. For reasons of space, we cannot include here any introductory material about NCG, but we suggest the interested readers to look at the survey paper [23] for a user-friendly introduction based on examples, as well as the books [21] and [22] for a more complete treatment.

We focus here on the NCG model obtained in [17]. The corresponding physical Lagrangian computed from the asymptotic expansion of the spectral action contains the full standard model Lagrangian with additional Majorana mass terms for right-handed neutrinos, as well as gravitational and cosmological terms coupled to matter. The presence of these terms and their relation to the particle physics content of the model make this approach of interest to theoretical cosmology. The gravitational terms include the Einstein–Hilbert action with a cosmological term, a topological term related to the Euler characteristic of the spacetime manifold, and, additionally, a conformal gravity term with the Weyl curvature tensor and a conformal coupling of the Higgs field to gravity. Without these last two contributions essentially one would be dealing with the usual minimal coupling of the standard model to gravity, but the presence of an additional nonminimal conformal coupling has relevance to various cosmological models that have been the object of recent investigation, especially in the context of modified theories of gravity. Another important way in which this model differs from the minimal coupling of gravity and matter is the dependence of the coefficients of the gravitational terms upon the Yukawa parameters of the particle physics content of the model. This feature, which is our main focus of investigation in the present paper, is unique to these NCG models and does not have an analog in other particle physics and cosmology models obtained from geometric settings such as string theories, extra dimensions, or brane worlds. Some conceptual similarities with these other approaches exist although, in the sense that in NCG models one typically modifies ordinary spacetime by taking a product with a noncommutative space and this may be thought of as another possible way to enrich it with extra dimensions.

The fact that, as shown in [17], the model lives naturally at unification scale, means that in cosmological terms it provides us with early universe models, hence it is interesting in terms of possible inflationary mechanisms. Extrapolations to lower energies are possible using renormalization group analysis, although extensions to cosmological models of the more recent universe only become possible when nonperturbative effects in the spectral action are also taken into account. The main motivation for considering these NCG models in a cosmological context is that the nontrivial dependence of the cosmological and gravitational parameters on the particle physics content is, as we mentioned above, significantly different from other physical models, hence likely to provide inflationary scenarios in the early universe that differ significantly from other models.

This paper is the first of a planned series dedicated to an investigation of the cosmological implications of the NCG models in particle physics. In the present paper, we concentrate on a renormalization group analysis of the coefficients of the gravitational terms in the action. In fact, the asymptotic formula for the spectral action used in [17] shows that these coefficients are functions of certain parameters, which in turn depend on the data of the Yukawa parameters of the standard model with Majorana mass terms for right-handed neutrinos. They also depend on three additional parameters of the model, one of which is fixed by a grand unification type condition on the coupling constants.

The Yukawa parameters run with the renormalization group equations (RGE) of the particle physics model. In particular, since the NCG model lives naturally at unification scale, one can input boundary conditions at that energy scale and follow the RGE towards lower energies and investigate the effect of this running on the gravitational part of the model. One expects that, when running towards lower energies, nonperturbative effects in the spectral action will progressively become nonnegligible. This can limit the range of validity of this type of argument based on the asymptotic expansion alone, and on renormalization group analysis to cosmological models for the very early universe, that is, for sufficiently high energies where the asymptotic expansion holds. Any extrapolation to the modern universe would then have to take into account the full spectral action and not only its asymptotic form.

In the present paper, we focus on early universe models and on the asymptotic form of the spectral action. For the renormalization analysis, we rely on a detailed study of the RGE for the extension of the standard model with right-handed neutrinos and Majorana mass terms carried out in [1], even though their choice of boundary conditions at unification is different from some of the boundary conditions assumed in [17]. The boundary conditions proposed in [1] are dictated by particle physics considerations, while some of the constraints considered in [17] came from analyzing particular geometries, such as the flat space case. For example, for simplicity the Majorana masses were assumed in [17] to be degenerate, all of them close to unification scale, while here we are mostly interested in the nondegenerate case, with three different see-saw scales between the electroweak and unification scales, which leads to a more interesting behavior of the gravitational terms in the model. We plan to return to a more general analysis of the RGE flow of [1] with a wider range of possible boundary conditions in follow-up work.

The RGE flow of [1] runs between a unification energy, taken there to be of the order of 2×10^{16} GeV, down to the electroweak scale of 100 GeV. In terms of cosmological timeline, we are looking at the behavior of the model between the unification and the electroweak era. This means that, in terms of matter content, only the Higgs field and its coupling to gravity is relevant, so we mostly concentrate on the part of the Lagrangian of [17] that consists of these terms. Since this era of the early universe is believed to include the inflationary epoch, we look especially at different possible inflationary scenarios provided by this NCG model.

Our main results in this first paper are to show that, using the information on the particle physics content, it is possible to obtain cosmological models of the early universe with variable gravitational and cosmological constant, hence providing a range of different mechanisms for inflation, realized by the running of the effective gravitational constant and by its coupling to the Higgs field, or by the running of the effective cosmological constant of the model, or by a combination of these. We also show phenomena where, near particular energy scales and for special geometries, the usual Einstein–Hilbert action ceases to be the dominant contribution and the model comes to be dominated, at certain scales, by conformal gravity and an emergent Hoyle–Narlikar cosmology. We discuss how the running of the gravitational parameters of the model influences the behavior of the evaporation of primordial black holes (PBHs) by Hawking radiation. While the type of effects that we see in this model, which depend on the presence of variable effective gravitational and cosmological constants, are qualitatively similar to scenarios of negative gravity in the early universe previously analyzed in theoretical cosmology [5, 6, 11, 24–26, 35–37, 39–41, 43, 48], the mechanism that produces these effects in the NCG model is substantially different from those described in these earlier references, which makes the quantitative behavior also different and distinguishable from other models. In fact, most of the effects we investigate in this paper depend directly on the expression of the coefficients of the gravitational and bosonic terms in the asymptotic expansion of the spectral action in terms of the Yukawa parameters of the underlying particle physics model. This is a *purely geometric* property of this model and it comes directly from the presence of the "small extra dimensions" in the form of the zero-dimensional (but K-theoretically six-dimensional) finite noncommutative space in addition to the extended spacetime dimensions.

While the energy range where the renormalization group analysis applies limits the results based only on the perturbative expansion of the spectral action to early universe models, if some of the results obtained in this paper persist when nonperturbative effects in the spectral action become significant, they may provide possible dark energy and dark matter predictions. For instance, the behavior of the variable effective cosmological constant may lead to dark energy scenarios in the more recent universe. Moreover, we show that the particle physics content of the model is consistent with dark matter models based on right-handed neutrinos with Majorana mass terms in [33, 44, 45]. In fact, the particle content is the same as in the νMSM model with three active and three sterile neutrinos. What is needed in order to relate the model of [17] to these dark matter models is a choice of boundary conditions that makes it possible for at least one of the Majorana masses to descent to somewhere near the electroweak scale, hence providing sterile neutrinos with the characteristics required to give acceptable dark matter candidates (see the analysis in [33]). We will return to a closer analysis of dark energy and dark matter implications of the NCG models in a planned continuation of this work.

In a related but different direction, recent work on some cosmological aspects of the model of [17] was also done in [38].

2 The asymptotic formula for the spectral action and the gravitational parameters

We recall here briefly the main aspects of the NCG model of particle physics derived in [17] that we need to use in the rest of the paper. We refer the reader to [17] and to Chapter 1 of [22] for a detailed treatment. The reader who wishes to skip this preliminary part can start directly with the asymptotic expansion of the spectral action recalled in Section 2.5, which is what we concentrate on in the rest of the paper, but we prefer to add a few words on the derivation of the model via NCG for the sake of completeness.

2.1 Spectral triples and the spectral action functional

The particle physics models based on NCG, both the original one of [19] and the new one of [17] that incorporates right-handed neutrinos and neutrino mixing with Majorana mass terms, are based on the formalism of *spectral triples*. These were introduced by Connes [18] as an extension of the notion of Riemannian manifold to NCG. The data $(\mathcal{A}, \mathcal{H}, D)$ defining a (real) spectral triple are summarized as follows:

- $\mathcal{A}$ is an involutive algebra with unit. Requiring the algebra to be unital corresponds to working with compact manifolds. (Extensions of the notion of spectral triple to nonunital cases have also been developed.)
- $\mathcal{H}$ is a separable Hilbert space endowed with a representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ of the algebra $\mathcal{A}$ by bounded linear operators.
- $D = D^\dagger$ is a linear self-adjoint operator acting on $\mathcal{H}$. Except for finite-dimensional cases, D is in general not a bounded operator, hence it is only defined on a dense domain.
- D has compact resolvent: $(1 + D^2)^{-1/2}$ is a compact operator.
- The commutators $[\pi(a), D]$ are bounded operators for all $a \in \mathcal{A}$.

- The spectral triple is even if there is on $\mathcal{H}$ a $\mathbb{Z}/2$-grading γ satisfying $[\gamma, \pi(a)] = 0$ and $D\gamma = -\gamma D$.
- The spectral triple has a real structure if there is an antilinear isomorphism $J : \mathcal{H} \to \mathcal{H}$ with $J^2 = \varepsilon$, $JD = \varepsilon' DJ$, and $J\gamma = \varepsilon'' \gamma J$, where the signs ϵ, ϵ', and ϵ'' determine the KO-dimension modulo 8 of the spectral triple, according to by the table

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

- The Hilbert space $\mathcal{H}$ has an $\mathcal{A}$-bimodule structure with respect to the action of $\mathcal{A}$ defined by $b^0 = Jb^* J^{-1}$ and satisfying the commutation condition $[a, b^0] = 0$ for all a and b in $\mathcal{A}$.
- The operator D satisfies the order one condition $[[D, a], b^0] = 0$, for all $a, b \in \mathcal{A}$.

Commutative geometries, which in this context means ordinary Riemannian manifolds, can be described as spectral triples: for a compact spin Riemannian manifold X the associated spectral triple $(C^\infty(X), L^2(X, S), D_X)$ is given by the algebra of smooth functions, the Hilbert space of square integrable spinors, and the Dirac operator. The metric tensor can be recovered from these data. For an even-dimensional manifold $\gamma_X = \gamma_5$ is the grading $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ on the spinor bundle given by the usual chirality operator γ_5 and the real structure J_X is the charge conjugation operator. Examples of spectral triples associated to objects that are not manifolds include a wide range of geometries such as quantum groups, fractals, or noncommutative tori. As we recall in Section 2.2 below, the spectral triples involved in the particle physics models are of a very special form which is *almost commutative*, namely a product of an ordinary manifold with a small noncommutative space.

It was shown by Chamseddine and Connes [14] that there is a natural action functional on a spectral triple. This *spectral action* functional is defined as $\mathrm{Tr}(f(D/\Lambda))$, where $f > 0$ is a cut-off function and Λ is the energy scale. There is an asymptotic formula for the spectral action, for large energy Λ, of the form

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{k \in \mathrm{DimSp}} f_k \Lambda^k {\int\!\!\!\!\!\!-} |D|^{-k} + f(0)\zeta_D(0) + o(1), \qquad (2.1)$$

where $f_k = \int_0^\infty f(v) v^{k-1} dv$ are the momenta of the function f and the noncommutative integration is defined in terms of residues of zeta functions

$$\zeta_{a,D}(s) = \mathrm{Tr}(a|D|^{-s}). \tag{2.2}$$

The sum in (2.1) is over points in the *dimension spectrum* of the spectral triple, which is a refined notion of dimension for noncommutative spaces, consisting of the set of poles of the zeta functions (2.2).

2.2 The noncommutative space of the model

The main result of [17] is a complete derivation of the full standard model Lagrangian with additional right-handed neutrino, lepton mixing matrix and Majorana mass terms, by a *calculation* starting from a very simple geometric input. The initial *ansatz* used in [17] is the choice of a finite dimensional algebra, the left–right symmetric algebra

$$\mathcal{A}_{\mathrm{LR}} = \mathbb{C} \oplus \mathbb{H}_{\mathrm{L}} \oplus \mathbb{H}_{\mathrm{R}} \oplus M_3(\mathbb{C}), \tag{2.3}$$

where $\mathbb{H}_{\mathrm{L}}$ and $\mathbb{H}_{\mathrm{R}}$ are two copies of the real algebra of quaternions.

The representation is then naturally determined by taking the sum $\mathcal{M}$ of all the inequivalent irreducible odd spin representations of $\mathcal{A}_{\mathrm{LR}}$, so that the only further input that one needs to specify is the number N of generations. The (finite-dimensional) Hilbert space is then given by N copies of $\mathcal{M}$,

$$\mathcal{H}_F = \oplus^N \mathcal{M}.$$

The Hilbert space with the $\mathcal{A}_{\mathrm{LR}}$ action splits as a sum $\mathcal{H}_F = \mathcal{H}_f \oplus \mathcal{H}_{\bar{f}}$ of matter and antimatter sectors, and an orthogonal basis of $\mathcal{H}_f$ gives all the fermions of the particle physics model

$$\begin{aligned} \nu_{\mathrm{L}} &= |\uparrow\rangle_{\mathrm{L}} \otimes \mathbf{1}^0, & \nu_{\mathrm{R}} &= |\uparrow\rangle_{\mathrm{R}} \otimes \mathbf{1}^0, \\ e_{\mathrm{L}} &= |\downarrow\rangle_{\mathrm{L}} \otimes \mathbf{1}^0, & e_{\mathrm{R}} &= |\downarrow\rangle_{\mathrm{R}} \otimes \mathbf{1}^0, \\ u_{\mathrm{L}} &= |\uparrow\rangle_{\mathrm{L}} \otimes \mathbf{3}^0, & u_{\mathrm{R}} &= |\uparrow\rangle_{\mathrm{R}} \otimes \mathbf{3}^0, \\ d_{\mathrm{L}} &= |\downarrow\rangle_{\mathrm{L}} \otimes \mathbf{3}^0, & d_{\mathrm{R}} &= |\downarrow\rangle_{\mathrm{R}} \otimes \mathbf{3}^0, \end{aligned} \tag{2.4}$$

respectively, giving the neutrinos, the charged leptons, the u/c/t quarks, and the d/s/b quarks in terms of the representation of $\mathcal{A}_{\mathrm{LR}}$. Here $|\uparrow\rangle$ and $|\downarrow\rangle$ are the basis of the $\mathbf{2}$ representation of $\mathbb{H}$ where the action of $\lambda \in \mathbb{C} \subset \mathbb{H}$ is, respectively, by λ or $\bar{\lambda}$, and $\mathbf{1}^0$ and $\mathbf{3}^0$ are the actions of $\mathbb{C}$ and $M_3(\mathbb{C})$, respectively, through the representation $a^0 = Ja^*J^{-1}$.

The $\mathbb{Z}/2$-grading γ_F exchanges the left and right chirality of fermions and the real structure operator J_F exchanges the matter and antimatter sectors and performs a complex conjugation. These properties of $\gamma_{\rm F}$ and $J_{\rm F}$ suffice to determine the KO-dimension modulo 8 of the resulting spectral triple and an interesting aspect is that, unlike in the earlier particle physics models based on NCG, in this case the KO-dimension is 6 modulo 8, although the metric dimension is zero.

The order one condition on the Dirac operator is seen in [17] as a coupled equation for a subalgebra $\mathcal{A}_F \subset \mathcal{A}_{\rm LR}$ and a Dirac operator and it is shown that there is a unique subalgebra of maximal dimension that allows for the order one condition to be satisfied. The algebra $\mathcal{A}_F$ is of the form

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \tag{2.5}$$

where the first summand embeds diagonally into $\mathbb{C} \oplus \mathbb{H}$ in $\mathcal{A}_{\rm LR}$, thus breaking the left–right symmetry. It is expected, although presently not known, that this symmetry breaking should be dynamical. This geometric argument identifying the maximal algebra on which the order one condition can be satisfied was later extended in [15] to more general *ansatz* algebras than $\mathcal{A}_{\rm LR}$, but with the same resulting $\mathcal{A}_F$.

2.3 Dirac operators: Yukawa parameters and Majorana masses

The selection of the subalgebra $\mathcal{A}_F$ for the order one condition for the Dirac operator is what produces geometrically in this model the Majorana mass terms for right-handed neutrinos. In fact, one has in [17] a complete classification of the possible Dirac operators on the noncommutative space $(\mathcal{A}_F, \mathcal{H})$ compatible with γ_F and J_F (see also [10] for a more general discussion of moduli spaces of Dirac operators for finite spectral triples). These are all of the form

$$D(Y) = \begin{pmatrix} \mathbb{S} & \mathbb{T}^\dagger \\ \mathbb{T} & \bar{\mathbb{S}} \end{pmatrix},$$

with $\mathbb{S} = \mathbb{S}_1 \oplus (\mathbb{S}_3 \otimes 1_3)$ and $\mathbb{T} = Y_{\rm R} : |\nu_{\rm R}\rangle \to J_F\, |\nu_{\rm R}\rangle$, and with $\mathbb{S}_1$ and $\mathbb{S}_3$, respectively, of the form

$$\mathbb{S}_1 = \begin{pmatrix} 0 & 0 & Y^\dagger_{(\uparrow 1)} & 0 \\ 0 & 0 & 0 & Y^\dagger_{(\downarrow 1)} \\ Y_{(\uparrow 1)} & 0 & 0 & 0 \\ 0 & Y_{(\downarrow 1)} & 0 & 0 \end{pmatrix},$$

$$\mathbb{S}_3 = \begin{pmatrix} 0 & 0 & Y_{(\uparrow 3)}^{\dagger} & 0 \\ 0 & 0 & 0 & Y_{(\downarrow 3)}^{\dagger} \\ Y_{(\uparrow 3)} & 0 & 0 & 0 \\ 0 & Y_{(\downarrow 3)} & 0 & 0 \end{pmatrix}.$$

Here the $N \times N$-matrices involved in the expression of $\mathbb{S}_1$ and $\mathbb{S}_3$ are the Yukawa matrices that give Dirac masses and mixing angles. These are matrices in $\mathrm{GL}_3(\mathbb{C})$ in the case of $N = 3$ generations: $Y_e = Y_{(\downarrow 1)}$ is the Yukawa matrix for the charged leptons, $Y_\nu = Y_{(\uparrow 1)}$ for the neutrinos, $Y_d = Y_{(\downarrow 3)}$ for the d/s/b quarks, and $Y_u = Y_{(\uparrow 3)}$ for the u/c/t quarks. Moreover, the remaining term $M = Y_{\mathrm{R}}^{\mathrm{T}}$, with T denoting transposition, gives the matrix $\mathbb{T}$ in $D(Y)$ and is the symmetric matrix of the Majorana mass terms for right-handed neutrinos.

Thus, the model of [17] has three active and three sterile neutrinos as in the νMSM model, see [33, 44, 45], although in [17], unlike in the νMSM model, it is assumed that the three sterile neutrinos all have masses well above the electroweak scale. The see-saw relation $Y_\nu^{\mathrm{T}} M^{-1} Y_\nu$ for neutrino masses is obtained in [17] geometrically from the fact that the restriction of the Dirac operator $D(Y)$ to the subspace of $\mathcal{H}_F$ spanned by ν_{R}, ν_{L}, $\bar{\nu}_{\mathrm{R}}$, $\bar{\nu}_{\mathrm{L}}$ is of the form

$$\begin{pmatrix} 0 & M_\nu^\dagger & \bar{M}_{\mathrm{R}}^\dagger & 0 \\ M_\nu & 0 & 0 & 0 \\ \bar{M}_{\mathrm{R}} & 0 & 0 & \bar{M}_\nu^\dagger \\ 0 & 0 & \bar{M}_\nu & 0 \end{pmatrix}, \tag{2.6}$$

where M_ν is the neutrino mass matrix, see Lemma 1.225 of [22]. We return to discuss the relation of the model of [17] to the νMSM model of [44, 45] and to other sterile neutrinos scenarios of [33] in the context of dark matter models in cosmology, see Section 5 below.

The spectral triple that determines the physical Lagrangian of the model through the asymptotic expansion of the spectral action is then the *product geometry* $X \times F$, of a four-dimensional spacetime X, identified with the spectral triple $(C^\infty(X), L^2(X, S), D_X)$, and the finite noncommutative space F defined by the spectral triple $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ with D_F of the form $D(Y)$ as above. The product is given by the cup product spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with sign γ and real structure J

- $\mathcal{A} = C^\infty(X) \otimes \mathcal{A}_F = C^\infty(X, \mathcal{A}_F)$
- $\mathcal{H} = L^2(X, S) \otimes \mathcal{H}_F = L^2(X, S \otimes \mathcal{H}_F)$
- $D = D_X \otimes 1 + \gamma_5 \otimes D_F$
- $J = J_X \otimes J_F$ and $\gamma = \gamma_5 \otimes \gamma_F$.

The action functional considered in [17] to obtain the physical Lagrangian has a bosonic and a fermionic part, where the bosonic part is given by the spectral action functional with inner fluctuations of the Dirac operator and the fermionic part by the pairing of the Dirac operator with fermions,

$$\mathrm{Tr}(f(D_A/\Lambda)) + \tfrac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle. \tag{2.7}$$

Here $D_A = D + A + \varepsilon' J A J^{-1}$ is the Dirac operator with inner fluctuations given by the gauge potentials of the form $A = A^\dagger = \sum_k a_k[D, b_k]$, for elements $a_k, b_k \in \mathcal{A}$. The fermionic term $\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$ should be seen as a pairing of classical fields $\tilde{\xi} \in \mathcal{H}^+ = \{\xi \in \mathcal{H} \mid \gamma\xi = \xi\}$, viewed as Grassman variables. This is a common way of treating Majorana spinors via Pfaffians, see Section 16.2 of [22].

While this fermionic part is very important for the particle physics content of the model, as it delivers all the fermionic terms in the Lagrangian of the standard model, for our purposes related to cosmological models of the early universe, it will suffice of concentrate only on the bosonic part of the action, given by the spectral action term $\mathrm{Tr}(f(D_A/\Lambda))$, since during a good part of the cosmological period between the unification and the electroweak epoch the Higgs field is the matter content that will be mostly of relevance, [30].

2.4 Parameters of the model

As we have recalled above, the geometric parameters describing the possible choices of Dirac operators on the finite noncommutative space F correspond to the Yukawa parameters of the particle physics model and the Majorana mass terms for the right-handed neutrinos. We recall here some expressions of these parameters that appear in the asymptotic expansion of the spectral action and that we are going to analyze more in detail later in this paper. We define functions $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, and $\mathfrak{e}$ of the matrices Y_u, Y_d, Y_ν, Y_e, and of the Majorana masses M in the following way:

$$\begin{aligned}
\mathfrak{a} &= \mathrm{Tr}(Y_\nu^\dagger Y_\nu + Y_e^\dagger Y_e + 3(Y_u^\dagger Y_u + Y_d^\dagger Y_d)),\\
\mathfrak{b} &= \mathrm{Tr}((Y_\nu^\dagger Y_\nu)^2 + (Y_e^\dagger Y_e)^2 + 3(Y_u^\dagger Y_u)^2 + 3(Y_d^\dagger Y_d)^2),\\
\mathfrak{c} &= \mathrm{Tr}(MM^\dagger),\\
\mathfrak{d} &= \mathrm{Tr}((MM^\dagger)^2),\\
\mathfrak{e} &= \mathrm{Tr}(MM^\dagger Y_\nu^\dagger Y_\nu).
\end{aligned} \tag{2.8}$$

In addition to these parameters, whose role we describe in Section 2.5, we see clearly from (2.1) that the asymptotic formula for the spectral action depends on parameters f_k given by the momenta of the cut-off function f in the spectral action. Since the noncommutative space here is of the simple form $X \times F$, the only contributions to the dimension spectrum, hence to the asymptotic formula for the spectral action come from three parameters f_0, f_2, f_4, where $f_0 = f(0)$ and for $k > 0$

$$f_k = \int_0^\infty f(v) v^{k-1}\, dv.$$

2.5 The asymptotic expansion of the spectral action

It was proved in [17] that the asymptotic formula (2.1) applied to the action functional $\mathrm{Tr}(f(D_A/\Lambda))$ of the product geometry $X \times F$ gives a Lagrangian of the form

$$\begin{aligned} S = {} & \frac{1}{\pi^2}\left(48\, f_4\, \Lambda^4 - f_2\, \Lambda^2\, \mathfrak{c} + \frac{f_0}{4}\, \mathfrak{d}\right) \int \sqrt{g}\, d^4x \\ & + \frac{96\, f_2\, \Lambda^2 - f_0\, \mathfrak{c}}{24\pi^2} \int R \sqrt{g}\, d^4x \\ & + \frac{f_0}{10\,\pi^2} \int \left(\frac{11}{6} R^* R^* - 3 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}\right) \sqrt{g}\, d^4x \\ & + \frac{(-2\,\mathfrak{a}\, f_2\, \Lambda^2 + \mathfrak{e}\, f_0)}{\pi^2} \int |\varphi|^2 \sqrt{g}\, d^4x \\ & + \frac{f_0 \mathfrak{a}}{2\,\pi^2} \int |D_\mu \varphi|^2 \sqrt{g}\, d^4x \\ & - \frac{f_0 \mathfrak{a}}{12\,\pi^2} \int R |\varphi|^2 \sqrt{g}\, d^4x \\ & + \frac{f_0 \mathfrak{b}}{2\,\pi^2} \int |\varphi|^4 \sqrt{g}\, d^4x \\ & + \frac{f_0}{2\,\pi^2} \int (g_3^2\, G^i_{\mu\nu} G^{\mu\nu i} + g_2^2\, F^\alpha_{\mu\nu}\, F^{\mu\nu\alpha} + \frac{5}{3}\, g_1^2\, B_{\mu\nu}\, B^{\mu\nu}) \sqrt{g}\, d^4x, \end{aligned} \tag{2.9}$$

We see from this expansion how the coefficients of all the terms in this resulting action functional depend on the Yukawa and Majorana parameters through their combinations of the form $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, and $\mathfrak{e}$ defined as in (2.8), and from the three additional parameters f_0, f_2, and f_4.

The term of (2.9) with the Yang–Mills action for the gauge bosons,

$$\frac{f_0}{2\,\pi^2} \int \left(g_3^2\, G^i_{\mu\nu}\, G^{\mu\nu i} + g_2^2\, F^\alpha_{\mu\nu}\, F^{\mu\nu\alpha} + \frac{5}{3} g_1^2 B_{\mu\nu}\, B^{\mu\nu}\right) \sqrt{g}\, d^4x,$$

contains the coupling constants g_1, g_2, g_3 of the three forces. As shown in [17], the standard normalization of these Yang–Mills terms gives the GUT (grand unified theory) relation between the three coupling constants and fixes the fact that this model lives naturally at a *preferred energy scale* given by the unification scale $\Lambda = \Lambda_{\rm unif}$. The normalization of the Yang–Mills terms fixes the value of the parameter f_0 to depend on the common value g at unification of the coupling constants: as shown in Sections 4.5 and 5.1 of [17] one obtains

$$\frac{g^2 f_0}{2\pi^2} = \frac{1}{4}. \tag{2.10}$$

One also normalizes the kinetic term for the Higgs as in [17] by the change of variables $H = \frac{\sqrt{\mathfrak{a} f_0}}{\pi}\varphi$ to get $\frac{1}{2}\int |DH|^2 \sqrt{g}\, d^4x$.

The normalization of the Yang–Mills terms and of the kinetic term of the Higgs then gives, at unification scale, an action functional of the form

$$\begin{aligned} S = {} & \frac{1}{2\kappa_0^2}\int R\sqrt{g}\, d^4x + \gamma_0 \int \sqrt{g}\, d^4x \\ & + \alpha_0 \int C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\sqrt{g}\, d^4x + \tau_0 \int R^*R^*\sqrt{g}\, d^4x \\ & + \frac{1}{2}\int |DH|^2\sqrt{g}\, d^4x - \mu_0^2 \int |H|^2\sqrt{g}\, d^4x \\ & - \xi_0 \int R|H|^2\sqrt{g}\, d^4x + \lambda_0 \int |H|^4\sqrt{g}\, d^4x \\ & + \frac{1}{4}\int (G^i_{\mu\nu}G^{\mu\nu i} + F^\alpha_{\mu\nu}F^{\mu\nu\alpha} + B_{\mu\nu}B^{\mu\nu})\sqrt{g}\, d^4x, \end{aligned} \tag{2.11}$$

where the coefficients are now

$$\begin{aligned} \frac{1}{2\kappa_0^2} &= \frac{96 f_2\Lambda^2 - f_0\mathfrak{c}}{24\pi^2}, \\ \gamma_0 &= \frac{1}{\pi^2}\left(48 f_4\Lambda^4 - f_2\Lambda^2\mathfrak{c} + \frac{f_0}{4}\mathfrak{d}\right), \\ \alpha_0 &= -\frac{3f_0}{10\pi^2}, \\ \tau_0 &= \frac{11 f_0}{60\pi^2}, \\ \mu_0^2 &= 2\frac{f_2\Lambda^2}{f_0} - \frac{\mathfrak{e}}{\mathfrak{a}}, \\ \xi_0 &= \frac{1}{12}, \\ \lambda_0 &= \frac{\pi^2\mathfrak{b}}{2f_0\mathfrak{a}^2}, \end{aligned} \tag{2.12}$$

again as a function of the Yukawa and Majorana parameters through the coefficients $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, and $\mathfrak{e}$ of (2.8), and of the two remaining free parameters of the model, f_2 and f_4, after the value of f_0 has been fixed by the unification condition.

3 Renormalization group and running parameters

All the Yukawa parameters Y_u, Y_d, Y_ν, Y_e, as well as the Majorana mass terms M are subject to running with the RGE dictated by the particle physics content of the model, in this case the standard model with additional right-handed neutrinos with Majorana mass terms. Consequently, also the parameters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, and $\mathfrak{e}$ of (2.8) run with the renormalization group flow as functions of Λ, with assigned initial conditions at $\Lambda = \Lambda_{\rm unif}$, which is the preferential energy scale of the model.

Some estimates based on renormalization group analysis were obtained already in [17], for the Higgs and the top quark masses, but those were based, in first approximation, on just the RGE at one-loop for the minimal standard model.

In this section, we analyze the running of the parameters of the model with the renormalization group flow, using the full RGE of the extension of the standard model by right-handed neutrinos and Majorana masses. There is an extensive literature available in particle physics on the relevant RGE analysis, see for instance [2,3,12]. We use here a more detailed analysis of the renormalization group flow, again to one-loop order, for the standard model with additional Majorana mass terms for right-handed neutrinos, as given in [1] and implemented by the authors of [1] in the Mathematica package `http://www.ph.tum.de/~rge/REAP/`.

The full RGE for this particle physics model have beta functions given by

$$16\pi^2 \beta_{g_i} = b_i g_i^3 \quad \text{with } (b_{SU(3)}, b_{SU(2)}, b_{U(1)}) = \left(-7, -\frac{19}{6}, \frac{41}{10}\right), \tag{3.1}$$

where [1] is using here a different normalization from [17] and the factor 5/3 has been now included in g_1^2. Thus, as for the minimal standard model, at one-loop order the RGE for the coupling constants uncouple from those of the other parameters. We then have for the Yukawa matrices

$$16\pi^2 \beta_{Y_u} = Y_u(\tfrac{3}{2}Y_u^\dagger Y_u - \tfrac{3}{2}Y_d^\dagger Y_d + \mathfrak{a} - \tfrac{17}{20}g_1^2 - \tfrac{9}{4}g_2^2 - 8g_3^2), \tag{3.2}$$

$$16\pi^2 \beta_{Y_d} = Y_d(\tfrac{3}{2}Y_d^\dagger Y_d - \tfrac{3}{2}Y_u^\dagger Y_u + \mathfrak{a} - \tfrac{1}{4}g_1^2 - \tfrac{9}{4}g_2^2 - 8g_3^2), \tag{3.3}$$

$$16\pi^2\beta_{Y_\nu} = Y_\nu(\tfrac{3}{2}Y_\nu^\dagger Y_\nu - \tfrac{3}{2}Y_e^\dagger Y_e + \mathfrak{a} - \tfrac{9}{20}g_1^2 - \tfrac{9}{4}g_2^2), \tag{3.4}$$

$$16\pi^2\beta_{Y_e} = Y_e(\tfrac{3}{2}Y_e^\dagger Y_e - \tfrac{3}{2}Y_\nu^\dagger Y_\nu + \mathfrak{a} - \tfrac{9}{4}g_1^2 - \tfrac{9}{4}g_2^2). \tag{3.5}$$

The RGE for the Majorana mass terms has beta function

$$16\pi^2\beta_M = Y_\nu Y_\nu^\dagger M + M(Y_\nu Y_\nu^\dagger)^{\mathrm{T}} \tag{3.6}$$

and the one for the Higgs self coupling λ is given by

$$16\pi^2\beta_\lambda = 6\lambda^2 - 3\lambda(3g_2^2 + \tfrac{3}{5}g_1^2) + 3g_2^4 + \tfrac{3}{2}(\tfrac{3}{5}g_1^2 + g_2^2)^2 + 4\lambda\mathfrak{a} - 8\mathfrak{b} \tag{3.7}$$

In the treatment of the renormalization group analysis given in the references mentioned above, one assumes that the Majorana mass terms are non-degenerate, which means that there are different see-saw scales at decreasing energies in between unification and the electroweak scale. In between these see-saw scales, one considers different *effective field theories*, where the heaviest right-handed neutrinos are integrated out when one passes below the corresponding see-saw scale. In practice, the procedure for computing the RGE for this type of particle physics models can be summarized as follows:

- Run the renormalization group flow down from unification energy Λ_{unif} to first see-saw scale determined by the largest eigenvalue of M, using assigned boundary conditions at unification.
- Introduce an effective field theory where $Y_\nu^{(3)}$ is obtained by removing the last row of Y_ν in the basis where M is diagonal and $M^{(3)}$ is obtained by removing the last row and column.
- Restart the induced renormalization group flow given by equations (3.2) to (3.7) with Y_ν replaced by $Y_\nu^{(3)}$ and M replaced by $M^{(3)}$ and with matching boundary conditions at the first see-saw scale. Run this renormalization group flow down to second see-saw scale.
- Introduce a new effective field theory with $Y_\nu^{(2)}$ and $M^{(2)}$ obtained by repeating the above procedure starting from $Y_\nu^{(3)}$ and $M^{(3)}$.
- Run the induced renormalization group flow for these fields with matching boundary conditions at the second see-saw scale, down until the first and lowest see-saw scale.
- Introduce again a new effective field theory with $Y_\nu^{(1)}$ and $M^{(1)}$ and matching boundary conditions at the first see-saw scale.
- Run the induced RGE down to the electroweak energy $\Lambda = \Lambda_{\mathrm{ew}}$.

The procedure illustrated here assumes that the three see-saw scales are all located between the unification and the electroweak scale, that is, that all the sterile neutrinos are heavy.

Notice that there is a difference between the boundary conditions assumed in [1] for M at unification energy and the assumption made in [17] on the Majorana mass terms. In fact, in Section 5.5 of [17] under the assumptions of flat space and with Higgs term $|H|^2$ sufficiently small, it is shown that one can estimate from the equations of motion of the spectral action that the largest Majorana mass can be as high as the unification energy, while in [1] the unification scale is taken at around 10^{16} GeV but the top see-saw scale is around 10^{14} GeV. For the purpose of the present paper we only work with the boundary conditions of [1], while a more extensive study of the same RGE with a broader range of boundary conditions will be considered elsewhere. We report the explicit boundary conditions of [1] at $\Lambda = \Lambda_{\text{unif}}$ in the appendix.

3.1 Running parameters and see-saw scales

In the following, we assume as in [17] that the value of the parameter f_0 is fixed by the relation (2.10). In terms of the boundary conditions at unification used in [1], this gives as value for f_0 either one of the following:

$$f_0 = \frac{\pi^2}{2g_1^2} = 8.52603, \quad f_0 = \frac{\pi^2}{2g_2^2} = 9.46314, \quad f_0 = \frac{\pi^2}{2g_3^2} = 9.36566. \quad (3.8)$$

These three different choices come from the fact that, as it is well known, the values for the three coupling constants do not exactly meet in the minimal standard model, nor in its variant with right-handed neutrinos and Majorana mass terms. Notice that here we are already including the factor of 5/3 in g_1^2, unlike in [17], where we have $g_3^2 = g_2^2 = \frac{5}{3}g_1^2$ at unification. We shall perform most of our explicit calculations in the following using the first value for f_0 and it is easy to check that replacing it with either one of the others will not affect significantly any of the results. The parameters f_2 and f_4 remain free parameters in the model and we discuss in Section 4 how they can be varied so as to obtain different possible cosmological models.

We concentrate here instead on the coefficients $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, and $\mathfrak{e}$ of (2.8) and on their dependence on the energy scale Λ through their dependence on the Yukawa parameters and the Majorana mass terms and the RGE (3.2) to (3.7). The renormalization group flow runs between the electroweak scale $\Lambda_{\text{ew}} = 10^2$ GeV and the unification scale, chosen as in [1] at 2×10^{16} GeV.

By solving numerically the equations and plotting the running of the coefficients (2.8) one finds that the coefficients $\mathfrak{a}$ and $\mathfrak{b}$ show clearly the effect of the first (highest) see-saw scale, while the effect of the two lower see-saw scales is suppressed (see Figures 1 to 4).

The running of the coefficients $\mathfrak{c}$ and $\mathfrak{d}$ exhibits the effect of all three see-saw scales, while the running of the remaining coefficient $\mathfrak{e}$ is the only one that exhibits a large jump at the highest see-saw scale.

Notice that the lack of differentiability at all the see-saw scales is inevitable, due to the procedure used in [1] and recalled here above for the construction of the effective field theories in between the different see-saw scales.

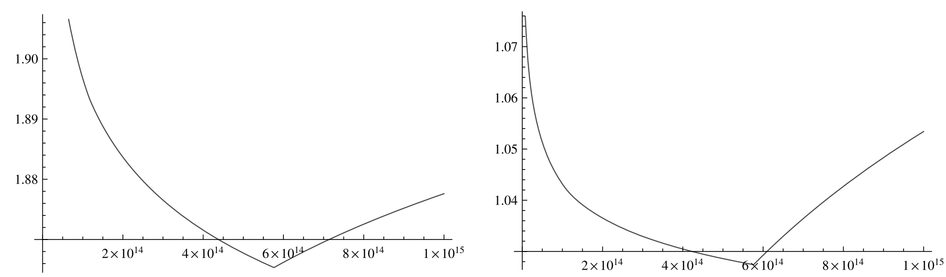

Figure 1: Coefficients $\mathfrak{a}$ and $\mathfrak{b}$ as functions of the energy scale Λ near the top see-saw scale.

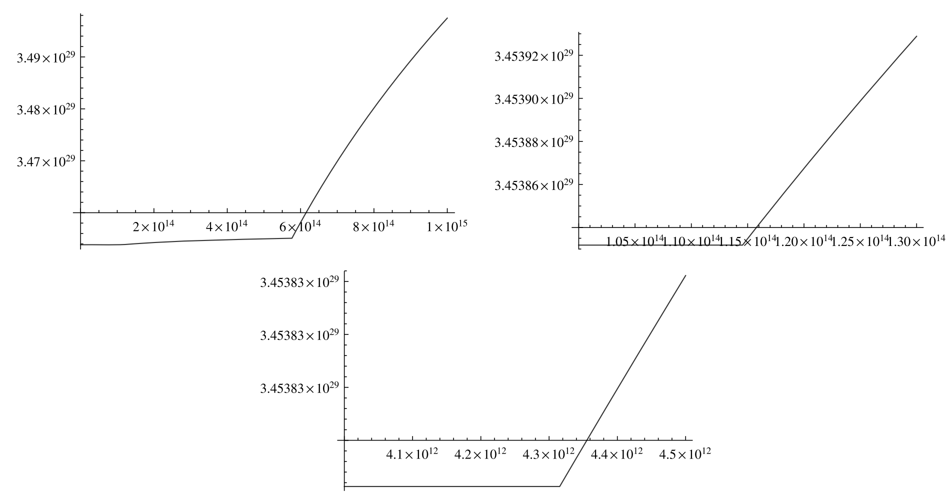

Figure 2: The coefficient $\mathfrak{c}$ as a function of the energy scale Λ near the three see-saw scales.

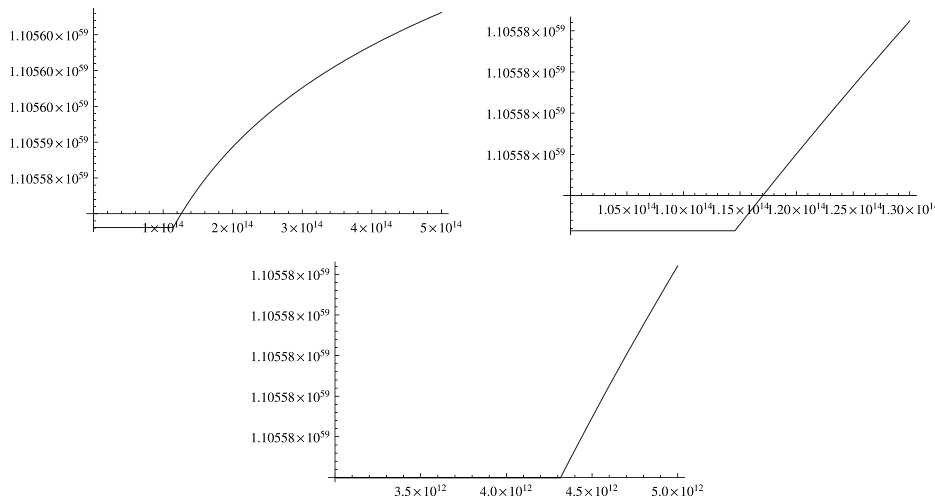

Figure 3: The coefficient $\mathfrak{d}$ as a function of the energy scale Λ near the three see-saw scales.

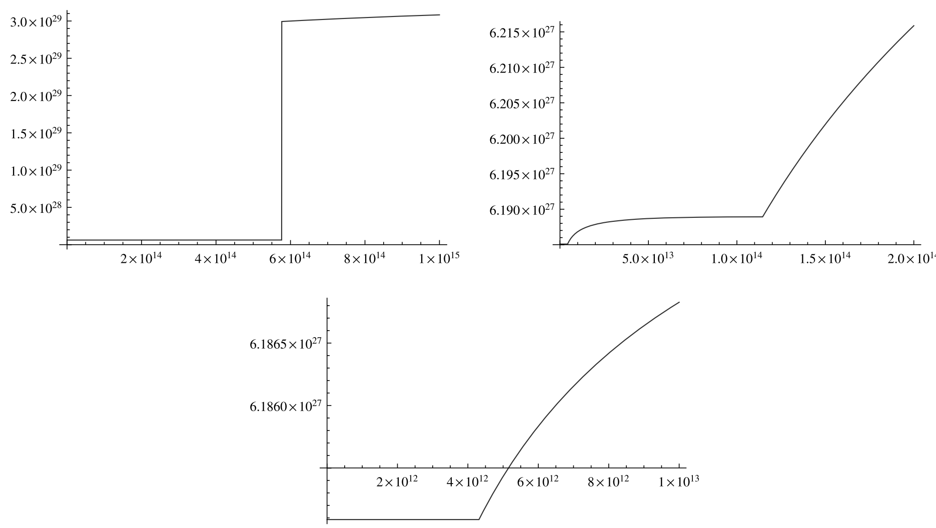

Figure 4: The coefficient $\mathfrak{e}$ as a function of the energy scale Λ near the three see-saw scales.

4 Cosmological implications of the model

We now use the information on the running of the coefficients (2.8) with the renormalization group to study the effect on the coefficients of the gravitational and Higgs terms in the asymptotic expansion of the spectral action.

We derive some information about cosmological models of the early universe that arise naturally in this NCG setting. In particular, we focus on the following aspects:

- Spontaneously arising Hoyle–Narlikar cosmologies in Einstein–Hilbert backgrounds.
- Linde's hypothesis of antigravity in the early universe, via running of the gravitational constant and conformal coupling to the Higgs.
- Gravity balls from conformal coupling to the Higgs field.
- Detectable effects on gravitational waves of the running gravitational constant, as in modified gravity theories.
- PBHs with or without gravitational memory.
- Higgs-based slow-roll inflation.
- Varying effective cosmological constant and vacuum-decay.
- Cold dark matter from Majorana masses of right-handed neutrinos.

The main features of the NCG model that will be discussed in the following and that lead to the effects listed above are summarized as follows:

- Variable effective gravitational constant.
- Variable effective cosmological constant.
- Conformal gravity.
- Conformal coupling of the Higgs field to gravity.

4.1 Einstein gravity and conformal gravity

The usual Einstein–Hilbert action (with cosmological term)

$$\frac{1}{16\pi G}\int R\sqrt{g}\, d^4x + \gamma_0 \int \sqrt{g}\, d^4x$$

minimally coupled to matter gives the Einstein field equations

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \gamma_0 g^{\mu\nu} = -8\pi G T^{\mu\nu},$$

where the energy momentum tensor $T^{\mu\nu}$ is obtained from the matter part of the Lagrangian, see Section 9.7 of [22] and Section 7.1.13 of [47]. (Here we use the notation γ_0 for the (variable) cosmological constant, which is more frequently denoted by Λ or λ in the cosmology and general relativity literature, but unfortunately both of these letters are already assigned other meanings here.) In addition to these terms, where both G and γ_0 will be running with the energy scale Λ and depending on the free parameters f_2 and

f_4, the asymptotic expansion for the spectral action also delivers conformal gravity terms. Conformal gravity is considered an alternative to the usual form of general relativity, where the Einstein–Hilbert action is replaced by an action based on the Weyl curvature tensor

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} - \tfrac{1}{2}(g_{\lambda\nu}R_{\mu\kappa} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}) + \tfrac{1}{6}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}).$$

This has the property of being conformally invariant, namely under a transformation of the form $g_{\mu\nu}(x) \mapsto f(x)^2\, g_{\mu\nu}(x)$ the Weyl tensor remains unchanged, $C_{\lambda\mu\nu\kappa} \mapsto C_{\lambda\mu\nu\kappa}$. The action functional of conformal gravity is of the form

$$\alpha_0 \int C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa}\sqrt{g}\, d^4x, \tag{4.1}$$

upon rewriting the above in terms of the Riemann curvature tensor

$$\alpha_0 \int (R_{\lambda\mu\nu\kappa}R^{\lambda\mu\nu\kappa} - 2R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{3}R^2)\sqrt{g}\, d^4x,$$

and using the fact that $R_{\lambda\mu\nu\kappa}R^{\lambda\mu\nu\kappa} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is a total divergence (see [37]) one can rewrite the conformal action functional as

$$2\alpha_0 \int (R_{\mu\nu}R^{\mu\nu} - \tfrac{1}{3}R^2)\sqrt{g}\, d^4x,$$

which gives field equations

$$W^{\mu\nu} = -\frac{1}{4\alpha_0}T^{\mu\nu},$$

where

$$\begin{aligned} W^{\mu\nu} = 2C^{\mu\lambda\nu\kappa}_{;\nu;\kappa} - C^{\mu\lambda\nu\kappa}R_{\lambda\kappa} &= \tfrac{1}{6}g_{\mu\nu}\nabla^2 R + \tfrac{1}{3}\nabla_\mu\nabla_\nu R - \tfrac{1}{3}R(2R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R) \\ &\quad + R^{\beta\rho}(\tfrac{1}{2}g_{\mu\nu}R_{\beta\rho} - 2R_{\beta\mu\rho\nu}). \end{aligned}$$

In addition to the Weyl curvature tensor itself being conformally invariant, one can add to the conformal gravity action a coupling to a field φ. Under a conformal transformation $g_{\mu\nu}(x) \mapsto f(x)^2\, g_{\mu\nu}(x)$, a field transforming like

$\varphi \mapsto f^{-1}\varphi$ gives

$$(\partial_\mu \varphi)^2 \mapsto f^{-4}((\partial_\mu\varphi)^2 + \varphi(\varphi \nabla_\mu \log f - 2\nabla_\mu \varphi)\nabla^\mu \log f),$$

while the scalar curvature transforms like

$$R \mapsto f^{-2}(R - (d-1)((d-2)\nabla_\mu \log f \nabla^\mu \log f + 2\nabla^\mu \nabla_\mu \log f),$$

so a nonminimal coupling of the field φ to gravity of the form

$$2\xi_0 \int R\varphi^2 \sqrt{g}\, d^4x$$

is conformally invariant in dimension $d = 4$ if $2\xi_0 = 1/6$. In addition to these terms a quartic potential

$$\lambda_0 \int \varphi^4 \sqrt{g}\, d^4x$$

also preserves conformal invariance. Thus, adding to the conformal action (4.1) terms of the form

$$\frac{1}{2}\int (\partial_\mu\varphi)^2 \sqrt{g}\, d^4x - \frac{1}{12}\int R\varphi^2 \sqrt{g}\, d^4x + \lambda_0 \int \varphi^4 \sqrt{g}\, d^4x \tag{4.2}$$

maintains the conformal invariance. The conformal gravity action (4.1) with an additional nonminimal conformal coupling to another field φ in the form (4.2) is the basis of the Hoyle–Narlikar cosmologies, which were proposed as possible models for steady-state cosmologies in [31], with the field φ related to the Mach principle. In fact, the presence of the field φ allows for a modification of the energy momentum tensor of the form $T^{\mu\nu} \mapsto T^{\mu\nu} - \xi_0\varphi^2(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R) - g^{\mu\nu}\lambda_0\varphi^4$, which was used as "creation field" in steady state models. While the steady-state cosmologies fail to account for major cosmological phenomena such as the background radiation, hence Hoyle–Narlikar cosmologies cannot be extrapolated towards the early universe, conformal gravity remains a valuable model (see [37] for a recent discussion). Within the NCG model we will see below that one typically has a dominant Einstein–Hilbert action, and only at certain scales where the behavior of the running effective gravitational constant presents phase transitions one finds that the subdominant terms of conformal gravity become dominant. This gives rise to emergent Hoyle–Narlikar cosmologies, for which the problem of extrapolating towards earlier times does not arise, as they become suppressed by the dominant Einstein–Hilbert term away from the energy scale of the phase transition. In the NCG model, the role of the field nonminimally conformally coupled to gravity is played by the Higgs field. Variants of the model of [17], discussed for instance in [13], allow for the presence of a further scalar field σ also conformally nonminimally coupled to R. Most of those arguments we describe in the following sections that are based on the coupling of the Higgs to gravity can be formulated also in terms of this other field σ, though we will not explicitly mention it.

4.2 Variable effective gravitational constant

According to the expressions for the coefficients (2.12) of the terms in the asymptotic expansion of the spectral action, we see that this model has an Einstein–Hilbert term, where the usual coefficient $\frac{1}{16\pi G}\int R\sqrt{g}\,d^4x$, with G the Newton constant $G \sim (10^{19}\,\mathrm{GeV})^{-2}$, with $1/\sqrt{G} = 1.22086 \times 10^{19}\,\mathrm{GeV}$ the Planck mass, is replaced by an effective gravitational constant of the form

$$G_{\mathrm{eff}} = \frac{\kappa_0^2}{8\pi} = \frac{3\pi}{192 f_2 \Lambda^2 - 2 f_0 \mathfrak{c}(\Lambda)}. \tag{4.3}$$

In this expression, the parameter f_0 is fixed by the unification condition (3.8) (we will use the first value for simplicity), the parameter f_2 is unconstrained in $\mathbb{R}_+^*$ and the function $\mathfrak{c}(\Lambda)$ is determined by the RGE.

Thus, one can best represent the effective gravitational constant of the model as a surface $G_{\mathrm{eff}}(\Lambda, f_2)$, which is a function of the energy scale in the range $\Lambda_{\mathrm{ew}} \leq \Lambda \leq \Lambda_{\mathrm{unif}}$ between the electroweak and the unification scales. It is in fact often preferable to consider the surface $G_{\mathrm{eff}}^{-1}(\Lambda, f_2)$ that describes the inverse effective gravitational constant, since it is in this form that it appears in the coefficients of the action functional, and it does not have singularities.

As an example of the form of the surface $G_{\mathrm{eff}}^{-1}(\Lambda, f_2)$, one sees in figure 5 that, for sufficiently small f_2, the surface exhibits a kink at the top see-saw scale.

As another example, if one requires in this model that the value of the effective gravitational constant at the lower end of the energy spectrum we are considering, that is, at the electroweak scale, already agrees with the usual Newton constant, this requires a very large fine tuning of the parameter f_2, which is fixed to have the value $f_2 = 7.31647 \times 10^{32}$. Notice that this is just an example. There is no physical reason to assume in this model that at the time of the electroweak epoch of the early universe the effective gravitational constant would have to be already equal to that of the modern universe. In this example, one then sees that the running of G_{eff}^{-1} becomes dominated entirely by the term $192 f_2 \Lambda^2/(3\pi)$, with the term $2 f_0 \mathfrak{c}(\Lambda)/(3\pi)$ remaining all the while several orders of magnitude smaller.

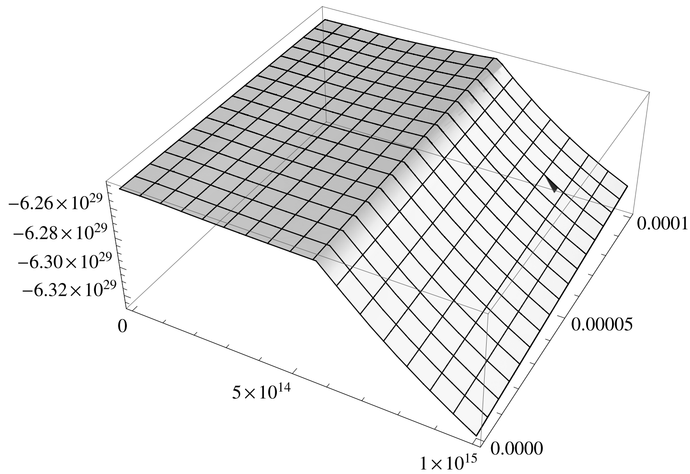

Figure 5: The region of the surface $G_{\rm eff}(\Lambda, f_0)^{-1}$ in the range $10^{14} \leq \Lambda \leq 10^{15}$ GeV and for $10^{-16} \leq f_2 \leq 10^{-4}$.

This is in contrast with the running of figure 5 where f_2 is small, and the running of $2f_0\mathfrak{c}(\Lambda)/(3\pi)$ comes to play a significant role, as one sees from the effect of the see-saw scales.

4.3 Emergent Hoyle–Narlikar cosmologies

We look more closely at cases with large f_2. For simplicity we illustrate what happens in this range by focusing on the example we mentioned above where f_2 is fixed so that $G_{\rm eff}(\Lambda_{\rm ew}) = G$, although qualitatively the results described in this section hold for a wider range of choices of f_2 sufficiently large. We show that, in the case of spaces with $R \sim 1$, a phase transition happens at the top see-saw scale, where the dominant Einstein–Hilbert action is suppressed and the action of the system is dominated by a spontaneously arising Hoyle–Narlikar type cosmology.

We first identify the dominant terms in the action, under the hypothesis of large f_2. Then we identify the conditions under which these terms are suppressed and the remaining terms become dominant.

Proposition 4.1. *For sufficiently large values of the parameter f_2 (for instance when $G_{\rm eff}(\Lambda_{\rm ew}) = G$), and in the range of energies where the effective cosmological term is kept small by the choice of the parameter f_4, the* *dominant terms in the expansion of the spectral action* (2.11) *are*

$$\Lambda^2 \left(\frac{1}{2\tilde{\kappa}_0^2} \int R\sqrt{g}\, d^4x - \tilde{\mu}_0^2 \int |H|^2 \sqrt{g}\, d^4x \right), \tag{4.4}$$

for $\tilde{\kappa}_0 = \Lambda\kappa_0$ *and* $\tilde{\mu}_0 = \mu_0/\Lambda$.

Proof. In this case, as we have seen already, the running of the effective gravitational constant is dominated by the term $G_{\rm eff}^{-1} \sim 192 f_2 \Lambda^2/(3\pi)$. Similarly, in the quadratic term of the Higgs μ_0^2, the term $2f_2\Lambda^2/f_0$ is dominant when f_2 is sufficiently large. In particular, in the example where $G_{\rm eff}(\Lambda_{\rm ew}) = G$, the term $2f_2\Lambda^2/f_0$ satisfies

$$\frac{2f_2\Lambda_{\rm ew}^2}{f_0} = 1.71627 \times 10^{36},$$

while the second term is several orders of magnitude smaller,

$$-\frac{\mathfrak{e}(\Lambda_{\rm ew})}{\mathfrak{a}(\Lambda_{\rm ew})} = -1.51201 \times 10^{27},$$

even though the coefficient $\mathfrak{e}$ varies more significantly than the other coefficients of (2.8).

We then proceed to estimate the remaining terms of (2.11) and show that they are all suppressed with respect to the dominant terms above, for this choice of f_2. The parameters α_0, τ_0, ξ_0 are not running with Λ, and we can estimate them to be

$$\alpha_0 \sim -0.25916, \quad \tau_0 \sim 0.158376,$$

while $\xi_0 = 1/12$ remains fixed at the conformal coupling value.

To estimate the running of the coefficient λ_0, we propose an *ansatz* on how it is related to the running of λ in the RGE (3.7) and to that of the coefficients $\mathfrak{a}$ and $\mathfrak{b}$. We also know that the boundary conditions at unification of (2.12) satisfy

$$\lambda_0|_{\Lambda=\Lambda_{\rm unif}} = \lambda(\Lambda_{\rm unif}) \frac{\pi^2 \mathfrak{b}(\Lambda_{\rm unif})}{f_0 \mathfrak{a}^2(\Lambda_{\rm unif})},$$

where in (3.7) one uses the boundary condition $\lambda(\Lambda_{\rm unif}) = 1/2$ as in [1]. Thus, we investigate here the possibility that the coefficient λ_0 runs with

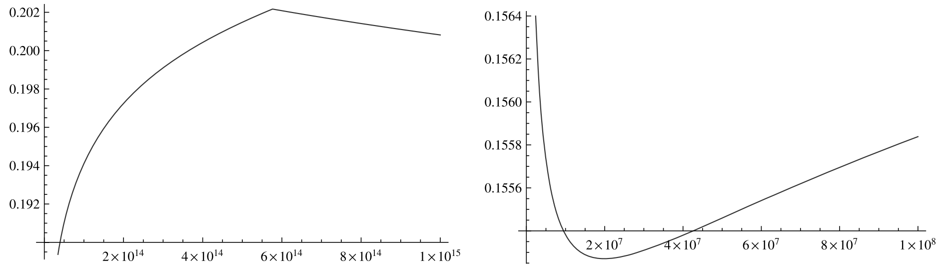

Figure 6: The running of λ_0 near the top see-saw scale and near the electroweak scale.

the RGE according to the relation

$$\lambda_0(\Lambda) = \lambda(\Lambda)\frac{\pi^2 \mathfrak{b}(\Lambda)}{f_0 \mathfrak{a}^2(\Lambda)}. \tag{4.5}$$

With this *ansatz* for the running of λ_0 one finds that the value of λ_0 varies between $\lambda_0(\Lambda_{\rm ew}) = 0.229493$ and $\lambda_0(\Lambda_{\rm unif}) = 0.184711$, along the curve shown in figure 6, with a maximum at $\lambda_0(\Lambda_{\rm ew})$ and a local maximum at the top see-saw scale with value 0.202167, and with a minimum near 2×10^7 GeV of value 0.155271. We discuss this *ansatz* more in details in Section 4.5 below.

Finally, for a given energy scale Λ it is always possible to eliminate the cosmological term at that energy scale by adjusting the coefficient f_4 (see Section 4.12). The vanishing condition for the effective cosmological constant at energy Λ is realized by the choice of

$$f_4 = \frac{(4 f_2 \Lambda^2 \mathfrak{c} - f_0 \mathfrak{d})}{192 \Lambda^4}. \tag{4.6}$$

Thus, if at a given energy scale the coefficient f_4 is chosen so that the effective cosmological constant vanishes, then the only terms that remain as dominant terms in the action are those of (4.4). $\square$

In this scenario then a new feature arises. Namely, the fact that one has comparable terms

$$\frac{1}{2\kappa_0^2} \sim \frac{96 f_2 \Lambda^2}{24\pi^2} \sim 2.96525 \times 10^{32} \Lambda^2$$

and

$$\mu_0^2 \sim \frac{2 f_2 \Lambda^2}{f_0} \sim 1.71627 \times 10^{32} \Lambda^2$$

can lead to cancellations under suitable geometric hypotheses.

Proposition 4.2. *Consider a space with $R \sim 1$. Then, for values $|H| \sim \sqrt{\mathfrak{a} f_0}/\pi$, the term* (4.4)*, which is the dominant term for f_2 sufficiently large, vanishes and is replaced by the sub-dominant*

$$\frac{\mathfrak{e}}{\mathfrak{a}}|H|^2 - \frac{f_0 \mathfrak{c}}{24\pi^2} R \sim \frac{f_0}{\pi^2}(\mathfrak{e} - \frac{\mathfrak{c}}{24}) \tag{4.7}$$

as the leading term in the formula (2.11) *for the spectral action. Near the top see-saw scale, the term* (4.7) *has a discontinuity, where the dynamics becomes dominated by a Hoyle–Narlikar cosmology given by the remaining terms of* (2.11).

Proof. As in [22], Corollary 1.219, we expand the Higgs field around $|H| \sim \sqrt{\mathfrak{a} f_0}/\pi$. Then, one can compare the two terms

$$\frac{1}{2\tilde{\kappa}_0^2} R - \tilde{\mu}_0^2 |H|^2. \tag{4.8}$$

This identifies a value for the constant curvature

$$R \sim \frac{2\tilde{\kappa}_0^2 \tilde{\mu}_0^2 \mathfrak{a} f_0}{\pi^2}$$

at which the dominant term

$$\frac{96 f_2 \Lambda^2}{24\pi^2} R - \frac{2 f_2 \Lambda^2}{f_0} \frac{\mathfrak{a} f_0}{\pi^2}$$

of (4.8) vanishes, leaving the smaller terms to dominate the dynamics. One can estimate that this gives a value for the scalar curvature very close to one, $R = 0.979907 \sim 1$, if we use the value of $\mathfrak{a}$ at unification energy and the first possible value of f_0 in (3.8).

The next smaller term in (4.8) is then of the form

$$\frac{\mathfrak{e}}{\mathfrak{a}}|H|^2 - \frac{f_0 \mathfrak{c}}{24\pi^2} R \sim \frac{\mathfrak{e}}{\mathfrak{a}} \frac{\mathfrak{a} f_0}{\pi^2} - \frac{f_0 \mathfrak{c}}{24\pi^2},$$

which gives (4.7). Near the top see-saw scale, at around 5.76405×10^{14} GeV the term $\mathfrak{e} - \mathfrak{c}/24$ has a jump and a sign change due to the large jump of the coefficient $\mathfrak{e}$ near the top see-saw scale (see figure 4). At this phase transition what is left of the dynamics of (2.11) are the remaining terms. One therefore sees an emergent behavior near the phase transition of the top see-saw scale, of the following form.

The coefficient f_4 can be chosen so that the cosmological term vanishes at this same top see-saw scale energy (see Section 4.12). The dynamics of the model is then dominated by the remaining terms, which recover a well-known treatment of gauge and Higgs field in *conformal gravity*, as discussed for instance in Section 2.2 of [26]. According to these models, the conformally invariant action for the gauge and Higgs bosons is given by the terms

$$\begin{aligned} S_c = \alpha_0 \int C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma} \sqrt{g}\, d^4x + \frac{1}{2}\int |DH|^2 \sqrt{g}\, d^4x \\ - \xi_0 \int R|H|^2 \sqrt{g}\, d^4x + \lambda_0 \int |H|^4 \sqrt{g}\, d^4x \\ + \frac{1}{4}\int (G^i_{\mu\nu} G^{\mu\nu i} + F^\alpha_{\mu\nu} F^{\mu\nu\alpha} + B_{\mu\nu} B^{\mu\nu}) \sqrt{g}\, d^4x. \end{aligned} \tag{4.9}$$

The additional topological term $\tau_0 \int R^* R^* \sqrt{g}\, d^4x$ is nondynamical and only contributes the Euler characteristic of the manifold. The action (4.9) is that of a Hoyle–Narlikar cosmology. ☐

In this model, the scalar curvature R, which is constant or near constant, provides an effective quadratic term for the Higgs and a corresponding symmetry breaking phenomenon, as observed in [26]. This produces in turn a breaking of conformal symmetry, via the Higgs mechanism giving mass to some of the gauge fields, thus breaking conformal invariance. In this range, with a constant curvature R and in the absence of the quadratic term in μ_0, the Higgs field is governed by a potential of the form $\mathcal{V}_{R=1}(H) = -\xi_0|H|^2 + \lambda_0|H|^4$, which has a minimum at $|H|^2 = \xi_0/(2\lambda_0)$.

We discuss in Section 4.7 below another instance of emergent Hoyle–Narlikar cosmology based on conformal gravity at a phase transition where the effective gravitational constant undergoes a sign change and an antigravity regime arises.

4.4 Effects on gravitational waves

The fact that we have a variable gravitational constant in the model has detectable effects on phenomena like gravitational waves whose propagation depends on the value of the gravitational constant.

Under the assumptions that the remaining terms in the asymptotic expansion of the spectral action are negligible with respect to the dominant (4.4), and further that $|H| \sim 0$, so that only the Einstein–Hilbert term dominates, one can show as in [38] that the equations of motion for (2.11) reduce to just

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \kappa_0^2 T^{\mu\nu}.$$

In [38] the authors conclude from this that, in the isotropic case, the NCG model has no effect on the gravitational waves that distinguish it from the usual Einstein–Hilbert cosmology. However, in fact, even in this case one can find detectable effects on the gravitational waves that distinguish the NCG model from the ordinary case of general relativity, because of the running of the effective gravitational constant.

We consider here two different scenarios for the time–energy relation, one which will be relevant close to the electroweak scale [30], where $\Lambda \sim t^{-1/2}$, and the other that refers to the inflationary period, with $\Lambda \sim \mathrm{e}^{-\alpha t}$. We show that in both cases the behavior of the gravitational waves differs from the behavior, with the same time–energy conversion, of the solutions in the classical case, thus detecting the presence of NCG.

Proposition 4.3. *In between the unification and the electroweak scale the gravitational waves propagate according to the Λ-dependent equation*

$$-3\left(\frac{\dot a}{a}\right)^2 + \frac{1}{2}\left(4\left(\frac{\dot a}{a}\right)\dot h + 2\ddot h\right) = \frac{12\pi^2}{96 f_2 \Lambda^2 - f_0 \mathfrak{c}(\Lambda)} T_{00}. \tag{4.10}$$

Upon rewriting the energy variable Λ as a function of time through $\Lambda = 1/a(t)$, one obtains

$$\ddot h + t^{-1}\dot h - \tfrac{3}{4} t^{-2} = \frac{12\pi^2 T_{00}}{96 f_2 t^{-1} - f_0 \mathfrak{c}(t^{-1/2})}. \tag{4.11}$$

in the radiation dominated era where $\Lambda \sim t^{-1/2}$, and

$$\ddot h + 2\alpha \dot h - 3\alpha^2 = \frac{12\pi^2 T_{00}}{96 f_2 \mathrm{e}^{-2\alpha t} - f_0 \mathfrak{c}(\mathrm{e}^{-\alpha t})}. \tag{4.12}$$

in the inflationary epoch where $\Lambda \sim \mathrm{e}^{-\alpha t}$.

Proof. For a metric of the form

$$g_{\mu\nu} = a(t)^2 \begin{pmatrix} -1 & 0 \\ 0 & \delta_{ij} + h_{ij}(x) \end{pmatrix}, \tag{4.13}$$

one separates the perturbation h_{ij} into a trace and traceless part, and the gravitational waves are then governed by the Friedmann equation, which gives

$$-3\left(\frac{\dot{a}}{a}\right)^2+\frac{1}{2}\left(4\left(\frac{\dot{a}}{a}\right)\dot{h}+2\ddot{h}\right)=\kappa_0^2 T_{00}. \tag{4.14}$$

This equation is *formally* the same as the usual equation for the gravitational waves, up to replacing κ_0^2 for $8\pi G$, as remarked in [38]. However, the dependence of κ_0^2 on the energy scale Λ leads to the result (4.10).

The change of variable between energy and time, for a cosmology of the form (4.13) is given by $\Lambda = 1/a(t)$. Thus, we can write the equation (4.14) by expressing the right-hand side also as a function of time in the form

$$-3\left(\frac{\dot{a}(t)}{a(t)}\right)^2+\frac{1}{2}\left(4\left(\frac{\dot{a}(t)}{a(t)}\right)\dot{h}(t)+2\ddot{h}(t)\right)=\frac{12\pi^2 T_{00}}{96 f_2(1/(a(t))^2)-f_0\mathfrak{c}(1/a(t))}. \tag{4.15}$$

In the radiation-dominated era, where the function $a(t)$ behaves like $a(t) = t^{1/2}$ we find

$$\frac{\dot{a}}{a}=\frac{1}{2}t^{-1},$$

which gives

$$-3\left(\frac{\dot{a}}{a}\right)^2+\frac{1}{2}\left(4\left(\frac{\dot{a}}{a}\right)\dot{h}+2\ddot{h}\right)=\ddot{h}+t^{-1}\dot{h}-\frac{3}{4}t^{-2}.$$

This gives (4.11). In the inflationary era, where the function $a(t)$ behaves exponentially $a(t) = \mathrm{e}^{\alpha t}$, one obtains instead $\dot{a}/a = \alpha$ and

$$-3\left(\frac{\dot{a}}{a}\right)^2+\frac{1}{2}\left(4\left(\frac{\dot{a}}{a}\right)\dot{h}+2\ddot{h}\right)=\ddot{h}+2\alpha h-3\alpha^2,$$

which gives (4.12). □

For a choice of the parameter f_2 sufficiently large (such as the one that gives $G_{\mathrm{eff}}(\Lambda_{\mathrm{ew}}) = G$) for which $\kappa_0^2(\Lambda) \sim \tilde{\kappa}_0^2/\Lambda^2$ we then obtain the following explicit solutions to equation (4.10).

Proposition 4.4. *We consider the case with the parameter f_2 sufficiently large (for instance in the example where $G_{\mathrm{eff}}(\Lambda_{\mathrm{ew}}) = G$). In the radiation-dominated era, where the metric* (4.13) *has $a(t) \sim t^{1/2}$, and the energy–time* *relation is given by* $\Lambda = 1/a(t)$ *equation* (4.11) *has solutions of the form*

$$h(t) = \frac{4\pi^2 T_{00}}{288 f_2} t^3 + B + A\log(t) + \tfrac{3}{8}\log(t)^2. \tag{4.16}$$

In the inflationary epoch, for a metric of the form (4.13) *with* $a(t) \sim e^{\alpha t}$, *for some* $\alpha > 0$, *equation* (4.12) *has solutions of the form*

$$h(x) = \frac{3\pi^2 T_{00}}{192 f_2 \alpha^2} e^{2\alpha t} + \frac{3\alpha}{2} t + \frac{A}{2\alpha} e^{-2\alpha t} + B. \tag{4.17}$$

Proof. We assume that the parameter f_2 is sufficiently large. For example, we take the case where f_2 is fine tuned so as to have an agreement between the value at the electroweak scale of the effective gravitational constant and the usual Newton constant, $\kappa_0^2(\Lambda_{\rm ew}) = 8\pi G$. One then finds at high energies a different effective Newton constant, which behaves like

$$\kappa_0^2 \sim \frac{12\pi^2}{96 f_2} \Lambda^{-2},$$

which then gives a modified gravitational waves equation

$$-3\left(\frac{\dot a}{a}\right)^2 + \frac{1}{2}\left(4\left(\frac{\dot a}{a}\right)\dot h + 2\ddot h\right) = \frac{\tilde\kappa_0^2}{\Lambda^2} T_{00}, \tag{4.18}$$

with $\tilde\kappa_0^2 = \frac{12\pi^2}{96 f_2}$.

We look first at the radiation-dominated case where $a(t) = t^{1/2}$. In the interval of energies that we are considering, this has relevance close to the electroweak scale, see [30]. We then have equation (4.11) in the form

$$\ddot h + t^{-1}\dot h - \frac{3}{4} t^{-2} = t\frac{12\pi^2 T_{00}}{96 f_2}. \tag{4.19}$$

Assuming T_{00} constant, the general solution of

$$t^{-1}\ddot h + t^{-2}\dot h - \frac{3}{4} t^{-3} = C$$

is of the form

$$h(t) = \frac{C}{9} t^3 + B + A\log(t) + \frac{3}{8}\log(t)^2,$$

for arbitrary integration constants A and B. With $C = (12\pi^2 T_{00})/(96 f_2)$, this gives (4.16).

In the inflationary epoch where one has $a(t) = e^{\alpha t}$, for some $\alpha > 0$, one can write equation (4.12) in the form

$$\ddot{h} + 2\alpha\dot{h} - 3\alpha^2 = e^{2\alpha t}\frac{12\pi^2 T_{00}}{96 f_2}.$$

Again assuming T_{00} constant, the general solution of

$$\ddot{h} + 2\alpha\dot{h} - 3\alpha^2 = Ce^{2\alpha t}$$

with a constant C is of the form

$$h(t) = \frac{3\alpha}{2}t + \frac{C}{8\alpha^2}e^{2\alpha t} - \frac{A}{2\alpha}e^{-2\alpha t} + B,$$

for arbitrary integration constants A and B. With $C = (12\pi^2 T_{00})/(96 f_2)$ as above, this gives (4.17). $\square$

The behavior of the solutions (4.16) and (4.17) should be compared with the analogous equations in the ordinary case where of equation (4.14) with $\kappa_0^2 = 8\pi G T_{00}$ independent of the energy scale Λ and equal to the ordinary Newton constant. In this case, one obtains, in the radiation-dominated case with $a(t) = t^{1/2}$ the equation

$$\ddot{h} + t^{-1}\dot{h} - \tfrac{3}{4}t^{-2} = 8\pi G T_{00},$$

which has general solution

$$h(t) = 2\pi G T_{00} t^2 + B + A\log(t) + \frac{3}{8}\log(t)^2,$$

which differs from (4.16) for the presence of a quadratic instead of cubic term. Similarly, in the case of the inflationary epoch, where one has $a(t) = e^{\alpha t}$, in the ordinary case one has the equation

$$\ddot{h} + 2\alpha\dot{h} - 3\alpha^2 = 8\pi G T_{00},$$

which has general solution

$$\left(\frac{4\pi G T_{00}}{\alpha} + \frac{3\alpha}{2}\right)t + \frac{A}{2\alpha}e^{-2\alpha t} + B,$$

which differs from what we have in (4.17) by the presence of an additional linear term instead of an exponential term.

Similar examples with different forms of the factor $a(t)$ in the metric (4.13) can easily be derived in the same way. Choices of the parameter f_2 for which the term $\mathfrak{c}(\Lambda)$ cannot be neglected will give rise to varying behaviors of the equations both in the radiation dominated era and during inflation. However, in those cases the equations cannot be integrated exactly so we cannot exhibit explicit solutions.

4.5 The λ_0-ansatz and the Higgs

We mention here briefly another consequence of the *ansatz* (4.5) on the running of the coefficient $\lambda_0(\Lambda)$ between the electroweak and the unification scales. This running is different from the one used in [17] to derive the Higgs mass estimate. In fact, the RGE equations themselves are different, since in [17] one only considers the RGE for the minimal standard model and the boundary conditions at unifications are also significantly different from the ones used in [1] that we use here (see the second appendix for a more detailed discussion of the boundary conditions in [17] and [1]). Also in [17] the coefficient λ_0 is assumed to run as the λ in the minimal standard model, with only the boundary condition at unification relating it to the values of $\mathfrak{a}$ and $\mathfrak{b}$ (which are in turn different from the values at unification according to the RGEs of [1]). Thus, one can check how adopting the *ansatz* (4.5) for $\lambda_0(\Lambda)$ together with the boundary conditions of [1] for λ, $\mathfrak{a}$ and $\mathfrak{b}$ affects the estimate for the Higgs mass. In [17] one obtains a heavy Higgs at around 170 GeV, by the value

$$\sqrt{2\lambda}\frac{2M}{g} \sim 170\,\mathrm{GeV},$$

where λ is the low energy limit of the RGE flow in the minimal standard model for the coefficient λ_0 and where $2M/g \simeq 246\,\mathrm{GeV}$ is the Higgs vacuum. If we replace the running used in [17] with the running of $\lambda_0(\Lambda)$ of (4.5) with the boundary condition of [1], the same estimate would deliver a much lower value

$$\sqrt{2\lambda_0(\Lambda_{\mathrm{ew}})}\frac{2M}{g} = \sqrt{\frac{2\lambda(\Lambda_{\mathrm{ew}})\pi^2\mathfrak{b}(\Lambda_{\mathrm{ew}})}{f_0\mathfrak{a}^2(\Lambda_{\mathrm{ew}})}}\frac{2M}{g} \sim 158\,\mathrm{GeV}.$$

This looks potentially interesting in view of the fact that the projected window of exclusion for the Higgs mass in [46] starts at 158 GeV (see also [32]). This gives only a first possible indication that a more detailed analysis of the RGEs for the standard model with Majorana mass terms, as in [1], and a careful discussion of the boundary conditions at unification and of the running of the coefficients in the asymptotic expansion of the spectral action may yield a wider spectrum of possible behaviors for the Higgs field within these NCG models. This topic deserves more careful consideration that is beyond the main focus of the present paper.

4.6 Antigravity in the early universe

Cosmological models exhibiting a sign change in the effective Newton constant in the early universe, due to the interactions of gravity and matter, were studied for instance in [35, 41, 48]. Those models of antigravity are based on the presence of a non-minimal conformal coupling of gravity to another field, with a Lagrangian of the form

$$\mathcal{L} = \int \frac{-1}{16\pi G} R\sqrt{g} + \frac{1}{12}\int R\varphi^2\sqrt{g} + \mathcal{L}(\varphi, A, \psi),$$

where the last term contains the kinetic and potential terms for the field φ and its interactions with other fields A, ψ. The conformal coupling of R and φ gives rise to an effective gravitational constant of the form

$$G_{\rm eff}^{-1} = G^{-1} - \frac{4}{3}\pi\varphi^2,$$

where φ is treated as a constant, which is estimated in [35] in terms of the excess of neutrino over antineutrino density, the quartic interaction coefficient λ of φ and the Weinberg angle. A decrease in $G_{\rm eff}$ produces a corresponding increase in the Planck density (see, however, the criticism to this model discussed in [41]). Antigravity sectors with negative effective gravitational constant as in [35] were recently considered within various approaches based on extra dimensions and brane world models, see for instance [28, 29], or from the point of view of moduli in heterotic superstring theory, as in [42].

Within the NCG model it is also possible to find scenarios where one has antigravity in the early universe. The same mechanism proposed in [35] can be reproduced within the NCG model, due to the presence of the conformal coupling of the Higgs field to gravity. However, there is another, independent mechanism that can also produce a sign change of the Newton constant between the unification and electroweak cosmological phase transitions and which is only due to the running of the effective Newton constant with the RGE equations of the particle physics content of the model. In fact, there are choices of the parameter f_2 of the model for which the effective gravitational constant undergoes a sign change.

An example of this behavior is obtained if one chooses the value of the effective gravitational constant $G_{\rm eff}$ to be equal to the Newton constant at unification scale, $G_{\rm eff}(\Lambda_{\rm unif}, f_2) = G$, and then runs it down with the RGE equations. Notice that the assumption $G_{\rm eff}(\Lambda_{\rm unif}, f_2) = G$ is the same that was proposed in [17], but due to the different RGE analysis considered here, the behavior we describe now is different from the one projected in [17].

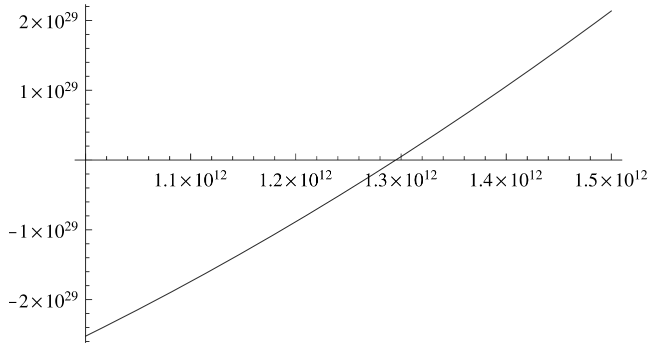

Figure 7: An example of transition to negative gravity in the running of $G_{\mathrm{eff}}^{-1}(\Lambda)$ with $G_{\mathrm{eff}}(\Lambda_{\mathrm{unif}}, f_2) = G$.

With the estimate used in Lemma 5.2 of [17] for the Majorana mass terms at unification (under the assumptions of flat space and negligible Higgs vacuum expectation) setting the effective gravitational constant of the model equal to the Newton constant at unification energy only requires f_2 of the order of at most $f_2 \simeq 10^2$, while using the boundary conditions of [1] one find a larger value of f_2. In fact, we see that setting

$$f_2 \simeq 18291.3 \tag{4.20}$$

gives $G_{\mathrm{eff}}(\Lambda_{\mathrm{unif}}) = G = (1.22086 \times 10^{19})^{-2}$.

One sees then, as in figure 7 that the resulting $G_{\mathrm{eff}}(\Lambda, f_2)$, for this choice of f_2 has a sign change at around 1.3×10^{12} GeV. Thus, with these boundary conditions one finds an example of a regime of negative gravity in the early universe. Other possible choices of f_2 lead to similar examples.

4.7 The conformal gravity regime

In this scenario, it is especially interesting to see what happens near the energy scale of 1.3×10^{12} GeV, where G_{eff}^{-1} vanishes, as in figure 7. This gives another example of an emergent conformal gravity regime at a phase transition (here the change from positive to negative gravity) of the system.

Proposition 4.5. *Let the parameter f_2 be chosen so that the inverse effective gravitational constant $G_{\mathrm{eff}}^{-1}(\Lambda)$ has a zero at some $\Lambda = \Lambda_0$. Assume the vanishing of the topological term and suppose that the parameter f_4 is chosen so that the effective cosmological constant also vanishes at Λ_0. Then near Λ_0 and for $|H|^2$ sufficiently small the dynamics of* (2.11) *is dominated by pure conformal gravity.*

Proof. At the singularity $\Lambda_{\rm sing}$ for $G_{\rm eff}$, assuming the vanishing of the topological term, the terms that remain in the bosonic part of the action are the cosmological term, the conformal gravity term with the Weyl curvature tensor, and the Higgs and gauge bosons terms. If the Higgs field is sufficiently near the $H = 0$ vacuum, and the parameter f_4 is chosen so that the cosmological term also vanishes at the same scale $\Lambda_{\rm sing}$, one finds that what remains of the bosonic action is just the conformal gravity action

$$S(\Lambda_{\rm sing}) = \alpha_0 \int C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \sqrt{g}\, d^4x \tag{4.21}$$

with $\alpha_0 = -3f_0/(10\pi^2) \simeq -0.25916$, and with the additional weakly coupled term of the gauge bosons. Thus, in this scenario, when running down the coefficients of the bosonic spectral action from unification scale towards the electroweak scale, a singularity of $G_{\rm eff}$ occurs at an intermediate scale of 1.3×10^{12} GeV. At the singularity, if the cosmological term also vanishes and the Higgs contribution is sufficiently small, the model becomes dominated by a conformal gravity action (4.21).

In this regime, the equations of motion will then be of the form

$$2C^{\mu\lambda\nu\kappa}_{;\lambda,;\kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa} = -\frac{1}{4\alpha_0} T^{\mu\nu}, \tag{4.22}$$

as in (188) of [37]. Thus one has in this regime pure conformal gravity with an effective gravitational constant, see Section 8.7 of [37]. □

The emergence of a conformal gravity regime was also observed in [38], although not in terms of renormalization group analysis, but near the special value of the Higgs field $|H| \to \sqrt{6}/\kappa_0$.

4.8 Gravity balls

We now analyze more carefully the second mechanism that produces negative gravity besides the running of $G_{\rm eff}(\Lambda)$, namely the nonminimal conformal coupling to the Higgs field. This will reproduce in this model a scenario similar to that of [35].

Let us assume for simplicity that the parameter f_2 is chosen so that $G_{\rm eff}(\Lambda, f_2) > 0$ for all $\Lambda_{\rm ew} \leq \Lambda \leq \Lambda_{\rm unif}$. We show that, even in this case, it is possible to have regions of negative gravity, due to the coupling to the Higgs. These behave like the gravity balls and nontopological solitons of [36, 43], but with a more elaborate behavior coming from the fact that the underlying gravitational constant is also changing with Λ according to the RGE flow.

Proposition 4.6. *Let f_2 be assigned so that the effective gravitational constant satisfies $G_{\rm eff}(\Lambda, f_2) > 0$ for all $\Lambda_{\rm ew} \le \Lambda \le \Lambda_{\rm unif}$. Then negative gravity regions, with $|H|$ near $|H|^2 \sim \mu_0^2/(2\lambda_0)$, arise in the range of energies Λ such that*

$$\ell_H(\Lambda, f_2) > \ell_G(\Lambda, f_2), \tag{4.23}$$

for

$$\ell_H(\Lambda, f_2) = \frac{(2f_2\Lambda^2\mathfrak{a}(\Lambda) - f_0\mathfrak{e}(\Lambda))\mathfrak{a}(\Lambda)}{\pi^2\lambda(\Lambda)\mathfrak{b}(\Lambda)}, \tag{4.24}$$

and

$$\ell_G(\Lambda, f_2) = \frac{192f_2\Lambda^2 - 2f_0\mathfrak{c}(\Lambda)}{4\pi^2}. \tag{4.25}$$

Proof. The presence of the conformal coupling term

$$-\frac{1}{12}\int R|H|^2\sqrt{g}\, d^4x$$

in the normalized asymptotic formula for the spectral action (2.11) means that, in regions with nearly constant $|H|^2$, the effective gravitational constant of the model is further modified to give

$$G_{{\rm eff},H} = \frac{G_{\rm eff}}{1 - (4\pi/3)G_{\rm eff}|H|^2}. \tag{4.26}$$

This is the same mechanism used in [35] for negative gravity models. This means that, assuming that $G_{\rm eff}(\Lambda) > 0$ for all $\Lambda_{\rm ew} \le \Lambda \le \Lambda_{\rm unif}$, one will have

$$\begin{cases} G_{{\rm eff},H} < 0, & \text{for } |H|^2 > \dfrac{3}{4\pi G_{\rm eff}(\Lambda)}, \\[2ex] G_{{\rm eff},H} > 0, & \text{for } |H|^2 < \dfrac{3}{4\pi G_{\rm eff}(\Lambda)}. \end{cases}$$

This means, for instance, that in the presence of an unstable equilibrium at $|H| = 0$ and a stable equilibrium at $|H|^2 = v^2$ satisfying $v^2 > \frac{3}{4\pi G_{\rm eff}(\Lambda)}$, one can have gravity balls near zeros of the field $|H|^2$, where gravity behaves in the usual attractive way, inside a larger-scale negative gravity corresponding to the true equilibrium $|H|^2 = v^2$.

In the action (2.11), the Higgs field has a quartic potential given by

$$-\mu_0^2 \int |H|^2 \sqrt{g}\, d^4x + \lambda_0 \int |H|^4 \sqrt{g}\, d^4x,$$

which has a minimum at $\mu_0^2/(2\lambda_0)$. Thus, to identify the negative gravity regime we need to compare the running of (4.24),

$$\begin{aligned} \ell_H(\Lambda, f_2) := \frac{\mu_0^2}{2\lambda_0}(\Lambda) &= \frac{2(f_2\Lambda^2/f_0) - (\mathfrak{e}(\Lambda)/\mathfrak{a}(\Lambda))}{\lambda(\Lambda)(\pi^2\mathfrak{b}(\Lambda)/f_0\mathfrak{a}^2(\Lambda))} \\ &= \frac{(2f_2\Lambda^2\mathfrak{a}(\Lambda) - f_0\mathfrak{e}(\Lambda))\mathfrak{a}(\Lambda)}{\pi^2\lambda(\Lambda)\mathfrak{b}(\Lambda)}, \end{aligned}$$

where we again used the *ansatz* (4.5) on the running of the coefficient $\lambda_0(\Lambda)$, and the running of the function (4.25),

$$\ell_G(\Lambda, f_2) := \frac{3}{4\pi G_{\mathrm{eff}}(\Lambda)} = \frac{3}{4\pi}\frac{192 f_2\Lambda^2 - 2f_0\mathfrak{c}(\Lambda)}{3\pi} = \frac{192 f_2\Lambda^2 - 2f_0\mathfrak{c}(\Lambda)}{4\pi^2}.$$

Thus, for different possible values of the parameter f_2, a negative gravity regime $G_{\mathrm{eff},H} < 0$ and gravitational balls are possible in the range where $\ell_H(\Lambda, f_2) > \ell_G(\Lambda, f_2)$. □

An example of such a transition to a negative gravity region is illustrated in figure 8, whereas the surface $\ell_H(\Lambda, f_2)$ near the top see-saw scale behaves as in figure 9.

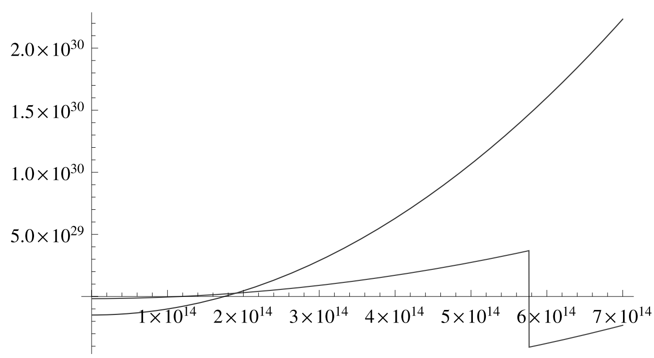

Figure 8: Transition to a negative gravity regime where $\ell_H(\Lambda, f_2) > \ell_G(\Lambda, f_2)$ in the region $10^{12} \leq \Lambda \leq 10^{15}$ with $f_2 = 1$.

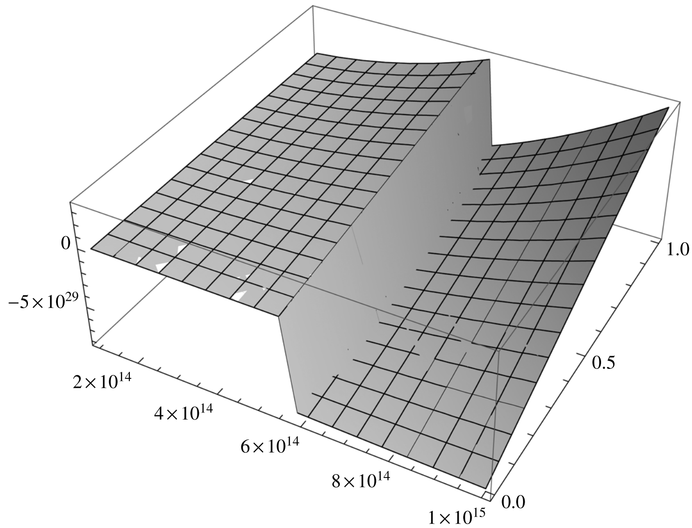

Figure 9: The surface $\ell_H(\Lambda, f_2)$ in the range $10^{14} \leq \Lambda \leq 10^{15}$ GeV, around the top see-saw scale, for $10^{-4} \leq f_2 \leq 1$.

4.9 Primordial black holes

The possibility of PBHs in the early universe was originally suggested by Zeldovich and Novikov in the late 1960s (see [39] for a survey). They originate from the collapse of overdense regions, as well as from other mechanisms such as phase transitions in the early universe, cosmic loops and strings, or inflationary reheating. The consequences on PBHs of the running of the gravitational constant in the early universe have been analyzed, for example, in [11]. In models where the gravitational constant in the early universe may be different from the value it has in the modern universe, PBHs, whose existence is conjectured but has not been presently confirmed, are seen as a possible source of information about the changing gravitational constant. In fact, the mass loss rate due to evaporation via Hawking radiation and the Hawking temperature of PBHs depends on the value of the Newton constant so that the evolution of such black holes depends on the change in the Newton constant. The evaporation of PBHs is often proposed as a mechanism underlying γ-ray bursts (see for instance [6, 9]). An especially interesting question regarding PBHs is that of *gravitational memory*, as described in [5]. It is usually assumed that two possible scenarios for the evolution of PBHs can happen: one where the evolution follows the changing gravitational constant and one with the possibility of "gravitational memory", namely where the evolution of the PBH is determined by an effective gravitational constant different from the one of surrounding space. This latter phenomenon can arise if the model has the possibility of having regions, as in the case of the gravity balls discussed in Section 4.8 above, where the effective gravitational constant has a value different from the one of surrounding space.

We discuss here the effect on PBHs of the running of the effective gravitational constant in the NCG model. For PBHs that formed at a time in the early universe when the gravitational constant $G_{\rm eff}(t)$ was different from the one of the modern universe and whose evolution in time reflects the corresponding evolution of the gravitational constant, the black hole "adjusts its size" to the changing $G_{\rm eff}(t)$ according to the equation

$$\frac{d\mathcal{M}(t)}{dt} \sim -(G_{\rm eff}(t)\mathcal{M}(t))^{-2}, \tag{4.27}$$

where $\mathcal{M}$ is the mass of the PBH. Correspondingly, the temperature varies with the changing gravitational constant as

$$T = (8\pi G_{\rm eff}(t)\mathcal{M}(t))^{-1}. \tag{4.28}$$

In the case with gravitational memory, one can have a black hole that evolves according to a gravitational constant that is different to the one of the surrounding space. In our setting, given the scenario described in Section 4.8, this second case occurs with equation (4.27) replaced by

$$\frac{d\mathcal{M}(t)}{dt} \sim -(G_{{\rm eff},H}(t)\mathcal{M}(t))^{-2}, \tag{4.29}$$

with $G_{{\rm eff},H}$ as in (4.26).

Proposition 4.7. *In the radiation dominated era, for a cosmology with metric tensor of the form* (4.13)*, the evaporation of PBHs by Hawking radiation in the NCG model is given by*

$$\mathcal{M}(\Lambda, f_2) = \sqrt[3]{\mathcal{M}^3(\Lambda_{\rm in}) - \frac{2}{3\pi^2}\int_\Lambda^{\Lambda_{\rm in}} \frac{(192 f_2 x^2 - 2 f_0 \mathfrak{c}(x))^2}{x^3}\, dx}, \tag{4.30}$$

in the case of PBHs without gravitational memory, while PBHs with gravitational memory evaporate according to

$$\mathcal{M}(\Lambda, f_2) = \sqrt[3]{\mathcal{M}^3(\Lambda_{\rm in}) - \frac{2}{3\pi^2}\int_\Lambda^{\Lambda_{\rm in}} \frac{(1-(4\pi/3)G_{\rm eff}(x)|H|^2)^2}{x^3 G_{\rm eff}(x)^2}\, dx}, \tag{4.31}$$

where

$$G^{-1}_{\mathrm{eff},H}(\Lambda) = \frac{(1-(4\pi/3)G_{\mathrm{eff}}(\Lambda)|H|^2)}{G_{\mathrm{eff}}(\Lambda)} = \frac{192 f_2\Lambda^2 - 2f_0\mathfrak{c}(\Lambda)}{3\pi} - \frac{4}{3\pi}\frac{(2f_2\Lambda^2\mathfrak{a}(\Lambda) - f_0\mathfrak{e}(\Lambda))\mathfrak{a}(\Lambda)}{\lambda(\Lambda)\mathfrak{b}(\Lambda)},$$

for $|H|^2 \sim \mu_0^2/(2\lambda_0)$. *Here* Λ_{in} *is the energy scale at which the radiation dominated phase begins, that is, where one starts to have* $a(t) = t^{1/2}$.

Proof. For a metric of the form (4.13), in the radiation dominated era one has $a(t) \sim t^{1/2}$, hence the energy–time change of variables $\Lambda = 1/a(t)$ is of the form $\Lambda = t^{-1/2}$. Thus, equation (4.27) can be rewritten in the variable Λ in the form

$$\frac{d\mathcal{M}(\Lambda)}{d\Lambda} = \frac{2}{\Lambda^3(G_{\mathrm{eff}}(\Lambda)\mathcal{M}(\Lambda))^2}, \tag{4.32}$$

since $dt/d\Lambda = -2\Lambda^{-3}$. We look at the case of a PBH that has a given mass $\mathcal{M}(\Lambda_{\mathrm{in}})$ at the initial Λ_{in}, and we look at the evolution of $\mathcal{M}(\Lambda)$ between Λ_{in} and electroweak scale. Equation (4.32), written in the form

$$\mathcal{M}(\Lambda)^2\, d\mathcal{M}(\Lambda) = 2\frac{d\Lambda}{\Lambda^3 G^2_{\mathrm{eff}}(\Lambda)}$$

gives (4.30). The case with gravitational memory is obtained similarly by replacing the effective gravitational constant $G_{\mathrm{eff}}(\Lambda)$ with the one locally modified by the interaction with the Higgs field, $G_{\mathrm{eff},H}(\Lambda)$. □

The expression for the evaporation by Hawking radiation simplifies in the cases where f_2 is sufficiently large that the term $192f_2\Lambda^2$ dominates over $2f_0\mathfrak{c}(\Lambda)$. This is the case, for example, when $G_{\mathrm{eff}}(\Lambda_{\mathrm{ew}}) = G$ and f_2 is chosen accordingly.

Corollary 4.1. *In the case where* f_2 *is sufficiently large that the term* $2f_0\mathfrak{c}(\Lambda)$ *is negligible with respect to* $192f_2\Lambda^2$ *for all* $\Lambda_{\mathrm{ew}} \leq \Lambda \leq \Lambda_{\mathrm{in}}$, *the evaporation law for a PBH that forms at* Λ_{in} *gives a bound on its mass by* $(\Lambda_{\mathrm{in}}64\sqrt{6}f_2/\pi)^{2/3}$.

Proof. Under the assumption that $192f_2\Lambda^2$ dominates over $2f_0\mathfrak{c}(\Lambda)$, the right-hand side of (4.32) can be approximated by the dominant term of

$G_{\rm eff}(\Lambda)$, which gives

$$\mathcal{M}^2\, d\mathcal{M} = 2\left(\frac{64 f_2}{\pi}\right)^2 \Lambda\, d\Lambda. \tag{4.33}$$

Let Λ_0 denote the scale at which the PBH evaporates. Then (4.33) gives

$$\frac{1}{3}\mathcal{M}^3(\Lambda) = 2\left(\frac{(64 f_2)}{\pi}\right)^2 (\Lambda^2 - \Lambda_0^2).$$

This sets a bound to the mass of a PBH formed at $\Lambda = \Lambda_{\rm in}$ from the condition that

$$\Lambda_0^2 = \Lambda_{\rm in}^2 - \left(\frac{\pi}{(64 f_2 \sqrt{6})}\right)^2 \mathcal{M}^3_{\rm unif},$$

which gives

$$\mathcal{M}_{\rm in} \leq \left(\frac{\Lambda_{\rm in} 64 f_2 \sqrt{6}}{\pi}\right)^{2/3}. \qquad \square$$

We can analyze similarly the equation of the Hawking radiation for PBHs during the inflationary epoch when one has $a(t) = {\rm e}^{\alpha t}$ for some $\alpha > 0$. We obtain the following result.

Proposition 4.8. *In the inflationary epoch, for a cosmology with metric tensor of the form* (4.13)*, the evaporation of PBHs by Hawking radiation in the NCG model is given by*

$$\mathcal{M}(\Lambda, f_2) = \sqrt[3]{\mathcal{M}^3(\Lambda_{\rm in}) - \frac{1}{3\alpha\pi^2}\int_\Lambda^{\Lambda_{\rm in}} \frac{(192 f_2 x^2 - 2 f_0 \mathfrak{c}(x))^2}{x}\, dx}, \tag{4.34}$$

in the case of PBHs without gravitational memory, while PBHs with gravitational memory evaporate according to

$$\mathcal{M}(\Lambda, f_2) = \sqrt[3]{\mathcal{M}^3(\Lambda_{\rm in}) - \frac{1}{3\alpha\pi^2}\int_\Lambda^{\Lambda_{\rm in}} \frac{(1-(4\pi/3)G_{\rm eff}(x)|H|^2)^2}{x G_{\rm eff}(x)^2}\, dx}. \tag{4.35}$$

where $\Lambda_{\rm in}$ is the energy scale at which the inflationary behavior $a(t) = {\rm e}^{\alpha t}$ begins.

Proof. The argument is completely analogous to the previous case. Here the energy–time change of variables $\Lambda = 1/a(t)$ gives $t = -\alpha^{-1}\log\Lambda$. Thus, in the energy variable equation (4.27) becomes

$$\mathcal{M}^2\, d\mathcal{M} = \frac{d\Lambda}{\alpha\Lambda(G_{\rm eff}(\Lambda))^2},$$

for the case without gravitational memory, or the same equation with $G_{\rm eff}(\Lambda)$ replaced by $G_{{\rm eff},H}(\Lambda)$ in the case with gravitational memory. This gives (4.34) and (4.35). □

4.10 Higgs-based slow-roll inflation

Recently, a mechanism for inflation within the minimal standard model physics was proposed in [24]. It is based on the presence of a nonminimal coupling of the Higgs field to gravity of the form

$$-\xi_0 \int R|H|^2\sqrt{g}\, d^4x$$

as we have in the asymptotic expansion of the spectral action in the NCG model, but where the value of ξ_0 is not set equal to the conformal coupling $\xi_0 = 1/12$, but is subject to running with the RGE flow. In [24] the running of ξ_0 is governed by the beta function given in [8], which in our notation we can write approximately as

$$16\pi^2\beta_{\xi_0}(\Lambda) = (-12\xi_0(\Lambda)+1)F(Y_u, Y_d, Y_\nu, Y_e, M, g_1, g_2, g_3, \lambda), \qquad (4.36)$$

where the function F of the running parameters of the model is computed explicitly in [8]. In [24] this running is only considered within the minimal standard model, without the right-handed neutrinos and Majorana mass terms M, but their argument can be adapted to this extension of the standard model, since the general derivation of the running of ξ_0 in [8] applies in greater generality. In this variable ξ_0 scenario, the dimensionless quantity that governs inflation is $\psi = \sqrt{\xi_0}|H|/m_{\rm P}$, where $m_{\rm P}$ is the reduced Planck mass, which in our notation is $m_{\rm P}^2 = 1/\kappa_0^2$. The inflationary period corresponds in [24] to the large values $\psi \gg 1$, the end of the inflation to the values $\psi \sim 1$ and the low-energy regime to $\psi \ll 1$.

At present we do not know whether a modification of the NCG model that allows for a variable $\xi_0(\Lambda)$, different from the conformal coupling $\xi_0 = 1/12$ is possible within the constraints of the model, but we show here that, even in the case where ξ_0 is constant in Λ and equal to the conformal $\xi_0 = 1/12$, the NCG model still allows for a similar inflation mechanism to occur, through the running of the effective gravitational constant.

Proposition 4.9. *A Higgs-based slow-roll inflation scenario arises in the NCG model with parameter*

$$\psi(\Lambda)^2 = \xi_0 \kappa_0^2(\Lambda)|H|^2,$$

and with potential

$$V_{\rm E}(H) = \frac{\lambda_0 |H|^4}{(1+\xi_0\kappa_0^2|H|^2)^2}. \tag{4.37}$$

Proof. In the NCG model, the coefficient $\kappa_0(\Lambda)$ is running with Λ according to

$$\kappa_0^2(\Lambda) = \frac{12\pi^2}{96 f_2 \Lambda^2 - f_0 \mathfrak{c}(\Lambda)}.$$

Then the parameter that controls inflation is given by

$$\psi(\Lambda)^2 = \xi_0(\Lambda)\kappa_0^2(\Lambda)|H|^2 = \frac{\pi^2}{96 f_2 \Lambda^2 - f_0 \mathfrak{c}(\Lambda)}|H|^2.$$

If one then proceeds as in [24], one finds that in the Einstein metric $g^{\rm E}_{\mu\nu} = f(H) g_{\mu\nu}$, for $f(H) = 1 + \xi_0\kappa_0|H|^2$ the Higgs potential becomes of the form (4.37). In the range where $\psi \gg 1$ this approaches the constant function (constant in H but not in Λ)

$$V_{\rm E} = \frac{\lambda_0(\Lambda)}{4\xi_0^2(\Lambda)\kappa_0^4(\Lambda)} = \frac{\lambda(\Lambda)\mathfrak{b}(\Lambda)(96 f_2\Lambda^2 - f_0\mathfrak{c}(\Lambda))^2}{4 f_0 \mathfrak{a}^2(\Lambda)},$$

where we used again the *ansatz* (4.5) for the running of the coefficient $\lambda_0(\Lambda)$, while at low values $\psi \ll 1$ the potential is well approximated by the usual quartic potential $V_{\rm E}(H) \sim \lambda_0 |H|^4$. □

The asymptotic value $V_{\rm E}$ in turn depends on the energy scale Λ and different behaviors are possible upon changing the values of the parameter f_2 of the model as the examples in figure 10 illustrate. One can see the effect of the top see-saw scale on the running.

Notice that a potential of the form as in figure 11, which is especially suitable for slow-roll inflationary models, also arises naturally in the NCG setting from nonperturbative effects in the spectral action, as in [16]. We plan to return in future work to describe how the nonperturbative approach can be applied to cosmological models beyond the very early universe.

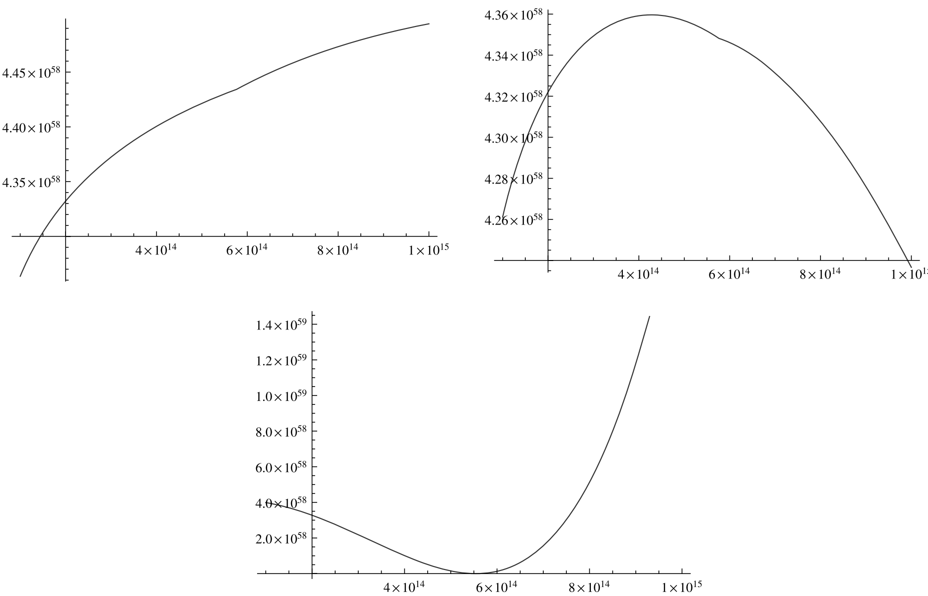

Figure 10: The behavior of the asymptotic value $V_{\rm E}$ of the potential $V_{\rm E}(H)$, plotted above for $f_2 = 10^{-4}$, $f_2 = 10^{-3}$, and $f_2 = 1/10$ and for $10^{14} \leq \Lambda \leq 10^{15}$

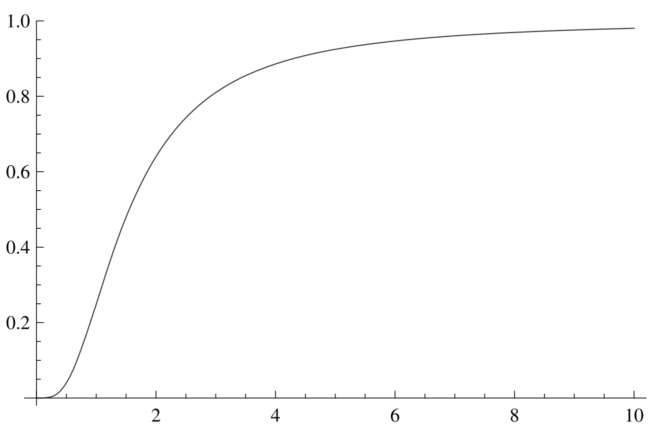

Figure 11: Plot of $V_{\rm E}(H)/V_{\rm E}$ against ψ.

4.11 Spectral index and tensor to scalar ratio

In a slow-roll inflation model, the first and second derivatives of the potential $V_{\rm E}$ together with a change of variable in the field that brings the action into a canonical form, determine the first and second slow-roll parameters ϵ and

η, see [34] Section 3.4 and also the analysis in [24]. We analyze here the form of the slow-roll parameters in the NCG model.

We write the potential $V_{\rm E}$ of (4.37) in the form

$$V_{\rm E}(s) = \frac{\lambda_0 s^4}{(1+\xi_0\kappa_0^2 s^2)^2}. \tag{4.38}$$

We will consider derivatives of this potential as a function of the variable s.

We also introduce the expression

$$C(s) := \frac{1}{2(1+\xi_0\kappa_0^2 s^2)} + \frac{3}{2\kappa_0^2}\frac{(2\xi_0\kappa_0^2 s)^2}{(1+\xi_0\kappa_0^2 s^2)^2}, \tag{4.39}$$

which corresponds to the $(d\sigma/d\phi)^2$ of equation (4) of [24] that gives the change of variable in the field that runs the inflation that puts the action in a canonical form.

The two slow-roll parameters are then given by the expressions

$$\epsilon(s) = \frac{1}{2\kappa_0^2}\left(\frac{V_{\rm E}'(s)}{V_{\rm E}(s)}\right)^2 C(s)^{-1}, \tag{4.40}$$

$$\eta(s) = \frac{1}{\kappa_0^2}\left(\frac{V_{\rm E}''(s)}{V_{\rm E}(s)}C(s)^{-1} - \frac{V_{\rm E}'(s)}{V_{\rm E}(s)}C(s)^{-3/2}\frac{d}{ds}C(s)^{1/2}\right), \tag{4.41}$$

which correspond to equations (6) and (7) of [24]. We then have the following result.

Proposition 4.10. *The first slow-roll coefficient is of the form*

$$\epsilon(s) = \frac{16\kappa_0^2}{s^2+\xi_0\kappa_0^2(1+(\kappa_0^2)^2)s^4}, \tag{4.42}$$

while the second slow-roll coefficient is given by

$$\eta(s) = \frac{8(3+\xi_0\kappa_0^2 s^2(1-2\xi_0\kappa_0^2(s^2+12\kappa_0^2(-1+\xi_0\kappa_0^2 s^2))))}{\kappa_0^2(s+\xi_0\kappa_0^2(1+(\kappa_0^2)^2)s^3)^2}. \tag{4.43}$$

Proof. The derivatives of the potential are

$$V_{\mathrm{E}}'(s) = -\frac{4\lambda_0\xi_0\kappa_0^2 s^5}{(1+\xi_0\kappa_0^2 s^2)^3} + \frac{4\lambda_0 s^3}{(1+\xi_0\kappa_0^2 s^2)^2},$$
$$V_{\mathrm{E}}''(s) = \frac{24\lambda_0\xi_0^2(\kappa_0^2)^2 s^6}{(1+\xi_0\kappa_0^2 s^2)^4} - \frac{36\lambda_0\xi_0\kappa_0^2 s^4}{(1+\xi_0\kappa_0^2 s^2)^3} + \frac{12\lambda_0 s^2}{(1+\xi_0\kappa_0^2 s^2)^2}.$$

One also has

$$C(s)^{-3/2} = \frac{\sqrt{8}(1+\xi_0\kappa_0^2 s^2)^3}{\sqrt{(1+\xi_0\kappa_0^2(1+12\xi_0\kappa_0^2)s^2)^3}},$$
$$\frac{d}{ds}C(s)^{1/2} = -\frac{\xi_0\kappa_0^2 s(1+\xi_0\kappa_0^2(s^2+12\kappa_0^2(-1+\xi_0\kappa_0^2 s^2)))}{(1+\xi_0\kappa_0^2 s^2)^2\sqrt{2(1+\xi_0\kappa_0^2(1+12\xi_0(\kappa_0^2)^2)s^2}}.$$

One then computes $\epsilon(s)$ and $\eta(s)$ directly from (4.40) and (4.41). □

Notice that both coefficients depend on the energy scale Λ and on the parameters f_2 and f_0 through the single parameter

$$\kappa_0^2(\Lambda) = \frac{12\pi}{96 f_2\Lambda^2 - f_0\mathfrak{c}(\Lambda)},$$

while no dependence through $\lambda_0(\Lambda)$ remains and in our model $\xi_0 = 1/12$.

The slow-roll coefficients provide expressions for the spectral index n_{s} and the tensor to scalar ratio r, which can be directly compared with cosmological data. Thus, a more detailed analysis of the behavior of these as functions of Λ and of the parameter f_2 of the model will give an exclusion curve for the parameter. We will provide a more detailed analysis elsewhere, but for the purpose of the present paper we derive the corresponding expression one has in this model for the spectral index and the tensor to scalar ratio.

Proposition 4.11. *The spectral index is of the form*

$$n_{\mathrm{s}} = 1 + \frac{32(216+\kappa_0^2(6s^2-\kappa_0^2(432+12\kappa_0^2(2+3(\kappa_0^2)^2)s^2+(1+(\kappa_0^2)^2)s^4)))}{\kappa_0^2(12s+\kappa_0^2(1+(\kappa_0^2)^2)s^3)^2}, \tag{4.44}$$

while the tensor to scalar ratio if given by

$$r = \frac{256\kappa_0^2}{s^2+(\kappa_0^2/12)(1+(\kappa_0^2)^2)s^4}. \tag{4.45}$$

Proof. One knows that the spectral index and the tensor to scalar ratio are related to the slow-roll coefficients by (see [34] Sections 7.5.2 and 7.6 and [24])

$$n_{\mathrm{s}} = 1 - 6\epsilon + 2\eta, \quad \text{and} \quad r = 16\epsilon. \tag{4.46}$$

The result then follows directly from Proposition 4.10. □

Once again, these parameters have a dependence on the choice of the free parameter f_2 of the model and a scaling behavior with the energy Λ, which depends on the running of $\kappa_0^2(\Lambda)$. Since the spectral index and the tensor to scalar ratio are heavily constrained by cosmological data from the WMAP (Wilkinson Microwave Anisotropy Probe) combined with baryon acoustic oscillations and supernovae data, this provides a way to impose realistic constraints on the parameter f_2 of the model based on direct confrontation with data of cosmological observations. We will provide a detailed analysis of the constraints imposed on the NCG model by the spectral index and tensor to scalar ratio in a separate paper. Since the parameters n_{s} and r are also providing information on the gravitational waves (see [34, Section 7.7]) a more detailed analysis of their behavior and dependence on the parameters of the model will give us a better understanding of the effects on gravitational waves of the presence of noncommutativity and may explain in a different way the amplification phenomena in the propagation of gravitational waves that we observed in Section 4.4.

4.12 Variable effective cosmological constant

The relation between particle physics and the cosmological constant, through the contribution of the quantum vacua of fields, is well known since the seminal work of Zeldovich [49]. The cosmological constant problem is the question of reconciling a very large value predicted by particle physics with a near zero value that conforms to the observations of cosmology. Among the proposed solutions to this problem are various models, starting with [49], with a varying effective cosmological constant, which would allow for a large cosmological constant in the very early universe, whose effect of negative pressure can overcome the attractive nature of gravity and result in accelerated expansion, and then a decay of the cosmological constant to zero (see also [40] for a more recent treatment of variable cosmological constant models). Often the effective cosmological constant is produced via a nonminimal coupling of gravity to another field, as in [25], similarly to what one does in the case of an effective gravitational constant.

In the present model, one can recover the same mechanism of [25] via the nonminimal coupling to the Higgs field, but additionally one has a running of the effective cosmological constant $\gamma_0(\Lambda)$ which already by itself may produce the desired effect of decaying cosmological constant. We illustrate in this section an example of how different choices of the parameter f_4, for a fixed choice of f_2, generate different possible decay behaviors of the cosmological constant. These can then be combined with the effect produced by the nonminimal coupling with the Higgs field, which behaves differently here than in the case originally analyzed by [25]. This still does not resolve the fine tuning problem, of course, because we are trading the fine tuning of the cosmological constant for the tuning of the parameters f_2 and f_4 of the model, but the fact that these parameters have a geometric meaning in terms of the spectral action functional may suggest geometric constraints.

As we have seen in the proof of Proposition 4.1 above, one can impose the vanishing of the effective cosmological constant γ_0 at a given energy scale Λ by fixing the parameter f_4 equal to the value given in (4.6). This means that, when the effective gravitational constant of the model varies as in the effective gravitational constant surface $G_{\rm eff}(\Lambda, f_2)$, depending on the value assigned to the parameter f_2, there is an associated surface that determines the value of the parameter f_4 that gives a vanishing effective cosmological constant. This is defined by the equation

$$\pi^2\gamma_0(\Lambda, f_2, f_4) = 48 f_4\Lambda^4 - f_2\Lambda^2\mathfrak{c}(\Lambda) + \tfrac{1}{4} f_0\mathfrak{d}(\Lambda) = 0. \tag{4.47}$$

The solutions to this equation determine the surface

$$f_4(\Lambda, f_2) = \frac{f_2\Lambda^2\mathfrak{c}(\Lambda) - (1/4) f_0\mathfrak{d}(\Lambda)}{48\Lambda^4}. \tag{4.48}$$

To illustrate the different possible behaviors of the system when imposing the vanishing of the effective cosmological constant at different possible energy scales Λ, we look at the examples where one imposes a vanishing condition at one of the two ends of the interval of energies considered, that is, for $\Lambda = \Lambda_{\rm ew}$ or for $\Lambda = \Lambda_{\rm unif}$. For simplicity we also give here an explicit example for a fixed assigned value of f_2, which first fixes the underlying $G_{\rm eff}(\Lambda)$. we choose, as above, the particular example where f_2 is chosen to satisfy $G_{\rm eff}(\Lambda_{\rm ew}) = G$. Under these conditions we impose the vanishing of the effective cosmological constant at either $\Lambda_{\rm ew}$ or $\Lambda_{\rm unif}$ through the constraint

$$\gamma_0(\Lambda_{\rm ew}, f_2, f_4) = 0 \quad \text{or} \quad \gamma_0(\Lambda_{\rm unif}, f_2, f_4) = 0, \tag{4.49}$$

which fixes the value of $f_4 = f_4(\Lambda_{\rm ew}, f_2)$. Then the behavior at higher energies of the effective cosmological constant is given by the surface

$$\gamma_0(\Lambda, f_2, f_4(\Lambda_{\rm ew}, f_2)) \quad \text{or} \quad \gamma_0(\Lambda, f_2, f_4(\Lambda_{\rm unif}, f_2)) \tag{4.50}$$

as a function of the energy Λ and the parameter f_2.

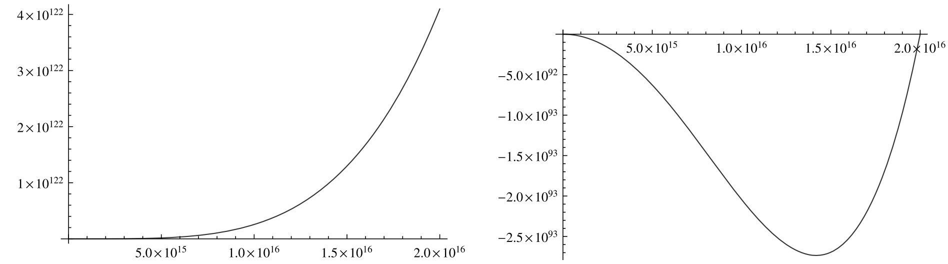

Figure 12: Running of $\gamma_0(\Lambda)$ under the assumptions of vanishing $\gamma_0(\Lambda_{\rm ew}) = 0$ or $\gamma_0(\Lambda_{\rm unif}) = 0$.

We find two very different behaviors in these two examples, which have different implications in terms of dark energy. Imposing the vanishing of γ_0 at $\Lambda_{\rm ew}$ produces a very fast growth of $\gamma_0(\Lambda)$ at larger energies $\Lambda_{\rm ew} \leq \Lambda \leq \Lambda_{\rm unif}$, while imposing the vanishing at $\Lambda_{\rm unif}$ produces a $\gamma_0(\Lambda)$ that quickly decreases to a large negative value and then slowly decreases to become small again near $\Lambda_{\rm ew}$, as illustrated in the two examples of figure 12. A negative cosmological constant adds to the attractive nature of gravity hence it counteracts other possible inflation mechanism that may be present in the model, while a positive cosmological constant counteracts the attractive nature of gravity and can determine expansion. Thus, we see in these two examples that different choices of the free parameter f_4 in the model can lead to very different qualitative behaviors of the effective cosmological constant.

The range of different variable cosmological constants can be further expanded by considering other possible values of the parameter f_4 in the surface $\gamma_0(\Lambda, f_4, f_2)$ associated to a chosen value of f_2.

Again, as we have seen in the case of the effective gravitational constant, we have here two distinct mechanisms for vacuum-decay: the running $\gamma_0(\Lambda)$ as described above, depending on the parameters f_2 and f_4 and on the RGE, and a further modification due to the non-minimal coupling with the Higgs field, as in [25].

Namely, if one has an unstable and a stable equilibrium for $|H|^2$, one can encounter for the cosmological constant the same type of phenomenon described for the effective gravitational constant by the gravity balls, where the value of $\gamma_{0,H}$ is equal to γ_0 at the unstable equilibrium $|H|^2 = 0$ on some large region, while outside that region it decays to the stable equilibrium $|H|^2 = v^2$ for which

$$\gamma_{0,H}(\Lambda) = \frac{\gamma_0(\Lambda)}{1 - 16\pi G_{\rm eff}(\Lambda)\xi_0 v^2}.$$

This behaves differently in the case of our system from the model of [25]. In fact, here the nonminimal coupling is always the conformal one $\xi_0 = 1/12$, while the gravitational constant is itself running, so we obtain the following.

Proposition 4.12. *The nonminimal conformal coupling of the Higgs field to gravity,*

$$-\xi_0 \int R|H|^2 \sqrt{g}\, d^4x$$

changes the running of the effective cosmological constant to

$$\gamma_{0,H}(\Lambda) = \frac{\gamma_0(\Lambda)}{1 - 16\pi G_{\rm eff}(\Lambda)\xi_0 |H|^2}. \tag{4.51}$$

This gives

$$\gamma_{0,H}(\Lambda) = \frac{\gamma_0(\Lambda)}{1 - (4\mathfrak{a}(\Lambda)(2f_2\Lambda^2\mathfrak{a}(\Lambda) - f_0\mathfrak{e}(\Lambda))/\lambda(\Lambda)\mathfrak{b}(\Lambda)(192 f_2\Lambda^2 - 2f_0\mathfrak{c}(\Lambda)))},$$

for $|H|^2 \sim \mu_0^2/(2\lambda_0)$.

Proof. Arguing as in [25] we obtain (4.51), which now depends on the running of both $\gamma_0(\Lambda)$ and $G_{\rm eff}(\Lambda)$, as well as on the Higgs field $|H|^2$. Assuming the latter to be nearly constant with $|H|^2 \sim \mu_0^2/(2\lambda_0)$, we obtain

$$\begin{aligned}\gamma_{0,H}(\Lambda) &= \gamma_0(\Lambda)\frac{G_{{\rm eff},H}(\Lambda)}{G_{\rm eff}(\Lambda)}\\ &= \frac{\gamma_0(\Lambda)}{1 - (4\pi/3)G_{\rm eff}(\Lambda)(2f_2\Lambda^2\mathfrak{a}(\Lambda) - f_0\mathfrak{e}(\Lambda)\mathfrak{a}(\Lambda)/\pi^2\lambda(\Lambda)\mathfrak{b}(\Lambda))},\end{aligned} \tag{4.52}$$

where, as in the treatment of the variable effective gravitational constant, we have

$$|H|^2 \sim \frac{\mu_0^2}{2\lambda_0} = \frac{2f_2\Lambda^2\mathfrak{a}(\Lambda) - f_0\mathfrak{e}(\Lambda)\mathfrak{a}(\Lambda)}{\pi^2\lambda(\Lambda)\mathfrak{b}(\Lambda)}$$

using the *ansatz* (4.5) for $\lambda_0(\Lambda)$, and $G^{-1}_{{\rm eff},H}(\Lambda) = G^{-1}_{\rm eff}(\Lambda) - \frac{4\pi}{3}|H|^2$. □

To see how the interaction with the Higgs field can modify the running of the effective cosmological constant, consider again an example that exhibits the same behavior as the second graph of figure 12. This is obtained, for instance, by choosing $f_2 = 1$ and f_4 satisfying the vanishing condition $\gamma_0(\Lambda_{\rm unif}) = 0$. Then the running of $\gamma_{0,H}(\Lambda)$ given in (4.52) exhibits an overall

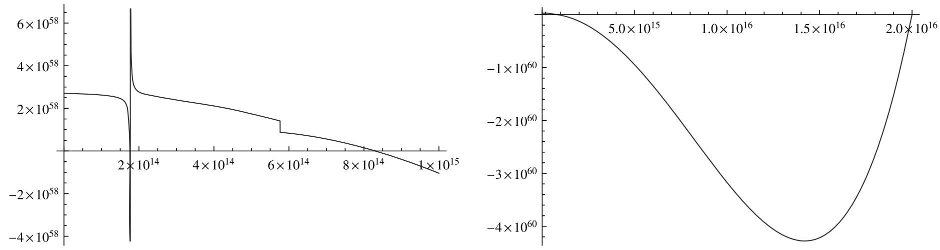

Figure 13: Running of $\gamma_{0,H}(\Lambda)$ with $f_2 = 1$ and with $\gamma_0(\Lambda_{\rm unif}) = 0$.

behavior similar to that of $\gamma_0(\Lambda)$ but with a change of sign and a singularity, as shown in figure 13.

4.13 Early time bounds estimate

In models with variable cosmological (and gravitational) constant, some strong constraints exist from "early time bounds" on the vacuum-energy density that are needed in order to allow nucleosynthesis and structure formation, see [7, 27].

In particular, in our model, this means that we can prefer choices of the parameters f_2 and f_4 for which the ratio

$$\frac{\gamma_0(\Lambda)}{8\pi G_{\rm eff}(\Lambda)} \tag{4.53}$$

is small at the electroweak end of the energy range we are considering, so that the resulting early time bound will allow the standard theory of big-bang nucleosynthesis to take place according to the constraints of [7, 27].

Notice that, since we have $\gamma_{0,H}(\Lambda)/G_{{\rm eff},H}(\Lambda) = \gamma_0(\Lambda)/G_{\rm eff}(\Lambda)$ the estimate on (4.53) is independent of further effects of interaction with the Higgs field, and it only depends on the choice of the parameters f_2 and f_4 of the model.

This type of estimate can be used to select regions of the space of parameters f_2 and f_4 that are excluded by producing too large a value of (4.53) at $\Lambda = \Lambda_{\rm ew}$. For example, consider the two cases of figure 12, where we set $f_2 = 1$ and we choose f_4 so that it gives vanishing $\gamma_0(\Lambda)$ at $\Lambda_{\rm ew}$ or at $\Lambda_{\rm unif}$. The first case gives $\frac{\gamma_0(\Lambda_{\rm ew})}{8\pi G_{\rm eff}(\Lambda_{\rm ew})} = 0$, while the second gives a very large negative value of (4.53) at $\Lambda_{\rm ew}$ of -5.93668×10^{86} with a running of (4.53) as in figure 14.

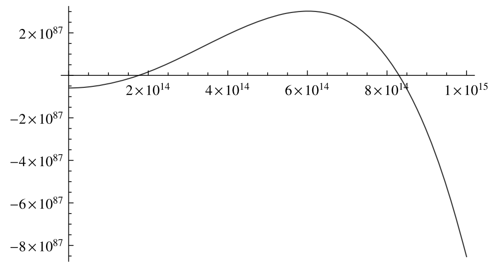

Figure 14: Running of $\gamma_0(\Lambda)/(8\pi G_{\rm eff}(\Lambda))$ with $f_2 = 1$ and with $\gamma_0(\Lambda_{\rm unif}) = 0$.

5 Dark matter

We have concentrated in this paper on early universe models, with an emphasis on various mechanisms for inflation and dark energy that arise within the NCG model, and their mutual interactions. Another main question about cosmological implications of the NCG models of particle physics is whether they can accommodate possible models for cold dark matter. This exits the domain of validity of the asymptotic expansion of the spectral action, as such models apply to a more modern universe than what is covered by the perturbative analysis of the spectral action functional. However, one can at least comment qualitatively on possible candidates within the NCG model for dark matter particles. In addition to the minimal standard model, one recovers form the computation of the asymptotic formula for the spectral action and from the fermionic part of the action in [17] additional right-handed neutrinos with lepton mixing matrix and Majorana mass terms. Thus, since this is at present the only additional particle content beyond the minimal standard model that can be accommodated in the NCG setting, it is natural to try to connect this model to existing dark matter models based on Majorana mass terms for right-handed neutrinos.

Those that seem more closely related to what one can get within the NCG model are the ones described by Shaposhnikov–Tkachev [45], Shaposhnikov [44], and Kusenko [33]. In these dark matter models, one has the usual active neutrinos, with very small masses, and an additional number of sterile neutrinos with Majorana masses. In the case of the νMSM model of [44], one has three active and three sterile neutrinos. In these models the sterile neutrinos provide candidate dark matter particles. However, for them to give rise to plausible dark matter models, one needs at least one (or more) of the sterile neutrino Majorana masses to be below the electroweak scale.

In the detailed discussion given in [33] one sees that, for example, one could have two of the three Majorana masses that remain very large, well above the electroweak scale, possibly close to unification scale, while a third one lowers below the electroweak scale, so that the very large Majorana masses still account for the see-saw mechanism, while the smaller one provides a candidate dark matter particle.

It is possible to obtain a scenario of this kind within the NCG model, provided that one modifies the boundary conditions of [1] in such a way that, instead of having three see-saw scales within the unification and the electroweak scale, with the smallest one already at very high energy around 10^{12} GeV, one sets things so that the lowest Majorana mass descends below the electroweak scale. A more detailed analysis of such models will be carried out in forthcoming work where we analyze different choices of boundary conditions for the RGE flow of the model.

6 Conclusions and perspectives

We have shown in this paper how various cosmological models arise naturally from the asymptotic expansion of the spectral action functional in the NCG model of particle physics of [17] and the running of the coefficients of this asymptotic expansion via the RGE of [1].

We have seen in particular the spontaneous emergence of conformal gravity and Hoyle–Narlikar cosmologies at phase transitions caused by the running of the effective gravitational constant. We described effects of this running on the gravitational waves and on primordial black holes (PBHs). We described mechanisms by which this running, combined with the conformal coupling of gravity to the Higgs field, can generate regions of negative gravity in the early universe. We discussed the running of the effective cosmological constant and slow-roll inflation models induced by the coupling of the Higgs to gravity. We discussed briefly the connection to dark matter models based on right-handed neutrinos and Majorana mass terms.

A planned continuation of this investigation will cover the following topics:

- Varying boundary conditions for the RGE flow.
- Dark matter models based on Majorana sterile neutrinos.
- Extensions of the NCG model with dilaton field.
- Nonperturbative effects in the spectral action.
- Exclusion curve from spectral index and tensor to scalar ratio.

Appendix A Boundary conditions for the RGE equations

We recall here the boundary conditions for the RGE flow of [1] and we discuss the compatibility with the condition assumed in [17]. A more detailed analysis of the RGE flow of [1] with different boundary conditions and its effect on the gravitational and cosmological terms will be the focus of a followup investigation.

A.1 The default boundary conditions

The boundary conditions for the RGE flow equations we used in this paper are the default boundary conditions assumed in [1]. These are as follows:

$$\lambda(\Lambda_{\rm unif}) = \tfrac{1}{2},$$
$$Y_u(\Lambda_{\rm unif}) = \begin{pmatrix} 5.40391\times 10^{-6} & 0 & 0 \\ 0 & 0.00156368 & 0 \\ 0 & 0 & 0.482902 \end{pmatrix}.$$

For $Y_d(\Lambda_{\rm unif}) = (y_{ij})$ they have

$$\begin{aligned}
y_{11} &= 0.0000482105 - 3.382\times 10^{-15}\mathrm{i}\\
y_{12} &= 0.000104035 + 2.55017\times 10^{-7}\mathrm{i}\\
y_{13} &= 0.0000556766 + 6.72508\times 10^{-6}\mathrm{i}\\
y_{21} &= 0.000104035 - 2.55017\times 10^{-7}\mathrm{i}\\
y_{22} &= 0.000509279 + 3.38205\times 10^{-15}\mathrm{i}\\
y_{23} &= 0.00066992 - 4.91159\times 10^{-8}\mathrm{i}\\
y_{31} &= 0.000048644 - 5.87562\times 10^{-6}\mathrm{i}\\
y_{32} &= 0.000585302 + 4.29122\times 10^{-8}\mathrm{i}\\
y_{33} &= 0.0159991 - 4.21364\times 10^{-20}\mathrm{i}
\end{aligned}$$

$$Y_e(\Lambda_{\rm unif}) = \begin{pmatrix} 2.83697\times 10^{-6} & 0 & 0 \\ 0 & 0.000598755 & 0 \\ 0 & 0 & 0.0101789 \end{pmatrix}$$

$$Y_{\rm nu}(\Lambda_{\rm unif}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.1Y \end{pmatrix}$$

$$M(\Lambda_{\rm unif}) = \begin{pmatrix} -6.01345\times 10^{14} & 3.17771\times 10^{12} & -6.35541\times 10^{11} \\ 3.17771\times 10^{12} & -1.16045\times 10^{14} & 5.99027\times 10^{12} \\ -6.35541\times 10^{11} & 5.99027\times 10^{12} & -4.6418\times 10^{12} \end{pmatrix}.$$

A.2 Constraints from NCG

There are constraints on the boundary conditions at unification in the NCG model. Those were described in [17] and we report them here below.

- A constraint on the value at unification of the parameter λ

$$\lambda(\Lambda_{\rm unif}) = \frac{\pi^2}{2f_0}\frac{\mathfrak{b}(\Lambda_{\rm unif})}{\mathfrak{a}(\Lambda_{\rm unif})^2}.$$

- A relation between the parameter $\mathfrak{a}$ and the Higgs vacuum

$$\frac{\sqrt{\mathfrak{a} f_0}}{\pi} = \frac{2M_{\rm W}}{g}.$$

- A constraint on the coefficient $\mathfrak{c}$ at unification, coming from the see-saw mechanism for the right-handed neutrinos

$$\frac{2f_2\Lambda_{\rm unif}^2}{f_0} \le \mathfrak{c}(\Lambda_{\rm unif}) \le \frac{6f_2\Lambda_{\rm unif}^2}{f_0}.$$

- The mass relation at unification

$$\sum_{\rm generations} (m_\nu^2 + m_{\rm e}^2 + 3m_u^2 + 3m_d^2)|_{\Lambda=\Lambda_{\rm unif}} = 8M_W^2|_{\Lambda=\Lambda_{\rm unif}}, \qquad (A.1)$$

  where m_ν, m_e, m_u, and m_d are the masses of the leptons and quarks, that is, the eigenvectors of the matrices $\delta_{\uparrow 1}$, $\delta_{\downarrow 1}$, $\delta_{\uparrow 3}$, and $\delta_{\downarrow 3}$, respectively, and $M_{\rm W}$ is the W-boson mass.

Clearly, not all of these constraints are compatible with the default boundary conditions of [1]. So either one relaxes some of these conditions, as we have been doing in the present paper, or one performs a wider search for more appropriate and fine tuned boundary conditions for the RGE flow, analyzing how different choices of boundary values affect the behavior analyzed in this paper. This is presently under investigation.

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References

[1] S. Antusch, J. Kersten, M. Lindner, M. Ratz and M. A. Schmidt, *Running neutrino mass parameters in see-saw scenarios*, J. High Energy Phys. **03**(024) (2005), `hep-ph/0501272v3`.

[2] S. Antusch, J. Kersten, M. Lindner and M. Ratz, *Neutrino mass matrix running for non-degenerate see-saw scales*, Phys. Lett. **B538** (2002), 87–95.

[3] K. S. Babu, C. N. Leung and J. Pantaleone, *Renormalization of the neutrino mass operator*, Phys. Lett. **B319** (1993), 191–198.

[4] J. Barrett, *Lorentzian version of the noncommutative geometry of the Standard Model of particle physics*, J. Math. Phys. **48**(1) (2007), 012303, 7.

[5] J. D. Barrow, *Gravitational memory?* Phys. Rev. **D46**(8) (1992) R3227, 4.

[6] A. A. Belyanin, V. V. Kocharovsky and Vl. V. Kocharovsky, *Gamma-ray bursts from evaporating primordial black holes*, Radiophys. Quantum Electronics, **41**(1) (1996), 22–27.

[7] M. Birkel and S. Sarkar, *Nucleosynthesis bounds on a time-varying cosmological 'constant'*, Astropart. Phys. **6** (1997), 197–203.

[8] I. L. Buchbinder, D. D. Odintsov and I. L. Shapiro, *Effective action in quantum gravity*, IOP Publishing, 1992.

[9] E. V. Bugaev, V. B. Petkov, A. N. Gaponenko, P. A. Klimai, M. V. Andreev, A. B. Chernyaev, I. M. Dzaparova, D. D. Dzhappuev, Zh. Sh. Guliev, N. S. Khaerdinov, N. F. Klimenko, A. U. Kudzhaev, A. V. Sergeev, V. I. Volchenko, G. V. Volchenko and A. F. Yanin, *Experimental search of bursts of gamma rays from primordial black holes using different evaporation models*, `arXiv:0906.3182`.

[10] B. Ćaćić, *Moduli spaces of Dirac operators for finite spectral triples*, in "Quantum groups and noncommutative spaces: perspectives on quantum geometry", eds. M. Marcolli and D. Parashar, Vieweg Verlag, Wiesbaden, 2010, pp. 9–68, `arXiv:0902.2068`.

[11] B. J. Carr, *Primordial black holes as a probe of the early universe and a varying gravitational constant*, `arXiv:astro-ph/0102390v2`.

[12] J. A. Casas, V. Di Clemente, A. Ibarra and M. Quirós, *Massive neutrinos and the Higgs mass window*, Phys. Rev. **D62** (2000), 053005, 4.

[13] A. Chamseddine, *Noncommutative geometry as the key to unlock the secrets of space-time*, "Quanta of Maths", Clay Math. Proc., **11**, American Mathematical Society, Providence, 2010, pp. 127–148, `arXiv:0901.0577`.

[14] A. Chamseddine and A. Connes, *The spectral action principle*, Comm. Math. Phys. **186**(3) (1997), 731–750.

[15] A. Chamseddine and A. Connes, *Why the standard model?* J. Geom. Phys. **58**(1) (2008), 38–47.

[16] A. Chamseddine and A. Connes, *The uncanny precision of the spectral action*, Comm. Math. Phys. **293**(3) (2010), 867–897, `arXiv:0812.0165`.

[17] A. Chamseddine, A. Connes and M. Marcolli, *Gravity and the standard model with neutrino mixing*, Adv. Theor. Math. Phys. **11**(6) (2007), 991–1089.

[18] A. Connes, *Geometry from the spectral point of view*, Lett. Math. Phys. **34**(3) (1995), 203–238.

[19] A. Connes, *Gravity coupled with matter and the foundation of non-commutative geometry*, Comm. Math. Phys. **182**(1) (1996), 155–176.

[20] A. Connes, *Noncommutative geometry and the standard model with neutrino mixing*, J. High Energy Phys. **081**(11) (2006), 19 (electronic).

[21] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, 1994.

[22] A. Connes and M. Marcolli, *Noncommutative geometry, quantum fields, and motives*, Colloquium Publications, **55**, Amer. Math. Soc., 2008.

[23] A. Connes and M. Marcolli, *A walk in the noncommutative garden*, in "An invitation to noncommutative geometry", pp. 1–128, World Scientific, Hackensack, 2008.

[24] A. De Simone, M. P. Hertzberg and F. Wilczek, *Running inflation in the Standard Model*, Phys. Lett. **B678** (2009), 1–8, `hep-ph/0812.4946v2`.

[25] A. D. Dolgov, *An attempt to get rid of the Cosmological constant*, in "The very early universe", eds. G. W. Gibbons, S. W. Hawking and S. T. S. Siklos, Cambridge University Press, Cambridge, 1983, p. 449

[26] A. Edery, L. Fabbri and M. B. Paranjape, *Spontaneous breaking of conformal invariance in theories of conformally coupled matter and Weyl gravity*, `hep-th/0603131`.

[27] K. Freese, F. C. Adams, J. A. Frieman and E. Mottola, *Cosmology with decaying vacuum energy*, Nuclear Phys. **B287** (1987), 797–814.

[28] R. Gregory, V. A. Rubakov and S. M. Sibiryakov, *Gravity and anti-gravity in a brane world with metastable gravitons*, Phys. Lett. **B489** (2000), 203–206.

[29] U. Guenther, A. Zhuk, V. Bezerra and C. Romero, *AdS and stabilized extra dimensions in multidimensional gravitational models with*

nonlinear scalar curvature terms $1/R$ *and* R^4, Class. Quant. Grav. **22** (2005), 3135–3167.

[30] A. H. Guth, *Phase transitions in the very early universe*, in "The very early universe", eds. G. W. Gibbons, S. W. Hawking and S. T. C. Siklos, Cambridge University Press, Cambridge, 1982, pp. 171–204.

[31] F. Hoyle and J. V. Narlikar, *A new theory of gravitation*, Proc. R. Soc. Lond. Ser. A Math. Phys. Sci., **282**(1389) (1964), 191–207.

[32] K. Jakobs and M. Schumacher, *Prospects for Higgs boson searches at the LHC*, in "Perspectives on LHC physics", eds. G. Kane and A. Pierce, World Scientific, Hackensack, 2008, pp. 179–204.

[33] A. Kusenko, *Sterile neutrinos: the dark side of the light fermions*, `arXiv:0906.2968`.

[34] A. R. Liddle and D. H. Lyth, *Cosmological inflation and large-scale structure*, Cambrige University Press, Cambridge, 2000.

[35] A. D. Linde, *Gauge theories, time-dependence of the gravitational constant and antigravity in the early universe*, Phys. Lett. **B93**(4) (1980), 394–396.

[36] D. Lohiya, *Non-topological solitons in non-minimally coupled scalar fields: theory and consequences*, Astrophys. Space Sci., **232**, (1995) 1–22.

[37] P. D. Mannheim, *Alternatives to dark matter and dark energy*, Progr. Part. Nucl. Phys. **56**(2) (2006), 340–445.

[38] W. Nelson and M. Sakellariadou, *Cosmology and the noncommutative geometry approach to the Standard Model*, Phys. Rev. **D81** (2010), 085038, `arXiv:0812.1657 [hep-th]`.

[39] I. D. Novikov, A. G. Polnarev, A. A. Starobinsky and Ya. B. Zeldovich, *Primordial black holes*, Astron. Astrophys. **80** (1979), 104–109.

[40] J. M. Overduin and F. I. Cooperstock, *Evolution of the scale factor with a variable comological term*, `astro-ph/9805260`.

[41] M. D. Pollock, *On the proposed existence of an anti-gravity regime in the early universe*, Phys. Lett. **B108** (1982), 386–388.

[42] M. D. Pollock, *Determination of the superstring moduli from the higher-derivative terms*, Phys. Lett. **B495** (2000), 401–406.

[43] M. V. Safonova and D. Lohiya, *Gravity balls in induced gravity models – 'gravitational lens' effects.* Gravit. Cosmol. (1) (1998), 1–10.

[44] M. Shaposhnikov, *The* ν*MSM, leptonic asymmetries, and properties of singlet fermions*, J. High Energy Phys. **08**(008) (2008), 56.

[45] M. Shaposhnikov and I. Tkachev, *The νMSM, inflation, and dark matter*, Phys. Lett. **B639** (2006), 414–417.

[46] Tevatron New Phenomena, Higgs working group, CDF Collaboration, D0 Collaboration, *Combined CDF and Dzero Upper Limits on Standard Model Higgs Boson Production at High Mass (155-200 GeV/c2) with 3 fb-1 of data*, `arXiv:0808.0534`.

[47] S. Weinberg, *Gravitation and Cosmology*, John Wiley and Sons, New York, 1972.

[48] A. Zee, *Broken-symmetry theory of gravity*, Phys. Rev. Lett. **42** (1979), 417–421.

[49] Ya. B. Zeldovich, *The cosmological constant and the theory of elementary particles*, Sov. Phys. Usp. **11** (1968), 381–393.