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Stability and decay rates for the five-dimensional Schwarzschild metric under biaxial perturbations

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Abstract

In this paper we prove the non-linear asymptotic stability of the five-dimensional Schwarzschild metric under biaxial vacuum perturbations. This is the statement that the evolution of $(SU(2) \times U(1))$-symmetric vacuum perturbations of initial data for the five-dimensional Schwarzschild metric finally converges in a suitable sense to a member of the Schwarzschild family. It constitutes the first result proving the existence of non-stationary vacuum black holes arising from asymptotically flat initial data dynamically approaching a stationary solution. In fact, we show quantitative rates of approach. The proof relies on vectorfield multiplier estimates, which are used in conjunction with a bootstrap argument to establish polynomial decay rates for the radiation on the perturbed spacetime. Despite being applied here in a five-dimensional context, the techniques are quite robust and may admit applications to various four-dimensional stability problems.

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1 Introduction

The existence of black holes features among the most fundamental predictions of general relativity. In the appropriate mathematical language of the theory, these objects correspond to solutions of the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.1}$$

possessing a regular event horizon and a complete null-infinity. General relativity admits an initial-value formulation suggesting that the appropriate setup to study black holes is in evolution from initial data. In this context, the main objective is to determine whether the maximal development associated to given data admits a complete null-infinity and a regular event horizon.

Some important special black hole solutions (hence their initial data) are known in closed form. They are static or stationary, with the well-known Schwarzschild and Kerr family of solutions among them, which are believed to play crucial roles as "final states" in gravitational collapse. It is fundamental for our understanding of the theory to investigate the stability of these explicit solutions, that is to say the global structure of the evolution arising from initial data close (in an appropriate sense) to that of the known reference solution. Due to the complexity of this non-linear problem, most rigorous studies have been focussed on special symmetry classes. Specifically, a paramount problem of black hole physics, the full non-linear stability of the Kerr-solution, remains open to date.

A model in which both the global spacetime structure associated to the evolution of general initial data and the stability of certain solutions in particular have been mathematically understood previously is that of the self-gravitating scalar field under spherical symmetry. The assumption of spherical symmetry casts the Einstein equations as a $(1+1)$-dimensional system of partial differential equations (PDEs), the inclusion of a massless scalar field being the simplest way to circumvent Birkhoff's theorem.[1] In the context of this model, Christodoulou [2] proved that generic initial data either disperse, i.e., asymptote to Minkowski space for late times, or collapse to regular black holes. His seminal work was extended by Dafermos and Rodnianski [6], who proved that the development of initial data collapsing to black holes in fact approaches a Schwarzschild-metric on the exterior of the black hole at a sufficiently fast polynomial rate. These decay rates [6] of

[1] Birkhoff's theorem implies that spherically symmetric vacuum solutions are either Minkowskian or Schwarzschildean.

the scalar field were first suggested on a heuristic level by Price [8], and are thought to be sharp. It is remarkable that [6] is a "large data" result. The initial data need not be assumed close to Schwarzschildean; all initial data containing a trapped surface are shown to approach a Schwarzschild metric.

1.1 The model

An alternative model allowing the study of gravitational collapse *in vacuo* was recently proposed by Bizon *et al.* [1]. To understand their idea we recall that, in view of the four-dimensional Birkhoff's theorem, gravitational collapse *in vacuo* ($T_{\mu\nu} = 0$ in (1.1)) cannot be studied under spherical symmetry. In axisymmetry on the other hand, the Einstein equations no longer reduce to a system of $(1+1)$-dimensional PDEs and the resulting problem does not seem tractable with current mathematical techniques. The way out of this dilemma suggested by Bizon *et al.* [1] is to study the Einstein vacuum equations under $SU(2)$-symmetry in *five* dimensions. This is motivated by the following observation: the analogue of spherical symmetry in four dimensions, i.e., an $SO(3)$ action on an orbital two-sphere, is clearly an $SO(4) \cong \left(SU(2)_L \times SU(2)_R\right)/\mathbb{Z}^2$ action on a three-sphere in five dimensions. However, via the latter isomorphism there exist subgroups of $SO(4)$, for instance $SU(2)_L$ and $\left(SU(2)_L \times U(1)_R\right)/\mathbb{Z}^2$ which still act transitively on the three-sphere.[2] Consequently, even within the class of the smaller symmetry groups (commonly called triaxial- or biaxial-Bianchi IX depending on the subgroup to which one restricts) the Einstein equations reduce to a system of $(1+1)$-dimensional PDEs. Moreover, Birkhoff's theorem is evaded by the introduction of one or two (in the triaxial case) dynamical degrees of freedom arising from the reduced symmetry.

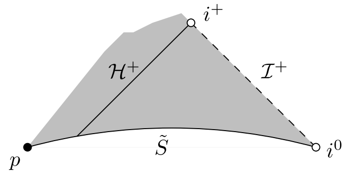
In the biaxial case this degree of freedom is manifest in a certain function B, which geometrically speaking corresponds to the "squashing" of the three sphere. B is normalized such that it is zero for the Schwarzschild–Tangherlini metric. From the point of view of the analysis it can be understood as the analogue of the massless scalar field in four dimensions. The Einstein equations (1.1) imply the following non-linear wave equation for the squashing field B:

$$\Box_g B = -\frac{4}{3r^2}\left(\mathrm{e}^{-8B} - \mathrm{e}^{-2B}\right). \tag{1.2}$$

[2]The subscripts L and R stand for the left and the right action, respectively.

In [1] the model outlined was investigated numerically, suggesting that small initial data will disperse, whereas large data will collapse to black holes, approaching some Schwarzschild–Tangherlini black hole for large times. The mathematical study of the model was initiated shortly thereafter by M. Dafermos in collaboration with the present author. In [5], the following statement[3] was proven:

Theorem. *Consider a triaxial-symmetric initial data set* (Σ, g, K)*, which is close in an appropriate norm*[4] *to an initial data set* (Σ, g_S, K_S) *evolving to the five-dimensional Schwarzschild–Tangherlini solution of mass* M*. Let the squashing fields* B_1*,* B_2 *which are identically zero for the five-dimensional Schwarzschild metric, be of compact support on the initial hypersurface. Let* $\mathcal{Q}$ *be the Lorentzian quotient of the future Cauchy development of the data. Then* $\mathcal{Q}$ *contains a subset with Penrose diagram:*



It particular, the quotient of the maximal development of the set (Σ, g, K) *admits a complete null-infinity with final Bondi mass* $\mathcal{M}_f$ *close to* M*, and a regular event horizon* $\mathcal{H}^+$ *on which the Penrose inequality* $r^2 \leq 2\mathcal{M}_f$ *holds. Here* r *is the area-radius function.*

The above theorem can be paraphrased as stating that perturbations of Schwarzschild–Tangherlini initial data again collapse to regular black holes close to the original Schwarzschild black hole. This result was termed *orbital stability* of the five-dimensional Schwarzschild metric in [5] and generated the first vacuum black hole solutions arising from asymptotically flat initial data that are not stationary.[5]

Crucial for the proof of the above theorem is the existence of good monotonicity properties for a function $m(u, v)$, called the Hawking mass, defined in (2.8). It converges to the ADM mass defined at the asymptotically flat end. It is shown to satisfy $\partial_u m \leq 0$ and $\partial_v m \geq 0$ on the domain of outer communications, leading to an a priori bound for the total mass fluctuation on the spacetime in terms of the initial data.

[3] Actually, it follows from a stronger statement proven in [5].

[4] See [5] for the precise definition.

[5] Solutions with a future complete, but not past complete, $\mathcal{I}^+$ have been constructed previously by Chruściel [4], by solving a certain parabolic problem.

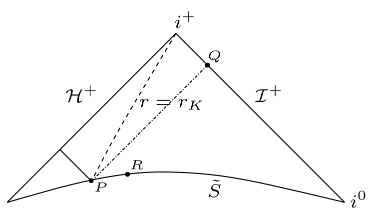
1.2 The main theorem

Orbital stability provides, of course, certain control over the global structure of the solution. Nevertheless, it leaves the details of the late-time behaviour unclear. In particular, solutions could exhibit unexpected features at late times with the squashing field B oscillating in some complicated manner and the geometry thus never settling down. This problem is finally addressed in the present paper. By proving appropriate decay rates we will show that the squashing field does decay for late times and hence that perturbations converge to another member of the Schwarzschild–Tangherlini family.

1.2.1 The statement

The main result is

Theorem 1.1. *Consider a biaxial-symmetric initial data set* (Σ, g, K), *which is close in the sense of the previous theorem to an initial data set* (Σ, g_S, K_S) *whose maximum development is the five-dimensional Schwarzschild–Tangherlini solution of mass* M. *Let* $\pi : \mathcal{M} \rightarrow \mathcal{Q}$ *denote the projection map of the maximal development of* (Σ, g, K) *to the two-dimensional Lorentzian quotient space* $\mathcal{Q}$ *and let* $\tilde{S} = \pi(\Sigma)$. *Fix a curve of constant area radius,* $r = r_K$, *away from the horizon, intersecting* $\tilde{S}$ *at* P *as depicted below.*



Assume furthermore that the initial data slice $\tilde{S}$ *coincides for* $r \geq r_K$ *with an integral curve of the globally defined vectorfield* ∇r *on* $\mathcal{Q}$ *and that the data is Schwarzschildean outside a compact set, i.e., that the squashing field* B *is of compact support.*

Define regular coordinates (u, v) *on the subset* $J^+(\tilde{S} \cap \{r \geq r_K\}) \cap J^-(\mathcal{I}^+)$ *of the Penrose diagram arising by the previous Theorem as follows. Let the point* R, *determined by the intersection of the curve* $r^2 = 4m$ *with* $\tilde{S}$, *have coordinates* $u = v = \sqrt{M}$. *Set* $r_{,v} = \frac{1}{2}(1-\mu)$, *with* $\mu = \frac{2m}{r^2}$, *along the null-ray* $\overline{PQ}$ *and* $r_{,u} = -\frac{1}{2}$ *along null-infinity. In these coordinates* $u \rightarrow \infty$ *along null-infinity as* i^+ *is approached. The horizon* $\mathcal{H}^+$ *is parametrized as* (∞, v). *Define* $t = \frac{v+u}{2}$ *and* $r^\star = \frac{v-u}{2}$.

Then there exists a dimensionless constant $\delta > 0$, depending only on the geometry of $\tilde{S}$ such that if the field B satisfies

$$M^{-\frac{3}{4}}\left[r^{\frac{3}{2}}|B| + r^{\frac{5}{2}}\left|\frac{B_{,u}}{r_{,u}}\right| + r^{\frac{5}{2}}\left|\frac{B_{,v}}{r_{,v}}\right|\right] \leq \delta \tag{1.3}$$

on $\tilde{S} \cap \{r \geq r_K\}$ and

$$\frac{1}{M}\int_{\tilde{S}\cap\{r\geq r_K\}}\left[u^2(\partial_u B)^2 + v^2(\partial_v B)^2 + (u^2+v^2)(-r_{,u})B^2\right]\frac{1}{\Omega}\,\mathrm{dvol}_3 \leq \delta^2, \tag{1.4}$$

as well as

$$M^{-\frac{3}{4}}\left[r^{\frac{3}{2}}|B| + r^{\frac{5}{2}}\left|\frac{B_{,u}}{r_{,u}}\right|\right] \leq \delta \tag{1.5}$$

on the ray $v = v\,(P) \cap \{r \leq r_K\}$, then the squashing function B satisfies

$$|B| + \sqrt{M}|B_{,v}| + \sqrt{M}\left|\frac{B_{,u}}{r_{,u}}\right| \leq \frac{C\sqrt{M}}{v_+} \qquad \textit{for } r \leq r_K, \tag{1.6}$$

where $v_+ = \max\,(1, v)$,

$$|B| + \sqrt{M}|B_{,v}| + \sqrt{M}\left|\frac{B_{,u}}{r_{,u}}\right| \leq \frac{C\sqrt{M}}{t} \qquad \textit{for } r \geq r_K, \tag{1.7}$$

$$|B| \leq \frac{C\ M^{\frac{3}{4}}}{r^{\frac{3}{2}}} \qquad \textit{for } r \geq r_K \tag{1.8}$$

on $\mathcal{D} = J^+\left(\tilde{S} \cap \{r \geq r_K\}\right) \cap J^-\left(\mathcal{I}^+\right)$ for a dimensionless constant C (which depends on the choice of r_K) computable from the initial data.

We will refer to this result as the *asymptotic stability* of the Schwarzschild–Tangherlini solution. In particular, Theorem 1.1 produces the first dynamical vacuum solutions arising from asymptotically flat initial data and converging to stationary black holes for late times.

1.2.2 Remarks

Restricting $\tilde{S}$ to coincide with a ∇r integral curve for $r \geq r_K$ is justified by Cauchy stability and the fact that the global properties of the Penrose diagram are already known by the orbital stability result of the previous theorem. It has been assumed to avoid some clumsy notation in the proof.

Cauchy stability also justifies stating the smallness assumptions (1.3)–(1.5) on the slice

$$\tilde{S}_{r_K} = \left(\tilde{S} \cap \{r \geq r_K\}\right) \cup \left(\{v = v\,(P)\} \cap \{r \leq r_K\}\right) \tag{1.9}$$

instead of $\tilde{S}$.[6] The advantage of doing it this way is that (1.3) and (1.5) do not depend on the choice of double-null coordinates on the Penrose diagram.[7] Assumption (1.4) on the contrary depends on the choice of coordinates. However, since both the u and the v coordinate are easily shown to be finite in the region where B is supported on $\tilde{S} \cap \{r \geq r_K\}$ in the given coordinate system, assumption (1.4) is automatically satisfied if we choose the δ in (1.3) small enough since B and is assumed to be of *compact* support initially. Hence it could be dropped by making δ even smaller. We have nevertheless included (1.4) for conceptual reasons which will become apparent later in the proof.[8] Condition (1.4) would also be required if one eventually drops the assumption of compact support, for then (1.4) imposes conditions on the decay of the fields near infinity.

Factors of $\sqrt{M}$ have been inserted in all formulae to make constants dimensionless.

1.3 Summary of the proof

Before we embark upon an outline and a discussion of the proof, it is perhaps illuminating to compare and contrast the situation with the proof of Price's law [6] for a self-gravitating spherically symmetric scalar field in $3+1$ dimensions. It turns out that the techniques developed in the latter paper to derive decay rates do not generalize to the system under consideration. The underlying reason can be traced back to two crucial estimates applied in [6]. The first of these, which allows one to extract decay directly from the horizon, relies heavily on the homogeneity of the non-linear wave equation satisfied by the field ϕ in the scalar field model. The second estimate is made possible by the existence of an almost Riemann invariant, a quantity admitting better decay properties than the scalar field ϕ itself, which can be exploited to derive uniform decay of the energy in the area radius r. This

[6] The smallness assumption (1.5) easily translates into an appropriate smallness assumption on $\tilde{S}$, depending on the geometry of $\tilde{S}$ for $r \leq r_K$, after extending the coordinate system to all of $J^+(\tilde{S}) \cap J^-\left(\mathcal{I}^+\right)$.

[7] This will become useful later because the bootstrap argument applied in the proof requires the definition of different coordinate systems.

[8] Quantity (1.4) is related to a boundary term in the vectorfield multiplier estimate associated with the vectorfield K.

decay played an important role in conjunction with the pigeonhole principle completing the argument in [6].

In the five-dimensional case there is no almost Riemann invariant and hence no apparent analogue to obtain decay in r for the energy in the asymptotic region. Moreover, the wave equation (1.2) satisfied by the dynamical field B has an inhomogeneous part, which in particular appears in the red-shift estimate. These obstacles necessitate a very different approach to proving decay. The path we choose here is based on exploiting energy currents arising from vectorfield multipliers. This method was already central in the proof of the non-linear stability of Minkowski space [3] and has recently been applied at the linear level in the black hole context for the first time [7]. In the latter paper, decay rates for a scalar field satisfying the homogeneous linear wave equation on a four-dimensional Schwarzschild spacetime are proven.[9] Key to establishing decay, at least away from the horizon, is the application of a so-called Morawetz vectorfield. A careful analysis reveals that the decay rates can be generalized to the linear problem associated with the non-linear problem studied here, namely the analysis of the linearized version of the wave equation (1.2) on a fixed Schwarzschild–Tangherlini background. What is more, the method of compatible currents being very geometric and robust in nature in fact carries over to the non-linear problem suggesting that the decay rates (1.6)–(1.8) may be established for the non-linear problem as well. However, in contrast to the linear case several non-linear error terms now enter the various estimates, which cannot be controlled a priori. This requires the introduction of a bootstrap argument to be applied in conjunction with the estimates obtained from the method of compatible currents.

It is noteworthy that the paper provides the first application of compatible currents techniques in a (non-linear) black hole context. The argument presented here is generally more robust than that of [6] but is of course restricted to small data. More precisely, the method presented is expected to be appropriate to eventually address non-linear problems without symmetry, most famously the non-linear stability of the Kerr-solution. In particular, since the technique is not bound to any dimension one should be able to reprove a version of "Price's law" [6] for small initial data along the lines of the present paper.

[9] Clearly, this is the associated linear problem to the model of the self-gravitating scalar field. Most notably, it can be treated without any symmetry assumptions on the scalar field, cf. [7].

1.3.1 Compatible currents

The basic idea behind the exploitation of energy currents based on vectorfield multipliers is quite simple. We construct a Lagrangian whose field equation generates the non-linear wave equation (1.2) satisfied by the squashing field B. The canonical energy momentum tensor $T_{\mu\nu}$ can be contracted with a vectorfield V^μ to produce a one-form $P_\nu = T_{\mu\nu}V^\mu$. Finally, Stokes' theorem relates the spacetime (or "bulk") integral of the divergence $\nabla^\nu P_\nu$ over a certain region to integrals along its boundary. This leads to the identity

$$\int_{\partial\mathcal{D}} P^\mu n_\mu = \int_{\mathcal{D}} \nabla_\mu P^\mu = \int_{\mathcal{D}} \left[T_{\mu\nu}\pi^{\mu\nu} + V^\mu \nabla^\nu T_{\mu\nu}\right], \tag{1.10}$$

where $\pi^{\mu\nu} = \frac{1}{2}\left(\nabla^\mu V^\nu + \nabla^\nu V^\mu\right)$ is the deformation tensor of the vectorfield V. One possible application of (1.10) is to estimate a future boundary integral from the past boundary and the spacetime term. On the other hand, for some vector fields we will estimate a bulk term from the boundary terms. The power of the method arises from an interplay between the identities associated with different vectorfields adapted to the geometry of particular regions. It is crucial that due to the Lagrangian structure both the boundary and the bulk term of (1.10) only depend on the 1-jet of B. Suitably applied, the method ultimately produces weighted L^2-bounds on the fields from which pointwise bounds on the fields follow in the standard manner.

1.3.2 The bootstrap

Before any bootstrap assumptions can be specified, coordinates have to be defined on the Penrose diagram. This turns out to be a rather subtle issue, intimately related to the bootstrap argument itself. The crucial observation is that the coordinates have to be normalized *to the future* of the bootstrap region, in order to capture the decay for late times in the estimates.[10] This is realized as follows. Consider the integral curves of the vectorfield ∇r, foliating the black hole exterior.[11] Each of these curves also intersects the curve of fixed area radius $r = 2\sqrt{M_f}$ (with M_f being the final Bondi mass the latter is comfortably away from the horizon). Hence we can associate a *geometric time* to any ∇r integral curve by using the affine parameter along the curve $r = 2\sqrt{M_f}$. Now for each such "time" $\tilde{\tau}$ on the curve, we construct a coordinate system $\mathcal{C}_{\tilde{\tau}}$ (depending on $\tilde{\tau}$!) on the black hole exterior by the following procedure. We find the point A on the ∇r curve associated to

[10] This is reminiscient of the situation in Christodoulou–Klainerman's proof of the stability of Minkowski space [3].

[11] Note that for convenience, we have assumed in Theorem 1.1 that the initial data are also defined on such a curve, at least up to its intersection with a curve $r = r_K$.

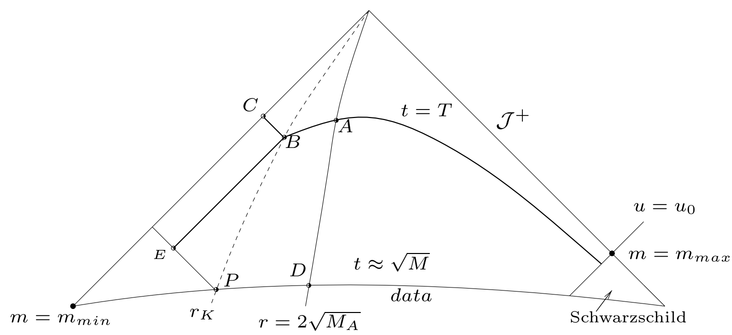


Figure 1: The choice of coordinates.

$\tilde{\tau}$, where $r^2 = 4m$. The $r = \text{const}$ curve through A will intersect the data at some point D. The affine length from A to D along that curve defines the *coordinate time* at A.[12] The actual coordinate system (u, v) is finally defined by imposing that $t = \frac{u+v}{2} = T$ holds on the integral curve of the vectorfield ∇r starting at the point A, at least up to the point B where the integral curve intersects a certain constant r_K-curve, fixed once and for all, which is chosen to lie close to the horizon. Moreover, we set $r^\star = \frac{v-u}{2} = 0$ at the point A and $r_{,v} = \frac{1}{2}(1-\mu)$ on $\overline{BE}$. There is some choice to complete the coordinate system by specifying $r_{,u}$ on $\overline{BC}$. For most practical calculations we will use Eddington–Finkelstein-type coordinates, setting $\nu = -\frac{1}{2}(1-\mu)$ on $\overline{BC}$. In any case, the bounds proven will be manifestly independent on the choice of u coordinate on $\overline{BC}$ (figure 1).

An important issue immediately arising from the way we define the coordinates is that the notion of a constant t slice differs in the different coordinate systems depending on the choice of $\tilde{\tau}$. (Of course, the analogous statement holds for the notion of timelike surfaces of constant $r^\star$.) Nevertheless, we will show that the coordinate systems remain uniformly close to each other in a suitable sense, in particular that the $t_{\tilde{\tau}}$ coordinate of the initial data slice between r_K and the support radius is always close to $\sqrt{M}$, however large we choose $\tilde{\tau}$. A detailed analysis is given in Section 8.3.

Every $\tilde{\tau}$ defines a T, which in turn defines a region $\mathcal{A}(T)$ depicted in figure 5. It is the region, enclosed by the $t = T$-curve up to some point B' with coordinates $(T, r_K^\star)$, the null-line $v = T + r_K^\star$ linking B' with the horizon, a horizon piece, the null-line $v = 2\sqrt{M} + r_K^\star$, the $t = 2\sqrt{M}$ piece and the $u = u_0$ null-line on which the field B is identically zero by the

[12]See definition (3.2). We add a factor of $\sqrt{M}$ in order to avoid dividing by zero when we state decay in t.

assumption of compact support.[13] Here $r^\star_K = \sup_{t<T} r^\star(t, r_K)$. Another curve, $r^\star = r^\star_{\text{cl}}$, located to the right of $r^\star = r^\star_K$ will also be introduced and fixed. We now choose a small constant c and define the bootstrap region to be the region associated to the largest time $\tilde{\tau}_B$, such that for any $\sqrt{M} \le \tilde{\tau} \le \tilde{\tau}_B$ the following "statement $\mathcal{P}$" holds in the associated region $\mathcal{A}(T(\tilde{\tau}))$ in the coordinate system $C_{\tilde{\tau}}$:

1. In the subregion $\{r^\star \ge r^\star_K\} \cap \mathcal{A}(T)$, the area radius satisfies

$$\left| r^\star - \left[r(t, r^\star) + \sqrt{\frac{M_A}{2}} \left(\log \left(\frac{r(t, r^\star) - \sqrt{2M_A}}{r(t, r^\star) + \sqrt{2M_A}} \right) + p \right) \right] \right| < c\sqrt{M} \quad (1.11)$$

   with

$$p = -2\sqrt{2} - \log \frac{2 - \sqrt{2}}{2 + \sqrt{2}} \quad (1.12)$$

   and M_A defined to be the Hawking mass at the point $(T, r^\star = 0)$.[14]
2. We have[15]

$$\tfrac{1}{2}\sqrt{M} < \sup_{\tilde{S} \cap \{r^\star \ge r^\star_K\} \cap \{u \ge u_0\}} t < \tfrac{3}{2}\sqrt{M}. \quad (1.13)$$

3. The weighted energy E^K_B defined in (7.3) satisfies $E^K_B\left(\tilde{T}\right) < cM$ on all arcs $\{t = \tilde{T} < T\} \cap \{r^\star \ge r^\star_K\} \subset \mathcal{A}(T)$.
4. The energy-flux satisfies $m(u_{\text{hoz}}, v_2) - m(u_{\text{hoz}}, v_1) < \frac{cM^2}{(v_{1+})^2}$ for any $v_1 \le v_2$ along the part of the horizon located in $\mathcal{A}(T)$, where $v_{i+} = \max(1, v_i)$.
5.

$$m\left(u_{r^\star_{\text{cl}}}, v\right) - m(u_{\text{hoz}}, v) < \frac{cM^2}{v_+^2} \quad (1.14)$$

   holds in $\mathcal{A}(T)$. Here $v_+ = \max(1, v)$.

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[13] Note that this null-line has a geometric significance by the assumption of compact support. The exact value of u_0 will depend on the coordinate system chosen.

[14] The reader should note that in Schwarzschild with $M = M_A$ the left-hand side of (1.11) is identically zero. The coordinate $r^\star$ is then the so-called Regge–Wheeler tortoise coordinate.

[15] This assumption states, in particular, that the initial data slice is both near and to the past of the bootstrap region. It ensures that the bootstrap region does not move away from the data.

6. The integral bound

$$\tilde{F}_B^Y = \int r^3 \frac{(B_{,u})^2}{\Omega^2} du < \frac{C_L M^2}{v_+^2} \quad \text{for} \quad C_L = \sup_{r^\star \geq r^\star_{\mathrm{cl}}} \frac{1}{1-\mu} \tag{1.15}$$

   holds along lines of constant v in the region $\{r^\star \leq r^\star_{\mathrm{cl}}\} \cap \{u \leq T - r^\star(T, r_K)\} \cap \mathcal{A}(T)$, corresponding to a decay of energy as measured by local observers near the horizon.[16]

We define the set

$$A = \left\{\tilde{\tau} \in \left[\sqrt{M}, \infty\right) \;\mid \mathcal{P}_{T(\hat{\tau})} \;\text{ holds in } \mathcal{A}(T(\hat{\tau})) \text{ for all } \hat{\tau} \leq \tilde{\tau}\right\} \subset \left[\sqrt{M}, \infty\right) \tag{1.16}$$

which will be shown to be open, closed and non-empty. This implies that the statement $\mathcal{P}$ holds on the entirety of the black hole exterior. The decay rates of Theorem 1.1 follow immediately after proving that the coordinate systems used in the bootstrap converge to one which is close to the one asserted by Theorem 1.1.

The openness of the set A follows from a straightforward continuity argument. The difficult part in closing the bootstrap therefore is to "improve" the statement $\mathcal{P}$ on the closure of the set $\mathcal{A}(T)$.

1.3.3 Closing the bootstrap

The third bootstrap assumption is shown to imply $\frac{1}{(t_i)^2}$ decay of the energy flux on the arcs $\{t = t_i\} \cap \{r^\star \geq r^\star_K\} \cap \{u \geq \frac{1}{11} t_i\}$, from which pointwise bounds on the field B and its v-derivative are obtained. Additionally, strong decay of B in the area radius r can be extracted from the boundedness of E_B^K. The assumptions also provide sufficient control over the coordinate functions at late times. In particular, one determines the relation between the area radius $r(u, v)$ and the coordinate $r^\star = \frac{v-u}{2}$, at least in the region where $r^\star \geq r^\star_K$. For late times this relation converges to the well-known formula expressing the area radius r in terms of the tortoise coordinate $r^\star$ of the five-dimensional Schwarzschild metric as captured by bootstrap assumption 1.3.2. It follows in particular that the value of r does not change much (the corrections are shown to be of order $\frac{1}{t}$) along a $r^\star = $ const-curve in the region $r^\star \geq r^\star_K$, allowing us to go back and forth between the two in the course of the paper. Moreover, bootstrap assumption 1.3.2 is improved.

[16] That is to say the quantity $\tilde{F}_B^Y$ measures exactly the energy which is not seen by the Hawking energy at the horizon.

Various constant r- and constant $r^\star$-curves in the region $r \geq r_K$ will play a crucial role, since certain integrands arising from the method of compatible currents admit good signs in appropriate regions.[17] The $r^\star = r^\star_{\mathrm{cl}}$-curve, occurring in the bootstrap for instance (along which $r \approx r_{\mathrm{cl}}$ by the previous remarks), is determined by various requirements defined later but is in any case located to the right of the aforementioned $r = r_K$. The latter curve on the other hand, can and will be chosen close to the horizon providing a source of smallness in the bootstrap argument. A second source of smallness arises from Cauchy stability: After picking some r_K we can choose a very late time t_0 up to which the fields are still small and after which terms like $\frac{C(r_K)}{t}$, with $C\left(r_K\right)$ a constant depending on the choice of r_K, are small.

We now turn to various energy currents arising from vectorfield multipliers and describe how the bootstrap is closed. The remarkable properties admitted by the Hawking mass for the system under consideration manifest themselves in identity (1.10) for the vectorfield

$$T = \frac{4r_{,v}}{\Omega^2}\partial_u - \frac{4r_{,u}}{\Omega^2}\partial_v. \tag{1.17}$$

The spacetime-term associated to the T-energy identity vanishes and one obtains a relation between boundary terms, which are precisely the associated energy fluxes. The monotonicity of the Hawking mass equips all boundary terms with signs when applied in the region[18] (cf. figure 4)

$$\begin{aligned} {}^{u_H}\mathcal{D}^{r^\star_{\mathrm{cl}},u_J}_{[t_1,t_2]} := &\left(\{t_1 \leq t \leq t_2\} \cap \{r^\star \geq r^\star_{\mathrm{cl}}\} \cap \{u \geq u_J\}\right) \\ &\cup \left\{\{t_1 + r^\star_{\mathrm{cl}} \leq v \leq t_2 + r^\star_{\mathrm{cl}}\} \cap \{r^\star \leq r^\star_{\mathrm{cl}}\} \cap \{u \leq u_H\}\right\}. \end{aligned} \tag{1.18}$$

Such regions arise from a dyadic decomposition of the bootstrap region between t_0 and T with $t_{i+1} = 1.1 t_i$ playing a crucial role later in the argument.

It can be shown that the boundary terms associated to the vectorfield

$$X = f\left(r^\star\right)\left(\partial_u - \partial_v\right), \tag{1.19}$$

for some carefully chosen bounded function f, are controlled by the energy flux (i.e., the T boundary terms) and the integral bound (1.15) when applied

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[17] By bootstrap assumption 1.3.2 constant r and constant $r^\star$-curves are close to one another in that region.

[18] From the vectorfield point of view this follows from the fact that T is timelike, that the normal to the region is non-spacelike and the positivity properties of $T_{\mu\nu}$. Cf. (1.10).

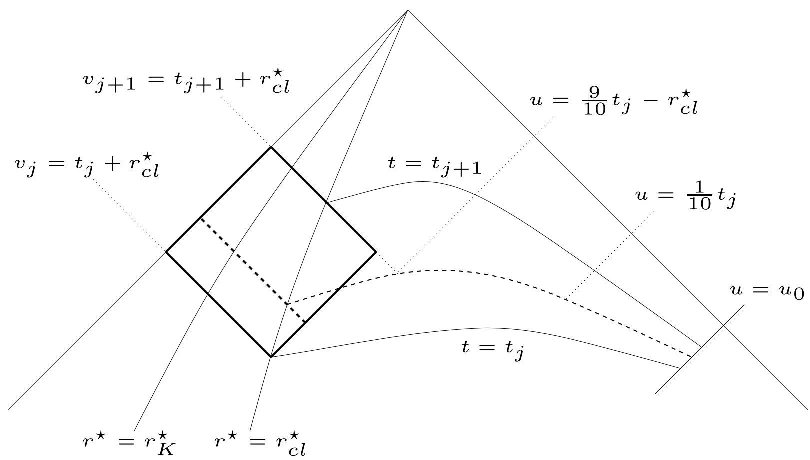


Figure 2: Closing the bootstrap.

in region (1.18). The function f is in turn chosen such that the spacetime-term of X admits a positive sign. In conjunction with the bootstrap assumptions this results in a $\frac{1}{(t_i)^2}$-decay bound for a positive spacetime integral in the dyadic region ${}^{t_{i+1}-r^\star_{\text{cl}}}\mathcal{D}^{r^\star_{\text{cl}},\frac{1}{10}t_i}_{[t_i,t_{i+1}]}$, which will prove useful in controlling the spacetime integrals of other vectorfields.

Close to the horizon, in a characteristic rectangle $[u_1 = t_1 - r^\star_{\text{cl}}, u_2 = u_{\text{hoz}}] \times [v_1 = t_1 + r^\star_{\text{cl}}, v_2 = t_2 + r^\star_{\text{cl}}]$ associated to the dyadic region ${}^{u_{\text{hoz}}}\mathcal{D}^{r^\star_{\text{cl}},u_J}_{[t_1,t_2]}$, we will apply the vectorfield

$$Y = \frac{\alpha\,(r^\star)}{\Omega^2}\partial_u + \beta\,(r^\star)\,\partial_v \tag{1.20}$$

for appropriately chosen functions α and β (cf. the bold rectangle in figure 2). The strategy is to control the future-null boundary integrals from the past boundary- and the associated spacetime term.[19] The integrand of the latter contains a part admitting a good sign, which can be used in combination with the spacetime term of X to control the remaining spacetime term of Y. Moreover, one ingoing boundary term being located completely in the region $r^\star \geq r^\star_{\text{cl}}$, is always controlled by the energy flux and hence decays like $\frac{1}{t^2}$. Applying the identity in the characteristic rectangle with the bottom being $v_0 = t_0 + r^\star_{\text{cl}}$, where an appropriate smallness assumption holds by Cauchy stability, and the top being $v = \tilde{v}$ for any $v_0 \leq \tilde{v} \leq T + r^\star_K$ immediately yields uniform boundedness for both the boundary terms and the good

[19]Physically, the boundary terms of the Y vectorfield correspond to the energy flux as measured by a local observer near the horizon.

spacetime term of Y. The argument can be improved by a pigeonhole principle applied in every characteristic rectangle. Namely, one extracts from the good spacetime term of Y a "good F_Y-slice", i.e., a slice on which the local Y-energy density decays like $\frac{1}{v_i}$ times the good spacetime term plus a contribution from the energy in the region $r^\star \geq r^\star_{\text{cl}}$. This is depicted as the dotted line in figure 2 above.[20] Applying the vectorfield identity for Y again in a region with the good slice as its past-boundary, one exports the $\frac{1}{v_i}$-decay to all dyadic rectangles. Iterating the procedure one obtains $\frac{C}{(v_i)^2}$ decay for all boundary terms and the good spacetime term of Y. The decay of the Y boundary terms leads to the pointwise bound $|r^{\frac{3}{2}}\frac{B_{,u}}{r_{,u}}| \leq \frac{C}{v}$ in the region $r^\star \leq r^\star_{\text{cl}}$, which can be exported to the region $r^\star_{\text{cl}} \leq r^\star \leq \frac{9}{10}t$ using the energy estimate and the decay in the central region.

With the pointwise bound on $r^{\frac{3}{2}}\frac{B_{,u}}{r_{,u}}$ at our disposal, we can finally make use of the Morawetz vectorfield

$$K = \frac{(u+a)^2}{M}\partial_u + \frac{(v-a)^2}{M}\partial_v \tag{1.21}$$

for a constant a.[21] As mentioned previously, its application is necessitated by the lack of an almost Riemann invariant and it proves crucial in the derivation of decay rates away from the horizon. The vectorfield identity for the region ${}^{u_H=\tilde{T}-r^\star_K}\mathcal{D}^{r^\star_K,u_0}_{[t_0,\tilde{T}]}$ associated to any $\tilde{T} \leq T$ and some large t_0 relates a future boundary term to a past boundary term, a horizon term and their associated spacetime term.

The boundary terms on the $\tilde{T}$-arc contain "good"-terms which are precisely the strongly weighted energies $E^K_B\left(\tilde{T}\right)$ of the second bootstrap assumption and error terms. The vectorfield identity is now exploited so as to estimate this "good" term on the future arc in terms of *all* other terms entering the identity. These latter quantities are in turn shown to be small or of good sign, which will finally improve assumption 1.3.2. To derive the smallness for the various terms, it will be necessary to subdivide the domain of integration and to apply different estimates in each region, carefully taking the geometry of the black hole into account.[22] It should be emphasized that these estimates belong to the most subtle ones in the paper. They make crucial use of the monotonicity manifest in the Raychaudhuri equations (2.2) and (2.3), and exploit an exponential decay associated with the redshift very

[20]Alternatively, one can extract a "good F_T-slice" on which the T-energy flux is improved. This will come in handy later.

[21]This suitably chosen constant defines the origin of the vectorfield.

[22]It is here where the pointwise bound on $r^{\frac{3}{2}}\frac{B_{,u}}{r_{,u}}$ established earlier enters.

close to the horizon by introducing an intermediate region between $r^\star = r^\star_K$ and the horizon.

For the boundary terms, there are two sources from which the smallness is finally obtained: One is the choice of the curve $r = r_K$, which can be chosen very close to the horizon. The other stems from the choice of a late time t_0 up to which the initial data has only changed by an amount as small as we may wish by Cauchy stability and after which the good decay estimates, i.e., the weight of $\frac{1}{t_0}$ carries over.

To establish smallness for the spacetime term appearing in the K-vector-identity, on the other hand, a further argument is needed. This term consists of a "main" term, which is the one that appears in the linear case, and error terms. The error terms can be dealt with very analogously to the treatment of the error–boundary terms. The main term is shown to admit a good sign for $r^\star \leq r^\star_{\text{cl}}$ and for some $r^\star \geq R^\star$ for some $R^\star$. The remaining piece in the central region is divided into dyadic regions, $t_{j+1} = 1.1 t_j$. Each K-integral of such a dyadic region can be controlled by t_{j+1} times the spacetime integral of the vectorfield X in that region. Since the X-bulk term decays like $\frac{1}{(t_{j+1})^2}$ as outlined above, summing up the dyadic regions yields smallness for the main K-spacetime term (arising from the large time t_0, where we start the dyadic decomposition). This improves bootstrap assumption 1.3.2.

With the third bootstrap assumption being improved on all arcs $\tilde{T} \leq T$ it follows that the decay of the energy has been improved on all arcs.[23] As a corollary, the same decay is obtained through any achronal hypersurface lying completely in the region $r^\star_K \leq r^\star \leq \frac{9}{10} t$. In the final step we find in each dyadic rectangle a "good F_T-slice" on which the energy flux is improved to $\frac{\epsilon}{(v_i)^2}$, very analogous to finding a "good F_Y-slice" as described above. Combining it with the improved decay on the associated arc (cf. the dotted slice in figure 2), the domain of dependence property improves the bootstrap assumptions 1.3.2 and 1.3.2. Additionally, we can finally find a good F_Y-slice in each characteristic rectangle (improving assumption 1.3.2 on that slice), which in conjunction with the energy decay now being improved to $\frac{\epsilon}{v^2}$ everywhere in $r^\star \leq \frac{9}{10} t$, can be exported to all v-slices. Hence assumption 1.3.2 is also retrieved with a better constant. This completes the proof that the set A is indeed closed and the main theorem follows in view of the previous remarks.

It should be noted that the decay rate that can be extracted in this argument is limited by the weights appearing in the K vectorfield, i.e., by the

[23] This is a consequence of the previously mentioned fact that the expression for E^K_B contains strong weights from which the decay can be extracted.

decay in the central region.[24] In particular, we cannot derive the stronger decay $\frac{1}{v^{3-\epsilon}}$ near the horizon obtained in [6] for the massless scalar field. It is an interesting question whether other methods can improve the decay rates proven in this paper.

1.4 Outline of the paper

We start by introducing the biaxial Bianchi IX model and some notation (Section 2) before defining the aforementioned future-normalized coordinate system $C_{\tilde{\tau}}$ in Section 3. Various a priori bounds, which can be obtained without invoking the main bootstrap argument and turn out to be helpful at many stages of the paper are derived in Section 4. An important point to keep in mind, however, is that the decay of the energy in the area radius *cannot* be obtained by these methods due to the lack of an almost Riemann invariant for the model under consideration. The method of compatible currents is explained in more detail in Section 5, where moreover the relevant identities associated with the regions considered later are derived. In particular, the Hawking mass is recovered as a potential of a certain vectorfield-current (Section 6). After defining the bootstrap assumptions (Section 7), various bounds for the fields are derived from them and the stability of the coordinate systems $\mathcal{C}_{\tilde{\tau}}$ defined in Section 3 is established (Section 8). The identities associated to the vectorfields Y and X are analysed in Sections 9 and 10. Here a somewhat lengthy argument is pursued to construct the function f implicit in the vectorfield X, which finally ensures that its spacetime term admits a positive sign. Section 11 reveals how to control the weighted energies produced by Y near the horizon with the help of the vectorfield X. The relevant version of the pigeonhole principle is also explained at this stage. Finally, in Section 12 the Morawetz vectorfield K is introduced and the necessary estimates to control the various error integrals, as outlined in the introduction, are performed. Everything is put together in Section 13, where the bootstrap is closed. The paper finishes with some final remarks and open questions.

2 Biaxial Bianchi IX

The class of biaxial Bianchi IX metrics was introduced in [1]. We recall that these spacetimes are topologically $\mathcal{M} = \mathcal{Q} \times SU(2)$, where $\mathcal{Q}$ is a two-dimensional manifold and that global coordinates (u, v) can be found on $\mathcal{Q}$

[24]Clearly, better decay in the central region could immediately be exported to the horizon by a reiteration of the pigeonhole principle in conjunction with the vectorfield Y.

expressing the metric of $\mathcal{M}$ in the form

$$g = -\Omega^2(u,v)\,du\,dv + \tfrac{1}{4}r^2(u,v)\left(\mathrm{e}^{2B(u,v)}\left(\sigma_1^2+\sigma_2^2\right) + \mathrm{e}^{-4B(u,v)}\sigma_3^2\right), \quad (2.1)$$

where B and r are functions $\mathcal{Q} \to \mathbb{R}$ and the σ_i form a basis of left invariant one-forms on $SU(2)$. Note that if $B = 0$, the symmetry is enhanced to $(SU(2)_L \times SU(2)_R)/\mathbb{Z}^2 = SO(4)$ and the metric reduces to the five-dimensional Schwarzschild–Tangherlini metric in view of the higher dimensional version of a well-known theorem due to Birkhoff.[25] In this sense, B is the dynamical degree of freedom ruling the model. See [5] for a more detailed discussion.

The vacuum Einstein equations for the above model reduce to a system of $(1+1)$-dimensional PDEs on the quotient manifold $\mathcal{Q}$:

$$\partial_u(\Omega^{-2}\partial_u r) = -\frac{2r}{\Omega^2}\left((B_{,u})^2\right), \quad (2.2)$$

$$\partial_v\left(\Omega^{-2}\partial_v r\right) = -\frac{2r}{\Omega^2}\left((B_{,v})^2\right), \quad (2.3)$$

$$r_{,uv} = -\frac{1}{3}\frac{\Omega^2\rho}{r} - \frac{2r_{,u}r_{,v}}{r} = -\frac{\Omega^2}{r^3}m - \frac{1}{3}\frac{\Omega^2}{r}\left(\rho - \frac{3}{2}\right), \quad (2.4)$$

$$\begin{aligned}\partial_u\partial_v \log\Omega &= \frac{\Omega^2\rho}{2r^2} + \frac{3}{r^2}r_{,u}r_{,v} - 3(B_{,v})(B_{,u})\\ &= \frac{3\Omega^2}{2r^4}m + \frac{\Omega^2}{2r^2}\left(\rho - \frac{3}{2}\right) - 3\frac{\theta\zeta}{r^3}, \quad (2.5)\end{aligned}$$

$$B_{,uv} = -\frac{3}{2}\frac{r_{,u}}{r}B_{,v} - \frac{3}{2}\frac{r_{,v}}{r}B_{,u} + \frac{\Omega^2}{3r^2}\left(\mathrm{e}^{-8B} - \mathrm{e}^{-2B}\right). \quad (2.6)$$

Here we have defined the quantity[26]

$$\rho = 2\mathrm{e}^{-2B} - \tfrac{1}{2}\mathrm{e}^{-8B} \le \tfrac{3}{2}, \quad (2.7)$$

with the inequality following from elementary calculus. Equality holds if and only if $B = 0$. Note that the non-linear wave equation (2.6) can be written as (1.2) with $\Box$ being the d'Alembertian of the metric (2.1).

[25] Note also the relation between the familiar round metric ($d\omega^2_{\mathbb{S}^3}$) and the bi-invariant metric on S^3, $d\omega^2_{\mathbb{S}^3} = \frac{1}{4}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right)$.

[26] The quantity ρ is related to the scalar curvature of the group orbit by $R = \frac{4}{r^2}\rho$.

A remarkable feature of the above system is the existence of a function $m\,(u,v)$ called the *Hawking mass* and defined by

$$m = \frac{r^2}{2}\left(1 + \frac{4r_{,u}r_{,v}}{\Omega^2}\right). \tag{2.8}$$

Since the inequalities $r_{,u} < 0$ and $r_{,v} \geq 0$ were shown [5] to hold everywhere on the black hole exterior,[27] the Hawking mass has the following monotonicity properties there:

$$\partial_u m = -4r^3 \frac{\lambda}{\Omega^2}\left(B_{,u}\right)^2 + r\nu\left(1 - \frac{2}{3}\rho\right) \leq 0, \tag{2.9}$$

$$\partial_v m = -4r^3 \frac{\nu}{\Omega^2}\left(B_{,v}\right)^2 + r\lambda\left(1 - \frac{2}{3}\rho\right) \geq 0. \tag{2.10}$$

This allows the derivation of energy estimates for the field B, which plays an important role at all stages of the present paper. The existence of these estimates was already an essential ingredient in the proof of the orbital stability [5].

We conclude this section recalling some notation introduced in [5]. We set

$$\lambda = r_{,v} \qquad \nu = r_{,u} \qquad \zeta = r^{\frac{3}{2}}B_{,u} \qquad \theta = r^{\frac{3}{2}}B_{,v} \tag{2.11}$$

and introduce the quantities

$$\kappa = \frac{\lambda}{1-\mu} = \frac{\Omega^2}{-4\nu} \qquad \text{and} \qquad \gamma = \frac{-\nu}{1-\mu} = \frac{\Omega^2}{4\lambda} \tag{2.12}$$

satisfying

$$\kappa_{,u} = \kappa\left(\frac{2}{r^2}\frac{\zeta^2}{\nu}\right), \tag{2.13}$$

$$\gamma_{,v} = \gamma\left(\frac{2}{r^2}\frac{\theta^2}{\lambda}\right), \tag{2.14}$$

as well as the auxiliary quantities

$$\varphi_1(B) = \left(1 - \frac{2}{3}\rho\right) - 8B^2 \qquad \text{and} \qquad \varphi_2(B) = \frac{4B}{3}\left(\mathrm{e}^{-8B} - \mathrm{e}^{-2B}\right) + 8B^2, \tag{2.15}$$

[27] They hold on the initial data for small perturbations of Schwarzschild–Tangherlini and are seen to be preserved by equations (2.2) and (2.3).

both of order B^3. The volume element associated with (2.1) is

$$d\mathrm{Vol} = \sqrt{g}\, du\, dv\, dw = r^3 \frac{\Omega^2}{2} du\, dv\, dA_{\mathcal{S}^3} = r^3 \Omega^2 dt\, dr^\star\, dA_{\mathcal{S}^3} \tag{2.16}$$

where in the standard Euler-coordinates on $SU(2)$ (cf. [5])

$$dA_{\mathcal{S}^3} = \frac{1}{8} \sin\theta\, d\omega = \frac{1}{8} \sin\theta\, d\theta\, d\phi\, d\psi \quad \text{and hence} \quad \int dA_{\mathbb{S}^3} = 2\pi^2. \tag{2.17}$$

The monotonicity of the Hawking mass justifies the definitions

$$m_{\min}, m_{\max} \quad \text{for the minimal and maximal Hawking mass.} \tag{2.18}$$

Furthermore, the quantity M_f will denote the final Bondi mass and M the mass of the perturbed Schwarzschild solution. The mass M determines the scaling of the problem and I have normalized all quantities appropriately using factors of M. In particular, "smallness" always refers to dimensionless quantities.

We write $C(\epsilon)$ for a constant satisfying $\lim_{\epsilon \to 0} C(\epsilon) = 0$. The notation $a \sim b$ is used if there exist uniform constants c_1, c_2 with $c_1 \leq \frac{a}{b} \leq c_2$. Finally, we define

$$v_+ = \max(1, v) \quad \text{and} \quad v_{i+} = \max(1, v_i). \tag{2.19}$$

3 Choice of coordinates

As mentioned in the introduction, the choice of coordinates is already a rather delicate issue for the problem under consideration. Although the final result does not depend on the choice of coordinates, the bootstrap techniques applied in the proof require the coordinates to be normalized to the future of the bootstrap region introduced in Section 7. If on the contrary one normalized the coordinates on the initial data, one would not be able to obtain the improved decay of the fields at late times from the estimates, roughly speaking because contributions from the initial data, which have not yet decayed, enter the estimates. This necessitates, after a purely geometric definition of "time" for ∇r integral curves on the black hole exterior, the introduction of a different coordinate system $\mathcal{C}_{\tilde{\tau}} = (u_{\tilde{\tau}}, v_{\tilde{\tau}})$ defined with respect to every such "time" $\tilde{\tau}$. All such coordinate systems $\mathcal{C}_{\tilde{\tau}}$ are defined on the set

$$\mathcal{D} = J^-\left(\mathcal{I}^+\right) \cap J^+\left(\tilde{S}_{r_K}\right) \tag{3.1}$$

of the black hole exterior. In Section 9 we shall exploit the bootstrap assumptions to establish that — in a certain region — these coordinate systems are uniformly close to each other in a suitable sense. It should be observed that the coordinate systems $\mathcal{C}_{\tilde{\tau}}$ are different from the coordinate system asserted in Theorem 1.1. In the last section of the paper we will show that the coordinate system $\mathcal{C}_{\tilde{\tau}}$ for $\tilde{\tau} \to \infty$ is close to the one asserted by Theorem 1.1.

We begin by considering the family of ∇r integral curves starting out from some $r = r_K$-curve which is chosen close to the horizon[28] such that still $1 - \mu \geq \tilde{c} > 0$ holds for a small $\tilde{c}$, and ending at spacelike infinity i^0. These curves foliate $\mathcal{D} \cap \{r \geq r_K\}$. Moreover, every curve admits a unique point where $r = 2\sqrt{m}$. We pick any such curve and label the corresponding point by A. Denote the mass at A by m_A and consider the curve $r^2 = 4m_A$ going through A and intersecting the initial data at some point D. Let τ_{AD} be the affine length of the constant r curve (with tangent vector normalized to one) connecting A and D. Finally, define

$$T = \sqrt{M} + \frac{\tau_{AD}}{\sqrt{1 - \frac{2m_A}{r^2}}} = \sqrt{M} + \sqrt{2}\, \tau_{AD} \tag{3.2}$$

to be the time associated with the ∇r curve under consideration. In this way we can assign a notion of time to any ∇r integral curve. Considering next the curve $r^2 = 4M_f$ with affine parameter $\tilde{\tau}$ starting from the initial data, we obtain a map

$$\vartheta : [0, \infty) \ni \tilde{\tau} \mapsto T \in \left[\sqrt{M}, \infty\right) \tag{3.3}$$

which is defined by taking $\tilde{\tau}$ to the time associated with the ∇r integral curve which intersects the curve $r^2 = 4M_f$ at $\tilde{\tau}$. The map ϑ is easily seen to be continuous and surjective.

For every $\tilde{\tau}$ a coordinate system (u, v) is defined as follows. Let A have coordinates $(u, v) = (T = \vartheta(\tilde{\tau}), T = \vartheta(\tilde{\tau}))$. Set $\kappa = \gamma = \frac{1}{2}$ along the ∇r integral curve up to $r = r_K$. Since

$$\nabla r\,(u + v) = (\nabla r)^u + (\nabla r)^v = \frac{2}{\Omega^2}(-\nu - \lambda) = \frac{2(1-\mu)}{\Omega^2}(\gamma - \kappa), \tag{3.4}$$

we have that $t = \frac{u+v}{2}$ (thus defining t) is indeed equal to the constant T on the ∇r integral curve through A. Moreover $r^\star = \frac{v-u}{2}$ is equal to 0 at A. Let the ∇r curve defining the coordinate system intersect $r = r_K$ at B. We erect the constant $u = u_B$-ray to the past of B and set $\kappa = \frac{1}{2}$ there. The coordinate system is completed by specifying the u-coordinate on $\overline{BC}$. We set $\nu = -\frac{1}{2}(1 - \mu)$ on $\overline{BC}$. This might send the horizon to $u = \infty$, namely

[28] The choice will provide a source of smallness later in the bootstrap argument.

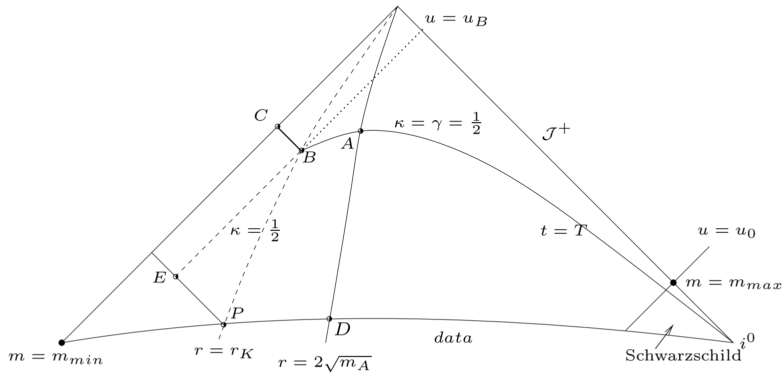


Figure 3: The choice of coordinates.

if $1 - \mu = 0$ at C.[29] We will see that in these coordinates $v \to \infty$ at $\mathcal{I}^+$. The coordinates thus defined will be referred to as Eddington Finkelstein coordinates. We also allow ourselves to move freely between the coordinates (u, v) and $\left(t = \frac{v+u}{2}, r^\star = \frac{v-u}{2}\right)$ (figure 3).

Clearly if $\tilde{\tau} = 0$, then the associated integral curve coincides for $r \geq r_K$ with the curve on which the initial data are defined, and $t = \sqrt{M}$ defines the initial data slice in $r \geq r_K$. Note that in any coordinate system associated with some $\tilde{\tau} > 0$, a slice on which $t =$ constant does in general *not* agree with a ∇r-slice. However, once we have introduced the bootstrap assumptions, we will be able to show that the two slices mentioned remain uniformly close to each other in $r^\star \geq r^\star_K$ for any $\tilde{\tau} > 0$. This argument is postponed to Section 8.3.2. Here we only introduce

Notation 3.1. *Let $t^{\tilde{\tau}_A}_{\tilde{\tau}_B}$ denote the t-coordinate, measured in the coordinate system defined by $\tilde{\tau}_A$, of the point defined by the intersection of the ∇r integral curve determined by $\tilde{\tau}_B$ and the curve $r^2 = 4m\left(\vartheta\left(\tilde{\tau}_A\right), r^\star = 0\right)$.*

We conclude with a remark on the differentiability of the coordinate systems. Due to the "cusp" at the point B the coordinate system is only C^1: The quantities κ and γ (and by definition (2.12) the first derivatives of the area radius function $r\,(u, v)$) are clearly continuous. The second derivatives $r_{,vv}$ and $r_{,uu}$ however, are discontinuous at the point B. This could be avoided by applying an appropriate smooth interpolating function in a small neighborhood around the point B. However, we will see later that the bootstrap involves only first derivatives (and hence continuous quantities) and that the regularity suffices to close the bootstrap.

[29] Of course, one does not expect this to be the case generically.

4 Basic estimates

In this section we are going to show that given an appropriate smallness assumption on the field B, namely (1.3) and (1.5), the field and its derivatives remain small on the entire $\mathcal{D}$. Since this "first round" is independent of the main bootstrap argument, it provides a good way-in to familiarize oneself with the basic estimates applied in different regions of the black hole exterior. The bounds in this section will be proven in the coordinate system $\mathcal{C}_{\tilde{\tau}}$ associated to *any* $\tilde{\tau} \geq 0$.[30] In this context it is crucial that the smallness assumptions (1.3) and (1.5) are manifestly independent of the coordinate choice. From [5] we recall that

$$1 - \frac{m_{\min}}{m_{\max}} < \epsilon\left(\delta\right) \qquad \text{with } \lim_{\delta \rightarrow 0} \epsilon\left(\delta\right) = 0 \tag{4.1}$$

can be chosen arbitrarily small by an appropriate assumption on the initial data. We will abbreviate $\epsilon\left(\delta\right)$ by ϵ in the following. In view of the monotonicity properties of the Hawking mass ((2.9) and (2.10)) the mass difference between any two points cannot exceed $m_{max} \cdot \epsilon\left(\delta\right)$. We note

Lemma 4.1. *If* (4.1) *holds, then on the horizon we have*

$$0 \leq 1 - \mu \leq \frac{2m_{\max}}{r^2}\epsilon. \tag{4.2}$$

Proof. From [5] we have both $1-\mu \geq 0$ on the black hole exterior, as well as the Penrose inequality $1 - \frac{2M_f}{r^2} \leq 0$ holding on the horizon with M_f the final Bondi mass. Combining this with (4.1) immediately yields (4.2). □

Corollary 4.1. *The area radius r satisfies*

$$|r_+ - r_-| \leq \frac{4m_{\max}}{\sqrt{m_{\min}}}\epsilon \quad \textit{on } \mathcal{H}^+ \tag{4.3}$$

with $r_\pm$ being the maximal (minimal) value of r on the horizon.

Corollary 4.2. *For any given $\eta > 0$ we can choose the δ of the initial data so small that for some $r = r_K$ curve located completely in $\mathcal{D}$ the estimate*

$$r_K - r \leq \eta \tag{4.4}$$

holds in $r \leq r_K$.

[30] Note that if $\tilde{\tau} = 0$ the coordinates are normalized on initial data.

For the estimates in this section only we will explicitly couple the location of the $r = r_K$ curve to the smallness of the initial data. In particular, we define the curve r_K by

$$1 - \frac{2m_{\max}}{(r_K)^2} = \epsilon^{\frac{1}{3}}. \tag{4.5}$$

It follows easily that the maximum r difference in the region $r \leq r_K$ satisfies

$$\Delta r \leq r_K - r_- \leq 3 \frac{m_{\max}}{2\sqrt{2m_{\min}}} \epsilon^{\frac{1}{3}}. \tag{4.6}$$

Proposition 4.1. *In any coordinate system $\mathcal{C}_{\tilde{\tau}}$ and with the assumptions of Theorem* 1.1 *on the initial data we have*

$$r|B| + \sqrt{r}|\theta| + M^{\frac{1}{4}} \left| \frac{\zeta}{\nu} \right| \leq \sqrt{M} \cdot C(\delta) \tag{4.7}$$

everywhere in $\mathcal{D}$. Moreover, the coordinate function κ satisfies

$$\left|\kappa - \tfrac{1}{2}\right| \leq C(\delta) \tag{4.8}$$

everywhere in $\mathcal{D}$. Here the constant $C(\delta)$ can be made arbitrarily small by choosing the δ of the initial data sufficiently small.

Proposition 4.1 will immediately follow from Propositions 4.2–4.5 (plus their associated corollaries) proven in the remainder of the section, each of them establishing the bounds in different regions of the black hole exterior. Note that the radial decay of B promised by Proposition 4.1 is weaker than that of Theorem 1.1.

Proposition 4.2. *In the region $\mathcal{D}$ we have*

$$|B(u,v)| \leq C_1(\epsilon(\delta), \delta), \tag{4.9}$$

where $C_1(\epsilon(\delta), \delta)$ can be made arbitrarily small by choosing the δ in (1.3) *small enough.*

Proof. We integrate from the initial data to any point in the region $r \geq r_K$ and estimate as follows:

$$
\begin{aligned}
|B(u,v)| &\leq \delta \left(\frac{\sqrt{M}}{r}\right)^{\frac{3}{2}} (u_{\text{data}}, v) + \int_{u_{\text{data}}}^{u} \frac{\zeta}{r^{\frac{3}{2}}} du \\
&\leq \delta \left(\frac{\sqrt{M}}{r}\right)^{\frac{3}{2}} + \sqrt{\int_{u_{\text{data}}}^{u} \frac{\zeta^2 (1-\mu)}{-\nu} du} \sqrt{\int_{u_{\text{data}}}^{u} \frac{-\nu}{(1-\mu) r^3} du} \\
&\leq \delta \left(\frac{\sqrt{M}}{r}\right)^{\frac{3}{2}} + \frac{1}{\sqrt{2}} \sqrt{m_{\max} - m_{\min}} \sup_{r \geq r_K} \left(\frac{1}{\sqrt{1-\mu}}\right) \frac{1}{r} \\
&\leq \delta \left(\frac{\sqrt{M}}{r}\right)^{\frac{3}{2}} + \epsilon^{\frac{1}{3}} \sqrt{\frac{m_{\max}}{r^2}},
\end{aligned}
\tag{4.10}
$$

which proves (4.9) in that region.
Next we turn to the region $\mathcal{D} \cap \{r \leq r_K\}$. We choose a constant $C > 2M^{\frac{3}{4}} r_-^{-\frac{3}{2}} \delta$ such that $|B| \leq C$ still implies that

$$
\frac{3}{2} - \rho = \frac{3}{2} - \left(2\mathrm{e}^{-2B} - \frac{1}{2}\mathrm{e}^{-8B}\right) \leq \frac{3}{2} \frac{m}{r^2}
\tag{4.11}
$$

in $\mathcal{D} \cap \{r \leq r_K\}$. For r_K sufficiently close to the horizon it is easily seen that $C = \frac{1}{10}$ is good enough. Define the region

$$
\mathcal{R} = \Big\{(u,v) \in \mathcal{D} \cap \{r \leq r_K\} \quad : \quad |B(\bar{u},\bar{v})| < C \ \text{ for all } \ (\bar{u},\bar{v}) \in J^-(u,v)\Big\}
\tag{4.12}
$$

which is clearly open and non-empty. We are going to apply a bootstrap argument: Pick a point in the closure of $\mathcal{R}$, where $B \leq C$ by continuity. We are going to improve this bound by showing that in the causal past of that point B is in fact smaller than $\frac{C}{2}$. By continuity it follows that the set $\mathcal{R}$ is also closed, hence must constitute the entirety of $\mathcal{D} \cap \{r \leq r_K\}$. The argument proceeds in two steps. First we make use of the redshift estimate integrating the equation

$$
\left(\frac{\zeta}{\nu}\right)_{,v} = -\frac{3}{2}\frac{\theta}{r} - \frac{4}{3}\frac{\kappa}{\sqrt{r}}\left(\mathrm{e}^{-8B} - \mathrm{e}^{-2B}\right) - \frac{\zeta}{\nu}\left(\frac{4\kappa}{r^3} m + \frac{4\kappa}{3r}\left(\rho - \frac{3}{2}\right)\right)
\tag{4.13}
$$

from the initial data yielding

$$\begin{aligned}\frac{\zeta}{\nu}(u,v) = {} & \frac{\zeta}{\nu}(u,v_i)\,\mathrm{e}^{-\int_{v_i}^{v}\left[\frac{4\kappa}{r^3}m+\frac{4\kappa}{3r}\left(\rho-\frac{3}{2}\right)\right](u,\bar v)\,d\bar v} \\ & + \int_{v_i}^{v} e^{-\int_{\bar v}^{v}\left[\frac{4\kappa}{r^3}m+\frac{4\kappa}{3r}\left(\rho-\frac{3}{2}\right)\right](u,\hat v)\,d\hat v} \\ & \times\left[-\frac{3}{2}\frac{\theta}{r}-\frac{4}{3}\frac{\kappa}{\sqrt{r}}\left(\mathrm{e}^{-8B}-\mathrm{e}^{-2B}\right)\right](u,\bar v)\;d\bar v \end{aligned} \tag{4.14}$$

and hence

$$\begin{aligned} & \left|\frac{\zeta}{\nu}(u,v)\right| \\ & \le \delta\frac{\left(\sqrt{M}\right)^{\frac{3}{2}}}{r_-}+\frac{3}{2}\sqrt{\int_{v_i}^{v}\mathrm{e}^{-\int_{\bar v}^{v}\left[\frac{4\kappa}{r^3}m\right](u,\hat v)d\hat v}\kappa(u,\bar v)\,d\bar v}\sqrt{\int_{v_i}^{v}\frac{\theta^2}{\kappa r^2}(u,\bar v)\,d\bar v} \\ & \quad+\frac{4}{3}\sup_{r\le r_K}\left[\left|\mathrm{e}^{-8B}-\mathrm{e}^{-2B}\right|\frac{1}{\sqrt{r}}\right]\int_{v_i}^{v}\mathrm{e}^{-\int_{\bar v}^{v}\left[\frac{2\kappa}{r^3}m\right](u,\hat v)d\hat v}\kappa(u,\bar v)\,d\bar v \\ & \le \delta\frac{\left(\sqrt{M}\right)^{\frac{3}{2}}}{r_-}+\frac{3}{2}\sqrt{\epsilon}\sqrt{\frac{m_{\max}\left(r_K\right)^3}{4m_{\min}r_-^2}}+\frac{2}{3}\frac{\left(r_K\right)^3}{\sqrt{r_-}m_{\min}}\left(\mathrm{e}^{8C}+\mathrm{e}^{2C}\right)\equiv\tilde{C}M^{\frac{1}{4}}. \end{aligned} \tag{4.15}$$

It follows that $|\frac{\zeta}{\nu}|$ is bounded (but note that the last term might not be small) in that region. In the second step we integrate from the $r = r_K$ curve, on which B is small by (4.10), or the initial data to obtain

$$B(u,v) - B(u_R,v) = \int_{u_{r_K}}^{u}\frac{\zeta}{r^{\frac{3}{2}}}(\bar u,v)\,d\bar u \tag{4.16}$$

and use the previous bound (4.10)

$$\begin{aligned} |B(u,v)| & \le \delta\frac{\sqrt{M}^{\frac{3}{2}}}{r^{\frac{3}{2}}}+\epsilon^{\frac{1}{3}}\sqrt{\frac{m_{\max}}{r^2}}+\tilde{C}M^{\frac{1}{4}}\int_{u_{r_K}}^{u}\frac{-\nu}{r^{\frac{3}{2}}} \\ & \le \delta\frac{\sqrt{M}^{\frac{3}{2}}}{r^{\frac{3}{2}}}+\epsilon^{\frac{1}{3}}\sqrt{\frac{m_{\max}}{r^2}}+\frac{\tilde{C}}{2}\frac{M^{\frac{1}{4}}}{\left(r_-\right)^{\frac{5}{2}}}\left(\left(r_K\right)^2-\left(r_-\right)^2\right). \end{aligned}$$

Now because the r difference is given by (4.6) in the region under consideration, we have indeed shown that B is smaller than $\frac{C}{2}$ in $\mathcal{R}$ for an appropriate choice of ϵ. By continuity the set $\mathcal{R}$ is also closed. Hence $\mathcal{R} = \mathcal{D} \cap \{r \leq r_K\}$. $\square$

Corollary 4.3. *In $r \geq r_K$ we have that*

$$|B(u,v)| \leq \sqrt{M}\frac{C_2(\epsilon,\delta)}{r}. \tag{4.17}$$

Proof. This is the statement of (4.10). $\square$

It is instructive to compare the $\frac{1}{r}$-decay of Corollary 4.3 with the analogous estimate derived for the massless scalar field in four dimensions [6]. In the latter case, one obtained by the above method $\frac{1}{\sqrt{r}}$-decay. There existed an almost Riemann invariant, i.e., a certain combination of the field and its derivatives, however, admitting better decay properties than the field or its derivatives alone. Via this quantity, it was possible to improve the decay in r of the field itself to $\frac{1}{r}$, which was in turn sufficient to extract energy decay in r. In five dimensions there is no almost Riemann invariant and energy decay in r will only be obtained from the application of the Morawetz vectorfield K in the context of the bootstrap argument pursued later.

Corollary 4.4. *In the region $\mathcal{R} = \mathcal{D} \cap \{r \leq r_K\}$ we have*

$$\left|\frac{\zeta}{\nu}\right| \leq M^{\frac{1}{4}} C_3(\epsilon,\delta). \tag{4.18}$$

Proof. This follows from revisiting the red-shift estimate (4.15) above, this time improving the estimate for the $\left(e^{-8B} - e^{-2B}\right)$- term by Proposition 4.2, which implies that $|e^{-8B} - e^{-2B}|$ is ϵ-small. In this way we obtain a smallness factor for all the terms involved in (4.15). $\square$

Proposition 4.3. *In $\mathcal{D}$ we have*

$$\left|\kappa - \tfrac{1}{2}\right| \leq C_4(\epsilon,\delta). \tag{4.19}$$

Proof. Integrating (2.13) from the $t = T \cap \{r \geq r_K\}$ surface to any point in the region $\{r \geq r_K\}$ yields

$$\kappa(u,v) = \kappa(u_T,v)\exp\left(\int_{u_T}^{u} \frac{2}{r^2}\frac{(1-\mu)\zeta^2}{(\nu)(1-\mu)}\right)(\bar{u},v)\, d\bar{u}. \tag{4.20}$$

If the point under consideration lies to the future of the $t = T$ hypersurface ($u \geq u_T$), the upper bound $\kappa \leq \frac{1}{2}$ follows from monotonicity, whereas the lower bound is obtained via

$$|\kappa(u,v)| \geq \frac{1}{2}\exp\left(\sup_{r\geq r_K}\left(\frac{2}{r^2(1-\mu)}\right)\int_{u_T}^{u}\frac{\zeta^2(1-\mu)}{\nu}(\bar{u},v)\,d\bar{u}\right)$$
$$\geq \frac{1}{2}\exp\left(\frac{-4m_{\max}\epsilon^{\frac{2}{3}}}{r_K^2}\right). \tag{4.21}$$

On the other hand, integrating (2.13) from the null line $u = T - r^\star(T, r_K)$ downwards the lower bound follows from monotonicity and the upper one by using (4.18)

$$|\kappa(u,v)| \leq \frac{1}{2}\exp\left((C_3(\epsilon,\delta))^2\right)\exp\left(-\frac{2\sqrt{M}}{r(u,v)}+\frac{2\sqrt{M}}{r_K}\right) \leq \frac{1}{2}+\tilde{c}_4(\epsilon,\delta). \tag{4.22}$$

Since the r-difference in the region $r \leq r_K$ is $\epsilon^{\frac{1}{3}}$-small by (4.6), we obtain the desired upper bound for κ in particular on all of $r = r_K$.

Now any point located in the past of the $t = T$ hypersurface and satisfying $r \geq r_K$ can be reached by integrating (2.13) from either $t = T$ or from $r = r_K$ where the upper bound (4.22) has already been established. The lower bound for κ at such a point follows from monotonicity, the upper one from

$$|\kappa(u,v)| \leq \left(\frac{1}{2}+\tilde{c}_4(\epsilon,\delta)\right)\exp\left(\sup_{r\geq r_K}\left(\frac{2}{r^2(1-\mu)}\right)\int_{u_T}^{u}\frac{\zeta^2(1-\mu)}{-\nu}(\bar{u},v)\,d\bar{u}\right)$$
$$\leq \left(\frac{1}{2}+\tilde{c}_4(\epsilon,\delta)\right)\exp\left(\frac{4m_{\max}\epsilon^{\frac{2}{3}}}{r_K^2}\right). \tag{4.23}$$

To extend the estimates to the entire region $r \leq r_K$ we integrate (2.13) from the $r = r_K$ curve (on which the lower bound (4.22) and the upper bound (4.21) has been established) to the horizon. Again the upper bound follows from monotonicity and for the lower one we write

$$\kappa(u,v) = \kappa(u_0,v)\exp\left(\int_{u_0}^{u}\frac{2\nu}{r^2}\frac{\zeta^2}{(\nu)^2}\right)(\bar{u},v)\,d\bar{u} \tag{4.24}$$

and estimate, using (4.18)

$$|\kappa(u,v)| \geq \frac{1}{2}\exp\left(\frac{-4m_{\max}\epsilon^{\frac{2}{3}}}{(r_K)^2}\right)\exp\left((C_3(\epsilon,\delta))^2\right)\exp\left(-\frac{2\sqrt{M}}{r(u,v)}+\frac{2\sqrt{M}}{r_K}\right). \tag{4.25}$$

Taking again (4.6) into account, we obtain the lower bound for κ also in that region. □

With the bound on κ established we also have good control over the quantity $\lambda = \kappa(1-\mu)$. In particular $\lambda < 1$ everywhere and λ becomes very small (perhaps zero) at the horizon. In particular, it follows that

$$|\theta| < \frac{|\theta|}{\kappa(1-\mu)} = \frac{|\theta|}{\lambda} \tag{4.26}$$

holds everywhere on $\tilde{S} \cap \{r \geq r_K\}$ and hence the $\frac{\theta}{\lambda}$-part of the smallness condition (1.3) implies smallness for θ as well. With this in mind we can prove.

Proposition 4.4. *In $\mathcal{D}$ we have*

$$|\theta| \leq C_5(\epsilon, \delta)\sqrt{\frac{M}{r}}. \tag{4.27}$$

Proof. We rewrite equation (2.6) as

$$\partial_u \theta = -\frac{3}{2}\frac{\lambda\zeta}{r} + \frac{\Omega^2}{3\sqrt{r}}\left(\mathrm{e}^{-8B} - \mathrm{e}^{-2B}\right) \tag{4.28}$$

and integrate it from $\tilde{S} \cap \{r \geq r_K\}$ to any point in $\mathcal{D}$. We note that for $|B|$ small we can find a constant K such that

$$\left(\mathrm{e}^{-8B} - \mathrm{e}^{-2B}\right)^2 \leq K\left(1 - \tfrac{2}{3}\rho\right) \tag{4.29}$$

holds. This constant approaches $\frac{9}{2}$ as $|B|$ goes to zero. We then estimate

$$\begin{aligned}
&|\theta(u,v)| \\
&\leq M^{\frac{3}{4}}\frac{\delta}{r}(u_{\text{data}}, v) + \frac{3}{2}\sqrt{\int_{u_{\text{data}}}^{u} \frac{\zeta^2(1-\mu)}{-\nu} d\bar{u}}\sqrt{\int_{u_{\text{data}}}^{u} \frac{\kappa\lambda(-\nu)}{r^2}(\bar{u}, v)\, d\bar{u}} \\
&\quad + \sup\left(\frac{4}{3}\kappa\right)\sqrt{K}\sqrt{\int_{u_{\text{data}}}^{u} r(-\nu)\left(1 - \frac{2}{3}\rho\right) d\bar{u}}\sqrt{\int_{u_{\text{data}}}^{u} \frac{-\nu}{r^2}(\bar{u}, v)\, d\bar{u}} \\
&\leq M^{\frac{3}{4}}\frac{\delta}{r} + \sqrt{\epsilon}\frac{\sqrt{m_{\max}}}{\sqrt{r}} + 8\sqrt{\epsilon}\frac{\sqrt{M}}{\sqrt{r}}.
\end{aligned} \tag{4.30}$$

□

Finally, we extend the bound on $\frac{\zeta}{\nu}$ to the region $r \geq r_K$.

Proposition 4.5. *We have*

$$\left|\frac{\zeta}{\nu}\right| \leq C_6\left(\epsilon, \delta\right) \tag{4.31}$$

in all of $\mathcal{D}$.

Proof. Integrate equation (4.13) from the $r = r_K$-curve, where $|\frac{\zeta}{\nu}| \leq C_3\left(\epsilon, \delta\right)$ by Corollary 4.4 out to infinity. Note that due to the estimate proven for the field B in Corollary 4.3 we may achieve (choosing δ small enough) that

$$\frac{3}{2} - \rho \leq \frac{3m}{r^2} \tag{4.32}$$

holds in the region $r \geq r_K$. Using again (4.29) we can follow the string of estimates

$$\begin{aligned}
\left|\frac{\zeta}{\nu}\left(u, v\right)\right| \leq& \left|\frac{\zeta}{\nu}\left(u, v_{r_K}\right)\right| + \frac{3}{2}\sqrt{\int_{v_{r_K}}^{v}\frac{\theta^2}{\kappa}\left(u, \bar{v}\right)d\bar{v}}\sqrt{\int_{v_{r_K}}^{v}\frac{\lambda}{\left(1-\mu\right)r^2}\left(u, \bar{v}\right)d\bar{v}} \\
&+ \frac{4}{3}\sqrt{\int_{v_{r_K}}^{v}\left(\mathrm{e}^{-8B} - \mathrm{e}^{-2B}\right)^2 r\left(u, \bar{v}\right)d\bar{v}}\sqrt{\int_{v_{r_K}}^{v}\frac{\kappa^2}{r^2}\left(u, \bar{v}\right)d\bar{v}} \\
\leq& \left|\frac{\zeta}{\nu}\left(u, v_{r_K}\right)\right| + \frac{3}{2}\sqrt{\epsilon}\frac{2}{\epsilon^{\frac{1}{6}}}\frac{\sqrt{m_{\max}}}{\sqrt{r_K}} \\
&+ \frac{4\sqrt{K}}{3\sqrt{2}}\sup_{r\geq r_K}\left(\frac{1}{\sqrt{\lambda}}\right)\sqrt{\int_{v_{r_K}}^{v}\left(1-\frac{2}{3}\rho\right)r\lambda\, d\bar{v}}\sqrt{\int_{v_{r_K}}^{v}\frac{\lambda}{\left(1-\mu\right)r^2}d\bar{v}} \\
\leq& \left|\frac{\zeta}{\nu}\left(u, v_{r_K}\right)\right| + \frac{3\sqrt{m_{\max}}}{\sqrt{r_K}}\epsilon^{\frac{1}{3}} \\
&+ \frac{4\sqrt{K}}{3\sqrt{2}}\sup_{r\geq r_K}\left(\frac{1}{\sqrt{\kappa}\left(1-\mu\right)}\right)\frac{\sqrt{m_{\max}}}{\sqrt{r_K}}\sqrt{\epsilon} \\
\leq& M^{\frac{1}{4}}C_3\left(\epsilon, \delta\right) + \frac{3\sqrt{m_{\max}}}{\sqrt{r_K}}\epsilon^{\frac{1}{3}} + 2\frac{4\sqrt{K}}{3\sqrt{2}}\frac{2\sqrt{M}}{\sqrt{r_K}}\epsilon^{\frac{1}{6}}
\end{aligned} \tag{4.33}$$

to conclude the result. □

So far we have shown that rB, $\frac{\zeta}{\nu}$, and $\sqrt{r}\theta$ are small and that κ is everywhere close to $\frac{1}{2}$ for the perturbed spacetime. Estimates for some higher derivative quantities will be required later. However, since all bounds can be considerably improved once the bootstrap assumptions have been introduced, we postpone the derivation of further pointwise estimates to Section 8.4. Here we only note.

Proposition 4.6. *On $\mathcal{D}$ we have, independent of the coordinate system $\mathcal{C}_{\tilde{\tau}}$, the bound*

$$\left| \frac{\Omega_{,v}}{\Omega} - \frac{m}{r^3} \right| \leq \frac{1}{\sqrt{M}} C_7 \left(\epsilon\right). \tag{4.34}$$

Proof. From the fact that $\kappa = \frac{1}{2}$ on $\{t = T\} \cap \{r^\star \geq r^\star \left(T, r_K\right)\}$ (hence $\kappa_{,r^\star} = 0$ there) and on $\{u = T - r^\star \left(T, r_K\right)\} \cap \{t \leq T\}$ (hence $\kappa_{,v} = 0$ on this null-line) the bound (4.34) follows on these sets. We can obtain the quantity $\frac{\Omega_{,v}}{\Omega}$ at any point on $\mathcal{D}$ by integrating equation (2.5) from the aforementioned set to the desired point. Inserting the estimates of Proposition 4.1 gives (4.34) everywhere. □

Remark. The quantity $\frac{\Omega_{,v}}{\Omega}$ is discontinuous at the point B in the coordinate system $\mathcal{C}_{\tilde{\tau}}$. This discontinuity is propagated along the null-line $v = v\left(B\right)$ when integrating the quantity $\partial_u \frac{\Omega_{,v}}{\Omega}$ (which is continuous! (cf. 2.5)) in u (cf. also Appendix A).

We conclude the section with a useful bound for the quantity γ in the region $\mathcal{D} \cap \{t \leq T\} \cap \{r \geq r_K\}$.

Proposition 4.7. *In $\mathcal{D} \cap \{t \leq T\} \cap \{r \geq r_K\}$ we have in the coordinate system $\mathcal{C}_{\tilde{\tau}}$*

$$C_8 \left(\epsilon, \delta\right) \leq \gamma - \tfrac{1}{2} \leq 0. \tag{4.35}$$

Proof. Integrate (2.14) from the $t = T$-slice in the past direction. By monotonicity $\gamma \leq \frac{1}{2}$ is obvious. The other direction is derived from

$$\gamma\left(u,v\right) = \gamma\left(u, v_T\right) \exp\left(\int_v^{v_T} -\frac{2}{r^2}\frac{\theta^2}{\lambda}\left(u, \bar{v}\right) d\bar{v}\right) \tag{4.36}$$

and the estimate

$$\begin{aligned} \gamma\left(u,v\right) &\geq \frac{1}{2} \exp\left[-\left(\sup_{\{r \geq r_K\} \cap \{t \leq T\}} \frac{2}{r^2\left(1-\mu\right)}\right) \int_v^{v_T} \frac{\theta^2}{\kappa}\left(u, \bar{v}\right) d\bar{v}\right] \\ &\geq \frac{1}{2} \exp\left[C_8\left(\epsilon\right)\right], \end{aligned} \tag{4.37}$$

which follows by choosing the mass fluctuation small enough. □

We close the section by emphasizing once more that the bounds proven in this section are independent of the particular coordinate system used, i.e., of how large we choose $\tilde{\tau}$ (and hence T). In this context it is important that the smallness assumptions (1.3) and (1.5) are invariant under a change of coordinates.

5 Compatible currents

5.1 The basic identity

Varying the Lagrangian

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu B \partial_\nu B + \frac{1}{2r^2}\left(1 - \frac{2}{3}\rho\right) \tag{5.1}$$

with respect to B leads to the non-linear wave equation (1.2) satisfied by the field B. We associate to (5.1) the energy momentum tensor

$$T_{\mu\nu} = \partial_\mu B \partial_\nu B - \frac{1}{2} g_{\mu\nu} (\partial B)^2 - \frac{1}{2r^2} g_{\mu\nu}\left(1 - \frac{2}{3}\rho\right) \tag{5.2}$$

satisfying the equation

$$\nabla^\mu T_{\mu\nu} = \frac{1}{r^3}\left(1 - \frac{2}{3}\rho\right)\nabla_\nu r. \tag{5.3}$$

Given any vectorfield V we can define its deformation tensor

$$\pi_V^{\mu\nu} = \tfrac{1}{2}\left(\nabla^\mu V^\nu + \nabla^\nu V^\mu\right) \tag{5.4}$$

and the vector

$$P^\alpha = g^{\alpha\beta} T_{\beta\delta} V^\delta. \tag{5.5}$$

The method of compatible currents is based on the following basic identity for an arbitrary vector field V:

$$-\nabla_\alpha P^\alpha = -\left(T_{\alpha\beta}\pi_V^{\alpha\beta} + \left(\nabla^\beta T_{\alpha\beta}\right) V^\alpha\right). \tag{5.6}$$

5.2 Useful formulae

In (u, v)-coordinates the components of the energy momentum tensor (5.2) read

$$
\begin{aligned}
T_{uu} &= (\partial_u B)^2 \, , \\
T_{vv} &= (\partial_v B)^2 \, , \\
T_{uv} &= -\frac{1}{2r^2} g_{uv} \left(1 - \frac{2}{3}\rho\right) = \frac{1}{4r^2}\Omega^2 \left(1 - \frac{2}{3}\rho\right) , \\
T_{ij} &= -\frac{1}{2} g_{ij} \left(\partial^\alpha B \partial_\alpha B + \frac{1}{r^2}\left(1 - \frac{2}{3}\rho\right)\right) . \qquad (5.7)
\end{aligned}
$$

The vectorfields V used in this paper have u and v components only and will furthermore depend only on these two variables. For such vectorfields we compute the components of their deformation tensor:

$$
\begin{aligned}
\pi^{uu} &= \frac{4}{\Omega^2} \partial_v \left(\frac{V_v}{\Omega^2}\right) , \\
\pi^{vv} &= \frac{4}{\Omega^2} \partial_u \left(\frac{V_u}{\Omega^2}\right) , \\
\pi^{uv} &= \frac{2}{\left(\Omega^2\right)^2} \left(\partial_v V_u + \partial_u V_v\right) , \\
g_{ij}\pi^{ij} &= -\frac{6}{r} \left(\frac{\nu}{\Omega^2} V_v + \frac{\lambda}{\Omega^2} V_u\right) . \qquad (5.8)
\end{aligned}
$$

Finally, the following explicit formulae for the contraction

$$
T_{\mu\nu}\pi^{\mu\nu} = T_{uu}\pi^{uu} + T_{vv}\pi^{vv} + 2T_{uv}\pi^{uv} + T_{ij}\pi^{ij} \qquad (5.9)
$$

will be useful:

$$
\begin{aligned}
T_{\mu\nu}\pi^{\mu\nu} = \frac{4}{\Omega^2} \Bigg(&(\partial_u B)^2 \partial_v \left(\frac{V_v}{\Omega^2}\right) + (\partial_v B)^2 \partial_u \left(\frac{V_u}{\Omega^2}\right) \\
&+ \frac{1}{4r^2} \left(\partial_v V_u + \partial_u V_v\right) \left(1 - \frac{2}{3}\rho\right)\Bigg) \\
&+ \frac{3}{r} \left(\frac{\nu}{\Omega^2} V_v + \frac{\lambda}{\Omega^2} V_u\right) \left(\partial^\alpha B \partial_\alpha B + \frac{1}{r^2}\left(1 - \frac{2}{3}\rho\right)\right) \qquad (5.10)
\end{aligned}
$$

and

$$
\begin{aligned}
&- T_{\mu\nu}\pi^{\mu\nu} - V^{\nu}\nabla^{\mu}T_{\mu\nu} \\
&= -\frac{4}{\Omega^2}\left((\partial_u B)^2\,\partial_v\left(\frac{V_v}{\Omega^2}\right) + (\partial_v B)^2\,\partial_u\left(\frac{V_u}{\Omega^2}\right)\right. \\
&\quad \left. + \frac{1}{4r^2}\left(\partial_v V_u + \partial_u V_v\right)\left(1-\frac{2}{3}\rho\right)\right) \\
&\quad - \frac{3}{r}\left(\frac{\nu}{\Omega^2}V_v + \frac{\lambda}{\Omega^2}V_u\right)\left(\partial^{\alpha}B\partial_{\alpha}B\right) - \frac{1}{r^3}\left(\frac{\nu}{\Omega^2}V_v + \frac{\lambda}{\Omega^2}V_u\right)\left(1-\frac{2}{3}\rho\right).
\end{aligned} \tag{5.11}
$$

5.3 Basic regions

In the course of the paper we shall apply the basic vectorfield identity (5.6) for different vector fields in adapted regions of the black hole exterior. Here the relevant formulae arising from (5.6) for these regions are derived.[31]

5.3.1 Characteristic rectangles

Writing out the identity (5.6) for a null-rectangle $\mathcal{R} = [u_1, u_2] \times [v_1, v_2]$ yields

$$
\begin{aligned}
-\int_{\mathrm{vol}} \nabla_{\alpha}P^{\alpha} &= -\int_{\mathbb{S}^3}\int_{v_1}^{v_2}\int_{u_1}^{u_2} \frac{1}{\sqrt{g}}\partial_{\alpha}\left(\sqrt{g}P^{\alpha}\right)\sqrt{g}\,du\,dv\,d\omega \\
&= -\int_{\mathbb{S}^3}\int_{v_1}^{v_2}\int_{u_1}^{u_2} \left[\partial_u\left(\sqrt{g}P^u\right) + \partial_v\left(\sqrt{g}P^v\right)\right] du\,dv\,d\omega.
\end{aligned} \tag{5.12}
$$

Defining the bulk term

$$
I_B^V = -\int_{\mathrm{vol}(\mathcal{R})}\left(T_{\alpha\beta}\pi_V^{\alpha\beta} + \left(\nabla^{\beta}T_{\alpha\beta}\right)V^{\alpha}\right)\frac{\Omega^2}{2}r^3 du\,dv\,dA_{\mathbb{S}^3} \tag{5.13}
$$

and the boundary terms

$$
\begin{aligned}
F_B^V\left([u_1,u_2]\times\{v\}\right) &= -\int_{\mathbb{S}^3}\int_{u_1}^{u_2}\sqrt{g}P^v\left(\bar{u},v\right)d\bar{u}\,d\omega \\
&= 2\pi^2\int_{u_1}^{u_2}\left[r^3\left(\partial_u B\right)^2 V^u + \frac{r\Omega^2}{4}\left(1-\frac{2}{3}\rho\right)V^v\right]du,
\end{aligned} \tag{5.14}
$$

[31] Since the coordinate system is only piecewise C^2, the justification of these formulae, which are easily derived formally, requires some care. A detailed discussion can be found in Appendix A.

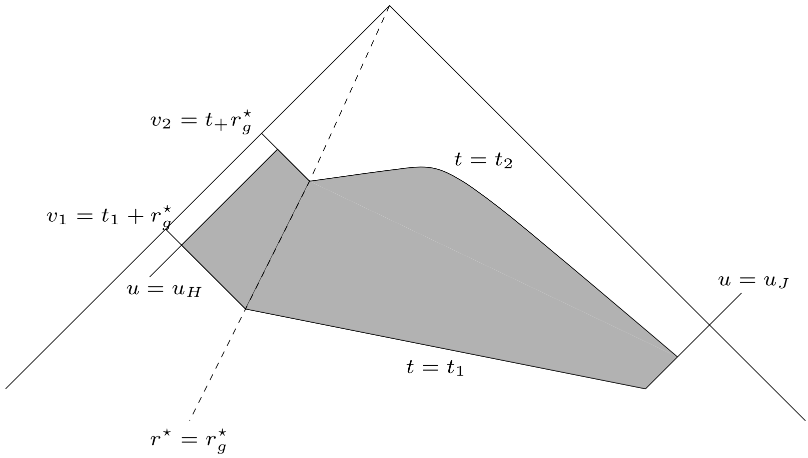


Figure 4: The region ${}^{u_H}\mathcal{D}^{r_g^\star,u_J}_{[t_1,t_2]}$.

$$\begin{aligned} F^V_B\left(\{u\}\times[v_1,v_2]\right) &= -\int_{\mathbb{S}^3}\int_{v_1}^{v_2}\sqrt{g}P^u\left(u,\bar{v}\right)d\bar{v}d\omega \\ &= 2\pi^2\int_{v_1}^{v_2}\left[r^3\left(\partial_v B\right)^2 V^v + \frac{r\Omega^2}{4}\left(1-\frac{2}{3}\rho\right)V^u\right]dv, \end{aligned} \tag{5.15}$$

we find the identity

$$\begin{aligned} &F^V_B\left(\{u_2\}\times[v_1,v_2]\right) + F^V_B\left([u_1,u_2]\times\{v_2\}\right) \\ &\quad = I^V_B\left(\mathcal{R}\right) + F^V_B\left(\{u_1\}\times[v_1,v_2]\right) + F^V_B\left([u_1,u_2]\times\{v_1\}\right) \end{aligned} \tag{5.16}$$

5.3.2 The region ${}^{u_H}\mathcal{D}^{r_g^\star,u_J}_{[t_1,t_2]}$

Another important region is (see figure 4)

$$\begin{aligned} {}^{u_H}\mathcal{D}^{r_g^\star,u_J}_{[t_1,t_2]} &:= \left(\{t_1\le t\le t_2\}\cap\{r^\star\ge r_g^\star\}\cap\{u_J\le u\le u_H\}\right) \\ &\cup\left(\{(u,v)\in\left[t_1-r_g^\star,u_H\right]\times\left[t_1+r_g^\star,t_2+r_g^\star\right]\}\right) \end{aligned} \tag{5.17}$$

for which one finds the basic identity

$$\hat{I}^V_B\left({}^{u_H}\mathcal{D}^{r_g^\star,u_J}_{[t_1,t_2]}\right) = \hat{F}^V_B\left(t_2\right) - \hat{F}^V_B\left(t_1\right) + \hat{H}_{u_H} - \hat{J}_{u_J}, \tag{5.18}$$

with the bulk term

$$\hat{I}^V_B\left({}^{u_H}\mathcal{D}^{r^\star_g,u_J}_{[t_1,t_2]}\right) = \int_{{}^{u_H}\mathcal{D}^{r^\star_g,u_J}_{[t_1,t_2]}} \left(-T_{\mu\nu}\pi^{\mu\nu}_V - \left(\nabla^\mu T_{\mu\nu}\right)V^v\right) d\mathrm{Vol} \tag{5.19}$$

and the boundary terms

$$\begin{aligned}\frac{1}{2\pi^2}\hat{F}^V(t) = \int_{r^\star_g}^{t-u_J} & -P^t\left(t,r^\star\right)\Omega^2 r^3 dr^\star \\ & + \int_{t-r^\star_g}^{u_H}\left[r^3\left(\partial_u B\right)^2 V^u + \frac{r\Omega^2}{4}\left(1-\frac{2}{3}\rho\right)V^v\right]du,\end{aligned} \tag{5.20}$$

where

$$\begin{aligned}-P^t &= \frac{V^u}{2\Omega^2}\left[2\left(\partial_u B\right)^2 + \frac{\Omega^2}{2r^2}\left(1-\frac{2}{3}\rho\right)\right] \\ &\quad + \frac{V^v}{2\Omega^2}\left[2\left(\partial_v B\right)^2 + \frac{\Omega^2}{2r^2}\left(1-\frac{2}{3}\rho\right)\right],\end{aligned} \tag{5.21}$$

$$\frac{1}{2\pi^2}\hat{H}^V_{u_H} = \int_{v_1}^{v_2}\left[r^3\left(\partial_v B\right)^2 V^v + \frac{r\Omega^2}{4}\left(1-\frac{2}{3}\rho\right)V^u\right]\left(u_{\mathrm{hoz}},v\right)dv \tag{5.22}$$

and

$$\frac{1}{2\pi^2}\hat{J}^V_{u_J} = \int_{2t_1-u_J}^{2t_2-u_J}\left[r^3\left(\partial_v B\right)^2 V^v + \frac{r\Omega^2}{4}\left(1-\frac{2}{3}\rho\right)V^u\right]\left(u_J,v\right)dv. \tag{5.23}$$

For the region under consideration we will also need to apply Green's identity to a term of the form $D\cdot\Box\left(B^2\right)$ for some function D.[32]

$$\begin{aligned}\hat{I}^V_B\left({}^{u_H}\mathcal{D}^{r^\star_g,u_J}_{[t_1,t_2]}\right) &= \cdots + \int\left[\left(\Box B^2\right)D\right]d\mathrm{Vol} = \cdots + \int\left[B^2\left(\Box D\right)\right]d\mathrm{Vol} \\ &\quad + G\left(t_2\right) - G\left(t_1\right) + N\left(t_2\right) - N\left(t_1\right) + H^G_{u_H} - J^G_{u_J},\end{aligned} \tag{5.24}$$

---

[32]The formula derived here is a priori valid only for $D \in C^2$. However it also holds for a D admitting less regularity, as is shown explicitly in Appendix A, where we demonstrate that for the cases where (5.24) is applied in the paper (equations (12.9) and (10.11)), D indeed satisfies these requirements.

where

$$\frac{1}{2\pi^2}G(t) = \int_{r_g^\star}^{t-u_0} \left[B^2\partial_t D - D\partial_t B^2\right] r^3 (t, r^\star)\, dr^\star, \tag{5.25}$$

$$\frac{1}{2\pi^2}N(t) = \int_{t-r_g^\star}^{u_H} \left[B^2\partial_u D - D\partial_u B^2\right] r^3 \left(u, t+r_g^\star\right) du, \tag{5.26}$$

$$\frac{1}{2\pi^2}H_{u_H}^G = \int_{t_1+r_g^\star}^{t_2+r_g^\star} \left[B^2\partial_v D - D\partial_v B^2\right] r^3 (u_H, v)\, dv, \tag{5.27}$$

$$\frac{1}{2\pi^2}J_{u_J}^G = \int_{t_1+r_g^\star}^{t_2+r_g^\star} \left[B^2\partial_v D - D\partial_v B^2\right] r^3 (u_J, v)\, dv. \tag{5.28}$$

We then define the renormalized bulk term

$$I_B^V\left({}^{u_H}\mathcal{D}_{[t_1,t_2]}^{r_g^\star,u_J}\right) := \cdots + \int_{{}^{u_H}\mathcal{D}_{[t_1,t_2]}^{r_g^\star,u_J}} \left[B^2(\Box D)\right] d\mathrm{Vol}, \tag{5.29}$$

for which the identity

$$I_B^V\left({}^{u_H}\mathcal{D}_{[t_1,t_2]}^{r_g^\star,u_J}\right) = F_B^V(t_2) - F_B^V(t_1) + H_{u_H} - J_{u_J} \tag{5.30}$$

with

$$F_B^V(t) = \hat{F}_B^V(t) - G(t) - N(t), \tag{5.31}$$

$$H_{u_H}^V = \hat{H}_{u_H}^V - H_{u_H}^G, \tag{5.32}$$

$$J_{u_J}^V = \hat{J}_{u_J}^V - J_{u_J}^G \tag{5.33}$$

holds. Note that for $u_J = u_0$, the boundary terms J_{u_J} all vanish, because B does not have any support on $u = u_0$ by the domain of dependence property.

Finally, for future reference we also define the subregion

$$\mathcal{B}_{[t_1,t_2]}^{r_g^\star,R_g^\star} = {}^{u_H}\mathcal{D}_{[t_1,t_2]}^{r_g^\star,u_J} \cap \{r_g^\star \leq r^\star \leq R_g^\star\} \tag{5.34}$$

and the slice

$$\Sigma_{\bar{t}} = \left(\{t=\bar{t}\} \cap \{r^\star \geq r_{\mathrm{cl}}^\star\}\right) \cup \left(\{v = \bar{t} + r_{\mathrm{cl}}^\star\} \cap \{r^\star \leq r_{\mathrm{cl}}^\star\}\right). \tag{5.35}$$

6 The vectorfield T and the Hawking mass

Recall that the Hawking mass m defined in (2.8) satisfies (2.9) and (2.10). The one-form dm is closed and by simple connectedness of the Penrose diagram, exact. It follows that energy is conserved. This fact can also be seen from the integral identity (5.6) applied to the the vectorfield

$$T = \frac{4\lambda}{\Omega^2}\partial_u - \frac{4\nu}{\Omega^2}\partial_v. \tag{6.1}$$

If we apply identity (5.6) in the region ${}^{u_H}\mathcal{D}^{r^\star_g, u_J}_{[t_1,t_2]}$, energy conservation translates into the following relation between the boundary terms:

$$F^T_B(t_2) = F^T_B(t_1) - H^T_{u_H} + J^T(u_J), \tag{6.2}$$

where

$$\begin{aligned} F^T_B(t) &= \int_{t-r^\star_{\text{cl}}}^{u_H} \left[4r^3\lambda\frac{(B_{,u})^2}{\Omega^2} - r\nu\left(1-\frac{2}{3}\rho\right)\right](u, t+r^\star_{\text{cl}})\,du \\ &\quad + \int_{r^\star_{\text{cl}}}^{t-u_0}\Bigg(r^3\frac{(B_{,v})^2}{\kappa} + 4r^3\frac{\lambda}{\Omega^2}(B_{,u})^2 \\ &\quad + r(\lambda-\nu)\left(1-\frac{2}{3}\rho\right)\Bigg)(t, r^\star)\,dr^\star, &(6.3)\\ H^T_{u_H} &= \int_{v_1}^{v_2}\left[r^3\frac{(B_{,v})^2}{\kappa} + r\lambda\left(1-\frac{2}{3}\rho\right)\right](u_H, v)\,dv, &(6.4)\\ J^T(u_J) &= \int_{2t_1-u_J}^{2t_2-u_J}\left[r^3\frac{(B_{,v})^2}{\kappa} + r\lambda\left(1-\frac{2}{3}\rho\right)\right](u_J, v)\,dv. &(6.5)\end{aligned}$$

We will sometimes use the notation $E(\Sigma)$, for the energy flux through an achronal slice Σ.

7 The bootstrap

The bootstrap is intimately related to the choice of coordinate systems defined in Section 3. We will use the notation introduced in that section.

7.1 The bootstrap region and the statement $\mathcal{P}$

Let
$$a = \sqrt{\frac{M}{2}}\left[-3\sqrt{2} - \log\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\right] \tag{7.1}$$
and c be some small constant. Define
$$S = t\partial_t + (r^\star - a)\,\partial_{r^\star} \qquad \underline{S} = t\partial_{r^\star} + (r^\star - a)\,\partial_t \tag{7.2}$$
and the quantity
$$\begin{aligned} E_B^K(t) = \frac{2\pi^2}{M}\int_{r_K^\star}^{t-u_0} &\Big[(1+2\nu)\left((SB)^2 + (\underline{S}B)^2\right) \\ &+ (-2\nu)\left(\left(SB + \frac{3}{2}\frac{r^\star - a}{r}B\right)^2 + \frac{(r^\star - a)^2}{r^2}B^2\right) \\ &+ (-2\nu)\left(\left(\underline{S}B + \frac{3}{2}\frac{t}{r}B\right)^2 + \frac{t^2}{r^2}B^2\right)\Big] r^3 dr^\star \end{aligned} \tag{7.3}$$

with
$$r_K^\star = \sup_{t<T} r^\star(t, r_K)\,. \tag{7.4}$$

To each $\tilde{\tau}$ we associate the region $\mathcal{A}(T(\tilde{\tau})) = {}^{u_{\text{hoz}}}\mathcal{D}^{r_K^\star, u_0}_{[2\sqrt{M},T)}$ (hence defining the T in (7.4)) (figure 5).

We define the statement $\mathcal{P}_{T(\tilde{\tau})}$ associated to a region $\mathcal{A}(T(\tilde{\tau}))$ to be[33]

1. In the subregion $\{r^\star \geq r_K^\star\} \cap \mathcal{A}(T)$, the area radius satisfies
$$\left| r^\star - \left[r(t,r^\star) + \sqrt{\frac{M_A}{2}}\left(\log\left(\frac{r(t,r^\star) - \sqrt{2M_A}}{r(t,r^\star) + \sqrt{2M_A}}\right) + p\right)\right]\right| < c\sqrt{M} \tag{7.5}$$
with
$$p = -2\sqrt{2} - \log\frac{2-\sqrt{2}}{2+\sqrt{2}} \tag{7.6}$$
and M_A defined to be the Hawking mass at the point $(T, r^\star = 0)$.
2. We have
$$\tfrac{1}{2}\sqrt{M} < \sup_{\tilde{S}\cap\{r^\star \geq r_K^\star\}\cap\{u \geq u_0\}} t < \tfrac{3}{2}\sqrt{M}. \tag{7.7}$$

[33] We will sometimes abbreviate $T(\tilde{\tau})$ by T, reminding the reader that any T arises from $\tilde{\tau}$ as described in Section 3.

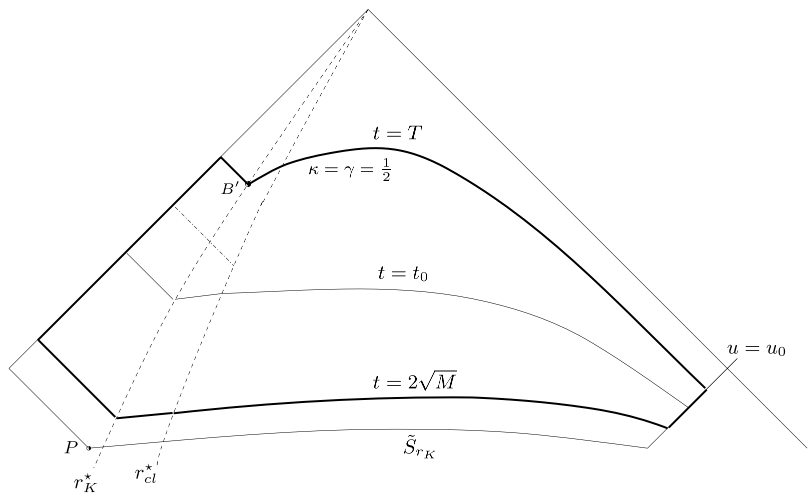


Figure 5: The bootstrap region.

3. The weighted energy density (7.3) satisfies

$$\frac{1}{M} E^K_B\left(\tilde{T}\right) < c \quad \text{on all arcs } \{2\sqrt{M} \leq t = \tilde{T} < T\} \cap \{r^\star \geq r^\star_K\} \cap \mathcal{A}(T)\ . \tag{7.8}$$

4. The energy flux satisfies

$$m(u_{\text{hoz}}, v_2) - m(u_{\text{hoz}}, v_1) < c\, M \frac{M}{(v_{1+})^2} \tag{7.9}$$

   for any $v_1 \leq v_2$ along the part of the horizon located in $\mathcal{A}(T)$ and

5.

$$m\left(u_{r^\star_{\text{cl}}}, v\right) - m(u_{\text{hoz}}, v) < c\, M \frac{M}{v_+^2} \tag{7.10}$$

   holds in $\mathcal{A}(T)$ for an $r^\star_{\text{cl}}$ defined in the subsection below.

6. The integral bound

$$\tilde{F}^Y_B = \int r^3 \frac{(B_{,u})^2}{-\nu} du < C_L M \frac{M}{v_+^2} \qquad \text{for} \quad C_L = \sup_{r^\star \geq r^\star_{\text{cl}}} \frac{1}{1-\mu} \tag{7.11}$$

   holds along lines of constant v in the region $\{r^\star \leq r^\star_{\text{cl}}\} \cap \{u \leq T - r^\star(T, r_K)\} \subset \mathcal{A}(T)$, corresponding to a decay of energy for local observers near the horizon.

Finally, we define the set

$$A = \left\{ \tilde{\tau} \in \left[\sqrt{M}, \infty\right) \mid \mathcal{P}_{T(\hat{\tau})} \text{ holds in } \mathcal{A}\left(T\left(\hat{\tau}\right)\right) \text{for all } \hat{\tau} \leq \tilde{\tau} \right\} \subset \left[\sqrt{M}, \infty\right). \tag{7.12}$$

Note that the lower bound on $\tilde{\tau}$ ensures that $T > 2\sqrt{M}$ (cf. (3.2)). The following key Theorem will close the bootstrap and is easily seen to imply the decay rates of Theorem 1.1. It will only be proven at the end of the paper.

Theorem 7.1. *The set A is non-empty, open and closed.*

A few remarks are in order. The first two bootstrap assumptions ensure that the different coordinate systems $\mathcal{C}_{\tilde{\tau}}$ do not move too far away from one another, at least in the region $r^\star \geq r^\star_K$. The first controls the deviation of the relation between the coordinate $r^\star$ and the area radius r from the familiar relation between the Regge–Wheeler coordinate and the area radius in the Schwarzschild metric. In particular, for Schwarzschild the left-hand side of (7.5) is zero. The second assumption ensures that the bottom of the bootstrap region (the $t = 2\sqrt{M}$ slice) does not move away too much from the geometrically defined initial data (and is moreover always located to the future of the data). In other words, the coordinates of $\tilde{S} \cap \{r^\star \geq r^\star_K\} \cap \{u \geq u_0\}$ are similar in all coordinate systems $\mathcal{C}_{\tilde{\tau}}$.

The open-part of Theorem 7.1 follows from a simple continuity argument:

Proposition 7.1. *The set A defined in* (7.12) *is open.*

Proof. We observe that the integral $E^K_B(t)$ and in fact all the quantities appearing in statement $\mathcal{P}$ of the bootstrap assumptions depend continuously on the choice of $\tilde{\tau}$. □

One should note in this context that all bootstrap assumptions involve only first derivatives of the fields and the area radius, and hence only continuous quantities (cf. the remarks on the differentiability of the coordinate systems at the end of Section 3).

The hard part of Theorem 7.1 consists in showing that A is closed. This will be accomplished by improving the constants appearing in the inequalities of the bootstrap assumptions.

7.2 The choice of $r^\star_{\text{cl}}$

In this subsection we define the quantity $r^\star_{\text{cl}}$ with respect to the coordinate system associated to the bootstrap region. Clearly, the location of $r^\star_{\text{cl}}$ will change between different coordinate systems when the bootstrap region is altered. However, by bootstrap assumption 1.3.2, it will always stay close to a geometrically defined curve of constant r, which is determined below.

By Propositions 4.6 and 4.1 we know that on $\mathcal{D}$ the bound

$$\sqrt{M}\left|\frac{\Omega_{,v}}{\Omega}-\frac{m}{r^3}\right|+\left|\kappa-\frac{1}{2}\right|\leq C\left(\epsilon\right) \tag{7.13}$$

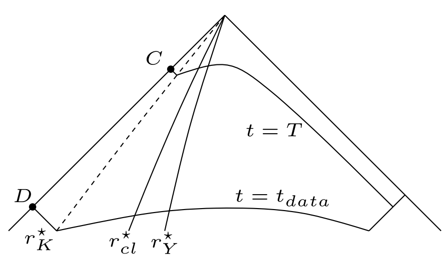
holds in any coordinate system $\mathcal{C}_{\tilde{T}}$. For any small number $\psi>0$ we can hence choose the initial data small enough such that there exists an $r_Y<\frac{3}{2}\sqrt{M}$ satisfying

$$\max_{r\leq r_Y}\left[\log\frac{r_Y}{r_-},r\frac{\Omega_{,v}}{\Omega}-\frac{1}{2},1-\mu\right]<\psi. \tag{7.14}$$

Here Corollary 4.2 has been used for the bound on the first factor. By bootstrap assumption 1.3.2 the curve $r^\star_Y:=\inf_{t<T}r^\star\left(t,r_Y\right)$ is always close to the geometrically defined curve r_Y. Hence we can additionally impose that

$$\frac{\sqrt{M}}{-r^\star_Y}<\psi \tag{7.15}$$

holds. Next we are going to determine how small ψ has to be. We define two functions $\alpha\left(r^\star\right)$ and $\beta\left(r^\star\right)$ in the coordinate system associated with the bootstrap region as follows:



The function α which is supported only for $r^\star\leq-\frac{1}{2}\sqrt{M}$ is everywhere non-negative and defined by setting $\alpha\left(r^\star_C=\frac{T+r^\star_K-u_{\text{hoz}}}{2}\right)=1$ and

$$\alpha'\left(r^\star\right)=\begin{cases}0 & \text{for } r^\star\leq r^\star_C,\\ \frac{1}{\sqrt{M}}\tilde{\chi}\left(r^\star\right) & \text{in } [r^\star_C,r^\star_K],\\ \frac{M^{\frac{1}{4}}}{\left(\sqrt{M}+|r^\star|\right)^{\frac{3}{2}}} & \text{in } [r^\star_K,r^\star_Y]\end{cases} \tag{7.16}$$

with $M = m(T, r^\star = 0)$ and $\tilde{\chi}$ a smooth positive interpolating function. In particular $\alpha = 1$ on $\overline{DC}$.

The non-negative function β, again with support only for $r^\star \leq -\frac{1}{2}\sqrt{M}$, is defined by setting $\beta\left(r^\star_D = \frac{2\sqrt{M}+r^\star_K - u_{\text{hoz}}}{2}\right) = 0$ and imposing that

$$\frac{24}{r(t,r^\star)}\Omega^2(t,r^\star) \geq \beta' \geq \frac{18}{r(t,r^\star)}\Omega^2(t,r^\star) \tag{7.17}$$

in all of $r^\star \leq r^\star_Y$. We can estimate the value of β on $r^\star_Y$ by

$$\begin{aligned}\beta(r^\star_Y) &= 0 + \int_D^{r^\star_Y} \beta_{,r^\star} dr^\star \leq \int_D^{r^\star_Y} 24\frac{\Omega^2}{r} dr^\star \\ &\leq \int_D^{r^\star_Y} 24\frac{\gamma\kappa}{\gamma+\kappa}\partial_{r^\star}\log r dr^\star \leq 12\log\frac{r_Y}{r_-}.\end{aligned}$$

Hence β remains controlled by the r-fluctuation in $r^\star \leq r^\star_Y$ and hence small by choosing ψ above suitably small. Note that α and β are in particular supported away from the curve $r^\star = 0$.

We finally choose the ψ of (7.14), (7.15) so small that the inequalities

$$\left(4\alpha\frac{\Omega_{,v}}{\Omega}r - \alpha' r\right) > \max\left[2\left(\frac{1}{4\kappa}\alpha - \beta\lambda\right)^2, \frac{r}{4\sqrt{M}}\alpha, \frac{\kappa(1-\mu)r}{\sqrt{M}}\right], \tag{7.18}$$

$$\alpha \geq \kappa\left(4\beta\lambda + 2r\beta' + 8r\beta\frac{\Omega_{,v}}{\Omega}\right) + \max\left[\frac{\kappa r\beta}{2\sqrt{M}}, \frac{r}{2\sqrt{M}}\right], \tag{7.19}$$

$$\left(-\frac{r^\star - a}{r}\right)\left[24\mu r\frac{\Omega_{,v}}{\Omega} + (1-\mu)(-70\kappa - 36\kappa\mu)\right] > 45 \tag{7.20}$$

hold in the region $r^\star \leq r^\star_Y$ and set $r^\star_{\text{cl}} = r^\star_Y - 2\sqrt{M}$.

Remark. The constant ψ and the corresponding r_Y (and the upper bound on initial data) can easily be computed explicitly and is fixed once and for all. In particular it does not depend on the size of the bootstrap region and the coordinate system that comes along with it. The curve $r^\star = r^\star_Y$ and hence $r^\star = r^\star_{\text{cl}}$ is then also fixed and always close to r_Y by bootstrap assumption 1.3.2 and the fact that r_K is chosen much closer to the horizon than r_Y. Smallness for the bootstrap on the other hand, will be exploited via the $r^\star_K$-curve and by choosing the initial data even smaller to "beat the constants" which are introduced by the choice of $r^\star_{\text{cl}}$.

7.3 Cauchy stability

For the closed part we will have to improve the constant c in the statement $\mathcal{P}$ (i.e., bounds (7.8–7.11)) in the region $\mathcal{A}(T)$. The argument constitutes the body of the paper. In this context, we note that within the process of improving the bootstrap assumptions there will be two sources of smallness. The first arises from the fact that $r = r_K$ can be chosen very close to the horizon. The second is obtained by selecting a ∇r-slice belonging to some large $\tilde{\tau}_0$ (and hence large associated time t_0) up to which Cauchy stability holds by a suitable smallness assumption on the data. This is expressed precisely by the following

Proposition 7.2. *For any small $\eta > 0$, $\tilde{\delta} > 0$, and any large $\tilde{\tau}_0$ (hence large associated time $T_0 = \vartheta(\tilde{\tau}_0)$, with ϑ defined in* (3.3)*) we can find an r_K and a $\delta > 0$ such that the following statement is true: If the smallness assumptions* (1.3) *and* (1.5) *of Theorem* 1.1 *hold for δ, then*

1) *the curve $r = r_K$ away from the horizon satisfies $r_K^2 - r_-^2 < \eta$;*
2) *in the coordinate system defined by $\tilde{\tau}_\bullet \in [0, \tilde{\tau}_0]$ the t-coordinate of the subset $\tilde{S} \cap \{r \geq r_K\} \cap \{u \geq u_0\}$ of the initial data satisfies*

$$|t - \sqrt{M}| < \tilde{\delta}\sqrt{M}, \tag{7.21}$$

3) *in the coordinate system defined by $\mathcal{C}_{\tilde{\tau}_0}$, the statement $\mathcal{P}$ holds with constant $\tilde{\delta}$ (instead of c) in the region ${}^{u_{\text{hoz}}}\mathcal{D}^{r_K^\star, u_0}_{[2\sqrt{M}, T_0]}$ and moreover, the pointwise bound*

$$|B| + M^{-\frac{1}{4}}\left|\frac{\zeta}{\nu}\right| + M^{-\frac{1}{4}}|\theta| \leq \sqrt{M}\frac{\tilde{\delta}}{v_+} \tag{7.22}$$

holds on any slice Σ_t (cf. (5.35)*) for $2\sqrt{M} \leq t \leq T_0$.*

Proof. The first assertion is the statement of Corollary 4.2. For the second statement consider the coordinate system $\mathcal{C}_{T=\vartheta(\tilde{\tau}_\bullet)}$ for a given $\tilde{\tau}_\bullet \in [0, \tilde{\tau}_0]$. The vectorfield ∇r introduced in Section 3 can be expressed in the associated $(t, r^\star)$ coordinates

$$\nabla r = \frac{1}{4\kappa\gamma}\left[(\gamma - \kappa)\,\partial_t + (\gamma + \kappa)\,\partial_{r^\star}\right] \tag{7.23}$$

as can the vectorfield $\nabla_{\perp} r$ which is defined to be orthogonal to ∇r and whose integral curves are the curves of constant area radius r:

$$\nabla_{\perp} r = \frac{1}{4\kappa\gamma} \left[(\gamma + \kappa) \partial_t + (\gamma - \kappa) \partial_{r^\star} \right]. \tag{7.24}$$

The rescaled vectorfields

$$R = \frac{1}{\sqrt{1-\mu}} \nabla r \quad \text{ and } \quad G = \frac{1}{\sqrt{1-\mu}} \nabla_{\perp} r \tag{7.25}$$

satisfy the orthonormality relations

$$g(R,R) = 1 \quad \text{ and } \quad g(G,G) = -1 \quad \text{ and } \quad g(R,G) = 0. \tag{7.26}$$

Let ϱ be the affine parameter along R and τ the affine parameter along G. In the following, we frequently refer to figure 3 of Section 3. At the point A we have $t = T_\bullet = \sqrt{M} + \sqrt{2}\tau_{AD}$ by definition. We would like to estimate the value of t at the point D and compare it to 1, which is the value of t if $\tilde{\tau}_\bullet = 0$, $T_\bullet = \sqrt{M}$ and the coordinates are defined on initial data. The rate at which t changes in affine parameter along the integral curve of $\nabla_{\perp} r$ going through A is given by

$$\frac{dt}{d\tau} = \frac{1}{4\kappa\gamma} (\kappa + \gamma) \frac{1}{\sqrt{1-\mu}}. \tag{7.27}$$

We will integrate (7.27) from $\tau = 0$ to $\tau = \tau_{AD}$ with initial condition $T_0 = \sqrt{M} + \frac{\tau_{AD}}{\sqrt{1-\mu_A}}$ at A. By Propositions 4.1 and 4.7 the estimates

$$|\kappa + \gamma - 1| \leq C(\epsilon) \quad \text{ and } \quad \left| \frac{1}{\sqrt{1-\mu}} - \sqrt{2} \right| \leq C(\epsilon) \tag{7.28}$$

hold along the curve. Given the fixed $\tilde{\tau}_0$ we choose the initial data so small that $\tilde{\tau}_0 \cdot C(\epsilon)$ is as small as we may wish. Hence $\tau_{AD} \cdot C(\epsilon)$ is small for any $\tau_{AD}(\tilde{\tau}_\bullet)$ with $\tilde{\tau}_\bullet \leq \tilde{\tau}_0$.[34] With these choices the estimate

$$|t(D) - \sqrt{M}| \leq C(\epsilon) \tag{7.29}$$

simply follows from integrating (7.27).

In a completely analogous fashion, by considering $\frac{dr^\star}{d\tau}$ and using that $|\gamma - \kappa| \leq C(\epsilon)$ we can show that the point $r^\star = 0$ on the initial data is close to $r = 2\sqrt{M}$: $|r(t_{\text{data}}, 0) - 2\sqrt{M}| \leq C(\epsilon)\sqrt{M}$.

[34] Note that $\tilde{\tau}_\bullet$ is close to τ_{AD}, since the curves $r^2 = 4M_F$ and $4m_A$ converge to one another for the initial data going to zero.

Before we finally estimate how t changes along the integral curve of R through D (i.e., the location of the initial data), we derive a rough estimate for the relation of r and $r^\star$. Consider the vectorfield

$$L = \frac{1}{\Omega}\partial_{r^\star}, \tag{7.30}$$

whose integral curves are the curves of constant t. The coordinate $r^\star$ changes along such a curve (affine parameter l) by

$$\frac{dr^\star}{dl} = \frac{1}{\Omega} = \frac{1}{\sqrt{4\kappa\gamma\,(1-\mu)}}. \tag{7.31}$$

Integrating from $r^\star = 0$, where $r \approx 2\sqrt{M}$ outwards to infinity noting that $1-\mu > \frac{4}{9}$ and that both κ and γ are close to $\frac{1}{2}$ in the region under consideration, we obtain

$$\frac{9}{10}l \leq r^\star \leq \frac{9}{4\sqrt{2}}l. \tag{7.32}$$

On the other hand, the area radius changes according to

$$\frac{dr}{dl} = \frac{1}{\Omega}\left(\lambda - \nu\right) = \frac{1}{2}\frac{\sqrt{1-\mu}}{\sqrt{\gamma\kappa}}\left(\kappa + \gamma\right) \tag{7.33}$$

leading to the estimate

$$2\sqrt{M} - C\left(\epsilon\right)\sqrt{M} + \tfrac{2}{5}l \leq r \leq 2\sqrt{M} + C\left(\epsilon\right)\sqrt{M} + \tfrac{11}{10}l. \tag{7.34}$$

Combining (7.32) and (7.34) yields the relation

$$\frac{9}{11}r - c_1 \leq r^\star \leq \frac{45}{8\sqrt{2}}r \qquad \text{with } c_1 = \frac{9}{11}\left(2\sqrt{M} - C\left(\epsilon\right)\sqrt{M}\right) \tag{7.35}$$

along any curve of constant time in the region $r \geq 2\sqrt{M}$. In particular, if a quantity decays in r in the asymptotic region, it decays in $r^\star$ as well.

Finally, we can consider the integral curve of R through D on which the initial data are defined. We want to prove that the value of t does not change much along that curve (at least up to the area radius $\tilde{R}$ where the support ends). First, we show that the horizon is a finite length of affine parameter along ∇r away from D. Namely, since the r-component of the vectorfield R

is given by $R^r = \sqrt{1-\mu}$, we have the equation

$$\frac{dr}{d\varrho} = \sqrt{1-\mu}. \tag{7.36}$$

Starting at $r(0) = 2\sqrt{m_A}$ and integrating inwards to the point where the curve intersects the horizon we find

$$\begin{aligned} 0 \leq -\varrho_{\text{hoz}} &= \int_{r_{\text{hoz}}}^{2\sqrt{m_A}} dr \frac{1}{\sqrt{1-\mu}} = \int_{r_{\text{hoz}}}^{2\sqrt{m_A}} dr 2 \cdot \partial_r \left(\sqrt{1-\mu}\right) \frac{r^3}{4m - 2rm_{,r}} dr \\ &\leq \left(4\sqrt{m_A} + C(\delta)\right) \left(\frac{1}{\sqrt{2}} - C(\delta)\right) \end{aligned} \tag{7.37}$$

for some small δ. On the other hand, we can integrate outwards from D along ∇r to a point where $r = \tilde{R}$. From (7.34) we know that the affine parameter is controlled by the r value along the curve, hence for large $\tilde{R}$

$$\varrho \leq \tfrac{6}{5}\tilde{R}. \tag{7.38}$$

Finally, t changes along the curve according to

$$\frac{dt}{d\varrho} = \frac{1}{4\gamma\kappa} (\gamma - \kappa) \frac{1}{\sqrt{1-\mu}} \leq 0. \tag{7.39}$$

Within $[r_K, 2\sqrt{m_A}]$ and $[2\sqrt{m_A}, \tilde{R}]$ we can use the pointwise bound

$$(\gamma - \kappa) \frac{1}{\sqrt{1-\mu}} \leq C(\epsilon) \tag{7.40}$$

following from the results of Section 4 and choose ϵ (hence the initial data) so small that $C(\epsilon)$ exceeds the support radius $\tilde{R}$:

$$|t_D - t| \leq C(\epsilon) \tfrac{6}{5}\tilde{R} \leq C(\epsilon). \tag{7.41}$$

In this way we can make the difference in t small in the region between $r = r_K$ and $r = \tilde{R}$ on the ∇r integral curve.

The pointwise bound of statement 3 follows directly from Proposition 4.1 together with the fact that the quantity v is finite in the region under consideration.

For (7.5) of statement $\mathcal{P}$ we observe that on $\{t = T_0\} \cap \{r^\star \geq r^\star_K\}$ we have $\partial_t r = 0$ by definition. From $\partial_{r^\star} r = (\kappa + \gamma)(1-\mu)$ we derive, using Propositions 4.1 and 4.7, estimate 1 of statement $\mathcal{P}$ on $t = T_0$ for an arbitrary good constant by a suitable smallness assumption on the data. However, along a curve of constant $r^\star \geq r^\star_K$, the value of r changes only by an amount which can be made small by suitable choice of initial data, as is seen from the estimate

$$|r(t_b, r^\star) - r(t_a, r^\star)| = \left| \int_{t_a}^{t_b} (\lambda + \nu)\, dt \right| \leq \int_{2\sqrt{M}}^{T_0} (1-\mu)(\kappa - \gamma)\, dt \leq C(\epsilon) \cdot T_0 \tag{7.42}$$

is small for any $t_a, t_b \in \left[2\sqrt{M}, T_0\right]$ if the data are small enough.

The second bootstrap assumption has been dealt with in statement 2 of Proposition 7.2 already.

The third bootstrap assumption involves integrals over compact intervals with the integrand containing B and its derivatives. The integral is small on $t = 2\sqrt{M}$ by assumption (1.3) and Cauchy stability. Again from Cauchy stability it follows that E^K_B will stay as small as we may wish up to the chosen $T = \vartheta(\tilde{\tau}_0)$ slice if we only chose the data small enough. This is perhaps most easily seen directly from the fact that u and v are always finite in the region of integration, and taking into account the pointwise bounds on B, $\frac{\zeta}{\nu}, \theta$ established in Proposition 4.1. Put together it follows that the quantity E^K_B can be made smaller than $\tilde{\delta}$ for a finite t slice by an appropriate assumption on the data.

The bootstrap assumptions involving the energy can be satisfied by choosing the data sufficiently small (recall the a priori bound on the mass fluctuation (4.1)). Finally, assumption (7.11) follows from the pointwise bound on $\frac{\zeta}{\nu}$ (cf. Proposition 4.1) and realizing that integrating the quantity ν in u yields a finite result. Hence, in the coordinate system defined by $\tilde{\tau}_0$, all inequalities in the statement $\mathcal{P}$ can be brought to hold with constant $\tilde{\delta}$ in the region ${}^{u_{\mathrm{hoz}}}\mathcal{D}^{r^\star_K, u_0}_{[2\sqrt{M}, T_0]}$. □

Corollary 7.1. *The set A defined in* (7.12) *is non-empty.*

Proof. By statement 2 of Proposition 7.2 for any $\tilde{\tau}_\bullet \leq \tilde{\tau}_0$ the coordinates of a point in the associated region $\mathcal{A}(\vartheta(\tilde{\tau}_\bullet))$ will be close to the coordinates of the same point in the coordinate system defined by $\tilde{\tau}_0$. Hence the statement $\mathcal{P}$ holds with constant $\tilde{\delta}$ in $\mathcal{A}(\vartheta(\tilde{\tau}_\bullet))$ for all $\tilde{\tau}_\bullet \leq \tilde{\tau}_0$ by choosing δ small enough. Therefore $[\sqrt{M}, \tilde{\tau}_0] \subset A$. □

In order to be useful in conjunction with the bootstrap, statement 3 of Proposition 7.2 has to hold in any coordinate system $\mathcal{C}_{\tilde{\tau}}$ associated to a

$\tilde{\tau} > \tilde{\tau}_0$ with $\tilde{\tau} \in A$. The argument is postponed to Proposition 8.8, after we have derived appropriate decay bounds from the bootstrap assumptions in the next section.

Proposition 7.2 also provides us with two sources of smallness. In particular, it justifies the following algebra for constants:

$$C\left(r^\star_{\text{cl}}\right)\eta = \tilde{\delta}, \tag{7.43}$$

$$\frac{C\left(r^\star_K\right)}{t_0} = \tilde{\delta}. \tag{7.44}$$

Namely, after we have chosen $\psi > 0$ (cf. (7.14) and (7.15)) to determine $r^\star_{\text{cl}}$, we can choose η so small that it "beats" any constant depending on $r^\star_{\text{cl}}$, and finally t_0 so large that $\frac{1}{\sqrt{M}}\frac{C\left(r^\star_K\right)}{t_0}$ is as small as we may wish. (Of course, the restrictions on the initial data get stronger and stronger in this process.) Consequently, everywhere that the formulation "we choose t_0 so large that" is used in the paper, we always have an application of Proposition 7.2 in mind.

8 Analysing the bootstrap assumptions

In this section we are going to derive certain decay bounds for the energy, the squashing field and some other quantities. These estimates will be useful for late times, i.e., they are to be understood in conjunction with Proposition 7.2 where we can choose such a late time. The time t_0 up to which Cauchy stability holds is chosen in particular so large that for $t \geq t_0$ we have $v \sim t$ in the region $r^\star_{\text{cl}} \leq r^\star \leq \frac{9}{10}t$ and that $v \sim t \sim r^\star$ in the region $r^\star \geq \frac{9}{10}t$. Moreover $v_0 = t_0 + r^\star_{\text{cl}} >> \sqrt{M}$. All statements about decay in this section are then valid in the subregion $\{t \geq t_0\} \cap \{v \geq v_0\}$ of the bootstrap region.

8.1 Energy decay

From assumption (7.8), we can directly derive $\frac{1}{t^2}$ decay of the energy in certain regions for late times.

Proposition 8.1. *On a hypersurface of constant t we have the bounds*

$$\frac{2\pi^2}{M}\int_{r^\star_K}^{t-u_0}\left(-2\nu\right)\frac{\left(r^\star - a\right)^2}{r^2}B^2 r^3 dr^\star \leq E^K_B\left(t\right), \tag{8.1}$$

$$\frac{2\pi^2}{M}\int_{r_K^\star}^{t-u_0}(-2\nu)\frac{t^2}{r^2}B^2r^3dr^\star \leq E_B^K(t) \tag{8.2}$$

and

$$\frac{2\pi^2}{M}\int_{r_K^\star}^{t-u_0}\left((u+a)^2(\partial_u B)^2+(v-a)^2(\partial_v B)^2\right)r^3dr^\star \leq 2E_B^K(t). \tag{8.3}$$

Proof. The first two bounds follow directly from (7.3). For the last inequality note that $2(\partial_u B)^2(u+a)^2+2(\partial_v B)^2(v-a)^2=(SB)^2+(\underline{S}B)^2$ and

$$\begin{aligned}(SB)^2+(\underline{S}B)^2&=(1+2\nu)\left((SB)^2+(\underline{S}B)^2\right)+(-2\nu)\left((SB)^2+(\underline{S}B)^2\right)\\&\leq(1+2\nu)\left((SB)^2+(\underline{S}B)^2\right)\\&\quad+4(-2\nu)\left(\left(SB+\frac{3(r^\star-a)}{2r}B\right)^2+\left(\underline{S}B+\frac{3t}{2r}B\right)^2\right)\\&\quad+3(-2\nu)\left(\frac{(r^\star-a)^2}{r^2}B^2+\frac{t^2}{r^2}B^2\right)\end{aligned} \tag{8.4}$$

and that we control all the terms on the right-hand side separately by (7.3). □

The following proposition is an immediate application of the latter and allows us to estimate the energy flux through certain slices for late times.

Proposition 8.2. *Let $(r_1^\star,t_1)$, $(\tilde{r}_1^\star,t_1)$ be such that $t_1-\tilde{r}_1^\star+a\geq\sqrt{M}$ and $t_1+r_1^\star-a\geq\sqrt{M}$ and let additionally $r_1^\star\geq r_K^\star$. Then we have*

$$\begin{aligned}&m(\tilde{r}_1^\star,t_1)-m(r_1^\star,t_1)\\&\quad\leq 3M\left((t_1-\tilde{r}_1^\star+a)^{-2}E_B^K(t_1)+(t_1+r_1^\star-a)^{-2}E_B^K(t_1)\right).\end{aligned}$$

Proof.

$$\begin{aligned}&m(\tilde{r}_1^\star,t_1)-m(r_1^\star,t_1)\\&\quad=\int_{r_1^\star}^{\tilde{r}_1^\star}\partial_{r^\star}m\,dr^\star=\int_{r_1^\star}^{\tilde{r}_1^\star}(-\partial_u m+\partial_v m)\,dr^\star\\&\quad\leq\int_{r_1^\star}^{\tilde{r}_1^\star}\left(\frac{(B_{,u})^2}{\gamma}r^3+\frac{(B_{,v})^2}{\kappa}r^3+B^2r(\lambda-\nu)\left(8+\frac{\varphi_1(B)}{B^2}\right)\right)dr^\star\end{aligned}$$

$$
\begin{aligned}
&\leq (t_1 - \tilde{r}_1^\star + a)^{-2} \int_{r_1^\star}^{\tilde{r}_1^\star} \Bigg((u+a)^2 \frac{(B_{,u})^2}{\gamma} \\
&\qquad -\nu (u+a)^2 \frac{B^2}{r^2} \left(8 + \frac{\varphi_1(B)}{B^2} \right) \Bigg) r^3 dr^\star \\
&\qquad + (t_1 + r_1^\star - a)^{-2} \int_{r_1^\star}^{\tilde{r}_1^\star} \Bigg((v-a)^2 \frac{(B_{,v})^2}{\kappa} \\
&\qquad +\lambda (v-a)^2 \frac{B^2}{r^2} \left(8 + \frac{\varphi_1(B)}{B^2} \right) \Bigg) r^3 dr^\star \\
&\leq 3M \Big((t_1 - \tilde{r}_1^\star + a)^{-2} E_B^K(t_1) + (t_1 + r_1^\star - a)^{-2} E_B^K(t_1) \Big),
\end{aligned}
$$

where we have used (2.9), (2.10) and Proposition 8.1 as well as the bounds (4.19) and (4.35). □

The previous proposition can be combined with the bootstrap assumptions (7.10) and (7.9). The fact that energy is conserved then immediately yields decay for any achronal slice in a certain subregion of $\mathcal{A}(T)$ as elaborated in the following

Proposition 8.3. *In the bootstrap-region $\mathcal{A}(T)$ the energy flux through any achronal surface*

$$S \subset \mathcal{A}(T) \cap \{r^\star \leq \tfrac{10}{11} t\} \tag{8.5}$$

with $v_- = \min_v S \geq t_0 + r^\star_{\mathrm{cl}} \geq \sqrt{M}$ and $\min_t S \geq t_0$ satisfies

$$E(S) \leq M^2 \frac{C(c)}{v_-^2}. \tag{8.6}$$

Proof. Dyadically decompose the region $\mathcal{A}(T)$ into regions ${}^{u_{\mathrm{hoz}}}\mathcal{D}^{r^\star_{\mathrm{cl}}, u_0}_{[t_i, t_{i+1}]}$ with $t_{i+1} = 1.1 t_i$.[35] Proposition 8.2 applied to any slice t_{i+1} with $r_1^\star = r_K^\star$ and $\tilde{r}_1^\star = \frac{10}{11} t_{i+1}$ yields (for late times, i.e., when $t - r^\star_{\mathrm{cl}} + a \approx t$, which is the case

[35] This decomposition implies that the width of each region is of the size of the t coordinate it is at. It should be noted that this decomposition may not fit exactly, i.e., the last of these dyadic tubes may have a smaller width. To keep the notation reasonably clean this fact is always to be understood implicitly. The results derived for each dyadic region in the paper are of course independent of the fact that the last region may be smaller.

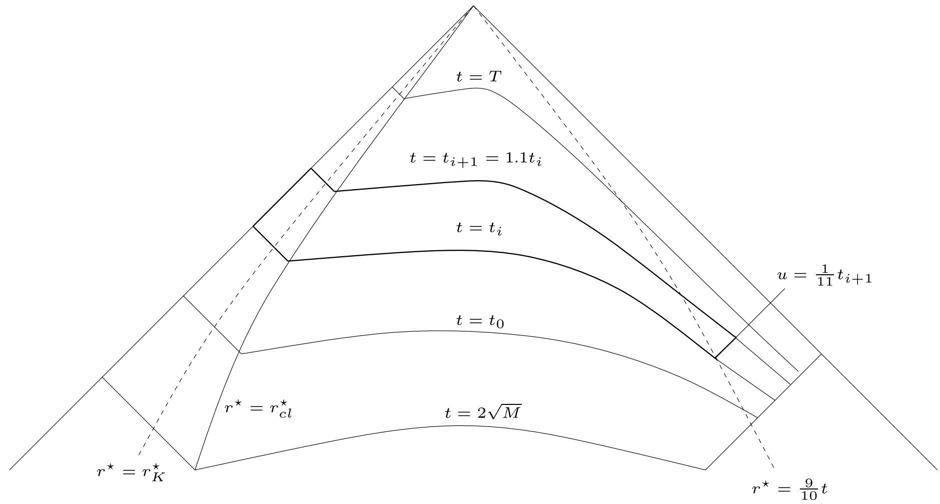


Figure 6: Energy decay from K.

for $t \geq t_0$, cf. Proposition 7.2)

$$m\left(t_{i+1}, r^\star = \frac{10}{11} t_{i+1}\right) - m\left(t_{i+1}, r^\star_K\right) \leq M^2 \frac{C(c)}{t_i^2}. \tag{8.7}$$

Combining this decay in the central region with the energy decay at the horizon (bootstrap assumptions (7.9) and (7.10)) we find from energy conservation that the energy must decay like $\frac{1}{v_-^2}$ through any achronal slice in the region where $r^\star \leq \frac{10}{11} t$. This shows (8.6), noting that for large times $t \geq t_0$ we have $t \sim v$ in the region $r^\star_{\mathrm{cl}} \leq r^\star \leq \frac{9}{10} t$. Note, in particular, that we have this decay of energy flux through the regions ${}^{u_{\mathrm{hoz}}}\mathcal{D}^{r^\star_{\mathrm{cl}}, u = \frac{1}{11} t_{i+1}}_{[t_i, t_{i+1}]}$ for large t_i (cf. figure 6, where such a region is depicted). □

8.2 Decay estimates for κ and γ

The following proposition establishes appropriate decay bounds on κ and γ sufficient to improve estimate (7.5) for the relation between $r^\star$ and r in the central region in the next section.

Proposition 8.4. *In the region $\mathcal{A}(T) \cap \{r^\star \leq r^\star_{\mathrm{cl}}\} \cap \{v \geq v_0\}$ we have*

$$\left|\kappa - \frac{1}{2}\right| \leq C_L \frac{M}{v^2}. \tag{8.8}$$

In the region $\mathcal{A}(T) \cap \{r^\star \geq r^\star_K\} \cap \{r^\star \leq \frac{9}{10}t\}$ *we have*

$$\frac{1}{2} \leq \kappa \leq \frac{1}{2} + C_L (2+c) \frac{M}{t^2}, \tag{8.9}$$

$$\frac{1}{2} \geq \gamma \geq \frac{1}{2} - C_K \, c\frac{M}{t^2} \tag{8.10}$$

with $C_K = \sup_{r^\star \geq r^\star_K} \frac{1}{1-\mu}$ *and* $C_L = \sup_{r^\star \geq r^\star_{\mathrm{cl}}} \frac{1}{1-\mu}$.

Proof. Integrating equation (2.13) from the set $\{u = T - r^\star(T, r_K)\} \cup (\{t = T\} \cap \{r^\star(T, r_K) \leq r^\star \leq r^\star_{\mathrm{cl}}\})$, where $\kappa = \frac{1}{2}$ by definition, to any point in the region $r^\star \leq r^\star_{\mathrm{cl}}$ yields after inserting bootstrap assumption (7.11)

$$\left| \kappa(t, r^\star_{\mathrm{cl}}) - \frac{1}{2} \right| \leq 2C_L \frac{M}{v^2} \tag{8.11}$$

in that region establishing (8.8). We can obtain κ at any point in the remaining region $\mathcal{A}(T) \cap \{r^\star \geq r^\star_{\mathrm{cl}}\} \cap \{r^\star \leq \frac{9}{10}t\}$ by integrating from the set $L = \{\{t = T\} \cap \{r^\star \geq r^\star_{\mathrm{cl}}\}\} \cup \{r^\star = r^\star_{\mathrm{cl}}\}$ on which either κ is equal to $\frac{1}{2}$ or satisfies estimate (8.11), to the desired point. An application of Proposition 8.3 then yields (8.9) in the region $\mathcal{A}(T) \cap \{r^\star_K \leq r^\star \leq \frac{9}{10}t\}$ as follows:

$$\begin{aligned} \kappa(t, r^\star) &= \kappa(u_L, v) \exp\left(- \int_{t-r^\star}^{u_L} \frac{2}{r^2} \frac{\zeta^2}{\nu}(\bar{u}, v)\, d\bar{u} \right) \\ &\leq \kappa(u_L, v) \exp\left(\sup\left[\frac{2}{r^2(1-\mu)} \right] \int_{t-r^\star}^{u_L} \frac{4\lambda}{\Omega^2} \zeta^2(\bar{u}, v)\, d\bar{u} \right) \\ &\leq \left[\frac{1}{2} + 2C_L \frac{M}{(t+r^\star)^2} \right] \left(1 + cC_L \frac{M}{t^2} \right). \end{aligned} \tag{8.12}$$

For estimate (8.10), we first note that $\gamma = \frac{1}{2}$ on $\{t = T\} \cap \{r \geq r_K\}$. On the $r^\star = \frac{9}{10}t$ curve we can obtain (8.10) by integrating (2.14) from $\{t = T\} \cap \{\frac{9}{10}T \leq r^\star \leq T - u_0\}$ downwards to any point in the region $\mathcal{A}(T) \cap \{r^\star \geq \frac{9}{10}t\}$. We use that λ is bounded below and $|\theta| \leq \frac{C(\epsilon)\sqrt{M}}{\sqrt{r}}$ in the integration region, both following from Proposition 4.1, to obtain

$$\gamma \geq \frac{1}{2} - \frac{C(\epsilon) M}{r^2} \tag{8.13}$$

there. Since r is controlled by $r^\star$ (cf. equation (7.35)) and $r^\star \geq \frac{9}{10}t$ in the region under consideration, we find the bound (8.10) in the region $\mathcal{A}(T) \cap \{r^\star \geq \frac{9}{10}t\}$, in particular on the $r^\star = \frac{9}{10}t$-curve. Finally, the value of γ at any point in the remaining region $\mathcal{A}(T) \cap \{r_K^\star \leq r^\star \leq \frac{9}{10}t\}$ can be obtained by integrating (2.14) in v from some point of the set $L' = \left\{\{t = T\} \cap \{r^\star \leq \frac{9}{10}t\}\right\} \cup \{r^\star = \frac{9}{10}t\}$ (on which γ already satisfies (8.10)). Using the decay of the energy flux we arrive at (8.10) in the remaining region:

$$\begin{aligned} \gamma(t, r^\star) &= \gamma(u, v_{L'}) \exp\left(-\int_{t+r^\star}^{v_{L'}} \frac{2}{r^2}\frac{\theta^2}{\lambda}(u, \bar{v})\, d\bar{v}\right) \\ &\geq \gamma(u, v_{L'}) \exp\left(-\sup\left[\frac{2}{r^2(1-\mu)}\right]\int_{t+r^\star}^{v_{L'}} \frac{\theta^2}{\kappa}(u, \bar{v})\, d\bar{v}\right) \\ &\geq \frac{1}{2}\left(1 - 2C_K\, c\frac{M}{t^2}\right). \end{aligned} \tag{8.14}$$

□

From the proof of the γ-estimate we deduce:

Corollary 8.1. *In the region $r^\star \geq r^\star_{\mathrm{cl}}$, estimate* (8.10) *holds with the constant C_K replaced by C_L.*

In the asymptotic region t is like r and the bounds extend:

Corollary 8.2. *In $\mathcal{A}_T \cap \{r^\star \geq \frac{9}{10}t\} \cap \{v \geq v_0\}$ we have*

$$\frac{1}{2} \leq \kappa \leq \frac{1}{2} + 2C_L(2+c)\frac{M}{r^2} \qquad \textit{and} \qquad \frac{1}{2} \geq \gamma \geq \frac{1}{2} + \frac{C(\epsilon)}{r^2}. \tag{8.15}$$

Proof. The bound for γ is the statement of (8.13). To obtain the bound for κ integrate (2.13) from $r^\star = \frac{9}{10}$ to the asymptotic region of $\mathcal{A}(T)$ in u using that $t \sim r^\star \sim r$ in the region $r^\star \geq \frac{9}{10}t$ (cf. again (7.35)) and that the energy estimate holds in the region under consideration. Note again that r could be replaced by t in that region. □

8.3 Stability of the coordinate systems

8.3.1 The relation between $r^\star$ and r.

We are now in a position to derive an estimate for the relation between the coordinate $r^\star = \frac{v-u}{2}$ and the function $r(u, v)$. This estimate in conjunction with Proposition 7.2 will automatically improve bootstrap assumption 1.3.2, which — modulo the error term — expresses precisely the relation of the tortoise coordinate $r^\star$ to the area radius in the five-dimensional Schwarzschild metric. For this section we will use $C((r_K, c)$ to denote a constant which depends on the weight of $\frac{1}{1-\mu}$ on $r = r_K$ and on the parameter c in the bootstrap assumptions.

Proposition 8.5. *The estimate*

$$\left| r^\star - \left[r\left(t, r^\star\right) + \sqrt{\frac{M_A}{2}} \left(\log\left(\frac{r\left(t, r^\star\right) - \sqrt{2M_A}}{r\left(t, r^\star\right) + \sqrt{2M_A}} \right) + p \right) \right] \right| \leq C\left(r_K, c\right) \frac{M}{t} \tag{8.16}$$

with p defined in (7.6) and M_A the Hawking mass at the point $(T, r^\star = 0)$, holds in the region $\mathcal{A}(T) \cap \{r^\star \geq r^\star_K\}$.

Proof. The estimates

$$|\partial_t r| = |\lambda + \nu| \leq C\left(r_K, c\right) \frac{M}{t^2}, \tag{8.17}$$

$$\partial_{r^\star} r = \lambda - \nu \leq (1 - \mu) + C\left(r_K, c\right) \frac{M}{t^2} \leq \left(1 - \frac{2M_A}{r^2}\right) + C\left(r_K, c\right) \frac{M}{t^2} \tag{8.18}$$

in the region $\mathcal{A}(T) \cap \{r^\star \geq r^\star_K\} \cap \{r^\star \leq \frac{9}{10}t\}$ are a direct consequence of Proposition 8.4. Since also $r\left(T, r^\star = 0\right) = 2\sqrt{M_A}$, relation (8.16) follows in the region $\mathcal{A}(T) \cap \{r^\star \geq r^\star_K\} \cap \{r^\star \leq \frac{9}{10}t\}$. An application of Corollary 8.2 finally extends the bound to the remaining region, $r^\star \geq \frac{9}{10}t$. □

This means that in the region $\mathcal{A}(T) \cap \{r^\star \geq r^\star_K\}$ we can go back and forth from r to $r^\star$ with an error-term of $\frac{1}{t}$, which is small at late times. In analogy with Corollary 8.1 we also have

Corollary 8.3. *In the region $\mathcal{A}(T) \cap \{r^\star \geq r^\star_{\mathrm{cl}}\}$ estimate (8.16) holds with constant $C\left(r_{\mathrm{cl}}, c\right)$ replacing $C\left(r_K, c\right)$.*

8.3.2 Stability of constant t slices

In this section we are going to study the relation of the different coordinate systems $\mathcal{C}_{\tilde{\tau}} = (u_{\tilde{\tau}}, v_{\tilde{\tau}})$ associated with different $\tilde{\tau} \in A$ (cf. Section 3). Instead of the smallness estimates entering the proof of Proposition 7.2, we will now exploit the decay estimates for the quantities κ and γ derived in Proposition 8.4. Recall Notation 3.1.

Proposition 8.6. *Let $\tilde{\tau}_A \in A$. In view of Corollary 7.1 assume $\tilde{\tau}_A \geq \tilde{\tau}_0$. Then in the coordinate system $\mathcal{C}_{\tilde{\tau}_A} = (u_{\tilde{\tau}_A}, v_{\tilde{\tau}_A})$ associated to $\tilde{\tau}_A$ the* *t-coordinate of the initial data slice satisfies the bound*

$$\sup_{\tilde{S}\cap\{r^\star \geq r^\star_K\}\cap\{u\leq u_0\}} |t-\sqrt{M}| \leq C(\epsilon) . \tag{8.19}$$

Proof. By Proposition 7.2 statement (8.19) already holds up to $\tilde{\tau}_0$ by a suitable smallness assumption on the initial data in all coordinate systems $\mathcal{C}_{\tilde{\tau}}$ with $\tilde{\tau} \leq \tilde{\tau}_0$. Consider now a coordinate system $\mathcal{C}_{\tilde{\tau}_A}$ for a $\tilde{\tau}_A \geq \tilde{\tau}_0$. Recall the vectorfields G and R defined in (7.25). In the following, we again frequently refer to figure 3 of Section 3. At the point A we have $t^{\tilde{\tau}_A}_{\tilde{\tau}_A} = T$ by definition. We would like to estimate the value $t^{\tilde{\tau}_A}_0$ at the point D and compare it to $\sqrt{M}$, which is the value of t if $\tilde{\tau}=0$ and the coordinates are defined on the initial data. The rate at which t changes in affine parameter τ along the integral curve of $\nabla_\perp r$ is given by (7.27). We first integrate (7.27) along the curve $r^2 = 4m_A$, from A to the point A', which is defined to be on the ∇r slice associated with $\tilde{\tau}_0$. Using the decay estimates

$$|\kappa+\gamma-1| \leq C_L(2+c)\frac{M}{t^2} \quad \text{and} \quad \left|\frac{1}{\sqrt{1-\mu}}-\frac{1}{\sqrt{1-\mu_A}}\right| \leq C(c)\frac{M}{t^2} \tag{8.20}$$

which hold along the curve by Proposition 8.4 (and its corollaries), we obtain an estimate

$$T - t^{\tilde{\tau}_A}_{\tilde{\tau}_0} \leq \frac{1}{\sqrt{1-\frac{2m_A}{r^2}}}(\tau_A-\tau_{A'}) + C(c, r^\star_{\mathrm{cl}})\frac{M}{t^{\tilde{\tau}_A}_{\tilde{\tau}_0}}, \tag{8.21}$$

where the last term is small and the constant $C(c, r^\star_{\mathrm{cl}})$ depends on the weight of $\frac{1}{1-\mu}$ on $r^\star_{\mathrm{cl}}$. Using the definition of $T = \vartheta(\tilde{\tau}_A) = \sqrt{M} + \frac{\tau_A}{\sqrt{1-\mu_A}}$ we derive

$$\left| t^{\tilde{\tau}_A}_{\tilde{\tau}_0} - \frac{\tau_{A'}}{\sqrt{1-\frac{2m_A}{r^2}}} - \sqrt{M}\right| \leq C(c, r^\star_{\mathrm{cl}})\frac{M}{t^{\tilde{\tau}_A}_{\tilde{\tau}_0}} \tag{8.22}$$

and with the bootstrap assumption on the energy

$$\left| t^{\tilde{\tau}_A}_{\tilde{\tau}_0} - \frac{\tau_{A'}}{\sqrt{1-\frac{2m_{A'}}{r^2}}} - \sqrt{M}\right| \leq C(c, r^\star_{\mathrm{cl}})\frac{M}{t^{\tilde{\tau}_A}_{\tilde{\tau}_0}}. \tag{8.23}$$

In the second step we integrate (7.27) from A' to D. In this region we can use the smallness estimates for κ and γ as in the proof of Proposition 7.2

obtaining

$$\left| t_{\tilde{\tau}_0}^{\tilde{\tau}_A} - t_0^{\tilde{\tau}_A} \right| \leq \frac{1}{\sqrt{1 - \frac{2m_{A'}}{r^2}}} \left(\tau_{A'}\right) + \sqrt{M} C\left(\epsilon\right), \tag{8.24}$$

where we used the fact $\frac{1}{\sqrt{M}} C(\epsilon) \cdot \tau_{A'}$ is small by a suitable choice of the initial data, which in turn follows from the smallness of $\frac{1}{\sqrt{M}} C(\epsilon)\, \tilde{\tau}_0$ by Proposition 7.2 and the estimate $|\tau_{A'} - \tilde{\tau}_0| \leq \sqrt{M} C(\epsilon)$. Putting together estimates (8.23) and (8.24) we obtain

$$\left| \sqrt{M} + \frac{\tau_{A'}}{\sqrt{1 - \frac{2m_{A'}}{r^2}}} - t_0^{\tilde{\tau}_A} \right| \leq \frac{\tau_{A'}}{\sqrt{1 - \frac{2m_{A'}}{r^2}}} + C\left(c, r^\star_{\text{cl}}\right) \frac{M}{t_{\tilde{\tau}_0}^{\tilde{\tau}_A}} + \sqrt{M} C\left(\epsilon\right) \tag{8.25}$$

from which it follows (choosing $\tilde{\tau}_0$ large enough and the initial data suitably small) that the t-coordinate at D is close to $\sqrt{M}$. In the second step, which is identical to the one in Proposition 7.2, one finally shows that t only changes by $C(\epsilon)$ along the ∇r-curve through D on which the initial data is defined. □

Corollary 8.4. *Bootstrap assumption* 2 *is improved.*

One easily generalizes the previous proposition to the statement that t does not change much along a ∇r integral curve located in the bootstrap region:

Proposition 8.7. *With the assumptions of Proposition* 8.6*, the t-coordinate along the ∇r integral curve associated to $\sqrt{M} \leq \tilde{\tau}_i \leq \tilde{\tau}_A$ satisfies*

$$\sup_{(\nabla r)_{\tilde{\tau}_i} \cap \{r^\star \geq r^\star_K\}} |t - \vartheta\left(\tilde{\tau}_i\right)| \leq C\left(\epsilon\right) \tag{8.26}$$

in the coordinate system $\mathcal{C}_{\tilde{\tau}_A} = (u_{\tilde{\tau}_A}, v_{\tilde{\tau}_A})$.

Proof. Repeat the proof of the previous proposition, now integrating only up to the ∇r integral curve associated with $\tilde{\tau}_i$. In the second step, when integrating equation (7.39) along the ∇r integral curve, one again uses the smallness estimate for $\kappa - \gamma$ in $[r^\star_K, \tilde{R}^\star]$. However the decay estimate

$$\left(\gamma - \kappa\right) \frac{1}{\sqrt{1 - \mu}} \leq \tilde{\epsilon} \frac{M^{\frac{3}{4}}}{r^{\frac{3}{2}}} \tag{8.27}$$

following from Proposition 8.2 can now be used in the region $[\tilde{R}, \infty)$. The $\tilde{\epsilon}$ arises because $\frac{M^{\frac{1}{4}}}{\sqrt{r}}$ is small in $\left[\tilde{R}, \infty\right)$. Inserting that the affine parameter

ϱ is proportional to r (cf. (7.34)), one concludes that

$$0 \leq -\frac{dt}{d\varrho} \leq \hat{\epsilon}\frac{M^{\frac{3}{4}}}{\varrho^{\frac{3}{2}}} \tag{8.28}$$

and hence the change in t along any ∇r integral curve is also small within the region $[\tilde{R}, \infty)$. □

Proposition 8.8. *Statement* 3 *of Proposition* 7.2 *holds in any coordinate system* $\mathcal{C}_{\tilde{\tau}}$ *for* $\tilde{\tau} \geq \tilde{\tau}_0$ *and* $\tilde{\tau} \in A$.

Proof. Bootstrap assumption 1 and the previous proposition implies that the location of the region ${}^{u_{\text{hoz}}}\mathcal{D}^{r^\star_K, u_0}_{[2\sqrt{M}, T_0]}$ only changes slightly between the different coordinate systems. In particular, the v coordinate of the region ${}^{u_{\text{hoz}}}\mathcal{D}^{r^\star_K, u_0}_{[2\sqrt{M}, T_0]}$ is uniformly bounded in the different coordinate systems, as is the t coordinate for $r^\star \geq r^\star_K$. Hence if statement 3 of Proposition 7.2 holds in the coordinate system $\mathcal{C}_{\tilde{\tau}_0}$ it also holds in the coordinate system $\mathcal{C}_{\tilde{\tau}}$ for $\tilde{\tau} \geq \tilde{\tau}_0$ and $\tilde{\tau} \in A$.[36] □

Finally, we conclude from Proposition 8.5

Corollary 8.5. *Bootstrap assumption* 1 *is improved.*

Proof. We apply Proposition 7.2, i.e., we choose t_0 large such that $C(r_K, c)$ $\frac{M}{t_0}$ is very small (in particular smaller than $\frac{c}{4}$) and the initial data so small that the bootstrap assumptions hold with constant $\frac{c}{2}$ at $t = t_0$. Then for $t > t_0$ the estimate of Proposition 8.5 takes over and improves the constant c in (7.5). □

8.4 Pointwise bounds

In this subsection we derive pointwise decay bounds on the squashing field B and its derivatives, as well as on some higher order quantities. The key idea is that these bounds hold up to some large time t_0 by Cauchy stability (cf. Propositions 7.2 and 8.8).[37] After that time the energy decay derived from the bootstrap assumptions in Proposition 8.3 ensures appropriate decay estimates for the fields.

[36] The δ of Proposition 7.2 may have to be chosen slightly smaller but the change is uniform in $\tilde{\tau}_A$ and hence the size of the bootstrap region!

[37] The location of $t = t_0$ might change slightly from coordinate system to coordinate system but the change is uniformly controlled by $C(\epsilon)$ as has just been established in Section 8.3.2.

8.4.1 The squashing field and its derivatives

Proposition 8.9. *The pointwise bound*

$$|B(t,r^\star)| \leq \sqrt{C_L}\; C(c)\,\frac{\sqrt{M}}{t} \tag{8.29}$$

holds everywhere in $\mathcal{A}(T) \cap \{r^\star_{\mathrm{cl}} \leq r^\star \leq \frac{9}{10}t\}$.

Proof. Estimate (8.29) holds for $t \in [2\sqrt{M}, t_0]$ (for some large but finite t_0) by Proposition 8.8 with an appropriate choice of the initial data. For $[t_0, T]$ we integrate out in the u-direction from the set $L = \{u = u_0\} \cup \left(\{t = t_0\} \cap \{r^\star \geq \frac{9}{10}t_0\}\right)$, where either $B \equiv 0$ by the assumption of compact support or the bound (8.29) holds by Cauchy stability, to the $r^\star = \frac{9}{10}t$ curve:

$$\begin{aligned} B\left(t, r^\star = \frac{9}{10}t\right) &= B\left(u_L, v = \frac{19}{10}t\right) + \int_{u_L}^{\frac{1}{10}t} B_{,u}\left(u, v = \frac{19}{10}t\right) du \\ &\leq \tilde{\delta}\frac{\sqrt{M}}{v} + \sqrt{\int_{u_L}^{\frac{1}{10}t} \frac{4\lambda}{\Omega^2}\zeta^2 du}\sqrt{\int_{u_L}^{\frac{1}{10}t} \frac{-4\kappa\nu}{4r^3\lambda}du} \leq C(\epsilon)\,\frac{\sqrt{M}}{t} \end{aligned} \tag{8.30}$$

since $r \sim r^\star \sim t$ on the curve. Note also that along a line of constant v in the region $r^\star \geq \frac{9}{10}t$ we have $v \sim t$. Integrating out further from a point $\left(t, \frac{9}{10}t\right)$ on the $r^\star = \frac{9}{10}t$-curve along the slice $t = \text{const}$ we obtain

$$\begin{aligned} |B| &\leq C(\epsilon)\,\frac{\sqrt{M}}{t} + \int_{r^\star_{\mathrm{cl}}}^{\frac{9}{10}t} |\partial_{r^\star} B|\, dr^\star \\ &\leq C(\epsilon)\,\frac{\sqrt{M}}{t} + \sqrt{\int_{r^\star_{\mathrm{cl}}}^{\frac{9}{10}t} (\partial_{r^\star} B)^2\, r^3 dr^\star}\sqrt{\int_{r^\star_{\mathrm{cl}}}^{\frac{9}{10}t} \left[-\frac{\partial}{\partial r^\star}\frac{1}{r^2}\right]\left(\frac{1}{2\frac{\partial r}{\partial r^\star}}\right) dr^\star} \\ &\leq C(\epsilon)\,\frac{\sqrt{M}}{t} + C(c)\,\frac{\sqrt{M}}{t}\sup\sqrt{\frac{1}{2\frac{\partial r}{\partial r^\star}}} \end{aligned} \tag{8.31}$$

yields (8.29) in the whole region $r^\star_{\mathrm{cl}} \leq r^\star \leq \frac{9}{10}t$ since $\sup\sqrt{\frac{1}{2\frac{\partial r}{\partial r^\star}}} \leq \frac{4}{5}C_L$ in the region $r^\star_{\mathrm{cl}}$. $\square$

Recall that in the region $r^\star \geq \frac{9}{10}t$ we were able to derive $\frac{1}{r}$-decay of the field B without involving the bootstrap assumptions (cf. Corollary 4.3). The next proposition shows that the boundedness of the quantity E_B^K improves this decay considerably:

Proposition 8.10. *In the region* $\mathcal{X} = \mathcal{A}(T) \cap \{r^\star \geq 3\sqrt{M}\} \cap \{u \geq \sqrt{M}\}$ *we have*

$$|B| \leq C(c)\,\frac{M}{r^{\frac{3}{2}} u^{\frac{1}{2}}}. \tag{8.32}$$

Proof. Choose a point $(t, r^\star)$ in the region $\mathcal{X}$ and a point $(t, \tilde{r}^\star)$ in the central region ($r^\star_{\mathrm{cl}} \leq \tilde{r}^\star \leq 3\sqrt{M}$). We have

$$r^3 B^2(t, r^\star) = r^3 B^2(t, \tilde{r}^\star) + \int_{\tilde{r}^\star}^{r^\star} \partial_{r^\star}\left(r^3 B^2\right) dr^\star. \tag{8.33}$$

By Proposition 8.9, $|B(t, \tilde{r}^\star)| \leq \frac{C}{t}$. Moreover, by Cauchy–Schwarz

$$\int_{\tilde{r}^\star}^{r^\star} \partial_{r^\star}\left(r^3 B^2\right) dr^\star \leq 2\sqrt{\int_{\tilde{r}^\star}^{r^\star} \left(\partial_{r^\star} B\right)^2 r^3 u^2 dr^\star}\,\frac{1}{t}\sqrt{\int_{\tilde{r}^\star}^{r^\star} t^2 B^2 r^3 \frac{1}{u^2} dr^\star} + \frac{1}{t^2}\int_{\tilde{r}^\star}^{r^\star} t^2 B^2 r^2 dr^\star. \tag{8.34}$$

We can finally insert inequalities (8.3) and (8.2) to find

$$B^2(t, r^\star) \leq C_L\ C(c)\,\frac{M\left(3\sqrt{M}\right)^3}{r^3 t^2} + 2C(c)\,\frac{M^2}{t r^2 u} + C(c)\,\frac{M^2}{r^2 t^2}. \tag{8.35}$$

Noting that in the region $u \geq 1$ we have for large times $t \geq \frac{r}{2}$ yields the desired result. □

For the region $u_0 \leq u \leq \sqrt{M}$ we can follow the same proof replacing u^2 by $u^2 + M$ (to avoid dividing by zero) to obtain

Proposition 8.11. *In the region* $\widetilde{\mathcal{X}} = \mathcal{A}(T) \cap \{r^\star \geq 3\sqrt{M}\}$ *we have*

$$|B| \leq C(c)\,\frac{M^{\frac{3}{4}}}{r^{\frac{3}{2}}}. \tag{8.36}$$

Having established better decay of B from the bootstrap assumptions we can also derive better decay of θ via an auxiliary quantity Θ, which is the analogue of the almost Riemann invariant in four dimensions. Note however that we cannot use Θ to improve the decay in B itself (as in the four-dimensional case) but only in its derivatives, once better decay in B has already been established.

Lemma 8.1. *On $\{r^\star = \frac{9}{10}t\}$, the quantity $\Theta = \theta + \frac{3}{2}\sqrt{r}\lambda B$ satisfies*

$$|\Theta\left(u,v\right)| \leq C\left(c\right)\frac{M^{\frac{3}{4}}}{v_+}. \tag{8.37}$$

Proof. Integrate the equation (recall definition (2.15))

$$\partial_u \Theta = B\left[\frac{35\lambda\nu}{4\sqrt{r}} - \frac{11\Omega^2}{2r^{\frac{5}{2}}}m\right] + \frac{1}{4}\frac{\Omega^2}{\sqrt{r}}\frac{\varphi_2\left(B\right)}{B} - \frac{\Omega^2}{2\sqrt{r}}B\left(\rho - \frac{3}{2}\right) \tag{8.38}$$

from the set $L = \{u = u_0\} \cup \left(\{t = t_0\} \cap \{r^\star \geq \frac{9}{10}t_0\right)\}$, where either $\Theta \equiv 0$ by the assumption of compact support or the bound (8.37) holds by Proposition 7.2 with constant $\tilde{\delta}$, to the $r^\star = \frac{9}{10}t$ curve. Since the right-hand side of equation (8.38) satisfies

$$\left|B\left[\frac{35\lambda\nu}{4\sqrt{r}} - \frac{11\Omega^2}{2r^{\frac{5}{2}}}m\right] + \frac{1}{4}\frac{\Omega^2}{\sqrt{r}}\frac{\varphi_2\left(B\right)}{B} - \frac{\Omega^2}{2\sqrt{r}}B\left(\rho - \frac{3}{2}\right)\right| \leq C\left(c\right)\frac{M}{r^2} \tag{8.39}$$

in the region $r^\star \geq \frac{9}{10}t$ following in turn from the decay of B derived in Proposition 8.10, we obtain the estimate

$$|\Theta\left(u,v\right)| \leq \tilde{\delta}\frac{M^{\frac{3}{4}}}{v_+} + C\left(c\right)\frac{M^{\frac{3}{4}}}{r} \leq C\left(c\right)\frac{M^{\frac{3}{4}}}{v_+}. \tag{8.40}$$

□

Corollary 8.6.

$$|\theta\left(u,v\right)| \leq C\left(c\right)\frac{M^{\frac{3}{4}}}{r} \tag{8.41}$$

holds in $r^\star \geq \frac{9}{10}t$.

Proof. Use Lemma 8.1 and take into account Proposition 8.10. □

Proposition 8.12. *The pointwise bound*

$$|\theta\left(t,r^\star\right)| \leq C\left(c\right)\frac{M^{\frac{3}{4}}}{t} \tag{8.42}$$

holds everywhere in $\mathcal{A}\left(T\right) \cap \{r^\star_{\text{cl}} \leq r^\star \leq \frac{9}{10}t\}$.

Proof. By the previous corollary

$$|\theta\,(u,v)\,| \le C\,(c)\,\frac{M^{\frac{3}{4}}}{t} \tag{8.43}$$

holds on $r^\star = \frac{9}{10}t$. We can integrate equation (4.28) from that curve or $t = t_0$, where the bound $|\theta| \le \tilde{\delta}\frac{M^{\frac{3}{4}}}{v}$ holds by Proposition 7.2, to any point in the region $\mathcal{A}\,(T) \cap \{r^\star \le \frac{9}{10}t\}$:

$$\theta\,(u,v) = \theta\,(u_i,v) - \frac{3}{2}\int_{u_i}^{u}\frac{\zeta}{r}\lambda d\bar{u} + \int_{u_i}^{u}\frac{\Omega^2}{3\sqrt{r}}\left(\mathrm{e}^{-8B} - \mathrm{e}^{-2B}\right)d\bar{u} \tag{8.44}$$

and hence

$$\begin{aligned} |\theta\,(u,v)\,| \le{}& \tilde{\delta}\frac{M^{\frac{3}{4}}}{v} + \frac{3}{2}\sqrt{\int_{u_i}^{u}\frac{4\zeta^2\lambda}{\Omega^2}d\bar{u}}\sqrt{\int_{u_i}^{u}\frac{-4\kappa\nu\lambda}{r^2}d\bar{u}} \\ &+ \frac{1}{3}\sqrt{\int_{u_i}^{u}-4\kappa\nu\left(\mathrm{e}^{-8B} - \mathrm{e}^{-2B}\right)^2 r d\bar{u}}\sqrt{\int_{u_i}^{u}\frac{-4\kappa\nu}{r^2}d\bar{u}} \le C\,(c)\,\frac{M^{\frac{3}{4}}}{v}. \end{aligned} \tag{8.45}$$

The energy estimate, Proposition 8.3 and the fact that $v \sim t$ in the region $r^\star_{\mathrm{cl}} \le r^\star \le \frac{9}{10}t$ yields the desired result. □

With the previous proposition and Proposition 4.1, Corollary 8.6 is easily extended to the entire bootstrap region:

Corollary 8.7.

$$|\theta\,(u,v)\,| \le C\,(c)\,\frac{M^{\frac{3}{4}}}{r} \tag{8.46}$$

holds in all of $\mathcal{A}\,(T)$.

Close to the horizon we have

Proposition 8.13. *The pointwise bounds*

$$|B\,(t,r^\star_{\mathrm{cl}})\,| + |\theta\,(t,r^\star_{\mathrm{cl}})\,| \le 2\sqrt{C_L}\;C\,(c)\,\frac{\sqrt{M}}{v_+}. \tag{8.47}$$

hold everywhere in $\mathcal{A}\,(T) \cap \{r^\star \le r^\star_{\mathrm{cl}}\} \cap \{u \le T - r^\star\,(T,r_K)\}$.

Proof. The decay for θ was already obtained in the proof of Proposition 8.12. From Proposition 8.9, we know that on $r^\star = r^\star_{\text{cl}}$ we have the decay

$$|B| \leq \sqrt{C_L}\; C(c)\, \frac{\sqrt{M}}{v_+}. \tag{8.48}$$

Consequently,

$$\begin{aligned} B(u,v) &= B\left(u_{r^\star_{\text{cl}}}, v\right) + \int_{u_{r^\star_{\text{cl}}}}^{u} \frac{\zeta}{r^{\frac{3}{2}}}(\bar{u}, v)\, d\bar{u} \\ &\leq \sqrt{C_L}\; C(c)\, \frac{\sqrt{M}}{v_+} + \sqrt{\int_{u_{r^\star_{\text{cl}}}}^{u} \frac{\zeta^2}{-\nu} d\bar{u}} \sqrt{\int_{u_{r^\star_{\text{cl}}}}^{u} \frac{-\nu}{r^3} d\bar{u}} \end{aligned} \tag{8.49}$$

and upon inserting bootstrap assumption (7.11) we obtain the result. $\square$

8.4.2 Higher order quantities

In this subsection various bounds on the derivatives of the quantity Ω^2 are proven. Since we have not yet established a pointwise bound on $\frac{\zeta}{\nu}$, the estimates will turn out to be suboptimal.[38] However, they suffice to estimate certain error terms in the X-vectorfield identity. An interplay between the X and the Y vectorfield will finally generate a pointwise bound on $\frac{\zeta}{\nu}$, which allows one to optimize the estimates (cf. Proposition 8.16). In particular, the new decay will then suffice to control the error terms occurring in identity (1.10) for the vectorfield K.

The first step is to improve Proposition 4.6 to a decay bound. In the following $C(r^\star_{\text{cl}}, c)$ denotes a constant whose weight is determined by C_L and which also depends on the c in the bootstrap assumptions.

It should be emphasized again that the quantity $\frac{\Omega_{,v}}{\Omega}$ is only piecewise continuous, with a discontinuity spreading along the null line $v = T + r^\star(T, r_K)$. The estimates below are valid because the quantity $\partial_u \frac{\Omega_{,v}}{\Omega}$, which is integrated along null-lines, is continuous. The same considerations are valid for the quantity $\frac{\Omega_{,u}}{\Omega}$ whose discontinuity is along the null line $u = T - r^\star(T, r_K)$.

[38]Such a pointwise bound could in principle be established via a bootstrap argument in the style of Proposition 4.2, with the pointwise decay bound on B (8.48) now entering the estimates.

Proposition 8.14. *In the region $\mathcal{A}(T) \cap \{r^\star \geq r^\star_K\} \cap \{r^\star \leq \frac{9}{10}t\}$ we have the one-sided bound*

$$\frac{\Omega_{,v}}{\Omega} - \frac{m}{r^3} \leq C\left(r^\star_{\mathrm{cl}}, c\right) \frac{\sqrt{M}}{t^2}. \tag{8.50}$$

In $\mathcal{A}(T) \cap \{r^\star \leq r^\star_{\mathrm{cl}}\}$ we have

$$\frac{\Omega_{,v}}{\Omega} - \frac{m}{r^3} \leq C\left(r^\star_{\mathrm{cl}}, c\right) \frac{\sqrt{M}}{v_+^2}. \tag{8.51}$$

Proof. Since $\kappa = \frac{1}{2}$ on the $u = T - r^\star(T, r_K)$ ray we have

$$\kappa_{,v} = 0 = 2\kappa \frac{\Omega_{,v}}{\Omega} - \kappa\left(\frac{4\kappa}{r^3}m + \frac{4}{3}\frac{\kappa}{r}\left(\rho - \frac{3}{2}\right)\right) \tag{8.52}$$

and, in view of Proposition 8.13, the estimate

$$\left|\frac{\Omega_{,v}}{\Omega}\left(u = T - r^\star(T, r_K), v\right)\right| \leq \frac{m}{r^3}\left(u = T - r^\star(T, r_K), v\right) + C\left(r^\star_{\mathrm{cl}}, c\right)\frac{\sqrt{M}}{v_+^2}. \tag{8.53}$$

Moreover, on $\{t = T\} \cap \{r^\star(T, r_K) \leq r^\star \leq \frac{9}{10}T\}$ we have by the constancy of κ

$$\kappa_{,r^\star} = 0 = 2\kappa \frac{\Omega_{,v}}{\Omega} - \kappa\left(\frac{4\kappa}{r^3}m + \frac{4}{3}\frac{\kappa}{r}\left(\rho - \frac{3}{2}\right)\right) - \kappa\frac{2}{r^2}\frac{\zeta^2}{\nu} \tag{8.54}$$

and hence

$$\frac{\Omega_{,v}}{\Omega} - \frac{m}{r^3} \leq C\left(r^\star_{\mathrm{cl}}, c\right) \frac{\sqrt{M}}{v_+^2} \tag{8.55}$$

following from the fact that $|B| \leq \frac{C}{v_+}$ in that region by Proposition 8.13. Note that inequality (8.55) would also be two-sided if we had the analogous pointwise bound on $\frac{\zeta}{\nu}$. Integrating (2.5) downwards from the set $L = \{u = T - r^\star(T, r_K)\} \cup (\{t = T\} \cap \{r^\star(T, r_K) \leq r^\star \leq r^\star_{\mathrm{cl}}\})$ to the $r^\star =$

$r^\star_{\text{cl}}$ curve yields

$$\frac{\Omega_{,v}}{\Omega}(u,v) = \frac{\Omega_{,v}}{\Omega}(u_L,v) + \int_{u_L}^{u}\left[-6\frac{\kappa m\nu}{r^4} - 2\kappa\frac{\nu}{r^2}\left(\rho-\frac{3}{2}\right) - 3\frac{\zeta\theta}{r^3}\right](\bar{u},v)\,d\bar{u} \tag{8.56}$$

and upon inserting the pointwise estimates on B and θ (Proposition 8.13), bootstrap assumption 1.3.2 and the estimate

$$\begin{aligned}\left|\int_{u}^{u_L}\left[-3\frac{\zeta\theta}{r^3}\right](\bar{u},v)\,d\bar{u}\right| &\leq 3C(c)\frac{M^{\frac{3}{4}}}{v_+}\sqrt{\int_{u_L}^{u}\frac{\zeta^2}{\Omega^2}d\bar{u}}\sqrt{\int_{u_L}^{u}\frac{-4\kappa\nu}{r^6}d\bar{u}} \\ &\leq C(c)\sqrt{C_L}\frac{M^{\frac{7}{4}}}{r_-^{\frac{5}{2}}}\frac{1}{v_+^2}\end{aligned}$$

for which (7.11) has been used, we finally find that

$$\frac{\Omega_{,v}}{\Omega} - \frac{m}{r^3} \leq C(r^\star_{\text{cl}},c)\frac{\sqrt{M}}{v_+^2} \tag{8.57}$$

holds everywhere in $r^\star \leq r^\star_{\text{cl}}$ establishing (8.51). Starting from this curve or from the curve $\{t=T\}\cap\{r^\star_{\text{cl}}\leq r^\star \leq \frac{9}{10}t\}$ we can integrate (2.5) further to any point in the region $\mathcal{A}(T)\cap\{r>r_K\}\cap\{r^\star\leq\frac{9}{10}t\}$, this time using the energy estimate instead of (7.11) to obtain (8.50). □

Proposition 8.15. *In the region* $\mathcal{A}(T)\cap\{r^\star\geq r^\star_K\}\cap\{r^\star\leq\frac{9}{10}t\}$

$$\left|\frac{\Omega_{,u}}{\Omega} + \frac{m}{r^3}\right| \leq \frac{C(\epsilon)}{t}, \tag{8.58}$$

in the region $\mathcal{A}(T)\cap\{r^\star\leq r^\star_K\}\cap\{u\leq T-r^\star_K\}$

$$0 > \frac{\Omega_{,u}}{\Omega} \geq -\frac{m}{r^3} - \frac{C(\epsilon)}{v_+}. \tag{8.59}$$

Proof. $\gamma=\frac{1}{2}$ on the set $L=\{t=T\}\cap\{r^\star(T,r_K)\leq\frac{9}{10}T\}$. From

$$-\gamma_{,r^\star} = 0 = 2\gamma\frac{\Omega_{,u}}{\Omega} - \gamma\left(-\frac{4\gamma}{r^3}m - \frac{4}{3}\frac{\gamma}{r}\left(\rho-\frac{3}{2}\right)\right) - \gamma\frac{2}{r^2}\frac{\theta^2}{\lambda} \tag{8.60}$$

we derive using the decay estimates (8.29), (8.47), (8.42) the bound

$$\left|\frac{\Omega_{,u}}{\Omega}+\frac{m}{r^3}\right| \leq C\left(r_K, c\right)\frac{M}{rT^2} \tag{8.61}$$

on L. We write the evolution equation (2.5) as

$$\partial_v\left(\frac{\Omega_{,u}}{\Omega}\right) = \gamma\left(6m\frac{\lambda}{r^4}+\frac{2\lambda}{r^2}\left(\rho-\frac{3}{2}\right)+3\frac{\theta}{\kappa}\frac{\zeta}{\nu}\frac{\lambda}{r^3}\right) \tag{8.62}$$

and integrate downwards in v. Using estimates (8.9), (8.10) and again the decay estimates for B and θ, the error terms are estimated:

$$\left|\int_v^{v_L}\left[3\gamma\frac{\theta}{\kappa}\frac{\zeta}{\nu}\frac{\lambda}{r^3}\right]\right|\left(\bar{u},v\right)d\bar{v} \leq C\left(c\right)\frac{M^{\frac{3}{4}}}{v_+}\sup\left|\frac{\zeta}{\nu}\right|\int_v^{v_L}\frac{\lambda}{r^3}d\bar{v} \leq \frac{C\left(\epsilon\right)}{v_+} \tag{8.63}$$

and

$$\left|\int_v^{v_L}\gamma\frac{2\lambda}{r^2}\left(\rho-\frac{3}{2}\right)\right| \leq \frac{C\left(\epsilon\right)}{v_+^2}. \tag{8.64}$$

This establishes estimate (8.58) in a subregion $(u \geq \frac{1}{10}T)$ of the region asserted in the proposition. For the remaining part, we derive the estimate

$$\left|\frac{\Omega_{,u}}{\Omega}+\frac{m}{r^3}\right| \leq C\left(\epsilon\right)\frac{M}{r^3} \tag{8.65}$$

valid on $\{t=T\}\cap\{r^\star \geq \frac{9}{10}T\}$ using the decay of B, θ (Corollary 8.7 and Proposition 8.11) in r. Integrating (8.62) downwards to any point in the region $\{r^\star \geq \frac{9}{10}t\}$ using again the estimates for B and θ one obtains (8.65) in the entire region $\{r^\star \geq \frac{9}{10}t\}$. Since $t \sim r^\star$ on the curve $r^\star = \frac{9}{10}t$ we obtain

$$\left|\frac{\Omega_{,u}}{\Omega}+\frac{m}{r^3}\right| \leq C\left(\epsilon\right)\frac{M}{rt^2} \tag{8.66}$$

on that curve. Finally, integrating (8.62) from the $r^\star = \frac{9}{10}t$ curve downwards up to any point in the region $r^\star \geq r^\star_K$ yields (8.58) for the entire region asserted in the proposition.

For estimate (8.59) we integrate (8.62) from $r^\star = r^\star_K$ where $t \sim v$ downwards. Clearly, the round bracket on the right-hand side of (8.62) is always positive and hence the upper bound of (8.59) follows immediately. For the lower bound we use the estimate $\gamma \leq \frac{1}{2}$ available in the region under consideration to estimate:

$$\begin{aligned}
\frac{\Omega_{,u}}{\Omega}(u,v) \geq & \frac{\Omega_{,u}}{\Omega}(u, v = u + 2r^\star_K) \\
& - \int_v^{u+2r^\star_K} \frac{1}{2}\left(6m\frac{\lambda}{r^4} + \frac{2\lambda}{r^2}\left(\rho - \frac{3}{2}\right) + 3\frac{\theta}{\kappa}\frac{\zeta}{\nu}\frac{\lambda}{r^3}\right) \\
\geq & -\frac{m}{r^3}(u, u+2r^\star_K) \\
& - m(u, u+2r^\star_K)\left(\frac{1}{r^3}(u,v) - \frac{1}{r^3}(u, u+2r^\star_K)\right) - \frac{C(\epsilon)}{v_+} \\
\geq & -\frac{m}{r^3}(u,v) - \frac{C(\epsilon)}{v_+}.
\end{aligned}$$

□

We easily extend the bounds to the asymptotic region:

Corollary 8.8. *In the region $\mathcal{A}(T) \cap \{r^\star \geq \frac{9}{10}t\}$ we have*

$$\frac{\Omega_{,v}}{\Omega} - \frac{m}{r^3} \leq C(r^\star_{\text{cl}}, c)\frac{\sqrt{M}}{r^2} \qquad \text{and} \qquad \left|\frac{\Omega_{,u}}{\Omega} + \frac{m}{r^3}\right| \leq C(\epsilon)\frac{M}{r^3}. \tag{8.67}$$

Proof. The first bound follows from integrating equation (2.5) outwards from $r^\star = \frac{9}{10}t$ using that $r \sim t$ in that region and the decay of the fields in r. The second bound was obtained in (8.65). □

As seen in the proof of Propositions 8.14 and 8.15 a pointwise *decay* bound on the quantity $\frac{\zeta}{\nu}$ would considerably improve the estimates on the higher order quantities. We summarize this as

Proposition 8.16. *Assume that*

$$\left|\frac{\zeta}{\nu}\right| \leq C\frac{M^{\frac{3}{4}}}{v_+} \text{ holds in } r^\star \leq r^\star_{\text{cl}} \quad \text{and} \quad \left|\frac{\zeta}{\nu}\right| \leq C\frac{M^{\frac{3}{4}}}{t} \quad \text{in } \frac{9}{10}t \geq r^\star \geq r^\star_{\text{cl}} \tag{8.68}$$

holds for a constant C depending only on $r^\star_{\text{cl}}$. Then in the region $\{r^\star \geq r^\star_g\}$ for any $r^\star_{\text{cl}} \geq r^\star_g \geq r^\star_K$ we have the bounds

$$\left|\frac{\Omega_{,v}}{\Omega} - \frac{m}{r^3}\right| \leq C(r^\star_{\text{cl}}, c)\frac{\sqrt{M}}{t^2} \qquad \text{and} \qquad \left|\frac{\Omega_{,u}}{\Omega} + \frac{m}{r^3}\right| \leq C\left(r^\star_g, c\right)\frac{\sqrt{M}}{t^2}. \tag{8.69}$$

Moreover, the one-sided bound (8.51) *in the region* $\mathcal{A}(T) \cap \{r^\star \le r^\star_{\text{cl}}\}$ *is extended to*

$$\left| \frac{\Omega_{,v}}{\Omega} - \frac{m}{r^3} \right| \le C\left(r^\star_{\text{cl}}, c\right) \frac{\sqrt{M}}{v_+^2}, \tag{8.70}$$

and the bound (8.59) *is refined to*

$$0 > \frac{\Omega_{,u}}{\Omega}(u,v) \ge -\frac{m}{r^3}(u,v) - C\left(r^\star_{\text{cl}}, c\right) \frac{\sqrt{M}}{v_+^2} \tag{8.71}$$

in the region $\mathcal{A}(T) \cap \{r^\star \le r^\star_{\text{cl}}\} \cap \{u \le T - r^\star_K\}$.

Proof. Revisit the proof of Propositions 8.14 and 8.15. Note that the constant in the u-estimate of (8.69) improves by moving away from the horizon since it depends on the weight of $\frac{1}{1-\mu}$ on $r^\star_g$. □

9 The vectorfield *Y*

Recall the functions α and β defined in Section 7. Close to the horizon we are going to apply the vectorfield

$$Y = \frac{2\alpha(r^\star)}{\Omega^2}\partial_u + 2\beta(r^\star)\partial_v \tag{9.1}$$

for which

$$Y^u = \frac{2\alpha}{\Omega^2}, \quad Y^v = 2\beta, \quad Y_u = -\beta\Omega^2, \quad Y_v = -\alpha. \tag{9.2}$$

The calculations will be carried out in the Eddington Finkelstein coordinates defined in Section 3. From (5.11) we derive the identity

$$\begin{aligned} &- T_{\mu\nu}\pi^{\mu\nu} - \left(\nabla^\beta T_{\beta\delta}\right) Y^\delta \\ &\quad = -\frac{2\left(\partial_u B\right)^2}{\Omega^4}\left(4\alpha\frac{\Omega_{,v}}{\Omega} - \alpha'\right) - 2\beta'\frac{\left(\partial_v B\right)^2}{\Omega^2} \\ &\qquad + \frac{1}{\Omega^2 r^2}\left(1 - \frac{2}{3}\rho\right)\left(-\frac{1}{2}\alpha' + \frac{\alpha\nu}{r} + \frac{\beta\lambda\Omega^2}{r} + \frac{1}{2}\beta'\Omega^2 + 2\beta\Omega^2\frac{\Omega_{,v}}{\Omega}\right) \\ &\qquad + \frac{12}{\Omega^2 r}\left(\frac{1}{4\kappa}\alpha - \lambda\beta\right)\partial_u B \partial_v B. \end{aligned} \tag{9.3}$$

In a characteristic rectangle $\mathcal{R} = [u_1, u_2] \times [v_1, v_2]$ the identity

$$F_B^Y\left(\{u_2\} \times [v_1, v_2]\right) + F_B^Y\left([u_1, u_2] \times \{v_2\}\right) \tag{9.4}$$
$$= I_B^Y\left(\mathcal{R}\right) + F_B^Y\left(\{u_1\} \times [v_1, v_2]\right) + F_B^Y\left([u_1, u_2] \times \{v_1\}\right) \tag{9.5}$$

follows, with the boundary terms given by

$$\frac{1}{2\pi^2} F_B^Y\left(\{u\} \times [v_1, v_2]\right) = 2\int_{v_1}^{v_2} \left(\beta \left(\partial_v B\right)^2 + \frac{1}{4r^2}\left(1 - \frac{2}{3}\rho\right)\alpha\right) r^3 dv, \tag{9.6}$$

$$\frac{1}{2\pi^2} F_B^Y\left([u_1, u_2] \times \{v\}\right) = 2\int_{u_1}^{u_2} \left(\frac{\alpha}{\Omega^2}\left(\partial_u B\right)^2 + \frac{\beta\Omega^2}{4r^2}\left(1 - \frac{2}{3}\rho\right)\right) r^3 du \tag{9.7}$$

and the spacetime term

$$\begin{aligned}\frac{1}{2\pi^2} I_B^Y\left(\mathcal{R}\right) &= \int_{v_1}^{v_2}\int_{u_1}^{u_2} \left(-T_{\mu\nu}\pi^{\mu\nu} - \left(\nabla^\beta T_{\beta\delta}\right) Y^\delta\right) \frac{1}{2}\Omega^2 r^3 du\, dv \\ &= \int_{v_1}^{v_2}\int_{u_1}^{u_2} \Bigg(-\Bigg[\frac{\left(\partial_u B\right)^2}{\Omega^2}\left(4\alpha\frac{\Omega_{,v}}{\Omega} - \alpha'\right) + \beta'\left(\partial_v B\right)^2 \\ &\quad + \frac{1}{2r^2}\left(1 - \frac{2}{3}\rho\right)\left(\frac{1}{2}\alpha' - \frac{\alpha\nu}{r} - \frac{\beta\lambda\Omega^2}{r} - \frac{1}{2}\beta'\Omega^2 - 2\beta\Omega^2\frac{\Omega_{,v}}{\Omega}\right)\Bigg] \\ &\quad + \frac{6}{r}\left(\frac{1}{4\kappa}\alpha - \lambda\beta\right)\partial_u B \partial_v B\Bigg) r^3 du\, dv. \end{aligned} \tag{9.8}$$

It will be useful to split the term into

$$I_B^Y\left(\mathcal{R}\right) = -\tilde{I}_B^Y\left(\mathcal{R}\right) + \widehat{I}_B^Y\left(\mathcal{R}\right), \tag{9.9}$$

where

$$\begin{aligned}\frac{1}{2\pi^2}\tilde{I}_B^Y\left(\mathcal{R}\right) &= \int_{v_1}^{v_2}\int_{u_1}^{u_2} \Bigg[\frac{\left(\partial_u B\right)^2}{\Omega^2}\left(4\alpha\frac{\Omega_{,v}}{\Omega} - \alpha'\right) + \beta'\left(\partial_v B\right)^2 + \frac{1}{2r^2}\left(1 - \frac{2}{3}\rho\right) \\ &\quad \times \left(\frac{1}{2}\alpha' - \frac{\alpha\nu}{r} - \frac{\beta\lambda\Omega^2}{r} - \frac{1}{2}\beta'\Omega^2 - 2\beta\Omega^2\frac{\Omega_{,v}}{\Omega}\right)\Bigg] r^3 du\, dv \end{aligned} \tag{9.10}$$

and

$$\frac{1}{2\pi^2}\widehat{I}_B^Y\left(\mathcal{R}\right) = \int_{v_1}^{v_2}\int_{u_1}^{u_2} \left(\frac{6}{r}\left(\frac{1}{4\kappa}\alpha - \lambda\beta\right)\partial_u B \partial_v B\right) r^3 du\, dv. \tag{9.11}$$

With the choices of the functions α and β made in Section 7, the integral $\tilde{I}^Y_B(\mathcal{R})$ is non-negative for $r^\star \le r^\star_Y$ and moreover, using (7.17),

$$\begin{aligned}
&\hat{I}^Y_B\left(r^\star \le r^\star_Y\right)\\
&\quad= 2\pi^2 \int_{v_1}^{v_2}\int_{u_1}^{u_2}\int_{\mathbb{S}^3}\left(\frac{6}{r}\left(\frac{1}{4\kappa}\alpha - \lambda\beta\right)\partial_u B \partial_v B\right) r^3 du\, dv\\
&\quad\le 2\pi^2 \int_{v_1}^{v_2}\int_{u_1}^{u_2}\frac{1}{2}\left(\frac{2\left(\partial_u B\right)^2}{\Omega^2}\frac{\left(\frac{1}{4\kappa}\alpha - \lambda\beta\right)^2}{r} + \frac{18}{r}\Omega^2\left(\partial_v B\right)^2\right) r^3 du\, dv\\
&\quad\le 2\pi^2 \int_{v_1}^{v_2}\int_{u_1}^{u_2}\frac{1}{2}\left(\frac{\left(\partial_u B\right)^2}{\Omega^2}\left(4\alpha\frac{\Omega_{,v}}{\Omega} - \alpha'\right) + \beta'\left(\partial_v B\right)^2\right) r^3 du\, dv\\
&\quad\le \frac{1}{2}\tilde{I}^Y_B\left(r^\star \le r^\star_Y\right)
\end{aligned} \tag{9.12}$$

holds in $r^\star \le r^\star_Y$. We conclude by rewriting identity (9.4) for a characteristic rectangle with one boundary being the horizon:

$$\begin{aligned}
&F^Y_B\left(\{u_{\text{hoz}}\}\times[v_1,v_2]\right) + F^Y_B\left([u_1,u_{\text{hoz}}]\times\{v_2\}\right) + \tilde{I}^Y_B(\mathcal{R})\\
&\quad= \hat{I}^Y_B(\mathcal{R}) + F^Y_B\left(\{u_1\}\times[v_1,v_2]\right) + F^Y_B\left([u_1,u_{\text{hoz}}]\times\{v_1\}\right).
\end{aligned} \tag{9.13}$$

10 The vectorfield X

All calculations in this section are performed in the Eddington Finkelstein coordinate system defined in Section 3.

10.1 The basic identity

The vectorfield X is defined as

$$X = 2f\left(r^\star\right)\partial_u - 2f\left(r^\star\right)\partial_v \tag{10.1}$$

for some function f chosen below and with u_J satisfying $u_J \ge t_1 - r^\star_{\text{cl}}$. It will be applied in the region

$$\mathcal{D}^{r^\star_{\text{cl}},u_J}_{[t_1,t_2]} := {}^{u_H = t_2 - r^\star_{\text{cl}}}\mathcal{D}^{r^\star_{\text{cl}},u_J}_{[t_1,t_2]} \tag{10.2}$$

for some $r^\star_{\text{cl}}$ also chosen below. We note

$$X^u = 2f, \quad X^v = -2f, \quad X_u = f\Omega^2, \quad X_v = -f\Omega^2. \tag{10.3}$$

From now on primes will denote a derivative with respect to $r^\star$, hence $\partial_v f(r^\star) = \frac{1}{2} f'$ and $\partial_u f(r^\star) = -\frac{1}{2} f'$. From (5.11) using

$$-\left(\partial_{r^\star} B\right)^2 = 2\partial_u B \partial_v B - \left(\partial_u B\right)^2 - \left(\partial_v B\right)^2, \tag{10.4}$$

we compute

$$\begin{aligned}
&- T_{\mu\nu}\pi^{\mu\nu} - \left(\nabla^\beta T_{\beta\delta}\right) X^\delta \\
&\quad = \frac{2}{\Omega^2} f' \left(\partial_{r^\star} B\right)^2 + \nabla^\alpha B \nabla_\alpha B \left(-f' - \frac{3}{r}\left(\lambda - \nu\right) f\right) \\
&\qquad + \frac{1}{r^2}\left(1 - \frac{2}{3}\rho\right)\left(-f' - \frac{\lambda - \nu}{r} f + \frac{\left(\Omega^2\right)_{,u} - \left(\Omega^2\right)_{,v}}{\Omega^2} f\right).
\end{aligned} \tag{10.5}$$

With the boundary terms

$$\begin{aligned}
\frac{1}{2\pi^2}\widehat{F}^X_B\left(t_i\right) = &-2\int_{r^\star_{\text{cl}}}^{t-u_J} f \partial_t B \partial_{r^\star} B\left(t_i, r^\star\right) r^3 dr^\star \\
&+ \int_{t-r^\star_{\text{cl}}}^{t_2 - r^\star_{\text{cl}}} \left[r^3\left(\partial_u B\right)^2 (2f) + \frac{r\Omega^2}{4}\left(1 - \frac{2}{3}\rho\right)(-2f)\right] du
\end{aligned} \tag{10.6}$$

and

$$\frac{1}{2\pi^2}\hat{H}^X_{u_H} = \int_{v_1}^{v_2}\left[r^3\left(\partial_v B\right)^2(-2f) + \frac{r\Omega^2}{4}\left(1 - \frac{2}{3}\rho\right)(2f)\right] dv, \tag{10.7}$$

$$\frac{1}{2\pi^2}\hat{J}^X_{u_J} = \int_{2t_1 - u_J}^{2t_2 - u_J}\left[r^3\left(\partial_v B\right)^2(-2f) + \frac{r\Omega^2}{4}\left(1 - \frac{2}{3}\rho\right)(2f)\right] dv, \tag{10.8}$$

one can state the identity

$$\int_{\mathcal{D}^{r^\star_{\text{cl}}, u_J}_{[t_1,t_2]}} [-T_{\mu\nu}\pi^{\mu\nu} - (\nabla^\beta T_{\beta\delta})X^\delta] d\,\text{Vol} = \widehat{F}^X_B(t_1) - \widehat{F}^X_B(t_0) + \hat{H}^X_{u_H} - \hat{J}^X_{u_J}. \tag{10.9}$$

Let us turn to the spacetime integral on the left of (10.9) with the integrand being given by (10.5). In view of definition (2.15) we can write

$$\nabla^\alpha B \nabla_\alpha B = \frac{1}{2}\Box B^2 + \frac{4B}{3r^2}\left(e^{-8B} - \mathrm{e}^{-2B}\right) = \frac{1}{2}\Box B^2 - 8\frac{B^2}{r^2} + \frac{1}{r^2}\varphi_2\left(B\right) \tag{10.10}$$

and the integrand (10.5) becomes

$$
\begin{aligned}
&- T_{\mu\nu}\pi^{\mu\nu} - \left(\nabla^{\beta} T_{\beta\delta}\right) X^{\delta} \\
&= \frac{2}{\Omega^2} f' \left(\partial_{r^\star} B\right)^2 + \frac{1}{2}\Box B^2 \left(-f' - \frac{3}{r}(\lambda - \nu) f\right) \\
&\quad + \frac{1}{r^2}\left(8B^2\right)\left(2\frac{\lambda - \nu}{r} f + \frac{\left(\Omega^2\right)_{,u} - \left(\Omega^2\right)_{,v}}{\Omega^2} f\right) \\
&\quad + \frac{1}{r^2}\left(\varphi_1(B)\right)\left(-f' - \frac{\lambda - \nu}{r} f + \frac{\left(\Omega^2\right)_{,u} - \left(\Omega^2\right)_{,v}}{\Omega^2} f\right) \\
&\quad \times \frac{\varphi_2(B)}{r^2}\left(-f' - \frac{3}{r}(\lambda - \nu) f\right).
\end{aligned} \tag{10.11}
$$

Finally, we apply Green's theorem to the $\Box B^2$-term.[39] Collecting the B^2-terms of (10.11) after the integration by parts we find

$$
-\frac{1}{2}B^2\left[-32 f\frac{\lambda - \nu}{r^3} + 16\frac{f}{r^2}\frac{(\Omega)^2_{,v} - (\Omega)^2_{,u}}{\Omega^2} + \Box\left(f' + \frac{3}{r}(\lambda - \nu) f\right)\right]. \tag{10.12}
$$

Since

$$
\Box w(u,v) = \left(-\frac{4}{\Omega^2}\partial_u\partial_v - \frac{6}{r}\frac{\nu}{\Omega^2}\partial_v - \frac{6}{r}\frac{\lambda}{\Omega^2}\partial_u\right) w(u,v) \tag{10.13}
$$

and moreover f depends only on $r^\star$ we arrive at

$$
\begin{aligned}
&\Box\left(f' + \frac{3}{r}(\lambda - \nu) f\right) \\
&= \frac{f'''}{\Omega^2} + \frac{6}{r}\frac{\lambda - \nu}{\Omega^2} f'' + f'\left[\frac{6}{\Omega^2}\left(\partial_{r^\star}\frac{\lambda - \nu}{r}\right) + \frac{9(\lambda - \nu)^2}{r^2\Omega^2}\right] \\
&\quad \times f\left[-\frac{12}{\Omega^2}\partial_u\partial_v\left(\frac{\lambda - \nu}{r}\right) - \frac{18}{r}\frac{\nu}{\Omega^2}\partial_v\left(\frac{\lambda - \nu}{r}\right) - \frac{18}{r}\frac{\lambda}{\Omega^2}\partial_u\left(\frac{\lambda - \nu}{r}\right)\right].
\end{aligned}
$$

[39] cf. the remarks in Appendix A.

Computing the derivatives explicitly for the expression in the square brackets of (10.12) yields the identity

$$
\begin{aligned}
&- 32\frac{\lambda-\nu}{r^3} + \frac{16}{r^2}\frac{(\Omega)^2_{,v} - (\Omega)^2_{,u}}{\Omega^2} \\
&- \frac{12}{\Omega^2}\partial_u\partial_v\left(\frac{\lambda-\nu}{r}\right) - \frac{18}{r}\frac{\nu}{\Omega^2}\partial_v\left(\frac{\lambda-\nu}{r}\right) - \frac{18}{r}\frac{\lambda}{\Omega^2}\partial_u\left(\frac{\lambda-\nu}{r}\right) \\
&= (\lambda-\nu)\left(-\frac{35}{r^3} - \frac{18\mu}{r^3}\right) + \frac{1}{r^2}\left(\frac{\Omega_{,v}}{\Omega} - \frac{\Omega_u}{\Omega}\right)(35+9\mu) + \mathcal{I}_7(B,\theta,\zeta)
\end{aligned}
\tag{10.14}
$$

with

$$
\begin{aligned}
\mathcal{I}_7(B) = {} & \frac{9}{r^4}\frac{\theta^2}{\kappa} + \frac{36}{r^4}\frac{\lambda\zeta^2}{\Omega^2} + \left(\rho - \frac{3}{2}\right)\frac{1}{r^3}\left[-14(\lambda-\nu) + 8r\left(\frac{\Omega_{,v}}{\Omega} - \frac{\Omega_{,u}}{\Omega}\right)\right] \\
& - \frac{16}{r^{\frac{7}{2}}}\left(\mathrm{e}^{-2B} - \mathrm{e}^{-8B}\right)(\theta-\zeta).
\end{aligned}
\tag{10.15}
$$

We summarize the remaining error terms as

$$
\begin{aligned}
\mathcal{I}_8(B) = {} & f\left(-\frac{\varphi_1(B)}{B^2r^2}\frac{(\Omega)^2_{,v} - (\Omega)^2_{,u}}{\Omega^2} - \frac{\lambda-\nu}{r^3}\left(\frac{\varphi_1(B) + 3\varphi_2(B)}{B^2}\right)\right) \\
& - f'\frac{\varphi_1(B) + \varphi_2(B)}{B^2r^2}
\end{aligned}
$$

and read off the pointwise estimate (cf. Corollary 4.3)

$$
\left|\mathcal{I}_7(B)\right| + \left|\mathcal{I}_8(B)\right| \leq C(\epsilon)\frac{\sqrt{M}}{r^4}. \tag{10.16}
$$

Taking care of the boundary terms arising from the application of Green's identity we can finally state the identity (cf. equation (5.30))

$$
I^X_B\left(\mathcal{D}^{r^\star_{\mathrm{cl}},u_J}_{[t_1,t_2]}\right) = F^X_B(t_2) - F^X_B(t_1) + H^X_{u_H} - J^X_{u_J} \tag{10.17}
$$

with the renormalized bulk term

$$
\begin{aligned}
&I_B^X\left(\mathcal{D}^{r^\star_{\mathrm{cl}},u_J}_{[t_1,t_2]}\right) \\
&\quad = \int_{\mathcal{D}^{r^\star_{\mathrm{cl}},u_J}_{[t_1,t_2]}} \left\{ \frac{2}{\Omega^2} f' \left(\partial_{r^\star} B\right)^2 - \frac{B^2}{2}\left[\frac{f'''}{\Omega^2} + \frac{6}{r}\left(\frac{\lambda-\nu}{\Omega^2}\right) f'' \right.\right. \\
&\qquad + f'\left(\frac{6}{\Omega^2}\left(\partial_{r^\star}\frac{\lambda-\nu}{r}\right) + \frac{9}{r^2}\frac{(\lambda-\nu)^2}{\Omega^2}\right) \\
&\qquad \left.\left. + \ f\left((\lambda-\nu)\left(-\frac{35}{r^3} - \frac{18\mu}{r^3}\right) + \frac{1}{r^2}\left(\frac{\Omega_{,v}}{\Omega} - \frac{\Omega_{,u}}{\Omega}\right)(35+9\mu)\right)\right]\right\} d\,\mathrm{Vol} \\
&\qquad + \int_{\mathcal{D}^{r^\star_{\mathrm{cl}},u_J}_{[t_1,t_2]}} B^2\left[-\frac{1}{2}\mathcal{I}_7(B) + \mathcal{I}_8(B)\right] d\,\mathrm{Vol}, \qquad (10.18)
\end{aligned}
$$

the new boundary terms

$$
\begin{aligned}
F_B^X(t) = \hat{F}_B^X(t) &- \int_{r^\star_{\mathrm{cl}}}^{t-u_J}\int_{\mathbb{S}^3}\left(f' + \frac{3}{r}(\lambda-\nu) f\right)(\partial_t B)\, B(t,r^\star)\, r^3 dr^\star dA_{\mathcal{S}^3} \\
&+ \int_{r^\star_{\mathrm{cl}}}^{t-u_J}\int_{\mathbb{S}^3}\frac{1}{2}\left(\partial_t\left(\frac{3}{r}(\lambda-\nu)\right) f\right) B^2(t_1,r^\star)\, r^3 dr^\star dA_{\mathcal{S}^3} \\
&- \int_{t-r^\star_{\mathrm{cl}}}^{\infty}\int_{\mathbb{S}^3}\left(f' + \frac{3}{r}(\lambda-\nu) f\right)(\partial_u B)\, B(u,t+r^\star)\, r^3 du dA_{\mathcal{S}^3} \\
&+ \int_{t-r^\star_{\mathrm{cl}}}^{\infty}\int_{\mathbb{S}^3}\frac{1}{2}\left(\partial_u\left(f' + \frac{3}{r}(\lambda-\nu)\right) f\right) B^2(u,t+r^\star_{\mathrm{cl}})\, r^3 du dA_{\mathcal{S}^3},
\end{aligned} \qquad (10.19)
$$

the horizon terms

$$
\begin{aligned}
H^X_{u_H} = \hat{H}^X_{u_H} &- \int_{t_1+r^\star_{\mathrm{cl}}}^{t_2+r^\star_{\mathrm{cl}}}\left[B\partial_v B\left(f' + \frac{3}{r}(\lambda-\nu) f\right)\right] r^3 (u_{\mathrm{hoz}},v)\, dv \\
&+ \int_{t_1+r^\star_{\mathrm{cl}}}^{t_2+r^\star_{\mathrm{cl}}}\left[\frac{B^2}{2}\partial_v\left(f' + \frac{3}{r}(\lambda-\nu) f\right)\right] r^3 (u_{\mathrm{hoz}},v)\, dv \qquad (10.20)
\end{aligned}
$$

and the J-terms

$$
\begin{aligned}
J^X_{u_J} = \hat{J}^X_{u_J} &- \int_{2t_1-u_J}^{2t_2-u_J}\left[B\partial_v B\left(f' + \frac{3}{r}(\lambda-\nu) f\right)\right] r^3 (u_J,v)\, dv \\
&+ \int_{2t_1-u_J}^{2t_2-u_J}\left[\frac{B^2}{2}\partial_v\left(f' + \frac{3}{r}(\lambda-\nu) f\right)\right] r^3 (u_J,v)\, dv. \qquad (10.21)
\end{aligned}
$$

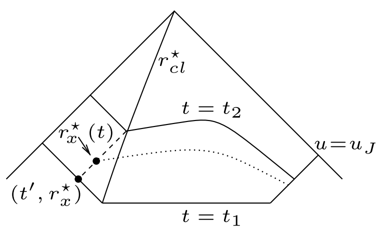
10.2 Analysing the X-bulk-term

10.2.1 Borrowing from the derivative-term

We would like the spacetime term (10.18) to have a sign. To achieve this we borrow from the term containing a derivative. Define

$$t' = \frac{t_1 + t_2}{2} \quad \text{and} \quad r^\star_x = r^\star_{\text{cl}} + \frac{t_1 - t_2}{2}, \tag{10.22}$$

$$r^\star_x(t) = \begin{cases} r^\star_{\text{cl}} + t_1 - t & \text{for } t_1 \le t \le t', \\ r^\star_{\text{cl}} + t - t_2 & \text{for } t' \le t \le t_2. \end{cases} \tag{10.23}$$



and compute[40]

$$\begin{aligned}
&\int_{t_1}^{t_2} dt \int_{r^\star_x(t)}^{t-u_J} \frac{f'}{\Omega^2} \left(\partial_{r^\star} B\right)^2 r^3 \Omega^2 dr^\star \\
&\quad = \int_{t_1}^{t_2} dt \int_{r^\star_x(t)}^{t-u_J} \frac{f'}{\Omega^2} \left(\partial_{r^\star} B + \xi B\right)^2 r^3 \Omega^2 dr^\star \\
&\quad + \int_{t_1}^{t_2} dt \int_{r^\star_x(t)}^{t-u_J} B^2 \left(\frac{f''\xi}{\Omega^2} + \frac{\xi' f'}{\Omega^2} + \frac{3}{r}\frac{\lambda - \nu}{\Omega^2} f'\xi\right) r^3 \Omega^2 dr^\star \\
&\quad - \int_{t_1}^{t_2} dt \int_{r^\star_x(t)}^{t-u_J} B^2 \left(\frac{f'}{\Omega^2}\xi^2\right) r^3 \Omega^2 dr^\star + J^{X,u_J}_{\text{error}} + H^{X,v_1 = t_1 + r^\star_{\text{cl}}}_{\text{error}} \\
&\quad + H^{X,u_2 = t_2 - r^\star_{\text{cl}}}_{\text{error}}
\end{aligned}$$

[40] Again care is necessary in the integration by parts because of the differentiability of the coordinate system. In any case it is sufficient to note that the integrands of the boundary terms are continuous and that the integrand of the bulk term is piecewise continuous (all terms except the ξ'-term are in fact continuous everywhere).

for some function ξ chosen in (10.28). The boundary terms are

$$J^{X,u_J}_{\text{error}} = -\int_{t_1}^{t_2} f'\xi B^2 r^3\,(t, t-u_J)\,dt = -\int_{2t_1-u_J}^{2t_2-u_J} f'\xi B^2 r^3\,(u_J, v)\,dv, \tag{10.24}$$

$$H^{X,v_1=t_1+r^\star_{\text{cl}}}_{\text{error}} = \int_{t_1}^{t'} dt f'\xi B^2 r^3\,(t, v_1-t) = \int_{t_1-r^\star_{\text{cl}}}^{t_2-r^\star_{\text{cl}}} f'\xi B^2 r^3\,(u, v_1)\,du \tag{10.25}$$

and

$$H^{X,u_2=t_2-r^\star_{\text{cl}}}_{\text{error}} = \int_{t'}^{t_2} dt f'\xi B^2 r^3\,(t, t-u_2) = \int_{t_1+r^\star_{\text{cl}}}^{t_2+r^\star_{\text{cl}}} f'\xi B^2 r^3\,(u_2, v)\,dv. \tag{10.26}$$

To keep the notation clean we write $M = m(T, r^\star = 0)$ in this section. For a sufficiently large constant σ we define the shifted coordinate x

$$x = r^\star - \sigma - \sqrt{M}. \tag{10.27}$$

We choose

$$\xi = \frac{3}{2}\frac{\lambda-\nu}{r} - \frac{nx}{x^2+\sigma^2} \tag{10.28}$$

for some $n \in \left(\frac{1}{2}, \infty\right)$ from which

$$\begin{aligned} -\xi' + \xi^2 - \frac{3}{r}(\lambda-\nu)\,\xi = &-\frac{9}{4}\frac{(\lambda-\nu)^2}{r^2} + \frac{x^2\left(n^2-n\right)+n\sigma^2}{\left(\sigma^2+x^2\right)^2} \\ &-\frac{3}{2}\left(\partial_{r^\star}\left(\frac{\lambda-\nu}{r}\right)\right) \end{aligned}$$

follows. Hence the integral (10.18) can be expressed as

$$\begin{aligned} &I^X_B\left(\mathcal{D}^{r^\star_{\text{cl}},u_J}_{[t_1,t_2]}\right) \\ &\quad= \int_{\mathcal{D}^{r^\star_{\text{cl}},u_J}_{[t_1,t_2]}} \frac{2f'}{\Omega^2}\left(\partial_{r^\star}B + \xi B\right)^2 d\,\text{Vol} \\ &\quad\quad - \frac{1}{2}\int_{\mathcal{D}_1}\left[\frac{f'''}{\Omega^2} + f''\left(\frac{4nx}{\Omega^2\left(\sigma^2+x^2\right)}\right) + f'\left(\frac{4x^2\left(n^2-n\right)+4n\sigma^2}{\left(\sigma^2+x^2\right)^2\Omega^2}\right)\right. \\ &\quad\quad \left. + f\left[(\lambda-\nu)\left(-\frac{35}{r^3}-\frac{18\mu}{r^3}\right) + \frac{1}{r^2}\left(\frac{\Omega_{,v}}{\Omega}-\frac{\Omega_{,u}}{\Omega}\right)(35+9\mu)\right]\right] B^2 d\,\text{Vol} \end{aligned}$$

$$+\int_{\mathcal{D}_1}\left(-\frac{1}{2}\mathcal{I}_7\left(B\right)+\mathcal{I}_8\left(B\right)\right)B^2 d\,\mathrm{Vol}$$
$$+J^{X,u_J}_{\mathrm{error}}+H^{X,v=t_1+r^\star_{\mathrm{cl}}}_{\mathrm{error}}+H^{X,u=t_2-r^\star_{\mathrm{cl}}}_{\mathrm{error}} \tag{10.29}$$

which we will write shorthand as

$$I^X_B\left(\mathcal{D}^{r^\star_{\mathrm{cl}},u_J}_{[t_1,t_2]}\right)=\bar{I}^X_B\left(\mathcal{D}^{r^\star_{\mathrm{cl}},u_J}_{[t_1,t_2]}\right)+J^{X,u_J}_{\mathrm{error}}+H^{X,v=t_1+r^\star_{\mathrm{cl}}}_{\mathrm{error}}+H^{X,u=t_2-r^\star_{\mathrm{cl}}}_{\mathrm{error}}, \tag{10.30}$$

where

$$\bar{I}^X_B\left(\mathcal{D}^{r^\star_{\mathrm{cl}},u_J}_{[t_1,t_2]}\right)=\int\Bigg\{\frac{2f'}{\Omega^2}\left(\partial_{r^\star}B+\xi B\right)^2$$
$$+B^2\left[F+f\cdot g-\frac{1}{2}\mathcal{I}_7\left(B\right)+\mathcal{I}_8\left(B\right)\right]\Bigg\}\,d\,\mathrm{Vol} \tag{10.31}$$

with the identifications

$$F=-\frac{1}{2\Omega^2}\left(f'''+\frac{4nxf''}{\sigma^2+x^2}+f'\left(\frac{4x^2\left(n^2-n\right)+4n\sigma^2}{\left(\sigma^2+x^2\right)^2}\right)\right), \tag{10.32}$$
$$g=-\frac{1}{2}\left[\left(\lambda-\nu\right)\left(-\frac{35}{r^3}-\frac{18\mu}{r^3}\right)+\frac{1}{r^2}\left(\frac{\Omega_{,v}}{\Omega}-\frac{\Omega_u}{\Omega}\right)\left(35+9\mu\right)\right]. \tag{10.33}$$

10.2.2 The choice of f

By Propositions 8.4, 8.14 and 8.15 we have the bounds

$$\lambda-\nu=\left(1-\mu\right)+C\left(r^\star_{\mathrm{cl}},c\right)\frac{M}{t^2}, \tag{10.34}$$
$$\frac{\Omega_{,v}}{\Omega}-\frac{\Omega_{,u}}{\Omega}\leq\frac{\mu}{r}+\frac{C\left(\epsilon\right)}{t}, \tag{10.35}$$
$$\left|\frac{\Omega_{,v}}{\Omega}-\frac{\Omega_{,u}}{\Omega}\right|\leq\frac{\mu}{r}+\frac{1}{\sqrt{M}}C\left(\epsilon\right) \tag{10.36}$$

in the region $r^\star_{\mathrm{cl}}\leq r^\star\leq\frac{9}{10}t$. This implies that

$$g\geq\frac{1}{2r^7}\left(35r^4-104Mr^2-108M^2\right)+\frac{C\left(\epsilon\right)}{tr^2} \tag{10.37}$$

and that

$$\left|g-\frac{1}{2r^7}\left(35r^4-104Mr^2-108M^2\right)\right|\leq\frac{C\left(\epsilon\right)}{\sqrt{M}r^2}, \tag{10.38}$$

where $M=m\left(T,r^\star=0\right)$. It is apparent (note Proposition 4.6 in particular) that the expression g is negative close to the horizon, positive far away from it with a single zero at some $r = r_{\text{zero}}(t)$ for which the estimate

$$\left| r_{\text{zero}} - \sqrt{2M}\sqrt{\tfrac{1}{35}\left(26 + \sqrt{1621}\right)} \right| \leq C(\epsilon)\sqrt{M} \tag{10.39}$$

can be derived. This in turn implies an estimate (with error of order $\frac{1}{t}$) for the $r^\star$ value along that curve via identification (8.16). It follows that g changes sign in a small interval around $r^\star_{\text{zero}}$. For future calculation the estimate

$$-\tfrac{1}{6}\sqrt{M} < r^\star_{\text{zero}} < -\tfrac{1}{10}\sqrt{M} \tag{10.40}$$

for $r^\star_{\text{zero}}$ will be sufficient.

We finally construct the function $f = f(x) = f\left(r^\star - \sigma - \sqrt{M}\right)$ by setting

$$f(x_{\text{zero}}) = f\left(-\sigma - \sqrt{M} - r^\star_{\text{zero}}\right) = 0 \tag{10.41}$$

and

$$f' = \frac{M^{n-\frac{1}{2}}}{(x^2+\sigma^2)^n}. \tag{10.42}$$

Note that f will be bounded for $n \in \left(\frac{1}{2}, \infty\right)$. Later we will set $n = \frac{3}{2}$. We compute

$$f'' = M^{n-\frac{1}{2}}\frac{-2xn}{(x^2+\sigma^2)^{n+1}}, \tag{10.43}$$

$$f''' = M^{n-\frac{1}{2}}\frac{2n\left(x^2+2nx^2-\sigma^2\right)}{(x^2+\sigma^2)^{n+2}} \tag{10.44}$$

to find

$$F = M^{n-\frac{1}{2}}\frac{n}{\Omega^2}\frac{x^2-\sigma^2}{(\sigma^2+x^2)^{n+2}}. \tag{10.45}$$

We will now show that there exists a positive constant $c(\sigma) > 0$ such that in (10.31)

$$F + f\cdot g - \frac{1}{2}\mathcal{I}_7(B) + \mathcal{I}_8(B) \geq \frac{1}{r^3}c(\sigma) \tag{10.46}$$

holds in the region of integration. Note that the above choice of f ensures that $f \cdot g$ is positive everywhere, except for an ϵ-small correction-term. In particular, in $\left[r^\star_{\text{zero}} - \frac{1}{10}\sqrt{M}, r^\star_{\text{zero}} + \frac{1}{10}\sqrt{M}\right]$ we have

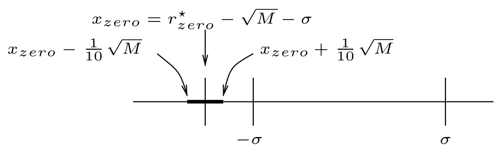
$$fg \geq -\frac{1}{M^{\frac{3}{2}}} C(\epsilon) \tag{10.47}$$

and outside of this set, using (10.16) and (10.37)

$$\frac{1}{8} fg - \frac{1}{2}\mathcal{I}_7(B) + \mathcal{I}_8(B) \geq \frac{c_2(\sigma)}{r^3}. \tag{10.48}$$

10.2.3 Estimating $\bar{I}^X_B\left(\mathcal{D}^{r^\star_{\text{cl}}, u_J}_{[t_1,t_2]}\right)$

The aim is to establish (10.46). We will do the computations in the shifted x-coordinate (10.27).



The region $x \leq -\sigma$ and $x \geq \sigma$: Note that F is already non-negative for $x \leq -\sigma$ and $x \geq \sigma$. In the subinterval $\left[x_{\text{zero}} - \frac{1}{10}\sqrt{M}, x_{\text{zero}} + \frac{1}{10}\sqrt{M}\right]$, the only subset where $f \cdot g$ might cause problems, we have the stronger bound

$$F \geq \frac{1}{M^{\frac{3}{2}}} c_1(\sigma) > 0, \tag{10.49}$$

which upon combination with (10.47) yields

$$F + fg - \frac{1}{2}\mathcal{I}_7(B) + \mathcal{I}_8(B) \geq \frac{1}{r^3} c(\sigma) \tag{10.50}$$

for the regions under consideration.

The region $[-\sigma, \sigma]$: We shall show that we can dominate the term F in the region $[-\sigma, \sigma]$ by the term $\frac{7}{8} f \cdot g$ in (10.31).[41] In conjunction with (10.48)

[41] Note that we save $\frac{1}{8} fg$ to obtain the positivity of (10.48).

this will yield the desired result (10.46). By Proposition 8.4

$$\left|\Omega^2-(1-\mu)\right|=\left|(4\gamma\kappa-1)(1-\mu)\right|\le C\left(r^\star_{\mathrm{cl}},c\right)\frac{M}{t^2} \tag{10.51}$$

holds in the region $-\sigma\le x\le\sigma$ and also f is positive there.[42] In view of (10.37) it suffices to show

$$M^{n-\frac12}\frac{n}{(1-\mu)}\frac{\sigma^2-x^2}{(\sigma^2+x^2)^{2+n}}\le\frac{1}{2r^3}\left(35-104\frac{M}{r^2}-108\frac{M^2}{r^4}\right)f\frac78 \tag{10.52}$$

in the region $-\sigma\le x\le\sigma$. There we can estimate

$$f(x)=\int_{-\sigma-\sqrt{M}+r^\star_{\mathrm{zero}}}^{x}f'>\int_{-\sigma-\sqrt{M}}^{x}f'\ge M^{n-\frac12}\frac{x+\sigma+\sqrt{M}}{\left(2\sigma^2+2\sigma\sqrt{M}+M\right)^n} \tag{10.53}$$

such that it suffices to establish

$$\begin{aligned}\frac{n}{(1-\mu)}\frac{\sigma^2-x^2}{(\sigma^2+x^2)^{2+n}}&<\frac{1}{2r^3}\left(35-\frac{104M}{r^2}-108\frac{M^2}{r^4}\right)\\&\quad\times\frac{x+\sigma+\sqrt{M}}{\left(2\sigma^2+2\sigma\sqrt{M}+M\right)^n}\frac78.\end{aligned} \tag{10.54}$$

Part I: Consider first the region $-\sigma<x\le-\frac23\sigma$ translating to $\sqrt{M}<r^\star\le\frac13\sigma+\sqrt{M}$. The lower bound on $r^\star$ implies a lower bound on r by identification (8.16). In particular, $r>\sqrt{M}\left(2+\frac35\right)$ in that region and hence

$$1-\mu>\frac{7}{10}\quad\text{and}\quad\frac{1}{1-\mu}<\frac{10}{7} \tag{10.55}$$

as well as

$$35-\frac{104M}{r^2}-108\frac{M^2}{r^4}>17 \tag{10.56}$$

hold in the region under consideration. Consequently, for σ sufficiently large we have

$$\begin{aligned}\frac{n}{(1-\mu)}\frac{\sigma^2-x^2}{(\sigma^2+x^2)^{2+n}}&<n\frac{10}{7}\frac{\sigma-x}{\sqrt{\sigma^2+x^2}}\frac{\sigma+x}{\left(\frac{13}{9}\sigma^2\right)^{3/2+n}}\\&\le n\sqrt2\frac{10}{7}\left(\frac{9}{13}\right)^{\frac32+n}\frac{x+\sigma}{\sigma^{3+2n}}.\end{aligned} \tag{10.57}$$

[42]Note that we can use the aforementioned proposition because the region under consideration lies in $r^\star\ge r^\star_{\mathrm{cl}}$.

The upper bound on $r^\star$ can be exploited for large σ to give

$$r \leq \tfrac{5}{12}\sigma. \tag{10.58}$$

Again choosing σ sufficiently large, this gives rise to the estimate

$$\begin{aligned}
&\frac{1}{2r^3}\left(35 - \frac{104M}{r^2} - 108\frac{M^2}{r^4}\right)\frac{x+\sigma+\sqrt{M}}{\left(2\sigma^2+2\sigma\sqrt{M}+M\right)^n}\frac{7}{8} \\
&\quad \geq \frac{17}{2}\left(\frac{12}{5\sigma}\right)^3\frac{x+\sigma}{\left(3\sigma^2\right)^n}\frac{7}{8} \geq \frac{116}{3^n}\frac{x+\sigma}{\sigma^{2n+3}}\frac{7}{8}.
\end{aligned} \tag{10.59}$$

Comparing (10.59) and (10.57) for $n = \frac{3}{2}$ we have shown the desired inequality (10.54) in the region under consideration.

Part II: Consider now the region $-\frac{2}{3}\sigma \leq x \leq \sigma$, which translates to $\frac{1}{3}\sigma + \sqrt{M} \leq r^\star \leq 2\sigma + \sqrt{M}$. We can choose σ so large that

$$1 - \mu \geq \tfrac{6}{7} \tag{10.60}$$

and

$$\frac{r}{x+\sigma} \leq \left(\frac{7}{6}\right)^{\frac{1}{3}} \tag{10.61}$$

hold in the region under consideration. We deduce that for large σ

$$\begin{aligned}
&\frac{7}{8}\frac{1}{2r^3}\left(35 - \frac{104m}{r^2} - 108\frac{m^2}{r^4}\right)\frac{x+\sigma+\sqrt{M}}{\left(2\sigma^2+2\sigma\sqrt{M}+M\right)^n} \\
&\quad \geq \frac{7}{8}35\frac{2}{5}\frac{x+\sigma}{\left(2\sigma^2\right)^n\left(x+\sigma\right)^3}\frac{6}{7}
\end{aligned}$$

and

$$\frac{n}{(1-\mu)}\frac{\sigma^2-x^2}{\left(\sigma^2+x^2\right)^{2+n}} \leq \frac{7}{6}n\frac{(\sigma-x)(\sigma+x)}{\left(\sigma^2+x^2\right)^{2+n}}. \tag{10.62}$$

Hence it suffices to show that, for large σ, we have the inequality

$$\frac{1}{9}n\left(2\sigma^2\right)^n\frac{(\sigma-x)(x+\sigma)^3}{\left(\sigma^2+x^2\right)^{2+n}} \leq 1. \tag{10.63}$$

For $n = \frac{3}{2}$ we obtain

$$\frac{1}{3}\sqrt{2}\frac{(\sigma-x)(x+\sigma)^3\sigma^3}{\left(\sigma^2+x^2\right)^{\frac{7}{2}}} < 1, \tag{10.64}$$

which is shown to be true by elementary arguments. Namely, for $x < 0$ we have

$$\frac{1}{3}\sqrt{2}\frac{(\sigma-x)\,(x+\sigma)^3\,\sigma^3}{(\sigma^2+x^2)^{\frac{7}{2}}} \le \frac{1}{3}\sqrt{2}\frac{(\sigma-x)\,\sigma^6}{(\sigma^2+x^2)^{\frac{7}{2}}} \le \frac{\sqrt{2}}{3}\frac{\sqrt{2\,(\sigma^2+x^2)}}{(\sigma^2+x^2)^{\frac{7}{2}}}\sigma^6 \le \frac{2}{3} < 1 \tag{10.65}$$

and for $x \ge 0$ we have

$$\begin{aligned}\frac{\sqrt{2}}{3}\frac{(\sigma-x)\,(x+\sigma)^3\,\sigma^3}{(\sigma^2+x^2)^{\frac{7}{2}}} &\le \frac{\sqrt{2}}{3}\frac{\sigma^4\,(x+\sigma)^3}{(\sigma^2+x^2)^{\frac{7}{2}}} \le \frac{\sqrt{2}}{3}\left(\frac{\sigma x+\sigma^2}{x^2+\sigma^2}\right)^3\frac{\sigma}{\sqrt{x^2+\sigma^2}}\\ &\le \frac{\sqrt{2}}{3}\left(\frac{\sigma x+\sigma^2}{x^2+\sigma^2}\right)^3 \le \frac{\sqrt{2}}{3}\left(\frac{1}{2}\left(1+\sqrt{2}\right)\right)^3\\ &< \frac{2\sqrt{2}}{3} < 1.\end{aligned} \tag{10.66}$$

This finally establishes that the integrand of $\bar{I}_B^X$ is non-negative and vanishes if and only if $B = 0$ everywhere.

Remark. The minimum size of the constant σ required for the estimates above to work can be determined explicitly. Choosing $\sigma = 40\sqrt{M}$ for instance is large enough.

10.2.4 Summary

We can write

$$\bar{I}_B^X\left(\mathcal{D}_{[t_1,t_2]}^{r^\star_{\mathrm{cl}},u_J}\right) = I_B^X\left(\mathcal{D}_{[t_1,t_2]}^{r^\star_{\mathrm{cl}},u_J}\right) - J_{\mathrm{error}}^{X,u_J} - H_{\mathrm{error}}^{X,v=t_1+r^\star_{\mathrm{cl}}} - H_{\mathrm{error}}^{X,u=t_2-r^\star_{\mathrm{cl}}} \tag{10.67}$$

and we have shown

Proposition 10.1. *The estimate*

$$\int_{\mathcal{D}_{[t_1,t_2]}^{r^\star_{\mathrm{cl}},u_J}} \frac{B^2}{r^3}d\mathrm{Vol} \le C(\sigma)\bar{I}_B^X\left(\mathcal{D}_{[t_1,t_2]}^{r^\star_{\mathrm{cl}},u_J}\right) \tag{10.68}$$

holds.

In the following subsection, we are going to estimate the error boundary terms on the right-hand side of (10.67).

10.3 Controlling the error-boundary terms

Lemma 10.1. *The error terms* (10.24)–(10.26) *satisfy*

$$|J^{X,u_J}_{\text{error}}| \leq C(\sigma)\left(m\left(u_J, 2t_2 - u_J\right) - m\left(u_J, 2t_1 - u_J\right)\right) \tag{10.69}$$

$$H^{X,v_1=t_1+r^\star_{\text{cl}}}_{\text{error}} \geq 0 \tag{10.70}$$

and

$$H^{X,u_2=t_2-r^\star_{\text{cl}}}_{\text{error}} \geq 0. \tag{10.71}$$

Proof. Statement (10.69) is immediate since we are away from the horizon and both ξ and f' decay sufficiently fast at infinity to retrieve the correct powers of r appearing in the energy. For the other two inequalities recall that $f' \geq 0$ always and that $x \leq -\sqrt{M}$ in the region of integration and hence $\xi \geq 0$. □

10.4 Controlling the boundary terms of $I^X_B\left(\mathcal{D}^{r^\star_{\text{cl}},u_J}_{[t_1,t_2]}\right)$

The following lemmata show that the boundary terms of the vectorfield X appearing in the vector field identity (10.17) are controlled by the energy plus a contribution from the term $\tilde{F}^Y_B$ appearing as bootstrap assumption 1.3.2. Together with the results of Lemma 10.1, identity (10.67) ultimately yields an estimate for the positive spacetime integral $\bar{I}^X_B$, manifest in Proposition 10.2.

Lemma 10.2. *We have, for any* $q \in \mathbb{R}^+$,

$$|F^X_B(t)| \leq \left[C\left(r^\star_{\text{cl}}\right) + C_f q\right] E_F(t) + C_f r^2_{\text{cl}} B^2\left(t, r^\star_{\text{cl}}\right) + \frac{2}{q} C_f \tilde{F}^Y_B\left(v = t + r^\star_{\text{cl}}\right) \tag{10.72}$$

with

$$E_F(t) = m(t, t - u_J) - m\left(u_H, t + r^\star_{\text{cl}}\right) \tag{10.73}$$

and $\tilde{F}^Y_B(v_1)$ *being the quantity appearing as bootstrap assumption* 1.3.2. *Moreover* $C_f = sup_{r \leq r^\star_{\text{cl}}}[f'r] \ll 1$ *is a small constant and* $C\left(r^\star_{\text{cl}}\right)$ *just depends on* $r^\star_{\text{cl}}$.

Proof. The F-boundary terms arising from energy (6.3) can be estimated

$$
\begin{aligned}
F_B^T(t) &= \int_{t-r^\star_{\mathrm{cl}}}^{t_2-r^\star_{\mathrm{cl}}} \left[4r^3\lambda\frac{(B_{,u})^2}{\Omega^2} - r\nu\left(1-\frac{2}{3}\rho\right)\right](u,t+r^\star_{\mathrm{cl}})\,du \\
&\quad + \int_{r^\star_{\mathrm{cl}}}^{t-u_J}\left(-4r^3\frac{\nu}{\Omega^2}(B_{,v})^2 + 4r^3\frac{\lambda}{\Omega^2}(B_{,u})^2\right. \\
&\quad \left. + r(\lambda-\nu)\left(1-\frac{2}{3}\rho\right)\right)(t,r^\star)\,dr^\star \\
&\geq \int_{t-r^\star_{\mathrm{cl}}}^{t_2-r^\star_{\mathrm{cl}}} \left[4r^3\lambda\frac{(B_{,u})^2}{\Omega^2} - r\nu\left(1-\frac{2}{3}\rho\right)\right](u,t+r^\star_{\mathrm{cl}})\,du \\
&\quad + \frac{1}{C_L}\int_{r^\star_{\mathrm{cl}}}^{t-u_J}\left((\partial_t B)^2 + (\partial_{r^\star}B)^2 + \frac{B^2}{r^2}\right)r^3(t,r^\star)\,dr^\star. \qquad (10.74)
\end{aligned}
$$

The F-boundary terms arising from the basic identity for the vectorfield X (10.19) are

$$
\begin{aligned}
\frac{1}{2\pi^2}F_B^X(t) &= -2\int_{r^\star_{\mathrm{cl}}}^{t-u_J} f\partial_t B\partial_{r^\star}B(t_i,r^\star)\,r^3 dr^\star \\
&\quad - \int_{r^\star_{\mathrm{cl}}}^{t-u_J}\left(f' + \frac{3}{r}(\lambda-\nu)f\right)(\partial_t B)\,B(t,r^\star)\,r^3 dr^\star \\
&\quad + \int_{r^\star_{\mathrm{cl}}}^{t-u_J}\frac{1}{2}\left(\partial_t\left(\frac{3}{r}(\lambda-\nu)\right)f\right)B^2(t_1,r^\star)\,r^3 dr^\star \\
&\quad + \int_{t-r^\star_{\mathrm{cl}}}^{t_2-r^\star_{\mathrm{cl}}}\left[r^3(\partial_u B)^2(2f) + \frac{r\Omega^2}{4}\left(1-\frac{2}{3}\rho\right)(-2f)\right]du \\
&\quad - \int_{t-r^\star_{\mathrm{cl}}}^{t_2-r^\star_{\mathrm{cl}}}\left(f' + \frac{3}{r}(\lambda-\nu)f\right)(\partial_u B)\,B(u,t+r^\star)\,r^3 du \\
&\quad + \int_{t-r^\star_{\mathrm{cl}}}^{t_2-r^\star_{\mathrm{cl}}}\frac{1}{2}\left(\partial_u\left(f' + \frac{3}{r}(\lambda-\nu)f\right)\right)B^2(u,t+r^\star_{\mathrm{cl}})\,r^3 du. \qquad (10.75)
\end{aligned}
$$

Spacelike F^X-terms: We estimate the first three terms of (10.75) (with the index *sl* denoting the restriction to the spacelike terms)

$$\begin{aligned}\left.\frac{|F_B^X(t)|}{2\pi^2}\right|_{\mathrm{sl}} &\leq \int_{r^\star_{\mathrm{cl}}}^{t-u_J} |f|\left((\partial_t B)^2 + (\partial_{r^\star} B)^2\right) r^3 dr^\star \\ &+ \int_{r^\star_{\mathrm{cl}}}^{t-u_J} \frac{f'}{2}\left((\partial_t B)^2\sqrt{M} + \frac{B^2}{\sqrt{M}}\right) r^3 dr^\star \\ &+ \int_{r^\star_{\mathrm{cl}}}^{t-u_J} |f|\frac{3}{2r}(\lambda-\nu)\left(r(\partial_t B)^2 + \frac{B^2}{r}\right) r^3 dr^\star \\ &+ \int_{r^\star_{\mathrm{cl}}}^{t-u_J} |f|\left[\frac{3}{2r^2}(\lambda-\nu)(\lambda+\nu) + \frac{3}{r}(r_{,vv} - r_{,uu})\right] B^2 r^3 dr^\star. \end{aligned} \tag{10.76}$$

Recall that f' is positive and that we have $|f| \leq C(\sigma)$ and $f' \leq C(\sigma)\frac{M}{r^3}$. Moreover $(\lambda+\nu)$ is clearly bounded everywhere, as is $r^2\frac{\Omega_{,v}}{\Omega}$ and $r^2\frac{\Omega_{,u}}{\Omega}$ (cf. Corollary 8.8). Using the equations

$$r_{,vv} = 2\frac{\Omega_{,v}}{\Omega}\lambda - \frac{2}{r^2}\theta^2 \quad \text{and} \quad r_{,uu} = 2\frac{\Omega_{,u}}{\Omega}\nu - \frac{2}{r^2}\zeta^2 \tag{10.77}$$

it becomes apparent that the terms in (10.76) are controlled by energy.

Null F^X-terms: For the last three terms of (10.75), which only arise for the $F^X(t_1)$-term by the geometry of the region, we observe that the first is manifestly controlled by the energy because f is bounded along the rays of integration. For the second term one estimates for any $q \in \mathbb{R}^+$

$$\begin{aligned}\int_{t_1-r^\star_{\mathrm{cl}}}^{t_2-r^\star_{\mathrm{cl}}} f'|\partial_u B||B| r^3 du &\leq C_f \int_{t_1-r^\star_{\mathrm{cl}}}^{t_2-r^\star_{\mathrm{cl}}} \left[qrB^2(-\nu) + \frac{1}{q}\frac{(B_{,u})^2}{-\nu} r^3\right] du \\ &\leq C_f\left(qE_F + \frac{1}{q}\tilde{F}_B^Y(v_1)\right), \end{aligned} \tag{10.78}$$

where bootstrap assumption (7.11) has been used, and

$$\begin{aligned}&\int_{t_1-r^\star_{\mathrm{cl}}}^{t_2-r^\star_{\mathrm{cl}}} \left(\frac{3}{r}(\lambda-\nu)|f|\right)|\partial_u B||B|(u, t+r^\star) r^3 du \\ &\leq \int_{t_1-r^\star_{\mathrm{cl}}}^{t_2-r^\star_{\mathrm{cl}}} \left(\frac{3}{r}(\lambda-\nu)|f|\right)\frac{1}{2}\left(\frac{(\partial_u B)^2}{-\nu}\sqrt{M} + \frac{1}{\sqrt{M}}B^2(-\nu)\right) r^3(u, v_1) du \\ &\leq C(\sigma) E_F. \end{aligned}$$

Finally, for the third term we note that $x \leq -\sqrt{M}$ and hence $f'' \geq 0$ in the integration region to estimate

$$\begin{aligned}
&\int_{t_1 - r^\star_{\mathrm{cl}}}^{t_2 - r^\star_{\mathrm{cl}}} \frac{1}{2}\left[-\partial_u f' - (\partial_u f)\,\frac{3}{r}\,(\lambda-\nu) + |f|\left|\partial_u\left(\frac{3}{r}\,(\lambda-\nu)\right)\right|\right] \\
&\times B^2\left(u, t + r^\star_{\mathrm{cl}}\right) r^3 du \\
&\leq C_f r^2_{\mathrm{cl}}\ B^2\left(t_1, r^\star_{\mathrm{cl}}\right) + C_f \int_{t_1 - r^\star_{\mathrm{cl}}}^{t_2 - r^\star_{\mathrm{cl}}} 2|B||B_{,u}|r^2 du \\
&+ C\left(m\left(u_1, v_1\right) - m\left(u_2, v_1\right)\right) \\
&\leq C_f r^2_{\mathrm{cl}}\ B^2\left(t_1, r^\star_{\mathrm{cl}}\right) + C_f q \cdot E_F + C_f \frac{1}{q} \tilde{F}^Y_B\left(v_1\right) + C E_F
\end{aligned}$$

for any $q \in \mathbb{R}^+$. □

Lemma 10.3. *We have*

$$|J^X_{u_J}| \leq C E_J\left(u_J\right) \tag{10.79}$$

for some constant C and

$$E_J\left(u_J\right) = m\left(u_J, 2t_2 - u_J\right) - m\left(u_J, 2t_1 - u_J\right). \tag{10.80}$$

Proof. For the terms in the first line of (10.21) apply the inequality $2BB_{,v} \leq \frac{B^2}{r} + r\left(B_{,v}\right)^2$ to retrieve the correct powers of r in the energy. For the second line observe that f', $r_{,vv}$ and $r_{,uu}$ decay sufficiently fast in r. □

Lemma 10.4. *We have*

$$H^X_{u_H} \leq C\ E_H\left(v_1, v_2\right) + C_f r^2_{\mathrm{cl}} B^2\left(t_2, r^\star_{\mathrm{cl}}\right) \tag{10.81}$$

for some constant C and

$$E_H\left(v_1, v_2\right) = m\left(u_{\mathrm{hoz}}, v_2\right) - m\left(u_{\mathrm{hoz}}, v_1\right). \tag{10.82}$$

Proof. The term $\hat{H}^X_{u_H}$ is manifestly controlled by the energy by the fact that f is bounded. For the second and third terms in (10.20), we observe that the terms that are multiplied by a λ- or a ν-factor (or derivatives thereof) are controlled by the energy. For the remaining terms we note that $x \leq -\sqrt{M}$ and hence $f'' > 0$ in this region and estimate (using a Hardy inequality)

$$\begin{aligned}
&\int_{v_1}^{v_2} \left[|B||\partial_v B| f' + \frac{B^2}{2} \left(f'\right)_{,v} \right] r^3 dv \\
&\quad \leq C_f r_{\mathrm{cl}}^2 B^2 \left(t_2, r_{\mathrm{cl}}^\star\right) + \frac{3}{2} \int_{v_1}^{v_2} B^2 f' r^2 \lambda dv + \int_{v_1}^{v_2} 2 |B||\partial_v B| f' dv \\
&\quad \leq C_f r_{\mathrm{cl}}^2 B^2 \left(t_2, r_{\mathrm{cl}}^\star\right) + C_f E_H + 16 \int_{v_1}^{v_2} \left(\partial_v B\right)^2 \frac{\left(f'\right)^2}{f''} r^3 dv \\
&\qquad + \int_{v_1}^{v_2} \left[\frac{B^2}{4} \left(f'\right)_{,v} \right] r^3 dv
\end{aligned} \tag{10.83}$$

from which we obtain

$$\int_{v_1}^{v_2} \left[|B||\partial_v B| f' + \frac{B^2}{4} \left(f'\right)_{,v} \right] r^3 dv \leq C_f r_{\mathrm{cl}}^2 B^2 \left(t_2, r_{\mathrm{cl}}^\star\right) + E_H \tag{10.84}$$

because

$$\frac{\left(f'\right)^2}{f''} = \frac{1}{-3x} \frac{M}{\sqrt{x^2 + \sigma^2}} \tag{10.85}$$

is small in the region under consideration allowing us to estimate the derivative term in the last line of (10.83) by the energy. □

From the previous lemmata and the identity (10.67) we conclude

Proposition 10.2. *In the region $\mathcal{D}_{[t_1,t_2]}^{r_{\mathrm{cl}}^\star, u_J}$ we have for any $q \in \mathbb{R}^+$*

$$\begin{aligned}
\bar{I}_B^X \left(\mathcal{D}_{[t_1,t_2]}^{r_{\mathrm{cl}}^\star, u_J} \right) \leq \left[C\left(r_{\mathrm{cl}}^\star\right) + C_f q \right] \Big[E_F\left(t_1\right) + E_F\left(t_2\right) \Big] + C\left(\epsilon\right) \frac{M^2}{t_1^2} \\
+ C E_H\left(v_1, v_2\right) + C E_J\left(u_J\right) + 2 C_f \frac{1}{q} \left(\tilde{F}_B^Y\left(v_1\right) \right).
\end{aligned} \tag{10.86}$$

Proof. Add up the estimates of the previous lemmata. Note that pointwise bounds for B^2 on the $r^\star = r_{\mathrm{cl}}^\star$-curve were obtained from the energy $E_f\left(t\right)$ in Proposition 8.9, including a small term $C\left(\epsilon\right) \frac{M^2}{t_1^2}$. Hence we can replace the terms appearing in Lemmata 10.2 and 10.4 by the energy, such that (10.86) is finally obtained. □

10.5 Controlling the time derivative from $I_B^X(\mathcal{B})$

For the statements of the next two propositions let $R^\star = -\frac{1}{3}\sqrt{M} < r^\star_{\text{zero}}$ and define the regions

$$\mathcal{B} = \{t_0 \leq t \leq t_1\} \cap \{r^\star_{\text{cl}} \leq r^\star \leq R^\star\} \tag{10.87}$$

and the slightly smaller region

$$\mathcal{B}_\varsigma = \{t_0 \leq t \leq t_1\} \cap \{r^\star_{\text{cl}} + \varsigma \leq r^\star \leq R^\star - \varsigma\}, \tag{10.88}$$

where $\varsigma \leq \frac{1}{6}\sqrt{M}$ is some positive number (in units of $\sqrt{M}$), say $\varsigma = \frac{1}{10}\sqrt{M}$. Define also a smooth cut off function χ supported in $[r^\star_{\text{cl}}, R^\star]$ and equal to 1 for $\{r^\star_{\text{cl}} + \varsigma \leq r^\star \leq R^\star - \varsigma\}$. Note that with the definition of $r^\star_{\text{cl}}$ (cf. Section 7.2) we have $r^\star_{\text{cl}} + \varsigma < r^\star_Y$ and also $R^\star - \varsigma \geq -\frac{1}{2}\sqrt{M}$.

Proposition 10.3. *We have, for large t_1,*

$$\frac{1}{M^{\frac{3}{2}}} \int_{\mathcal{B}} B^2 d\text{Vol} \leq C(\sigma)\, \bar{I}^X_B\left(\mathcal{B}^{r^\star_{\text{cl}}, R^\star}_{[t_1,t_2]}\right) \tag{10.89}$$

and

$$\frac{1}{\sqrt{M}} \int_{\mathcal{B}} (\partial_{r^\star} B)^2\, d\text{Vol} \leq C(r^\star_{\text{cl}}, \sigma)\, \bar{I}^X_B\left(\mathcal{B}^{r^\star_{\text{cl}}, R^\star}_{[t_1,t_2]}\right). \tag{10.90}$$

Furthermore, we can control the time derivative

$$\begin{aligned}\frac{1}{\sqrt{M}} \int_{\mathcal{B}_\varsigma} (\partial_t B)^2\, d\text{Vol} \leq C(r^\star_{\text{cl}}, \sigma, \chi) \Big[&\bar{I}^X_B\left(\mathcal{B}^{r^\star_{\text{cl}}, R^\star}_{[t_1,t_2]}\right) \\ &+ m(t_2, R^\star) - m(t_2, r^\star_{\text{cl}}) + m(t_1, R^\star) - m(t_1, r^\star_{\text{cl}}) \Big].\end{aligned} \tag{10.91}$$

Proof. Inequality (10.89) is the statement of Proposition 10.1. Equation (10.90) follows from an application of the triangle inequality to the first term in (10.31) and the previous bound on the B^2-integral, noting that $\frac{f'}{\Omega^2}$ is bounded above and below in the region under consideration. Finally, (10.91) is obtained via Green's identity[43]

$$\int_{\mathcal{B}} \Box\chi \left(-\frac{1}{2}B^2\right) = \int_{\mathcal{B}} \chi \left(-\frac{1}{2}\Box B^2\right) - \int \chi B \partial_t B r^3 dr^\star dA_{\mathbb{S}^3}\Big|_{t=t_1}^{t=t_2} \tag{10.92}$$

for the χ defined above, which can be written

$$\begin{aligned}\int_{\mathcal{B}} \chi \frac{1}{\Omega^2} \left(\partial_t B\right)^2 d\mathrm{Vol} = &\int_{\mathcal{B}} B^2 \left[-\frac{1}{2}\Box\chi + \frac{\chi}{r^2}\left(8 - \frac{\varphi_2\left(B\right)}{B^2}\right)\right] d\mathrm{Vol} \\ &+ \int_{\mathcal{B}} \chi \left(\partial_{r^\star} B\right)^2 \frac{1}{\Omega^2} d\mathrm{Vol} + \int \chi B \partial_t B r^3 dr^\star dA_{\mathbb{S}^3}\Big|_{t=t_1}^{t=t_2}.\end{aligned}$$

The spacetime integrals on the right-hand side are controlled by (10.89) and (10.90). For the boundary term in the second line we estimate

$$\begin{aligned}\int \chi B \partial_t B \left(t, \bar{r}^\star\right) \bar{r}^3 d\bar{r}^\star dA_{\mathbb{S}^3} &\leq \int_{r^\star_{\mathrm{cl}}}^{R^\star} \left(\frac{B^2}{\sqrt{M}} + \sqrt{M}\left(\partial_t B\right)^2\right)\left(t, \bar{r}^\star\right) \bar{r}^3 d\bar{r}^\star dA_{\mathbb{S}^3} \\ &\leq \sqrt{M} C\left(r^\star_{\mathrm{cl}}\right)\left(m\left(t, R^\star\right) - m\left(t, r^\star_{\mathrm{cl}}\right)\right).\end{aligned} \tag{10.93}$$

Putting all this together we obtain

$$\begin{aligned}\frac{1}{\sqrt{M}} \int_{\mathcal{B}_\varsigma} \left(\partial_t B\right)^2 d\mathrm{Vol} &\leq \frac{1}{\sqrt{M}} \int_{\mathcal{B}} \chi \frac{2}{\Omega^2} \left(\partial_t B\right)^2 d\mathrm{Vol} \\ &\leq C\left(r^\star_{\mathrm{cl}}, \sigma, \chi\right)\Big[\bar{I}^X_B\left(\mathcal{B}^{r^\star_{\mathrm{cl}}, R^\star}_{[t_1, t_2]}\right) + m\left(t_2, R^\star\right) - m\left(t_2, r^\star_{\mathrm{cl}}\right) \\ &\quad + m\left(t_1, R^\star\right) - m\left(t_1, r^\star_{\mathrm{cl}}\right)\Big].\end{aligned}$$

$\Box$

We can summarize this as

[43] There are no problems with the differentiability here.

Proposition 10.4. *The quantity*

$$I_B(\mathcal{W}) = \int_W \left[\frac{1}{\sqrt{M}}(\partial_t B)^2 + \frac{1}{\sqrt{M}}(\partial_{r^\star} B)^2 + \frac{1}{M^{\frac{3}{2}}} B^2\right] d\mathrm{Vol} \tag{10.94}$$

satisfies

$$\begin{aligned} I_B(\mathcal{B}_\varsigma) \leq C(r^\star_{\mathrm{cl}}, \sigma, \chi) \Big[& \bar{I}^X_B\left(\mathcal{B}^{r^\star_{\mathrm{cl}}, R^\star}_{[t_1,t_2]}\right) \\ & + m(t_2, R^\star) - m(t_2, r^\star_{\mathrm{cl}}) + m(t_1, R^\star) - m(t_1, r^\star_{\mathrm{cl}}) \Big]. \end{aligned} \tag{10.95}$$

Proof. This is a direct consequence of Proposition 10.3. □

11 Combining X and Y: horizon estimates

For the next two propositions recall the choice of $r^\star_Y$ made in Section 9, which implied in particular that Y is supported only in $r^\star \leq -\frac{1}{2}\sqrt{M}$.

11.1 Controlling I^Y_B from I^X_B and energy

Proposition 11.1. *Consider the characteristic rectangle $\mathcal{R} = [u_1, u_{\mathrm{hoz}}] \times [v_1, v_2]$ together with the $r^\star = r^\star_{\mathrm{cl}}$ curve intersecting (u_1, v_1). Define $u(v_2)$ by $r(u(v_2), v_2) = r^\star_{\mathrm{cl}}$ and $r(u, v(u)) = r^\star_{\mathrm{cl}}$. We have the estimates*

$$F^Y_B(\{u_1\} \times [v_1, v_2]) \leq C(r^\star_{\mathrm{cl}})(m(u_1, v_2) - m(u_1, v_1)), \tag{11.1}$$

$$\int_{u(v_2)}^{u_{\mathrm{hoz}}} \frac{(\partial_u B)^2}{\Omega^2} du \leq C F^Y_B([u_1, u_{\mathrm{hoz}}] \times \{v_2\}), \tag{11.2}$$

$$\int_{v_1}^{v(u)} r B^2 dv \leq C F^Y_B(\{u\} \times [v_1, v(u)]) \quad \textit{for all } u \geq u_1. \tag{11.3}$$

Proof. This follows from definitions (9.6) and (9.7) □

Proposition 11.2. *Recall the basic dyadic regions $\mathcal{D}^{r^\star_{\mathrm{cl}}, u_J}_{[t_1,t_2]}$ for the vectorfield X (10.2) and erect the characteristic rectangle*

$$\mathcal{R} = [u_1, u_{\mathrm{hoz}}] \times [v_1, v_2] \tag{11.4}$$

associated with such a region as depicted in figure 7. More precisely, let $u_1 = t_1 - r^\star_{\mathrm{cl}}$, $v_1 = t_1 + r^\star_{\mathrm{cl}}$, $v_2 = t_2 + r^\star_{\mathrm{cl}}$.

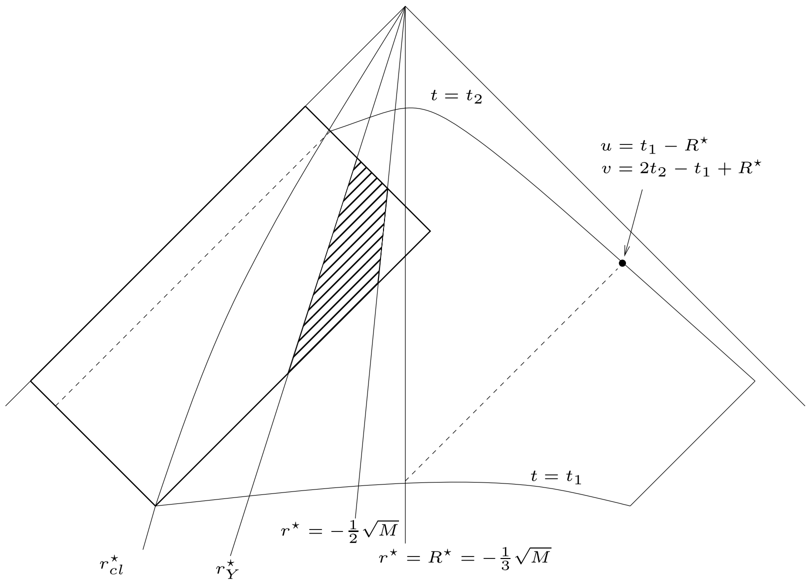


Figure 7: The horizon estimate.

Let also

$$\mathcal{T} = \{r^\star \geq r^\star_Y\} \cap \{v \leq v_2\} \cap \{u \geq u_1\}, \tag{11.5}$$

and recall that $R^\star = -\frac{1}{3}\sqrt{M}$. *We have the inequality*

$$\begin{aligned} &F^Y_B l(\{u_{\text{hoz}}\} \times [v_1, v_2]) + F^Y_B([u_1, u_{\text{hoz}}] \times \{v_2\}) + \tfrac{1}{2}\tilde{I}^Y_B(\mathcal{R} \setminus \mathcal{T}) \\ &\quad \leq C(r^\star_{\text{cl}}, \sigma)\left[\bar{I}^X_B\left(\mathcal{B}^{r^\star_{\text{cl}}, R^\star}_{[t_1,t_2]}\right) + m(t_2, R^\star) - m(t_2, r^\star_{\text{cl}}) + m(t_1, R^\star)\right. \\ &\qquad \left. -m(t_1, r^\star_{\text{cl}})\right] + C(r^\star_{\text{cl}})[m(u_1, v_2) - m(u_1, v_1)] + F^Y_B([u_1, u_{\text{hoz}}] \times \{v_1\}). \end{aligned} \tag{11.6}$$

Proof. Recall identity (9.13):

$$\begin{aligned} &F^Y_B(\{u_{\text{hoz}}\} \times [v_1, v_2]) + F^Y_B([u_1, u_{\text{hoz}}] \times \{v_2\}) + \tilde{I}^Y_B(\mathcal{R}) \\ &\quad = \hat{I}^Y_B(\mathcal{R}) + F^Y_B(\{u_1\} \times [v_1, v_2]) + F^Y_B([u_1, u_{\text{hoz}}] \times \{v_1\}). \end{aligned} \tag{11.7}$$

By Proposition 11.1 we control

$$F^Y_B(\{u_1\} \times [v_1, v_2]) \leq C(r^\star_{\text{cl}})(m(u_1, v_2) - m(u_1, v_1)). \tag{11.8}$$

To establish (11.6) we will show

$$\begin{aligned}\hat{I}^Y_B(\mathcal{R}) &\le \tfrac{1}{2}\tilde{I}^Y_B(\mathcal{R}\setminus\mathcal{T}) + C(r^\star_{\rm cl},\sigma)\Big[\bar{I}^X_B\Big(\mathcal{B}^{r^\star_{\rm cl},R^\star}_{[t_1,t_2]}\Big) + m(t_2,R^\star)\\ &\quad - m(t_2,r^\star_{\rm cl}) + m(t_1,R^\star) - m(t_1,r^\star_{\rm cl})\Big], \qquad (11.9)\\ \tilde{I}^Y_B(\mathcal{T}) &\le C(r^\star_{\rm cl},\sigma)\Big[\bar{I}^X_B\Big(\mathcal{B}^{r^\star_{\rm cl},R^\star}_{[t_1,t_2]}\Big)\\ &\quad + m(t_2,R^\star) - m(t_2,r^\star_{\rm cl}) + m(t_1,R^\star) - m(t_1,r^\star_{\rm cl})\Big]. \qquad (11.10)\end{aligned}$$

To see this decompose

$$\hat{I}^Y_B(\mathcal{R}) = \hat{I}^Y_B(\mathcal{R}\setminus\mathcal{T}) + \hat{I}^Y_B(\mathcal{T}). \qquad (11.11)$$

Since in $\mathcal{R}\setminus\mathcal{T}$ we have by definition $r^\star < r^\star_Y$ one can apply (9.12) to obtain

$$\hat{I}^Y_B(\mathcal{R}\setminus\mathcal{T}) \le \tfrac{1}{2}\tilde{I}^Y_B(\mathcal{R}\setminus\mathcal{T}). \qquad (11.12)$$

On the other hand, in the region $\mathcal{T}$ we have

$$\hat{I}^Y_B(\mathcal{T}) \le C(r^\star_Y) I_B(\mathcal{T}\cap\{r^\star \le R^\star = -\tfrac{1}{2}\sqrt{M}\}), \qquad (11.13)$$

which follows from the fact that Y is only supported for $r^\star \le -\frac{1}{2}\sqrt{M}$. An application of Proposition 10.4 to the term on the right-hand side of (11.13) will produce the required second term on the right-hand side of (11.9). Estimate (11.10) is obtained completely analogous to (11.13). □

Proposition 11.3. *With assumptions and geometry as in Proposition* 11.2 *we also have*

$$\begin{aligned}&F^Y_B(\{u_{\rm hoz}\}\times[v_1,v_2]) + F^Y_B([u_1,u_{\rm hoz}]\times\{v_2\}) + \tfrac{1}{2}\tilde{I}^Y_B(\mathcal{R}\setminus\mathcal{T})\\ &\quad\le C(r^\star_{\rm cl},\sigma)\big[m(u=t_1-R^\star, v=\tfrac{12}{11}t_2+R^\star)\\ &\qquad - m(u=t_2-r^\star_{\rm cl}, v=t_1+r^\star_{\rm cl})\big] + \tfrac{11}{10}F^Y_B([u_1,u_{\rm hoz}]\times\{v_1\}). \qquad (11.14)\end{aligned}$$

Proof. Use the inequality

$$\bar{I}^X_B\Big(\mathcal{B}^{r^\star_{\rm cl},R^\star}_{[t_1,t_2]}\Big) \le \bar{I}^X_B\Big({}^{u=t_2-r^\star_{\rm cl}}\mathcal{D}^{r^\star_{\rm cl},t_1-R^\star}_{[t_1,t_2]}\Big) \qquad (11.15)$$

which is obvious from the positivity of the integrand (compare the dashed lines in the previous figure for the regions). Inserting the estimate of Proposition 10.2 into inequality (11.6) we obtain the result by an appropriate choice of q. □

It is of crucial importance that the constant $C(r^\star_{\text{cl}})$ just depends on the choice of $r^\star_{\text{cl}}$ and not on $r^\star_K$.

11.2 Controlling F^Y_B from $\tilde{I}^Y_B$ and energy, on a good slice

Finally, we are going to control the boundary terms F^Y by $\tilde{I}^Y$ and the energy on a "good" null-slice.

Proposition 11.4. *With $\mathcal{R}$ and $\mathcal{T}$ as before pick a $\hat{v} \in [v_1, v_2]$ that satisfies*

$$F^Y_B\left([u_1, u_{\text{hoz}}] \times \{\hat{v}\}\right) = \inf_{v_1 \leq v \leq v_2} F^Y_B([u_1, u_{\text{hoz}}] \times \{v\}). \tag{11.16}$$

Then

$$\begin{aligned} &F^Y_B([u_1, u_{\text{hoz}}] \times \{\hat{v}\}) \\ &\quad \leq C(v_2 - v_1)^{-1} \tilde{I}^Y_B\left(\mathcal{R} \setminus \mathcal{T}\right) + C\left(r^\star_{\text{cl}}\right)\left(m(u_1, v_2) - m(u_1, v_1)\right). \end{aligned} \tag{11.17}$$

Proof. Recall that the expression (9.7) is manifestly positive. Set $u(v) = v - 2r^\star_{\text{cl}}$ and estimate

$$\begin{aligned} &F^Y_B([u_1, u_{\text{hoz}}] \times \{\hat{v}\}) \\ &\quad \leq \inf_{v_1 \leq v \leq v_2} F^Y_B([u(v), u_{\text{hoz}}] \times \{v\}) + F^Y_B([u_1, u(\tilde{v})] \times \{\tilde{v}\}) \\ &\quad \leq \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} F^Y_B([u(v), u_{\text{hoz}}] \times \{v\}) dv + C(r^\star_{\text{cl}})[m(u_1, v_2) - m(u_1, v_1)], \end{aligned}$$

where $\tilde{v}$ is the v-slice determined by taking the infimum of F^Y_B in the region $[u(v), u_{\text{hoz}}]$. For the integrand of the first term in the last line we have

$$\begin{aligned} &F^Y_B\left([v - 2r^\star_{\text{cl}}, u_{\text{hoz}}] \times \{v\}\right) \\ &\quad \leq 2\pi^2 \int_{v - 2r^\star_{\text{cl}}}^{u_{\text{hoz}}} du\, r^3 4\sqrt{M}\left[\frac{(\partial_u B)^2}{\Omega^2}\left(4\alpha\frac{\Omega_{,v}}{\Omega} - \alpha'\right)\beta'(\partial_v B)^2\right. \\ &\quad \left. + \frac{1}{2r^2}\left(1 - \frac{2}{3}\rho\right)\left(\frac{\alpha'}{2} - \frac{\alpha\nu}{r} - \frac{\beta\lambda\Omega^2}{r} - \frac{1}{2}\beta'\Omega^2 - 2\beta\Omega^2\frac{\Omega_{,v}}{\Omega}\right)\right], \end{aligned} \tag{11.18}$$

following from the fact that inequalities (7.18)–(7.20) hold in $r^\star \leq r^\star_{\text{cl}}$. Comparing (11.18) with (9.10) produces the first term in (11.17). $\square$

We will also need a related version of the previous proposition, which provides one with a good energy slice instead of a good F^Y-slice:

Proposition 11.5. *With $\mathcal{R}$ and $\mathcal{T}$ as before pick a $\hat{v} \in [v_1, v_2]$ that satisfies*

$$E\left([u_1, u_{\text{hoz}}] \times \{\hat{v}\}\right) = \inf_{v_1 \leq v \leq v_2} E([u_1, u_{\text{hoz}}] \times \{v\}). \tag{11.19}$$

Then

$$E\left([u_1, u_{\text{hoz}}] \times \{\hat{v}\}\right) \leq C\left(v_2 - v_1\right)^{-1} \tilde{I}^Y_B\left(\mathcal{R} \setminus \mathcal{T}\right) + \left(m(u_1, v_2) - m(u_1, v_1)\right). \tag{11.20}$$

Proof. Recall that

$$\begin{aligned} &E\left([u_1, u_{\text{hoz}}] \times \{v\}\right) \\ &= \int_{u_1}^{u_{\text{hoz}}} \int_{\mathbb{S}^3} \left(\frac{4\lambda}{\Omega^2}\left(\partial_u B\right)^2 + \frac{1}{r^2}\left(1 - \frac{2}{3}\rho\right)(-\nu)\right) r^3 du\, dA_{\mathbb{S}^3} \end{aligned}$$

is manifestly positive. With $u(v) = v - 2r^\star_{\text{cl}}$ estimate

$$\begin{aligned} &E([u_1, u_{\text{hoz}}] \times \{\hat{v}\}) \\ &\quad \leq \inf_{v_1 \leq v \leq v_2} E\left([u\left(v\right), u_{\text{hoz}}] \times \{v\}\right) + E\left([u_1, u\left(\tilde{v}\right)] \times \{\tilde{v}\}\right) \\ &\quad \leq \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} E\left([u\left(v\right), u_{\text{hoz}}] \times \{v\}\right) dv + [m(u_1, v_2) - m(u_1, v_1)]. \end{aligned}$$

The integrand of the first term in the last line can be controlled by

$$\begin{aligned} &E\left([v - 2r^\star_{\text{cl}}, u_{\text{hoz}}] \times \{v\}\right) \\ &\quad \leq 2\pi^2 \int_{v - 2r^\star_{\text{cl}}}^{u_{\text{hoz}}} du\ r^3 4\sqrt{M}\left[\frac{(\partial_u B)^2}{\Omega^2}\left(4\alpha\frac{\Omega_{,v}}{\Omega} - \alpha'\right) + \beta'(\partial_v B)^2\right. \\ &\quad \left. + \frac{1}{2r^2}\left(1 - \frac{2}{3}\rho\right)\left(\frac{\alpha'}{2} - \frac{\alpha\nu}{r} - \frac{\beta\lambda\Omega^2}{r} - \frac{1}{2}\beta'\Omega^2 - 2\beta\Omega^2\frac{\Omega_{,v}}{\Omega}\right)\right], \end{aligned} \tag{11.21}$$

following from the fact that inequalities (7.18)–(7.20) hold in $r^\star_{\text{cl}}$. Comparing (11.21) with (9.10) produces the first term in (11.20). $\square$

The results of this section are already sufficient to obtain a pointwise decay bound for the quantity $\frac{\zeta}{\nu}$. For reasons of presentation this is postponed to Section 13.1 but the reader impatient to see the argument can turn to the latter section at this point.

12 The vectorfield K

12.1 The basic identity

The vectorfield K is defined as

$$K = \frac{2}{M}(u+a)^2\partial_u + \frac{2}{M}(v-a)^2\partial_v. \tag{12.1}$$

It is the analogue of the Morawetz vector field in four dimensions. In particular, it is conformally Killing in five-dimensional Minkowski space.[44] We note

$$K^u = \frac{2}{M}(u+a)^2, \quad K^v = \frac{2}{M}(v-a)^2, \quad K_u = -\frac{\Omega^2}{M}(v-a)^2,$$
$$K_v = -\frac{\Omega^2}{M}(u+a)^2 \tag{12.2}$$

and

$$u = t - r^\star, \quad v = t + r^\star, \quad (v-a)^2 - (u+a)^2 = 4t\left(r^\star - a\right). \tag{12.3}$$

From (5.11) we compute the identity

$$\begin{aligned} &M\left(-T_{\mu\nu}\pi^{\mu\nu} - \nabla^\beta T_{\beta\delta}K^\delta\right) \\ &\quad = \frac{3}{2r}\left(\nu(u+a)^2 + \lambda(v-a)^2\right)\Box B^2 + 32\frac{B^2}{r^2}\left(t - \frac{1}{2r}(\nu(u+a)^2\right. \\ &\qquad \left. + \lambda(v-a)^2) + \frac{1}{4\Omega^2}\left(\left(\Omega^2\right)_{,u}(u+a)^2 + \left(\Omega^2\right)_{,v}(v-a)^2\right)\right) \\ &\qquad + \frac{\varphi_1(B)}{\Omega^2 r^2}\left(\left(\Omega^2\right)_{,v}(v-a)^2 + \left(\Omega^2\right)_{,u}(u+a)^2\right) + \frac{4t}{r^2}\varphi_1(B) \\ &\qquad + \frac{3}{r^3}(\nu(u+a)^2 + \lambda(v-a)^2)\left(\varphi_1(B) + \varphi_2(B)\right) \\ &\qquad - \frac{2}{r^3}\varphi_1(B)(\nu(u+a)^2 + \lambda(v-a)^2) \end{aligned} \tag{12.4}$$

with φ_1 and φ_2 defined in (2.15). We shall apply the basic vectorfield identity in the region (cf. figure 8)

$$\mathcal{D}^K_{[t_0,\tilde{T}]} = {}^{\tilde{T}-r^\star_K}\mathcal{D}^{r^\star_K,u_0}_{[t_0,\tilde{T}]} \tag{12.5}$$

[44] The vectorfield field has been shifted by a for reasons which will become apparent later.

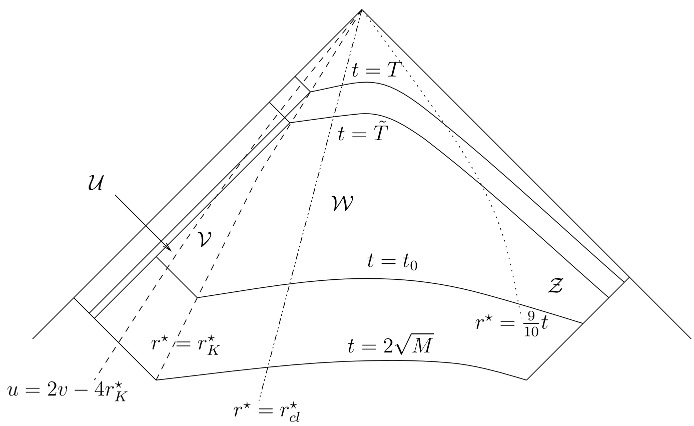


Figure 8: Different regions to control the error terms of K.

for any $\tilde{T} < T$ producing the identity

$$\widehat{I}^K_B\left(\mathcal{D}^K_{[t_0,\tilde{T}]}\right) = \widehat{F}^K_B(\tilde{T}) - \widehat{F}^K_B(t_0) + \hat{H}^K_{u_H=\tilde{T}-r^\star_K} + 0, \tag{12.6}$$

where

$$\widehat{I}^K_B\left(\mathcal{D}^K_{[t_0,\tilde{T}]}\right) = \int_{\mathcal{D}^K_{[t_0,\tilde{T}]}} (-T_{\mu\nu}\pi^{\mu\nu} - \nabla^\beta T_{\beta\delta}K^\delta) d\mathrm{Vol}, \tag{12.7}$$

$$\begin{aligned}
\frac{\widehat{F}^K_B(t)}{2\pi^2} &= \frac{1}{M}\int_{r^\star_K}^{t-u_0} \Big((\partial_u B)^2\, 2(u+a)^2 + (\partial_v B)^2 2(v-a)^2 \\
&\quad + \left((u+a)^2 + (v-a)^2\right)\frac{\Omega^2}{2r^2}\left(1 - \frac{2}{3}\rho\right)\Big)\, r^3 dr^\star \\
&\quad + \frac{1}{M}\int_{t-r^\star_K}^{\tilde{T}-r^\star_K}\Big[2(u+a)^2 r^3 \left(\partial_u B\right)^2 \\
&\quad + \frac{r\Omega^2}{2}(v-a)^2\left(1 - \frac{2}{3}\rho\right)\Big]\,(u, t + r^\star_K)\, du
\end{aligned} \tag{12.8}$$

and

$$\begin{aligned}
\frac{\hat{H}^K_{\tilde{T}-r^\star_K}}{2\pi^2} &= \frac{1}{M}\int_{t_0+r^\star_K}^{\tilde{T}+r^\star_K}\Big[2(v-a)^2 r^3 \left(\partial_v B\right)^2 \\
&\quad + \frac{r\Omega^2}{2}(u+a)^2\left(1 - \frac{2}{3}\rho\right)\Big]\left(\tilde{T} - r^\star_K, v\right) dv.
\end{aligned}$$

Note that the J-term vanishes in view of the assumption of compact support. We are now going to define the renormalized quantities I_B^K and E_B^K that arise from an application of Green's theorem to the $\Box B^2$ term in the spacetime integral (12.8). The D in the basic identity (5.24) is here given by (cf. Appendix A)

$$D = \frac{3}{2}\left(\frac{\nu(u+a)^2 + \lambda(v-a)^2}{r}\right). \tag{12.9}$$

We compute

$$\begin{aligned}
&\frac{3}{2}r^2\Box\left(\frac{\nu(u+a)^2+\lambda(v-a)^2}{r}\right) \\
&\quad = t\left(-24r\frac{r_{,uv}}{\Omega^2} - 12\frac{\lambda\nu}{\Omega^2}\right) + t\left(\frac{r^\star - a}{r}\right)\left(12r\frac{\lambda}{\Omega^2}r_{,uu} + 12rr_{,uv}\frac{\lambda}{\Omega^2}\right. \\
&\qquad \left. -24\frac{r^2}{\Omega^2}(r_{,uv})_{,v} + 24\frac{\nu\lambda^2}{\Omega^2}\right) + (u+a)^2\left([\lambda+\nu]\left(\frac{3r_{,uv}}{\Omega^2} + \frac{6\nu\lambda}{\Omega^2 r}\right)\right. \\
&\qquad \left. -\frac{6r}{\Omega^2}\left((r_{,uv})_{,v} + (r_{,uv})_{,u}\right)\right) + (v-a)^2\left(-\frac{3\lambda}{\Omega^2}r_{,uu} - \frac{3\nu}{\Omega^2}r_{,vv}\right)
\end{aligned} \tag{12.10}$$

and define the bulk term

$$\begin{aligned}
&I_B^K\left(\mathcal{D}^K_{[t_0,\tilde{T}]}\right) \\
&\quad = \frac{1}{M}\int\int\frac{1}{2}r^3\Omega^2 du\, dv\frac{B^2}{r^2}\left\{t\left[32 - 24r\frac{r_{,uv}}{\Omega^2} - 12\frac{\lambda\nu}{\Omega^2} + 4\frac{\varphi_1(B)}{B^2}\right]\right. \\
&\qquad + t\left(\frac{r^\star - a}{r}\right)\left[-64\lambda + 24\frac{\lambda^2\nu}{\Omega^2} + 12\frac{rr_{,uv}\lambda}{\Omega^2} - 24\frac{r^2}{\Omega^2}(r_{,uv})_{,v} + 12r\frac{\lambda}{\Omega^2}r_{,uu}\right. \\
&\qquad \left. -64\frac{\Omega_{,u}}{\Omega}r + \frac{\varphi_1(B)}{B^2}\left(-8r\frac{\Omega_{,u}}{\Omega} + 4\lambda\right) + 12\lambda\frac{\varphi_2(B)}{B^2}\right] \\
&\qquad + (u+a)^2\left[(\lambda+\nu)\left(3\frac{r_{,uv}}{\Omega^2} + \frac{6\nu\lambda}{\Omega^2 r} - \frac{16}{r} + \frac{1}{r}\frac{\varphi_1(B) + 3\varphi_2(B)}{B^2}\right)\right. \\
&\qquad \left. -6\frac{r}{\Omega^2}\left((r_{,uv})_{,v} + (r_{,uv})_{,u}\right)\right] + (v-a)^2\left[\left(16 + 2\frac{\varphi_1(B)}{B^2}\right)\left(\frac{\Omega_{,v}}{\Omega} + \frac{\Omega_{,u}}{\Omega}\right)\right. \\
&\qquad \left.\left. -3\frac{\lambda}{\Omega^2}r_{,uu} - 3\frac{\nu}{\Omega^2}r_{,vv}\right]\right\}.
\end{aligned} \tag{12.11}$$

In order for the identity (cf. equation (5.30))

$$I_B^K\left(\mathcal{D}^K_{[t_0,\tilde{T}]}\right) = F_B^K(\tilde{T}) - F_B^K(t_0) + H^K_{\tilde{T}-r^\star_K} \tag{12.12}$$

to hold, the boundary terms have to be

$$\begin{aligned}
&\frac{F_B^K(t)}{2\pi^2}\\
&\quad=\frac{\hat{F}_B^K(t)}{2\pi^2}+\frac{1}{M}\int_{r_K^\star}^{t-u_0}2B(\partial_t B)\left(\frac{3}{2r}(\nu(u+a)^2+\lambda(v-a)^2)\right)r^3(t,r^\star)dr^\star\\
&\qquad-\frac{1}{M}\int_{r_K^\star}^{t-u_0}B^2\partial_t\left(\frac{3}{2r}(\nu(u+a)^2+\lambda(v-a)^2)\right)r^3\,(t,r^\star)\,dr^\star\\
&\qquad+\frac{1}{M}\int_{t-r_K^\star}^{\tilde{T}-r_K^\star}2B(\partial_u B)\left(\frac{3}{2r}(\nu(u+a)^2+\lambda(v-a)^2)\right)r^3(u,t+r_K^\star)du\\
&\qquad-\frac{1}{M}\int_{t-r_K^\star}^{\tilde{T}-r_K^\star}B^2\partial_u\left(\frac{3}{2r}(\nu(u+a)^2+\lambda(v-a)^2)\right)r^3\,(u,t+r_K^\star)\,du
\end{aligned}\tag{12.13}$$

and

$$\begin{aligned}
\frac{H^K_{\tilde{T}-r_K^\star}}{2\pi^2}&=\frac{\hat{H}^K_{\tilde{T}-r_K^\star}}{2\pi^2}-\frac{1}{M}\int_{t_0+r_K^\star}^{\tilde{T}+r_K^\star}B^2\partial_v\left(\frac{3}{2r}(\nu(u+a)^2+\lambda(v-a)^2)\right)\\
&\qquad r^3(\tilde{T}-r_K^\star,v)dv+\frac{1}{M}\int_{t_0+r_K^\star}^{\tilde{T}+r_K^\star}2B(\partial_v B)\left(\frac{3}{2r}(\nu(u+a)^2\right.\\
&\qquad\quad\left.+\lambda(v-a)^2)\right)r^3(\tilde{T}-r_K^\star,v)dv.
\end{aligned}\tag{12.14}$$

12.2 The spacetime integral

Let us turn to an analysis of the integral (12.11). Besides formulae (10.77), (2.5) and (2.4) the following identities will be useful:

$$\begin{aligned}
\frac{(r_{,uv})_{,v}}{\Omega^2}&=-\frac{\Omega_{,v}}{\Omega}\frac{\mu}{r}+\frac{3\lambda\mu}{2r^2}-\frac{1}{r^3}\left(\frac{\theta^2}{\kappa}+r\lambda\left(1-\frac{2}{3}\rho\right)\right)\\
&\quad-\frac{\Omega_{,v}}{\Omega}\frac{2}{3r}\left(\rho-\frac{3}{2}\right)+\frac{\lambda}{3r^2}\left(\rho-\frac{3}{2}\right)+\frac{4}{3r^{\frac{5}{2}}}\theta(\mathrm{e}^{-2B}-\mathrm{e}^{-8B}),\\
\frac{(r_{,uv})_{,u}}{\Omega^2}&=-\frac{\Omega_{,u}}{\Omega}\frac{\mu}{r}+\frac{3\nu\mu}{2r^2}-\frac{1}{r^3}\left(-4\frac{\lambda}{\Omega^2}\zeta^2+r\nu\left(1-\frac{2}{3}\rho\right)\right)\\
&\quad-\frac{\Omega_{,u}}{\Omega}\frac{2}{3r}\left(\rho-\frac{3}{2}\right)+\frac{\nu}{3r^2}\left(\rho-\frac{3}{2}\right)+\frac{4}{3r^{\frac{5}{2}}}\zeta(\mathrm{e}^{-2B}-\mathrm{e}^{-8B}).
\end{aligned}$$

The bulk integral can be written

$$I_B^K\left(\mathcal{D}^K_{[t_0,\tilde{T}]}\right) = I^K_{B,\mathrm{main}}\left(\mathcal{D}^K_{[t_0,\tilde{T}]}\right) + I^K_{B,\mathrm{error}}\left(\mathcal{D}^K_{[t_0,\tilde{T}]}\right) \tag{12.15}$$

with

$$\begin{aligned}
&I^K_{B,\mathrm{main}}\left(\mathcal{D}^K_{[t_0,\tilde{T}]}\right)\\
&\quad= \frac{1}{M}\int\int \frac{1}{2} r^3\Omega^2 du\, dv \frac{B^2}{r^2}\left\{ t\left[35 + 9\mu + 4\frac{\varphi_1(B)}{B^2} + 8\left(\rho - \frac{3}{2}\right)\right]\right.\\
&\qquad + t\left(\frac{r^\star - a}{r}\right)\left[24\mu r\frac{\Omega_{,v}}{\Omega} - 64r\frac{\Omega_{,u}}{\Omega} + (1-\mu)\left[-70\kappa - 36\kappa\mu\right.\right.\\
&\qquad \left.\left.\left. -6r\frac{\Omega_{,u}}{\Omega}\right] + P(B)\right]\right\}, \qquad (12.16)\\
&I^K_{B,\mathrm{error}}\left(\mathcal{D}^K_{[t_0,\tilde{T}]}\right)\\
&\quad= \frac{1}{M}\int\int \frac{1}{2} r^3\Omega^2 du\, dv \frac{B^2}{r^2}\left\{\frac{(u+a)^2}{2}\left(Q(B) + \left(\frac{\Omega_{,v}}{\Omega} + \frac{\Omega_{,u}}{\Omega}\right)\right.\right.\\
&\qquad \times\left[12\mu + 8\left(\rho - \frac{3}{2}\right)\right] + \frac{(\lambda+\nu)}{r}\left[-35 - 18\mu - 14\left(\rho - \frac{3}{2}\right)\right.\\
&\qquad \left.\left. +2\frac{\varphi_1(B) + 3\varphi_2(B)}{B^2}\right]\right) + (v-a)^2\left(\left[\frac{\Omega_{,v}}{\Omega} + \frac{\Omega_{,u}}{\Omega}\right]\left(16 + \frac{3}{2}(1-\mu)\right.\right.\\
&\qquad \left.\left.\left. +2\frac{\varphi_1(B)}{B^2}\right) + 6\frac{\lambda\zeta^2}{r^2\Omega^2} + 6\frac{\nu\theta^2}{r^2\Omega^2}\right)\right\}, \qquad (12.17)
\end{aligned}$$

where

$$\begin{aligned}
P(B) = &-8r\frac{\Omega_{,u}}{\Omega}\frac{\varphi_1(B)}{B^2} + 4\kappa(1-\mu)\frac{\varphi_1(B) + 3\varphi_2(B)}{B^2} + \frac{24\theta^2}{\kappa r} - 24\frac{\lambda\zeta^2}{\Omega^2 r}\\
&- \frac{32}{\sqrt{r}}\theta(\mathrm{e}^{-2B} - \mathrm{e}^{-8B}) + \left(\rho - \frac{3}{2}\right)\left[-28\kappa(1-\mu) + 16\frac{\Omega_{,v}}{\Omega}r\right]
\end{aligned} \tag{12.18}$$

and

$$Q(B) = \frac{12\theta^2}{r^2\kappa} - \frac{48\lambda}{r^2\Omega^2}\zeta^2 - 16(\mathrm{e}^{-2B} - \mathrm{e}^{-8B})\frac{\theta + \zeta}{r^{\frac{3}{2}}}. \tag{12.19}$$

Note that

$$|P(B)| \leq C(\epsilon)\frac{\sqrt{M}}{r} \tag{12.20}$$

by the pointwise bounds of Section 8.4.

12.2.1 Estimating $I^K_{B,\text{main}}$

We start with the observation that $I^K_{B,\text{main}}$ has a good sign near the horizon and near infinity:

Lemma 12.1. *One can find $\hat{R}^\star$ such that the integrand of $I^K_{B,\text{main}}$ is negative for $r^\star \geq \hat{R}^\star$. It is also negative for $r^\star \leq r^\star_{\text{cl}}$.*

Proof. The second statement is a consequence of (7.20) and the inequality $\frac{\Omega_{,u}}{\Omega} < 0$ which follows from Proposition 8.15. For large $r^\star$ on the other hand, we can expand the integrand of (12.16) in powers of $\frac{1}{r}$ using the results of Section 8.4:

$$|B| \leq C(\epsilon)\frac{\sqrt{M}}{r} \quad \text{and} \quad \kappa = \frac{1}{2} + \mathcal{O}\left(\frac{1}{r^2}\right) \quad \text{and} \quad \frac{r\Omega_{,u}}{\mu\Omega} \approx -\frac{1}{2} \quad \text{and}$$
$$\frac{\Omega_{,v}}{\Omega} = \mathcal{O}\left(\frac{1}{r^2}\right)$$

and (cf. identification (8.16))

$$\begin{aligned}\frac{r^\star}{r} \sim 1 - \frac{\tilde{p} \pm \epsilon}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \quad & \text{where } \tilde{p} = \sqrt{\frac{M_A}{2}}p \\ &= \sqrt{\frac{M_A}{2}}\left[2\sqrt{2} + \log\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\right]\end{aligned}$$

to find

$$I^K_{B,\text{main}}\left(\mathcal{D}^K_{[t_0,\tilde{T}]}\right) = \frac{1}{M}\int\int \frac{1}{2}r^3\Omega^2 du\, dv \frac{B^2}{r^2} t\left\{\frac{35(a+\tilde{p})+\epsilon}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right\}.$$

With the chosen centre a of the K vector field ($a = -\tilde{p} - 1$ by equation (7.6)), the integrand will be negative in $r^\star \geq \hat{R}^\star$ for some suitably chosen $\hat{R}^\star$.[45] $\square$

Remark. In particular, we will choose t_0 so large that $\hat{R}^\star \leq \frac{9}{10}t_0$ holds.

[45] Note that the rest terms are all controlled by $\frac{C(r_{\text{cl}},c)}{r^2}$.

The idea in estimating the spacetime integral $I^K_{B,\text{main}}\left(\mathcal{D}^K_{[t_0,\tilde{T}]}\right)$ is to decompose the region of integration into dyadic pieces (cf. footnote 35)

$$\mathcal{D}^K_{[t_0,\tilde{T}]} = \sum_{j=0}^{N-1} \hat{\mathcal{D}}^K_{[t_j,t_{j+1}]} \quad \text{with } t_N = \tilde{T}, \tag{12.21}$$

$$\hat{\mathcal{D}}^K_{[t_j,t_{j+1}]} = \mathcal{D}^K_{[t_0,\tilde{T}]} \cap \{t_j \leq t \leq t_{j+1}\}. \tag{12.22}$$

For each piece $\hat{\mathcal{D}}^K_{[t_j,t_{j+1}]}$ we can control the bulk term $I^K_{B,\text{main}}$ by the bulk term $\bar{I}^X_B$ losing a power of t:

Proposition 12.1. *In the region $\mathcal{D}^K_{[t_0,\tilde{T}]}$ we have, for each dyadic piece*

$$\begin{aligned} I^K_{B,\text{main}}\left(\hat{\mathcal{D}}^K_{[t_j,t_{j+1}]}\right) &\leq \frac{1}{\sqrt{M}} C(r^\star_{\text{cl}}, \hat{R}^\star, \sigma) t_{j+1} \bar{I}^X_B \left(\mathcal{B}^{[r^\star_{\text{cl}},\hat{R}^\star]}_{[t_j,t_{j+1}]}\right) \\ &\leq \frac{1}{\sqrt{M}} C(r^\star_{\text{cl}}, \hat{R}^\star, \sigma) t_{j+1} \bar{I}^X_B \left({}^{u=\tilde{T}-r^\star_{\text{cl}}}\mathcal{D}^{r^\star_{\text{cl}},u=\frac{1}{11}t_{j+1}]}_{[t_j,t_{j+1}]}\right). \end{aligned} \tag{12.23}$$

Proof. By the previous lemma it suffices to show (12.23) with $\hat{\mathcal{D}}^K_{[t_1,t_2]}$ replaced by $\mathcal{B}^{[r^\star_{\text{cl}},\hat{R}^\star]}_{[t_1,t_2]}$ because the integrand of $I^K_{B,\text{main}}$ admits a good sign to the left of $r^\star_{\text{cl}}$ and to the right of $\hat{R}^\star$. For the compact $r^\star$-interval the first part of inequality (12.23) follows from Proposition 10.1, the second from $\hat{R}^\star \leq \frac{9}{10}t_0$ and the positivity of $\bar{I}^X_B$. □

12.2.2 Estimating $I^K_{B,\text{error}}$

In this subsection we are going to show that the contribution of the integral $I^K_{B,\text{error}}$ can be made as small as we may wish for late times by choosing $r^\star_K$ sufficiently close to the horizon and the initial data sufficiently small. To achieve this we shall split the integration into different regions $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{Z}$ defined as follows:

$$\mathcal{U} = \mathcal{D}^K_{[t_0,\tilde{T}]} \cap \{r^\star \leq r^\star_K\} \cap \{u \geq 2v - 4r^\star_K\}, \tag{12.24}$$

$$\mathcal{V} = \mathcal{D}^K_{[t_0,\tilde{T}]} \cap \{r^\star \leq r^\star_K\} \cap \{u \leq 2v - 4r^\star_K\}, \tag{12.25}$$

$$\mathcal{W} = \mathcal{D}^K_{[t_0,\tilde{T}]} \cap \{r^\star_K \leq r^\star \leq \tfrac{9}{10}t\}, \tag{12.26}$$

$$\mathcal{Z} = \mathcal{D}^K_{[t_0,\tilde{T}]} \cap \{r^\star \geq \tfrac{9}{10}t\}. \tag{12.27}$$

An immediate observation is

Lemma 12.2. *In all regions we have*

$$\kappa + \gamma \geq 0. \tag{12.28}$$

Proof. This is a consequence of the choice of coordinates and the monotonicity of κ in u and of γ in v manifest in equations (2.13) and (2.14). □

The next lemma establishes appropriate bounds to control the error terms of $I^K_{B,\mathrm{error}}$ in equation (12.17).

Lemma 12.3. *Recall that by Propositions* 8.9, 8.12 *and* 8.13 *the bound*

$$|B| + M^{-\frac{1}{4}}|\theta| \leq C(r^\star_{\mathrm{cl}})\frac{\sqrt{M}}{t} \tag{12.29}$$

holds in $\mathcal{W}$ and

$$|B| + M^{-\frac{1}{4}}|\theta| \leq C(r^\star_{\mathrm{cl}})\frac{\sqrt{M}}{v} \tag{12.30}$$

holds in $\mathcal{U} \cup \mathcal{V}$. Assume also

$$\left|\frac{\zeta}{\nu}\right| \leq C(r^\star_{\mathrm{cl}})\frac{M^{\frac{3}{4}}}{t} \ \text{ in } \mathcal{W} \quad \text{and} \quad \left|\frac{\zeta}{\nu}\right| \leq C(r^\star_{\mathrm{cl}})\frac{M^{\frac{3}{4}}}{v} \ \text{ in } \mathcal{U} \cup \mathcal{V}. \tag{12.31}$$

Then we have the following estimates

- *In region $\mathcal{W}$*

$$|Q(B)| + \left|6\frac{\lambda\zeta^2}{r^2\Omega^2}\right| + \left|6\frac{\nu\theta^2}{r^2\Omega^2}\right| \leq C(r^\star_K, c)\frac{M^{\frac{7}{4}}}{r^{\frac{3}{2}}t^2}, \tag{12.32}$$

$$\left|\frac{\Omega_{,v}}{\Omega} + \frac{\Omega_{,u}}{\Omega}\right| \leq C(r^\star_K, c)\frac{\sqrt{M}}{t^2}. \tag{12.33}$$

- *In region $\mathcal{V}$*

$$|Q(B)| + \left|6\frac{\lambda\zeta^2}{r^2\Omega^2}\right| + \left|6\frac{\nu\theta^2}{r^2\Omega^2}\right| \leq \frac{C(r^\star_{\mathrm{cl}}, c)}{\sqrt{M}u^2}, \tag{12.34}$$

$$\frac{\Omega_{,v}}{\Omega} + \frac{\Omega_{,u}}{\Omega} + \frac{C(r^\star_{\mathrm{cl}}, c)\sqrt{M}}{u^2} \geq 0. \tag{12.35}$$

- *In region $\mathcal{U}$*

$$-\nu \leq d_1 \exp\left(-\frac{d_2}{2\sqrt{M}}u\right) \tag{12.36}$$

  for positive constants $d_1 > 0$, $d_2 > 0$.

- *In region $\mathcal{Z}$*

$$\left|\frac{\Omega_{,v}}{\Omega} + \frac{\Omega_{,u}}{\Omega}\right| \leq C(r^\star_{\text{cl}}, c)\frac{\sqrt{M}}{r^2}. \tag{12.37}$$

Proof. The region $\mathcal{W}$: Bound (12.33) is the statement of Proposition 8.16. Bound (12.32) follows directly from the decay properties (12.29) and (12.30).

The region $\mathcal{V}$: From Proposition 8.16 we derive the bound

$$\left|\frac{\Omega_{,v}}{\Omega}(u,v) - \frac{m}{r^3}(u,v)\right| \leq C(r^\star_{\text{cl}}, c)\frac{\sqrt{M}}{u^2} \tag{12.38}$$

by observing that u is like v in the region $\mathcal{V}$.

The quantity $\frac{\Omega_{,u}}{\Omega}$ is obtained by integrating (2.5) written as

$$\partial_v\left(\frac{\Omega_{,u}}{\Omega}\right) = \gamma\left(6m\frac{\lambda}{r^4} + \frac{2\lambda}{r^2}\left(\rho - \frac{3}{2}\right) + 3\frac{\theta}{\kappa}\frac{\zeta}{\nu}\frac{\lambda}{r^3}\right) \tag{12.39}$$

from the set $L = \{\{t = T\} \cap \{r^\star_K \leq r^\star \leq r^\star_{\text{cl}}\}\} \cup \{r^\star = r^\star_{\text{cl}}\}$ downwards. On L itself we have by Proposition 8.16

$$\left|\frac{\Omega_{,u}}{\Omega}(u,v) - \frac{m}{r^3}(u,v)\right| \leq C(r^\star_{\text{cl}}, c)\frac{\sqrt{M}}{u^2}. \tag{12.40}$$

Since $\gamma \leq \frac{1}{2}$ in $\mathcal{V}$ by monotonicity and moreover $\frac{\Omega_{,u}}{\Omega}$ will always be negative, we can derive the bound

$$\begin{aligned}
\frac{\Omega_{,u}}{\Omega}(u,v) &= \frac{\Omega_{,u}}{\Omega}(u,v_R) - \int_v^{v_R} \gamma\left(6m\frac{\lambda}{r^4} + \frac{2\lambda}{r^2}\left(\rho - \frac{3}{2}\right) + 3\frac{\theta}{\kappa}\frac{\zeta}{\nu}\frac{\lambda}{r^3}\right)(u,\bar{v})d\bar{v} \\
&\geq -\frac{m}{r^3}(u,v_R) - C(r^\star_{\text{cl}}, c)\frac{\sqrt{M}}{u^2} + m(u,v_R)\left(\frac{1}{r^3(u,v_R)} - \frac{1}{r^3(u,v)}\right) \\
&\geq -\frac{m}{r^3}(u,v) - C(r^\star_{\text{cl}}, c)\frac{\sqrt{M}}{u^2}
\end{aligned} \tag{12.41}$$

in $\mathcal{V}$, where in the last step we used that in $\mathcal{V}$ the Hawking mass decays like $\frac{1}{u^2}$. Putting bounds (12.41) and (12.38) together yields (12.35). Bound (12.34) on $Q(B)$ follows directly from the pointwise bound (12.30).

The region $\mathcal{U}$: Integrating the quantity $\nu = \gamma(1-\mu)$ from the spacelike $t = T$ curve downwards to any point in the region $\mathcal{U}$ we obtain

$$\begin{aligned} -\nu(u,v) &= \frac{1}{2}(1-\mu)(u_T, v_T)\exp\left(-\int_v^{v_T} \tilde{f}(u,v)(u_T,\bar{v})d\bar{v}\right) \\ &= \frac{1}{2}(1-\mu)(u_T, v_T)\exp\left(-\int_{v_{r_K^\star}}^{v_T} \tilde{f}(u,v)(u,\bar{v})d\bar{v}\right. \\ &\quad \left. -\int_v^{v_{r_K^\star}} \tilde{f}(u,v)(u,\bar{v})d\bar{v}\right) \end{aligned} \tag{12.42}$$

with

$$\tilde{f}(u,v) = \frac{4\kappa}{r^3}m + \frac{4}{3}\frac{\kappa}{r}\left(\rho - \frac{3}{2}\right). \tag{12.43}$$

In both regions $\mathcal{V}$ and $\mathcal{U}$ the quantity $\tilde{f}$ is clearly positive, bounded below by some $d_2 > 0$. We can estimate, for a point (u,v) in region U

$$\begin{aligned} -\nu(u,v) &\leq \frac{1}{2}(1-\mu)(u_T, v_T)\exp\left(-\int_v^{v_{r_K^\star}} \tilde{f}(u,v)(u,\bar{v})d\bar{v}\right) \\ &\leq \frac{1}{2}(1-\mu)(u_T, v_T)\exp(-d_2(v_{r_K^\star} - v)) \\ &\leq \frac{1}{2}(1-\mu)(u_T, v_T)\exp\left(-\frac{d_2}{2\sqrt{M}}u\right). \end{aligned} \tag{12.44}$$

Here we have used that $v \leq v_{r_K^\star} - \frac{1}{2}u$ by definition of the region $\mathcal{U}$.

The region $\mathcal{Z}$: The estimate is the statement of Corollary 8.8. □

With the necessary bounds in place we can prove the following:

Proposition 12.2. *Assume* (12.31) *holds. Then the error term* $I^K_{B,\mathrm{error}}$ *satisfies*

$$\begin{aligned} I^K_{B,\mathrm{error}}\left(\mathcal{D}^{r_K^\star}_{[t_0,\tilde{T}]}\right) \leq \frac{1}{\sqrt{M}}C(r^\star_{\mathrm{cl}},\sigma)\sum_{j=0}^{N-1} t_{j+1}\left[\bar{I}^X_B\left({}^{u=t_{j+1}-r^\star_{\mathrm{cl}}}\mathcal{D}^{r^\star_{\mathrm{cl}},u=\frac{1}{11}t_{j+1}}_{[t_j,t_{j+1}]}\right)\right] \\ + M\tilde{\epsilon}(r_K^\star) + M\tilde{\delta}(t_0) \end{aligned} \tag{12.45}$$

with

$$\lim_{r_K^\star \to -\infty} \tilde{\epsilon}(r_K^\star) = 0 \quad \textit{as well as} \quad \lim_{t_0\to\infty} \tilde{\delta}(t_0) = 0. \tag{12.46}$$

Proof. By Lemma 12.2 we have $\lambda + \nu \geq 0$. Hence the term multiplying $(\lambda + \nu)$ in $I^K_{B,\text{error}}$ has a good (negative) sign in all regions and can be ignored. For the other two terms we look at the different regions:

Region $\mathcal{W}$: Note that u and v are controlled by t in this region. We insert (12.33) and (12.32) into the integral $I^K_{B,\text{error}}$. The resulting term, which has to be controlled is

$$\sqrt{M}C \int_{\mathcal{W}} du\ dv\ \Omega^2 r^3 \frac{B^2}{r^2}. \tag{12.47}$$

We split the region of integration into $\mathcal{W}^1 = \mathcal{W} \cap \{r^\star \geq r^\star_{\text{cl}}\}$ and $\mathcal{W}^2 = \mathcal{W} \cap \{r^\star \leq r^\star_{\text{cl}}\}$. The region $\mathcal{W}^1$ is partitioned into dyadic slices as was the bulk term:

$$I^K_{B,\text{error}}(\mathcal{W}^1) = \sum_{j=0}^{N-1} I^K_{B,\text{error}}\left(\mathcal{W}^1_{[t_j,t_{j+1}]}\right). \tag{12.48}$$

We can control each dyadic tube by $\bar{I}^X_B$ losing a power of t (arising from a missing power of r in (12.47))

$$\begin{aligned} I^K_{B,\text{error}}\left(\mathcal{W}^1_{[t_j,t_{j+1}]}\right) &\leq \frac{1}{\sqrt{M}} C(r^\star_{\text{cl}}, \sigma) \sum_{j=0}^{N-1} t_{j+1} \left[\bar{I}^X_B\left(\mathcal{W}^1_{[t_j,t_{j+1}]}\right)\right] \\ &\leq \frac{1}{\sqrt{M}} C(r^\star_{\text{cl}}, \sigma) \sum_{j=0}^{N-1} t_{j+1} \left[\bar{I}^X_B\left({}^{u=t_{j+1}-r^\star_{\text{cl}}}\mathcal{D}^{r^\star_{\text{cl}},u=\frac{1}{11}t_{j+1}}_{[t_j,t_{j+1}]}\right)\right]. \end{aligned}$$

In the region $\mathcal{W}^2$ we can ignore the factors of r. It suffices to estimate $B^2 \leq \frac{C(r^\star_K,c)}{t^2}$ from Proposition 8.9 and hence

$$\int_{\mathcal{W}^2} dt\, dr^\star\ \Omega^2 r^3 \frac{B^2}{r^2} \leq \int_{t_0}^{\tilde{T}} dt \frac{C(r^\star_K,c)}{t^2} \int_{r^\star_{\text{cl}}}^{r^\star_K} dr^\star \frac{4\kappa\gamma}{\kappa+\gamma} \frac{\partial r}{\partial r^\star} \leq C(r^\star_K, c)\frac{M^2}{t_0}. \tag{12.49}$$

Region $\mathcal{V}$: In this region u is like v at late times. We insert the bound (12.34) into $I^K_{B,\text{error}}$ and estimate the resulting term ($v_0 = t_0 + r^\star_{\text{cl}}$, $\tilde{V} = \tilde{T} + r^\star_{\text{cl}}$)

$$\frac{1}{\sqrt{M}} C \int_{\mathcal{V}} dv\, du \Omega^2 B^2 \leq \int_{v_0}^{\tilde{V}} \frac{1}{\sqrt{M}} \frac{C}{v^2} \int_{t_0 - r^\star_K}^{2v-4r^\star_K} du(-4\kappa\nu) \leq C\frac{M}{v_0}. \tag{12.50}$$

For the remaining terms, i.e., those containing a $\left(\frac{\Omega_{,v}}{\Omega} + \frac{\Omega_{,v}}{\Omega}\right)$ factor, we insert the one-sided bound (12.35) to control the error term

$$\int_{\mathcal{V}} dt\, dr^\star r^3 \frac{B^2}{r^2} u^2 (\partial_t \Omega^2) \tag{12.51}$$

for large times (in particular $t_0 + r^\star_K \geq 1$) as follows ($u_H = \tilde{T} - r^\star_K$). First define

$$r^\star_+ = -\frac{t_0}{2} + \frac{3}{2} r^\star_K \quad \text{and} \quad r^\star_- = r^\star_K - \frac{\tilde{T} - r^\star_K}{4} \tag{12.52}$$

and

$$\bar{t}(r^\star) = \begin{cases} t_0 + r^\star_K - r^\star & \text{for } r^\star \geq r^\star_+, \\ -3r^\star + 4r^\star_K & \text{for } r^\star \leq r^\star_+. \end{cases} \tag{12.53}$$

Then the term (12.51) can be estimated using Proposition 8.13

$$\begin{aligned}
&\frac{1}{M} \int_{r^\star_-}^{r^\star_K} dr^\star \int_{\bar{t}(r^\star)}^{\tilde{T} - r^\star_K + r^\star} dt\, r^3 \frac{B^2}{r^2} u^2 \left(\partial_t \Omega^2 + C\Omega^2 \frac{\sqrt{M}}{u^2} \right) \\
&\quad \leq C_L C(c) \sqrt{M} \int_{r^\star_-}^{r^\star_K} dr^\star \int_{\bar{t}(r^\star)}^{\tilde{T} - r^\star_K + r^\star} dt \left(\partial_t \Omega^2 + C\Omega^2 \frac{\sqrt{M}}{u^2} \right) \\
&\quad \leq C_L C(c) \sqrt{M} \int_{r^\star_-}^{r^\star_K} dr^\star \Omega^2 \left(\tilde{T} - r^\star_K + r^\star, r^\star \right) \\
&\qquad + C_L C(c) M \int_{t_0 + r^\star_K}^{\tilde{T} + r^\star_K} dv \int_{v - 2r^\star_K}^{\min(u_H, 2v - 4r^\star_K)} du \frac{\Omega^2}{u^2} \\
&\quad \leq C_L C(c) \sqrt{M} \left[r(\tilde{T}, r^\star_K) - r\left(\tilde{T} - r^\star_K + r^\star_-, r^\star_- \right) \right] \\
&\qquad + C_L C(c) M \int_{t_0 + r^\star_K}^{\tilde{T} + r^\star_K} dv \frac{1}{v^2} \int_{v - 2r^\star_K}^{u_H} du \Omega^2 \\
&\quad \leq C_L C(c) M \cdot \epsilon(r^\star_K) + C_L C(c) M \frac{\sqrt{M}}{t_0} \epsilon(r^\star_K),
\end{aligned} \tag{12.54}$$

where in the first step we have used that the round bracket in the first line is positive. The constant $C(r^\star_{\text{cl}})$ may have different values in each line. We also used that

$$\partial_{r^\star} r = \lambda - \nu = \frac{1}{4} \Omega^2 \left(\frac{1}{\gamma} + \frac{1}{\kappa} \right) \tag{12.55}$$

and therefore

$$\Omega^2 = 4 \partial_{r^\star} r \frac{\gamma \kappa}{\gamma + \kappa} \leq 4 \partial_{r^\star} r \tag{12.56}$$

holds. In summary, smallness for this error term arises from the smallness of the r-difference between any two points in the region $r^\star \leq r^\star_K$. The crucial point is that only $C(r^\star_{\text{cl}})$ enters the above estimate, such that the r-difference can “beat” the constant.

Region $\mathcal{U}$: To control the error terms in region $\mathcal{U}$ estimate the curly bracket of $I^K_{B,\text{error}}$ by some constant times u^2 and B^2 by something small (cf. Corollary 4.3). The resulting integral can be controlled via (12.36) as follows:

$$\begin{aligned}
&|I^K_{B,\text{error}}(\mathcal{U})| \\
&\leq \frac{1}{M} C(\epsilon) \int_{t_0+r^\star_K}^{\frac{\tilde{T}}{2}+\frac{3}{2}r^\star_K} dv \int_{2v-4r^\star_K}^{\tilde{T}-r^\star_K} du(-\nu)u^2(u,v) \\
&\leq \frac{1}{M} C(\epsilon) \int_{t_0+r^\star_K}^{\frac{\tilde{T}}{2}+\frac{3}{2}r^\star_K} dv \int_{2v-4r^\star_K}^{\tilde{T}-r^\star_K} du \exp\left(-\frac{d}{2\sqrt{M}}u\right) u^2 \\
&\leq MC(\epsilon)C\mathrm{e}^{-\frac{d}{2\sqrt{M}}t_0}.
\end{aligned}$$

The region $\mathcal{Z}$: On the one hand, we have to establish smallness for

$$\begin{aligned}
&\frac{1}{M}\int_{\mathcal{Z}} du\, dv r^3 \Omega^2 \frac{B^2}{r^2}[(u+a)^2+(v-a)^2]\left[Q(B)+6\frac{\lambda\zeta^2}{r^2\Omega^2}+\frac{3\theta^2}{2\kappa r^2}\right] \\
&\leq C\sqrt{M}\int_{\mathcal{Z}} du\, dv \left[Q(B)+6\frac{\lambda\zeta^2}{r^2\Omega^2}+\frac{3\theta^2}{2\kappa r^2}\right], \qquad (12.57)
\end{aligned}$$

where we used that r controls v and u in the region under consideration and Proposition 8.11. From Proposition 4.1 it is apparent that the critical term to control is

$$\int_{\mathcal{Z}} \frac{1}{2}\frac{1}{r^2}\zeta^2 du\, dv \leq C\int dv \frac{1}{v^2}\left(\int \zeta^2 du\right) \leq C(\epsilon)\frac{M}{t_0}. \qquad (12.58)$$

Namely, the remaining terms in the square bracket of (12.57) all decay like $\frac{\epsilon}{r^3}$ by Proposition 4.1 such that direct integration will already lead to a smallness factor.

The other critical term to control is

$$\frac{1}{M}\int du\, dv \frac{1}{2} r^3\Omega^2\frac{B^2}{r^2}((u+a)^2+(v-a)^2)\left[\frac{\Omega_{,v}}{\Omega}+\frac{\Omega_{,u}}{\Omega}\right] \qquad (12.59)$$

which upon inserting (12.37) and using the fact that r controls u and v in the region under consideration reduces to controlling the term

$$C(r^\star_{\text{cl}}, c)\frac{1}{\sqrt{M}}\int_{t_0}^{\tilde{T}} dt \int_{\frac{9}{10}t}^{\tilde{T}-u_0} dr^\star B^2 r(-\nu). \qquad (12.60)$$

Using that $r \sim t$ in region $\mathcal{Z}$ we can estimate (12.60) by

$$\begin{aligned}
&\leq C(r^\star_{\mathrm{cl}}, c)\frac{1}{\sqrt{M}}\int_{t_0}^{\tilde{T}} dt \frac{1}{t^2}\int_{\frac{9}{10}t}^{\tilde{T}-u_0} dr^\star B^2 r^3(-\nu) \\
&\leq C(r^\star_{\mathrm{cl}}, c)\sqrt{M}\int_{t_0}^{\tilde{T}} dt \frac{1}{t^2}E^K_B(t) \leq C(r^\star_{\mathrm{cl}}, c)\frac{M^{\frac{3}{2}}}{t_0},
\end{aligned} \tag{12.61}$$

where we have used bootstrap assumption (7.8). □

12.3 The boundary terms

We write the boundary terms (12.13) as

$$F^K_B(t) = F^K_{B,\mathrm{main}}(t) + F^K_{B,\mathrm{errorarc}}(t) + F^K_{B,\mathrm{errorline}}(t), \tag{12.62}$$

where

$$\begin{aligned}
&\frac{F^K_{B,\mathrm{main}}(t)}{2\pi^2} \\
&\quad = \frac{1}{M}\int_{r^\star_K}^{t-u_0}\left(-12t\frac{r^\star - a}{r}\nu B\partial_t B + 6B^2\nu\frac{r^\star - a}{r}\right) r^3 dr^\star \\
&\qquad + \frac{1}{M}\int_{r^\star_K}^{t-u_0}\left((\partial_u B)^2 2(u+a)^2 + (\partial_v B)^2 2(v-a)^2\right. \\
&\qquad \left. + ((u+a)^2 + (v-a)^2)\frac{\Omega^2}{2r^2}\left(1 - \frac{2}{3}\rho\right)\right) r^3 dr^\star \\
&\qquad + \frac{1}{M}\int_{t-r^\star_K}^{\tilde{T}-r^\star_K}\left[2(u+a)^2 r^3(\partial_u B)^2 + \frac{r\Omega^2}{2}(v-a)^2\right. \\
&\qquad \left. \times\left(1 - \frac{2}{3}\rho\right)\right](u, t + r^\star_{\mathrm{cl}})du,
\end{aligned} \tag{12.63}$$

$$\begin{aligned}
&\frac{F^K_{B,\mathrm{errorarc}}(t)}{2\pi^2} \\
&\quad = -\frac{1}{M}\int_{r^\star_K}^{t-u_0}\frac{3}{2}B^2\left(\frac{(u+a)^2}{r}(r_{,uu} + r_{,uv}) + \frac{(v-a)^2}{r}(r_{,vv} + r_{,uv})\right) r^3 dr^\star \\
&\qquad + \frac{1}{M}\int_{r^\star_K}^{t-u_0}\left(+3B\partial_t B\frac{(v-a)^2}{r}(\lambda + \nu) - 3\frac{v-a}{r}B^2(\lambda + \nu)\right) r^3 dr^\star \\
&\qquad + \frac{1}{M}\int_{r^\star_K}^{t-u_0}\frac{3}{2}B^2\left(\frac{\lambda + \nu}{r^2}(\nu(u+a)^2 + \lambda(v-a)^2)\right) r^3 dr^\star
\end{aligned} \tag{12.64}$$

and

$$\begin{aligned}
&\frac{F^K_{B,\mathrm{errorline}}(t)}{2\pi^2} \\
&\quad = \frac{1}{M}\int_{t-r^\star_K}^{\tilde{T}-r^\star_K} 2B(\partial_u B)\left(\frac{3}{2r}(\nu(u+a)^2+\lambda(v-a)^2)\right) r^3(u,t+r^\star)du \\
&\qquad - \frac{1}{M}\int_{t-r^\star_K}^{\tilde{T}-r^\star_K} B^2\partial_u\left(\frac{3}{2r}(\nu(u+a)^2+\lambda(v-a)^2)\right) r^3(u,t+r^\star)du.
\end{aligned} \tag{12.65}$$

and

$$\begin{aligned}
&\frac{H^K_{\tilde{T}-r^\star_K}}{2\pi^2} \\
&\quad = \frac{1}{M}\int_{t_0+r^\star_K}^{\tilde{T}+r^\star_K}\left[2(v-a)^2r^3(\partial_v B)^2 + \frac{r\Omega^2}{2}(u+a)^2\left(1-\frac{2}{3}\rho\right)\right](T-r^\star_K,v)dv \\
&\qquad - \frac{1}{M}\int_{t_0+r^\star_K}^{\tilde{T}+r^\star_K} B^2\partial_v\left(\frac{3}{2r}(\nu(u+a)^2+\lambda(v-a)^2)\right) r^3(T-r^\star_K,v)dv \\
&\qquad + \frac{1}{M}\int_{t_0+r^\star_K}^{\tilde{T}+r^\star_K} 2B(\partial_v B)\left(\frac{3}{2r}(\nu(u+a)^2+\lambda(v-a)^2)\right) r^3(T-r^\star_K,v)dv.
\end{aligned} \tag{12.66}$$

Note that $F^K_{B,\mathrm{errorline}}(\tilde{T}) = 0$ and that the last term of $F^K_{B,\mathrm{main}}$ also vanishes for $t = \tilde{T}$.

12.3.1 Estimating $F^K_{B,\mathrm{main}}(t)$

We are going to show that the boundary term $F^K_{B,\mathrm{main}}(t)$ comes with a sign. This is obviously the case for the integral in u, so it remains to establish non-negativity of the spacelike integrals. Define

$$S = (v-a)\partial_v + (u+a)\partial_u, \tag{12.67}$$

$$\underline{S} = (v-a)\partial_v - (u+a)\partial_u. \tag{12.68}$$

Note that

$$(SB)^2 + (\underline{S}B)^2 = 2(u+a)^2(\partial_u B)^2 + 2(v-a)^2(\partial_v B)^2 \tag{12.69}$$

and

$$S = t\partial_t + (r^\star - a)\partial_{r^\star}, \quad \underline{S} = t\partial_{r^\star} + (r^\star - a)\partial_t, \tag{12.70}$$

respectively

$$t\partial_t = S - (r^\star - a)\partial_{r^\star}, \quad t\partial_t = \frac{t}{(r^\star - a)}\underline{S} - \frac{t^2}{(r^\star - a)}\partial_{r^\star}. \tag{12.71}$$

We can insert these expressions into the boundary term (12.63) and integrate the second term by parts using S:

$$\begin{aligned}
&\int_{r_K^\star}^{\tilde{T}-u_0} Bt\partial_t B\frac{1}{r}(-2\nu)(r^\star - a)r^3 dr^\star \\
&= \int_{r_K^\star}^{\tilde{T}-u_0} \frac{-2\nu}{r}(r^\star - a)B((SB) - (r^\star - a)\partial_{r^\star}B)r^3 dr^\star \\
&= \nu r^2(r^\star - a)^2 B^2|_{r^\star=r_K^\star}^{r^\star=\tilde{T}-u_0} + \int_{r_K^\star}^{\tilde{T}-u_0}(-2\nu)r^3\left[\frac{r^\star - a}{r}B(SB)\right. \\
&\qquad \left. + B^2\left(\frac{r^\star - a}{r} + \frac{(r^\star - a)^2}{r^2}\left[\kappa - \left(\nu + r\frac{\Omega_{,u}}{\Omega}\right) + P_1(B)\right]\right)\right] dr^\star,
\end{aligned} \tag{12.72}$$

where

$$P_1(B) = \frac{\zeta^2}{r\nu} + \frac{2}{3}\kappa\left(\rho - \frac{3}{2}\right) \approx C(\epsilon) \tag{12.73}$$

is very small by Proposition 4.1. Note that the boundary term near infinity vanishes in view of the assumption of compact support on the initial data and the domain of dependence. The term at $r^\star = r_K^\star$ is manifestly positive since $\nu < 0$ in the integration region.

Alternatively, using $\underline{S}$, we obtain

$$\begin{aligned}
&\int_{r_K^\star}^{\tilde{T}-u_0} Bt\partial_t B\frac{1}{r}(-2\nu)(r^\star - a)r^3 dr^\star \\
&= \int_{r_K^\star}^{\tilde{T}-u_0} \frac{-2\nu}{r}(r^\star - a)B\left(\frac{t}{(r^\star - a)}\underline{S}B - \frac{t^2}{(r^\star - a)}\partial_{r^\star}B\right)r^3 dr^\star \\
&= \nu r^2 t^2 B^2\Big|_{r^\star=r_K^\star}^{r^\star=\tilde{T}-u_0} + \int_{r_K^\star}^{\tilde{T}-u_0}(-2\nu)r^3 \\
&\quad \times \left(\frac{t}{r}(\underline{S}B)B + \frac{t^2}{r^2}B^2\left[\kappa - \left(\nu + r\frac{\Omega_{,u}}{\Omega}\right) + P_1(B)\right]\right) dr^\star.
\end{aligned}$$

Again the boundary term has a positive sign at $r^\star = r_K^\star$.

If we split the relevant term in (12.63) into two equal pieces and collect terms we can write

$$\begin{aligned}
\frac{1}{2\pi^2}F^K_{B,\mathrm{main}}\left(\tilde{T}\right) &= \frac{1}{M}\int_{r^\star_K}^{\tilde{T}-u_0}\Bigg[\left(\partial_u B\right)^2 2\left(u+a\right)^2 + \left(\partial_v B\right)^2 2\left(v-a\right)^2 \\
&\quad + \left(\left(u+a\right)^2 + \left(v-a\right)^2\right)\frac{\Omega^2}{2r^2}\left(1-\frac{2}{3}\rho\right) \\
&\quad - 12t\frac{r^\star - a}{r}\nu B\partial_t B + 6B^2\nu\frac{r^\star-a}{r}\Bigg] r^3 dr^\star \\
&\geq \frac{1}{M}\int_{r^\star_K}^{\tilde{T}-u_0} dr^\star r^3\Bigg[\left(1+2\nu\right)\left(\left(SB\right)^2 + \left(\underline{S}B\right)^2\right) \\
&\quad + \left(-2\nu\right)\left(\left(SB + \frac{3}{2}\frac{r^\star-a}{r}B\right)^2 + \frac{\left(r^\star-a\right)^2}{r^2}B^2\right. \\
&\quad \times\left.\left(\frac{23}{4}+\delta+3\left[\kappa - \left(\nu + r\frac{\Omega_{,u}}{\Omega}\right)\right]\right)\right) \\
&\quad + \left(-2\nu\right)\left(\left(\underline{S}B + \frac{3}{2}\frac{t}{r}B\right)^2 + \frac{t^2}{r^2}B^2\right. \\
&\quad \times\left.\left(\frac{23}{4}+\delta+3\left[\kappa - \left(\nu + r\frac{\Omega_{,u}}{\Omega}\right)\right]\right)\right)\Bigg] r^3
\end{aligned}$$

which is manifestly positive. Furthermore,

$$\frac{1}{2\pi^2}F^K_{B,\mathrm{main}}\left(\tilde{T}\right) \geq E^K_B\left(\tilde{T}\right) + \frac{4}{M}\int_{r^\star_K}^{\tilde{T}-u_0}\left(-\nu\right)\frac{B^2}{r^2}\left(t^2 + \left(r^\star - a\right)^2\right) r^3 dr^\star \tag{12.74}$$

with E^K_B being the quantity appearing in bootstrap assumption 1.3.2.

12.3.2 Estimating $F^K_{B,\mathrm{errorarc}}\left(\tilde{T}\right)$

Lemma 12.4. *Under assumption (*12.31*) we have that*

$$\frac{1}{2\pi^2}F^K_{B,\mathrm{main}}\left(\tilde{T}\right) + F^K_{B,\mathrm{errorarc}}\left(\tilde{T}\right) \geq E^K_B\left(\tilde{T}\right) + M\hat{\epsilon} \tag{12.75}$$

with the $\hat{\epsilon}$ arising from the fact that $\tilde{T}$ is large.

Proof. We have

$$|\lambda+\nu| \leq C(r_K^\star, c)\frac{M}{\tilde{T}^2} \quad \text{for } r_K^\star \leq r^\star \leq \tfrac{9}{10}\tilde{T} \tag{12.76}$$

by Proposition 8.4 and

$$|\lambda+\nu| \leq C\left(r_K^\star, c\right)\frac{M}{r^2} \quad \text{for } r^\star \geq \tfrac{9}{10}\tilde{T} \tag{12.77}$$

by Corollary 8.2. Recalling the bound (8.16) we can estimate the expression

$$r_{,vv} + r_{,uv} = 2\lambda\frac{\Omega_{,v}}{\Omega} - \frac{2}{r^2}\theta^2 + 2\kappa\frac{\mu}{r}\nu + \frac{4\kappa\nu}{3r}\left(\rho - \frac{3}{2}\right) \tag{12.78}$$

by

$$|r_{,vv} + r_{,uv}| \leq \frac{\mu}{r}\left(\lambda+\nu\right) + C\left(r_K^\star, c\right)\frac{\sqrt{M}}{\tilde{T}^2} \leq C\left(r_K^\star, c\right)\frac{\sqrt{M}}{\tilde{T}^2} \tag{12.79}$$

and similarly

$$|r_{,uu} + r_{,uv}| \leq C\left(r_K^\star, c\right)\frac{\sqrt{M}}{\tilde{T}^2}\,, \tag{12.80}$$

both in the region $r_K^\star \leq r^\star \leq \frac{9}{10}\tilde{T}$. Analogously, we obtain

$$|r_{,vv} + r_{,uv}| + |r_{,uu} + r_{,uv}| \leq \frac{C}{r^2} \quad \text{in the region } r^\star \geq \frac{9}{10}\tilde{T}. \tag{12.81}$$

Inserting these estimates into (12.64) it becomes clear that we have to establish smallness for the terms

$$\int_{r_K^\star}^{\tilde{T}-u_J}\left[\sqrt{M}B^2r^2 + M^{\frac{3}{2}}\frac{\theta^2}{r} + M^{\frac{3}{2}}\frac{\zeta^2}{r}\right]\left(\tilde{T}, r^\star\right)dr^\star\,. \tag{12.82}$$

We split the integral into the region $r_K^\star \leq r^\star \leq \frac{9}{10}\tilde{T}$ and the region $r^\star \geq \frac{9}{10}\tilde{T}$. In the first region the derivative terms of (12.82) are manifestly controlled by the energy, decaying like $\frac{1}{\tilde{T}^2}$ by Proposition 8.3. The B^2 term can be estimated as

$$\int B^2r^2dr^\star \leq C\tilde{T}\int B^2rdr^\star \leq C\tilde{T}\left[m\left(\tilde{T}, \frac{9}{10}\tilde{T}\right) - m\left(\tilde{T}, r_K^\star\right)\right] \leq C\frac{M^2}{\tilde{T}}\,. \tag{12.83}$$

In both cases smallness is obtained from the fact that t_0 is chosen very large. In the region $r^\star \geq \frac{9}{10}t$ on the other hand, the derivative terms in (12.82) can be controlled by pulling out the $\frac{1}{r}$ as a smallness factor and use the energy estimate for the rest. For the B^2 term we have to borrow from the good last term of (12.74):

$$\int_{\frac{9}{10}\tilde{T}}^{\tilde{T}-u_J} B^2 r^2 dr^\star \leq \frac{10}{9\tilde{T}} \int_{\frac{9}{10}\tilde{T}}^{\tilde{T}-u_0} B^2 r^3 dr^\star \,. \tag{12.84}$$

Hence a tiny contribution from the last term of (12.74) will control this term and we finally arrive at (12.75). □

12.3.3 Estimating $H^K_{\tilde{T}-r^\star_K}$

Lemma 12.5. *Under assumption (*12.31*) we have*

$$-H^K_{u=T-r^\star_K} \leq M\tilde{\epsilon}\left(r^\star_K\right), \tag{12.85}$$

where $\tilde{\epsilon}\left(r^\star_K\right) \to 0$ for $r^\star_K \to \infty$.

Proof. The first term of (12.66) is clearly positive and can be neglected. For the other terms split the integration $\mathcal{I} = [v_1, v_2] = \left[t_0 + r^\star_K, \tilde{T} + r^\star_K\right]$ into the part which lies in $\mathcal{U}$ (where we can use estimate (12.36)) and the part in $\mathcal{V}$ (where we are going to exploit the fact that the r-difference is small). See figure 8. Following this line of thought we estimate the (negative of the) second term of (12.66)

$$\begin{aligned}
&\int_{\mathcal{I}} B^2 \partial_v \left(\frac{3}{2r}\left(\nu\left(u+a\right)^2 + \lambda\left(v-a\right)^2\right)\right) r^3 \left(\tilde{T} - r^\star_K, v\right) dv \\
&\quad \leq \int_{\mathcal{I}} \frac{3B^2}{2}\left(r\left(-\lambda\right)\nu(u+a)^2 + r^2 r_{,vv}(v-a)^2 + 2r^2\lambda(v-a)\right) \\
&\qquad \times \left(\tilde{T} - r^\star_K, v\right) dv \leq \left[\int_{\mathcal{I}\cap\mathcal{U}} + \int_{\mathcal{I}\cap\mathcal{V}}\right] \frac{3B^2}{2}\left(r\left(-\lambda\right)\nu\left(u+a\right)^2\right) \\
&\qquad \times \left(\tilde{T} - r^\star_K, v\right) dv + M^2 C\left(r^\star_{\text{cl}}, c\right) \int_{\mathcal{I}} \left[\frac{\lambda}{v_0} + r_{,vv}\right] \left(\tilde{T} - r^\star_K, v\right) dv \\
&\quad \leq C e^{-\frac{d}{2}u}\left(u+a\right)^2 \int_{\mathcal{I}\cap\mathcal{U}} B^2 r\lambda \left(\tilde{T} - r^\star_K, v\right) dv + CM^{\frac{3}{2}} \\
&\qquad \times \int_{\mathcal{I}\cap\mathcal{V}} \lambda\left(-\nu\right)\left(\tilde{T} - r^\star_K, v\right) dv + M^2 C\left(r^\star_{\text{cl}}, c\right)\epsilon\left(r^\star_K\right) \leq M^2\tilde{\epsilon}\left(r^\star_K\right),
\end{aligned} \tag{12.86}$$

where we used that $r_{,uv} \leq 0$, that u is like v in region $\mathcal{V}$, and assumptions (12.30). For the (negative of the) third term we obtain (C just depends on $r^\star_{\mathrm{cl}}$)

$$
\begin{aligned}
-\int_{t_0+r^\star_K}^{\tilde{T}+r^\star_K} & 2B\left(\partial_v B\right)\left(\frac{3}{2r}\left(\nu\left(u+a\right)^2+\lambda\left(v-a\right)^2\right)\right) r^3\left(\tilde{T}-r^\star_K, v\right) dv \\
& \leq \left[\int_{\mathcal{I}\cap\mathcal{U}}+\int_{\mathcal{I}\cap\mathcal{V}}\right]\frac{3}{2}r^2\left(\frac{B^2}{r}+r\left(B_{,v}\right)^2\right)\left(-\nu\right)\left(u+a\right)^2\left(\tilde{T}-r^\star_K, v\right) dv \\
& \quad +\int_{\mathcal{I}}\frac{3}{2}r^2\left(\frac{B^2}{r}+r\left(B_{,v}\right)^2\right)\lambda\left(v-a\right)^2\left(\tilde{T}-r^\star_K, v\right) dv \\
& \leq C\mathrm{e}^{-\frac{d}{2}u}\left(u+a\right)^2\frac{M^{\frac{3}{2}}}{v_0}+CM^{\frac{3}{2}}\int_{\mathcal{I}\cap\mathcal{V}}\lambda\left(\tilde{T}-r^\star_K, v\right) dv+CM^{\frac{3}{2}} \\
& \quad \times\int_{\mathcal{I}}\lambda\left(\tilde{T}-r^\star_K, v\right) dv \leq \tilde{\epsilon}\left(r^\star_K\right),
\end{aligned} \tag{12.87}
$$

where we again used that u is like v in region $\mathcal{V}$, and inequality (12.30), as well as the fact that $(-\nu) \leq \lambda$ (cf. Lemma 12.2). These estimates together yield (12.85). □

12.3.4 Estimating $F^K_{B,\mathrm{errorline}}(t)$

Clearly $F^K_{B,\mathrm{errorline}}\left(\tilde{T}\right) = 0$ since there is no upper null boundary for the region in which we apply K. Hence we only have to estimate $F^K_{B,\mathrm{errorline}}(t_0)$. This is done in the same manner as for the horizon term: Splitting the integral into a part lying in $\mathcal{V}$ and a part in $\mathcal{U}$, using estimate (12.31) in the former and applying (12.36) in the latter region.

12.4 Summary

We have shown the following:

Proposition 12.3. *Assume (*12.31*) holds. It follows that*

$$
\begin{aligned}
E^K_B\left(\tilde{T}\right) \leq \frac{1}{\sqrt{M}}C\left(r^\star_{\mathrm{cl}},\sigma\right)\sum_{j=0}^{N-1} t_{j+1}\bar{I}^X_B\left({}^{u=t_{j+1}-r^\star_{\mathrm{cl}}}\mathcal{D}^{r^\star_{\mathrm{cl}},\frac{1}{11}t_{j+1}}_{[t_j,t_{j+1}]}\right) \\
+F^K_B\left(t_0\right)+M\hat{\epsilon}\left(r^\star_K, t_0\right)
\end{aligned}
$$

and $\hat{\epsilon}$ can be made arbitrarily small by choosing both $-r^\star_K$ and then t_0 sufficiently large.

Proof. Write (12.12) as

$$F^K_{B,\text{main}}\left(\tilde{T}\right)+F^K_{B,\text{errorarc}}\left(\tilde{T}\right)=I^K_{B,\text{main}}\left(\mathcal{D}^{r^\star_K,u_0}_{[t_0,\tilde{T}]}\right)+I^K_{B,\text{error}}\left(\mathcal{D}^{r^\star_K,u_0}_{[t_1,t_2]}\right)$$
$$+F^K_B\left(t_0\right)-H^K_{u_H=\tilde{T}-r^\star_K} \tag{12.88}$$

and apply the estimates of Lemmata 12.4 and 12.5, as well as Propositions 12.1 and 12.2. □

13 Closing the bootstrap

With the required estimates now in place we are in a position to prove the closed part of Theorem 7.1, i.e., to improve the remaining bootstrap assumptions.[46]

We start with the observation that the X-bulk-term decays.

Proposition 13.1. *We have*

$$\bar{I}^X_B\left(\mathcal{D}^{r^\star_{\text{cl}},t_i-R^\star}_{t_i,t_{i+1}}\right)\leq\bar{I}^X_B\left(\mathcal{D}^{r^\star_{\text{cl}},\frac{1}{11}t_{i+1}}_{t_i,t_{i+1}}\right)\leq M^2\frac{C\left(r^\star_{\text{cl}}\right)}{t_i^2}\,. \tag{13.1}$$

Proof. Apply Proposition 10.2 in combination with Proposition 8.3 and the bootstrap assumption (7.11). □

With the help of the propositions proven in Section 11 we can derive the pointwise bound (12.31), which was assumed for most of the propositions established in section 12.[47]

13.1 A pointwise estimate for $\frac{\zeta}{\nu}$ using Y

Proposition 13.2. *In the region $\mathcal{A}\left(T\right)\cap\{r^\star\leq r^\star_{\text{cl}}\}\cap\{v\geq t_0+r^\star_{\text{cl}}\}$ we have*

$$\left|\frac{\zeta}{\nu}\right|\leq C\left(r^\star_{\text{cl}},c\right)\frac{M^{\frac{3}{4}}}{v} \tag{13.2}$$

[46] Recall that the first two have been improved already in Corollaries 8.4 and 8.5.

[47] The reader is assured that none of the results of Section 12 will be used in the following subsection. The argument has been placed in this section because it is also used to improve the integral bound (7.11).

and in $\mathcal{A}(T) \cap \{r^\star \geq r^\star_{\rm cl}\} \cap \{r^\star \leq \frac{9}{10}t\}$ *the estimate*

$$\left|\frac{\zeta}{\nu}\right| \leq C\left(r^\star_{\rm cl}, c\right)\frac{M^{\frac{3}{4}}}{t}. \tag{13.3}$$

Proof. Starting from the slice Σ_{t_0} (cf. definition (5.35)) erect the characteristic rectangle to any Σ_t, $t_0 \leq t \leq T$. By Cauchy stability (Proposition 7.2), we have that

$$\frac{1}{M}F^Y_B\left([u_1, u_{\rm hoz}] \times \{v_0 = t_0 - r^\star_{\rm cl}\}\right) \leq \tilde{\delta} \tag{13.4}$$

and hence an application of Proposition 11.3 together with (13.1) immediately yields

$$\frac{1}{M}F^Y_B\left([u_1, u_{\rm hoz}] \times \{v = t - r^\star_{\rm cl}\}\right) \leq \frac{11}{10}\tilde{\delta} + \tilde{\epsilon} \tag{13.5}$$

for any $t \leq T$. This estimate and Proposition 11.2 immediately imply the uniform estimate

$$\frac{1}{M}\tilde{I}^Y_B\left(\mathcal{R}_i \setminus \mathcal{T}_i\right) \leq \frac{11}{10}\tilde{\delta} + \tilde{\epsilon} \tag{13.6}$$

for the region $\mathcal{R}_i \setminus \mathcal{T}_i$ of any characteristic dyadic rectangle. Next we apply Proposition 11.4 to each dyadic rectangle to find a slice $\hat{v}$ satisfying

$$\frac{1}{M}F^Y_B\left([u_i, u_{\rm hoz}] \times \{\hat{v}\}\right) \leq C\frac{\tilde{\epsilon}\sqrt{M}}{v_{i+1} - v_i} + C\frac{\sqrt{M}}{(v_i)^2} \leq \frac{C\sqrt{M}}{t_i}. \tag{13.7}$$

Proposition 11.2 applied to the rectangle enclosed by the good slice in $[v_i, v_{i+1}]$ and $v = v_{i+1}$ yields, again using (13.1)

$$\frac{1}{M}F^Y_B\left([u_i, u_{\rm hoz}] \times \{v_{i+1}\}\right) \leq C\frac{\sqrt{M}}{v_{i+1}}. \tag{13.8}$$

Having exported the better decay to all late slices in this fashion, we can erect the characteristic rectangle again and apply Proposition 11.2, which produces the uniform decay estimate

$$\frac{1}{M}I^Y_B\left(\mathcal{R}_i \setminus \mathcal{T}_i\right) \leq C\frac{\sqrt{M}}{v_i}. \tag{13.9}$$

One may repeat the procedure, i.e., apply Proposition 11.4 again, which now provides one with a good slice (with the Y-flux decaying like $\frac{1}{(v_{i+1})^2}$). After application of Proposition 11.2 this leads to the decay

$$\frac{1}{M} F_B^Y \left([u_i, u_{\mathrm{hoz}}] \times \{v_{i+1}\} \right) \leq C \left(r^\star_{\mathrm{cl}}, c \right) \frac{M}{\left(v_{i+1} \right)^2} \tag{13.10}$$

on any late slice v_i. Finally, one may export the decay to any v-slice by choosing appropriate regions:

$$\frac{1}{M} F_B^Y \left([u\left(r^\star_{\mathrm{cl}}\right), u_{\mathrm{hoz}}] \times \{v\} \right) \leq C \left(r^\star_{\mathrm{cl}}, c \right) \frac{M}{v^2}, \tag{13.11}$$

$$\frac{1}{M} F_B^Y \left(u \times [v, \hat{v}] \right) \leq C \left(r^\star_{\mathrm{cl}}, c \right) \frac{M}{v^2}, \tag{13.12}$$

$$\frac{1}{M} \tilde{I}_B^Y \left(\mathcal{R} \setminus \mathcal{T} \right) \leq C \left(r^\star_{\mathrm{cl}}, c \right) \frac{M}{v^2} \tag{13.13}$$

everywhere. Note that we have assumed a better bound than (13.11) in the bootstrap assumption (7.11), however the bound (13.12) is new and essential to derive the pointwise bound for $\frac{\zeta}{\nu}$. Namely, integrating (4.13) upwards in a characteristic rectangle yields

$$\begin{aligned} \frac{\zeta}{\nu}\left(u, v_{i+1}\right) = & \frac{\zeta}{\nu}\left(u, v_i\right) \mathrm{e}^{-\int_{v_i}^{v_{i+1}} \left[\frac{4\kappa}{r^3} m + \frac{4\kappa}{3r}\left(\rho - \frac{3}{2}\right)\right](u,\bar{v}) d\bar{v}} \\ & + \int_{v_i}^{v_{i+1}} \mathrm{e}^{-\int_{\bar{v}}^{v} \left[\frac{4\kappa}{r^3} m + \frac{4\kappa}{3r}\left(\rho - \frac{3}{2}\right)\right](u,\hat{v}) d\hat{v}} \\ & \times \left[-\frac{3}{2}\frac{\theta}{r} - \frac{4}{3}\frac{\kappa}{\sqrt{r}} \left(\mathrm{e}^{-8B} - \mathrm{e}^{-2B} \right) \right] \left(u, \bar{v}\right) d\bar{v} \end{aligned} \tag{13.14}$$

and hence

$$\begin{aligned} \left| \frac{\zeta}{\nu}\left(u, v_{i+1}\right) \right| \leq & \left| \frac{\zeta}{\nu}\left(u, v_i\right) \right| \mathrm{e}^{-.1 \cdot d \cdot v_i} + \frac{3}{2} \frac{1}{r_{\mathrm{min}}} \\ & \times \sqrt{\int_{v_i}^{v_{i+1}} \mathrm{e}^{-\int_{\bar{v}}^{v} \left[\frac{3}{2}\frac{4\kappa}{r^3} m\right](u,\hat{v}) d\hat{v}} \kappa\left(u, \bar{v}\right) d\bar{v}} \sqrt{\int_{v_i}^{v_{i+1}} \frac{\theta^2}{\kappa}\left(u, \hat{v}\right) d\hat{v}} \\ & + C \left(\sup_{r \leq r^\star_{\mathrm{cl}}} \frac{1}{\sqrt{\alpha}} \right) \sqrt{\int_{v_i}^{v_{i+1}} \mathrm{e}^{-\int_{\bar{v}}^{v} \left[\frac{3}{2}\frac{4\kappa}{r^3} m\right](u,\hat{v}) d\hat{v}} \kappa\left(u, \bar{v}\right) d\bar{v}} \\ & \times \sqrt{\int_{v_i}^{v_{i+1}} \alpha r B^2 \left(u, \hat{v}\right) d\hat{v}} \leq C \frac{M^{\frac{3}{4}}}{v_i} + \left| \frac{\zeta}{\nu}\left(u, v_i\right) \right| \mathrm{e}^{-.1 \cdot d \cdot v_i}. \end{aligned}$$

Reiterating from the first to any chosen late rectangle we find for any (u, v_i) in the region $r^\star \leq r^\star_{\mathrm{cl}}$

$$\left|\frac{\zeta}{\nu}(u, v_i)\right| \leq C\left(r^\star_{\mathrm{cl}}, c\right)\frac{M^{\frac{3}{4}}}{v_i} \quad \text{and hence} \quad \left|\frac{\zeta}{\nu}(u, v)\right| \leq C\left(r^\star_{\mathrm{cl}}, c\right)\frac{M^{\frac{3}{4}}}{v} \tag{13.15}$$

which is (13.2). Integrating (4.13) from the set $L = \{r^\star = r^\star_{\mathrm{cl}}\} \cup \{\{t = t_0\} \cap \{r^\star \geq r^\star_{\mathrm{cl}}\}\}$, where the bound (13.3) holds by Cauchy stability and the estimate just established, we obtain (13.3) in the complete region using the energy estimate and the fact that $u \sim t$ in the region where $r^\star_{\mathrm{cl}} \leq r^\star \leq \frac{9}{10}t$. $\square$

13.2 Improving assumption (7.8)

With the pointwise bound on $\frac{\zeta}{\nu}$ established we can improve assumption (7.8) for any late boundary term $E^K_B\left(\tilde{T}\right)$ via Proposition 12.3. One applies the K-estimate in the region ${}^{u=t_N - r^\star_K}\mathcal{D}^{r^\star_K, u_0}_{[t_0, t_N]}$ some large $-r^\star_K$, late t_0 and $t_N = \tilde{T}$. Using (13.1) we have

$$\bar{I}^X_B\left({}^{u=t_{j+1}-r^\star_{\mathrm{cl}}}\mathcal{D}^{r^\star_{\mathrm{cl}}, u=\frac{1}{11}t_{j+1}}_{[t_j, t_{j+1}]}\right) \leq C\left(r^\star_{\mathrm{cl}}, c\right)\frac{M^2}{\left(t_{j+1}\right)^2} \leq \epsilon\left(t_0\right)\frac{M^{\frac{7}{4}}}{\left(t_{j+1}\right)^{\frac{3}{2}}} \tag{13.16}$$

with the ϵ arising from the fact that t_0 can be chosen as large as we may wish (at the cost of making the data smaller). Consequently,

$$\sum_{j=0}^{N-1} t_{j+1}\bar{I}^X_B\left({}^{u=t_{j+1}-r^\star_{\mathrm{cl}}}\mathcal{D}^{r^\star_{\mathrm{cl}}, u=\frac{1}{11}t_{j+1}}_{[t_j, t_{j+1}]}\right) \leq \epsilon\left(t_0\right) M^{\frac{3}{2}} \sum_{j=0}^{N-1}\frac{1}{\sqrt{1.1}^j} \leq \epsilon\left(t_0\right) M^{\frac{3}{2}} \tag{13.17}$$

in view of the finiteness of the geometric series

$$\sum_{n=0}^{\infty}\frac{1}{\sqrt{1.1}^n} \leq K\,. \tag{13.18}$$

Combining (13.17) with the fact that $F^K_B\left(t_0\right)$ is small by Cauchy stability, Proposition 12.3 yields

$$E^K_B\left(t_N\right) \leq M\epsilon\left(r^\star_K, t_0, \tilde{\delta}\right) \tag{13.19}$$

for any late t_N, which improves assumption (7.8).

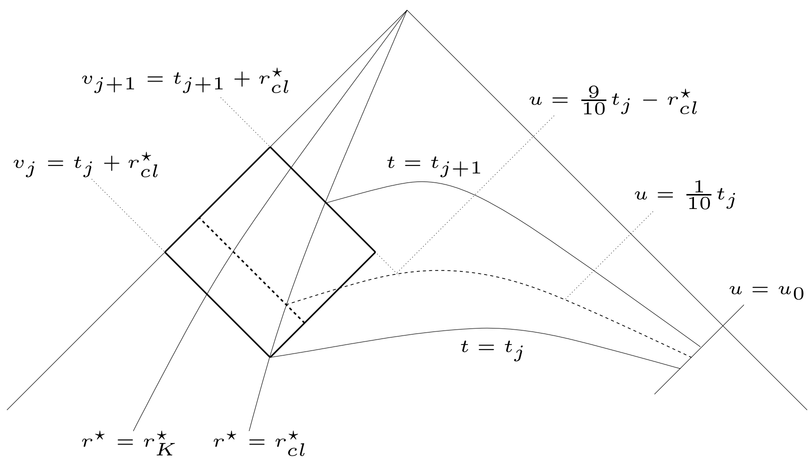


Figure 9: Closing the bootstrap.

13.3 Improving assumptions (7.9)–(7.11)

We apply Proposition 8.3 again, inserting the better bound for the K-boundary terms (13.19) at late times to improve the decay of the energy on any arc-part of late slices, $\Sigma_t \cap \{\frac{10}{11} t \geq r^\star \geq r^\star_{\mathrm{cl}}\}$. From Proposition 8.9 we also obtain improved decay for the field B in the region $r^\star \geq r^\star_{\mathrm{cl}}$.

Proposition 11.5 applied in *each* characteristic dyadic rectangle produces after inserting the better energy decay in the region $r^\star \geq r^\star_{\mathrm{cl}}$ a slice $\hat{v}_i = \tau_i + r^\star_{\mathrm{cl}}$ with improved energy-flux decay, $\frac{c}{(t_i)^3} + \frac{\epsilon}{(t_i)^2}$. By the domain of dependence property the decay of energy flux is improved on the horizon piece $v \in [\hat{v}_i, v_{i+1}]$ and on the ceiling part of the characteristic region, as indicated in figure 9. This retrieves in particular assumptions (7.9) and (7.10) with a better constant. In view of the energy flux decaying now like $\frac{\epsilon}{v^2}$ on all achronal slices in the region $\mathcal{A}(T) \cap \{r^\star \leq \frac{9}{10} t\}$, we can apply Proposition 11.4 to find a good F^Y_B-slice in each characteristic rectangle. Proposition 11.3 exports this good decay of the F^Y_B-term to all constant v-slices and hence the last outstanding bootstrap assumption (7.11) is finally retrieved with a better constant (figure 9).

What we have shown is that $\overline{A} = A$, so A is closed. This completes the proof of Proposition 7.1. The set A must therefore constitute the entire $[0, \infty]$ and hence the decay rates of Theorem 1.1 are proven in the entire $\mathcal{D}$, albeit in a different coordinate system than the one stated in the theorem. The final subsection shows that the coordinate systems used in the bootstrap are indeed close to the null-coordinate system defined in Theorem 1.1.

13.4 Convergence of coordinate systems

What we have already shown in Section 8.3.2 is that the coordinates of a region $\mathcal{A}(\vartheta(\tilde{\tau}_\bullet)) \cap \{r^\star \geq r^\star_K\}$ (a priori defined in the coordinate system $\mathcal{C}_{\tilde{\tau}_\bullet}$), are uniformly bounded in any coordinate system $\mathcal{C}_{\tilde{\tau}}$ for $\tilde{\tau} \geq \tilde{\tau}_\bullet$.[48] It is important to observe that the u-coordinate in the region $r^\star \leq r^\star_K$ is not uniformly close between the different coordinate systems. Indeed, in the coordinate system of Theorem 1.1 the horizon is located at $u = \infty$, whereas in any coordinate system $C_{\tilde{\tau}}$ it generically resides at a finite u value (eventually converging to $u \to \infty$ for $\tilde{\tau} \to \infty$).

We finally establish the relation of the $C_{\tilde{\tau}}$ to the coordinate system defined in Theorem 8.3.2. First recall that we have already shown that the geometrically defined point R of Theorem 1.1 (which features as an "origin" of the coordinate system) has coordinates uniformly close to $(\sqrt{M}, \sqrt{M})$ in any coordinate system $C_{\tilde{\tau}}$, cf. Section 8.3. In the second step we compare the scaling of the coordinates between the coordinate systems $C_{\tilde{\tau}}$ and the one asserted by Theorem 1.1. For this pick a point P on null-infinity. The value of γ at this point in the coordinate system $\mathcal{C}_{\tilde{\tau}}$ can (for large enough $\tilde{\tau}$) be estimated by integrating (2.14) from $t = T(\tilde{\tau})$ along a line of constant u:

$$\frac{1}{2} \leq \gamma(P) \leq \frac{1}{2} + \frac{C(\epsilon)}{r_N^2}, \tag{13.20}$$

where r_N is the area radius at the intersection of the ∇r integral curve defining the coordinate system $\mathcal{C}_{\tilde{\tau}}$ and the null line $u(P)$.[49] In the limit $\tilde{\tau} \to \infty$ we have $r_N \to \infty$ and hence $\gamma(P) \to \frac{1}{2}$. It follows that the scaling of the u-coordinate of $\mathcal{C}_{\tilde{\tau}}$ indeed converges to the one defined in Theorem 1.1.

The function κ on the other hand satisfies $|\kappa - \frac{1}{2}| \leq C(\epsilon)$ on $\mathcal{D}$ in both the coordinate systems $\mathcal{C}_{\tilde{\tau}}$ (cf. Proposition 4.1) and the one of Theorem 1.1. It is easy to show that with this bound holding on the null curve $u = \sqrt{M}$, the v coordinate of any two coordinate systems always satisfies $v \sim \bar{v}$ for $v \geq \sqrt{M}$, which is all what is needed to generalize decay statements in v to all coordinate systems. Namely integrating from the point W where the initial data intersect the null-line $u = \sqrt{M}$ ($v \approx \sqrt{M}$ there by previous

[48]Note again that $r = r^\star_K$ may change its location in the different coordinate systems but remains always close to the geometrically defined curve $r = r_K$ of constant area radius.

[49]For this estimate only the smallness of θ of Proposition 4.1 is used.

remarks) to a point Q we have

$$v_Q = v_W + \int_{v_W}^{v_Q} dv \leq v_W + (2 + C(\epsilon)) \sup_{u=\sqrt{M}} \frac{1}{1-\mu} \times \int_{v_W}^{v_Q} r_{,v} dv \leq \frac{3}{2}\sqrt{M} + 4(r_Q - r_W)$$

and

$$v_Q = v_W + \int_{v_W}^{v_Q} dv \geq v_W + (2 - C(\epsilon)) \inf_{u=\sqrt{M}} \frac{1}{1-\mu} \times \int_{v_W}^{v_Q} r_{,v} dv \geq \frac{\sqrt{M}}{2} + \frac{3}{2}(r_Q - r_W).$$

Hence $v \sim \bar{v}$ for any two coordinate systems.

We have shown that the limit of the coordinate systems $C_{\tilde{\tau}}$ is a coordinate system in which the origin is slightly shifted compared to the one of Theorem 1.1 and whose v scaling may be stretched or squeezed. It is now apparent that the decay rates stated also hold in the coordinate system of Theorem 1.1.

14 Final comments and open questions

Theorem 1.1 leaves room for generalizations. An obvious one is the treatment of the *triaxial case*, which at least conceptually is not expected to pose any difficulty. In fact the same vectorfields are expected to produce the required estimates for the fields B and C when contracted with an appropriate tensor $T_{\mu\nu}$ – with the only additional catch coming from the coupling of B and C. A much more challenging problem is the derivation of better decay rates than the ones established here. As mentioned previously, in the context of compatible currents, the maximal decay rate is limited by the weights appearing in the K-vectorfield. It is an interesting question whether an additional vectorfield (or an entirely different idea) can extract stronger decay, which might be expected from the four-dimensional case [6]. An even more ambitious problem concerns the large data regime of the five-dimensional Bianchi IX model. The numerical studies of [1] suggest that a similar result to the one proven here should hold. In fact, it may be possible to find an elaborate refinement of the ideas in [6], which will allow an analysis of the large data regime within the symmetry class.

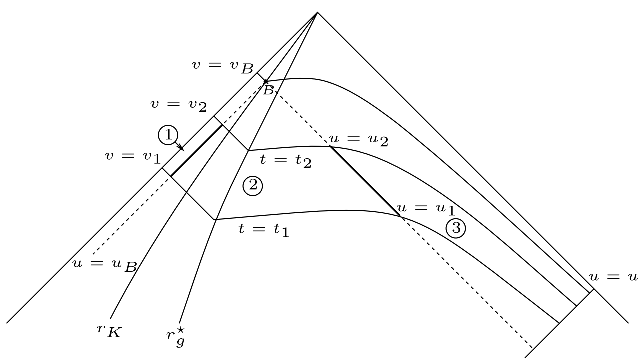
Finally, there should exist various applications of the techniques to four-dimensional problems. As already mentioned in the introduction, the present paper may serve as a blueprint to obtain a small-data version of [6] for the self-gravitating scalar field. For genuinely novel results, the case of a conformally coupled scalar field could be investigated.

Appendix A Regularity and Green's identity

It was remarked in Section 3 that the coordinate systems $\mathcal{C}_{\tilde{\tau}}$ are C^1. More precisely, it was shown that they are piecewise C^2 with a discontinuity in $\frac{\Omega_{,v}}{\Omega}$ spreading along the null-line $v\,(B)$ and a discontinuity in $\frac{\Omega_{,u}}{\Omega}$ along $u\,(B)$. This discontinuity could be avoided by the introduction of a smooth interpolating function in the region around the cusp at the point B (cf. figure 3). However, as this would burden the notation even further, we will show here that the regularity is sufficient to carry out the calculations involving the vectorfields.

Observing that the quantity P^α defined in (5.5) is continuous and $\nabla_\alpha P^\alpha$ at least piecewise continuous (cf. (5.11)), the basic identity (1.10) is valid for the vectorfields X, Y, K in the coordinate systems $\mathcal{C}_{\tilde{\tau}}$.

For the vectorfields X and K we also make use of Green's identity (5.24) in a region ${}^{u_H}\mathcal{D}^{r_g^\star, u_J}_{[t_1,t_2]}$.



As depicted, the region may contain part of the null-line $v\,(B)$ along which $\frac{\Omega_{,v}}{\Omega}$ could be discontinuous and part of the null-line $u\,(B)$ along which $\frac{\Omega_{,u}}{\Omega}$ could be discontinuous. The functions D for which (5.24) is applied are given by (10.11) and (12.9). In both cases, $D\,(u,v)$ is seen to be piecewise differentiable and such that $\Box D$ is piecewise continuous. To derive the identity (5.24) for these cases in our coordinate system, one should split the integration region ${}^{u_H}\mathcal{D}^{r^\star_g,u_J}_{[t_1,t_2]}$ into three pieces, along the null lines $u\,(B)$ and $v\,(B)$, introducing additional boundary terms from the bold lines. Green's identity is then clearly valid in each subregion because all functions admit appropriate regularity there, i.e., in particular D is differentiable and $\Box D$ is continuous in the interior. The integrand of the additional boundary term along the null-line $v\,(B)$ however

$$\int \left[B^2 \partial_u D - D \partial_u B^2\right] r^3 du \tag{A.1}$$

is continuous because the u derivative of D, which involves only the term $\frac{\Omega_{,u}}{\Omega}$ (but not its v-analogue!), is continuous there. It is also bounded and the above integral will appear with a different sign for the two subregions. Analogously, the integrand of the other boundary term

$$\int \left[B^2 \partial_v D - D \partial_v B^2\right] r^3 dv \tag{A.2}$$

is continuous because the v derivative of D is continuous there. Hence adding up the three subregions the additional boundary terms cancel and identity (5.24) indeed holds as stated.

Appendix B Different curves of constant $r^\star$

$r^\star_K$	very large and negative (close to the horizon), features as a source of smallness in the bootstrap
$r^\star_{\text{cl}}$	$r^\star_{\text{cl}} = r^\star_Y - 2\sqrt{M}$
$r^\star_Y$	negative, chosen in Section 7.2 to make a certain bulk term of the Y vectorfield positive in the region $r^\star \leq r^\star_Y$
$-\frac{1}{2}\sqrt{M}$	functions α and β are supported in $r^\star \leq -\frac{1}{2}\sqrt{M}$ only
$R^\star$	$R^\star = -\frac{1}{3}\sqrt{M}$ defined in Proposition 10.3
$r^\star_{\text{zero}}$	defined in Section 10.2.2, $-\frac{1}{6}\sqrt{M} \leq r^\star_{\text{zero}} \leq -\frac{1}{10}\sqrt{M}$
0	$r^2 \approx 4M$ (photon sphere for 5 dim. Schwarzschild)
$\tilde{R}^\star$	squashing field on initial data is not supported for $r^\star \geq \tilde{R}^\star$
$\hat{R}^\star$	defined in Lemma 12.1, equips a certain integrand with a sign in a particular region

Appendix C Glossary

α	function depending on $r^\star$, used in the definition of the vectorfield Y
β	function depending on $r^\star$, used in the definition of the vectorfield Y
B	squashing field
γ	defined in (2.12)
$\mathcal{D}$	defined in (3.1)
δ, $\tilde{\delta}$	smallness parameters
ϵ, $\tilde{\epsilon}$	smallness parameters
ζ	$\zeta = r^{\frac{3}{2}} B_{,u}$
η	smallness parameter (cf. Corollary 4.2 and Proposition 7.2)
θ	$\theta = r^{\frac{3}{2}} B_{,v}$
ϑ	function used for the definition of the coordinate systems $C_{\tilde{\tau}}$, cf. (3.3)
κ	defined in (2.12)
λ	$\lambda = r_{,v}$
μ	$\mu = \frac{2m}{r^2}$
ν	$\nu = r_{,u}$
ξ	function depending on $r^\star$ defined in (10.28)
m	Hawking mass (2.8)
M_f	final Bondi mass
M_A	Hawking mass at the point A, cf. Section 3
r	$r\,(u,v)$ area radius
ρ	defined in (2.7)
$\tilde{S}_{r_K}$	defined in (1.9)
Σ_t	defined in (5.35)
σ	parameter, chosen in the Section on the vectorfield X, cf. (10.27)
τ	affine parameter along $r^2 = 4M_A$, Section 3
$\tilde{\tau}$	affine parameter along $r^2 = 4M_f$, Section 3
φ_1, φ_2	defined in (2.15)
$\chi, \tilde{\chi}$	smooth interpolating functions, cf. (7.16) and Proposition 10.3
ψ	smallness parameter, Section 7.2
Ω^2	metric function

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