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Complex counterpart of Chern–Simons–Witten theory and holomorphic linking

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Abstract

In this paper we are begining to explore the complex counterpart of the Chern–Simon–Witten theory. We define the complex analogue of the Gauss linking number for complex curves embedded in a Calabi–Yau 3-fold using the formal path integral that leads to a rigorous mathematical expression. We give an analytic and geometric interpretation of our holomorphic linking following the parallel with the real case. We show in particular that the Green kernel that appears in the explicit integral for the Gauss linking number is replaced by the Bochner–Martinelli kernel. We also find canonical expressions of the holomorphic linking using the Grothendieck–Serre duality in local cohomology, the latter admits a generalization for an arbitrary field.

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1 Introduction

The relation between loop groups and their central extensions, Wess–Zumino–Novikov–Witten (WZNW) two-dimensional conformal field theories and Chern–Simons–Witten (CSW) three-dimensional topological theories, emerged as a unifying principle among various areas in mathematics and theoretical physics. It has gradually become clear that the three components and the relation itself admit a remarkable complexification, which combines in a profound way some further areas in both disciplines. It was shown in [10] that the classification of coadjoint orbits for a new class of two-dimensional current groups on Riemann surfaces can be viewed as a classification of stable vector bundles over these surfaces. In [11], an analogue of WZNW construction for two-dimensional current groups was obtained by means of Leray's residue theory in complex analysis cf [23].

In this paper, we begin to explore a complex counterpart of CSW theory. We consider a generalization of the simplest invariant of two curves in S^3, namely the Gauss linking number, which arises in the abelian CSW theory as a correlation function of holonomy functionals corresponding to two curves. In the abelian case one can give a precise meaning to the formal path integral and derive the familiar formula for the Gauss linking number. The "complexification" of loop group theory discovered in [10] and extended in [11] leads to a complex counterpart of the abelian CSW path integral, which we turn into a rigorous expression for the holomorphic linking. Then we show that the Green kernel that appears in the classical integral for the Gauss linking number is replaced by the Bochner–Martinelli kernel and has deep relation to the theory of Green currents in Arakelov geometry. The integral formula for the Gauss linking number leads to its topological realization as an intersection number and we derive its algebro-geometric analogue in the complex case. It turns out that the notion of holomorphic linking is related to many structures of complex and algebraic geometry, which can be viewed as complementary aspects of one unified picture. At one end the holomorphic linking is presented by the complex CSW path integral, at the other end, it is expressed via the Grothendieck–Serre duality in local cohomology. The goal of this paper is to demonstrate different realizations of the holomorphic linking and its connections with various structures of mathematics and mathematical physics.

Let us now explain the notion of holomorphic linking in some detail. By definition, the holomorphic linking of two complex curves Σ_1 and Σ_2 in a Calabi–Yau (CY) manifold M is a linear map on the product of the spaces of holomorphic differentials on Σ_1 and Σ_2. It is no longer a topological invariant but depends only on the complex structure on Σ_1 and Σ_2 and their embedding into a CY manifold M and not on the metric. To illustrate our notion of holomorphic linking, we consider an example of the simplest non-compact CY manifold $\mathbb{C}^3$ and two affine curves Σ_1 and Σ_2 embedded in it. Then the analogue of the Gauss formula is the following expression for the holomorphic linking:

$$\#((\Sigma_1,\theta_1),(\Sigma_2,\theta_2)) = \int_{\Sigma_1\times\Sigma_2} \frac{\varepsilon^{ijk}\overline{(z_i - w_i)}}{|z-w|^6} \wedge d\overline{z_j} \wedge d\overline{w_k} \wedge \theta_1 \wedge \theta_2,$$

where ε^{ijk} is the sign of the permutation (i,j,k) and θ_1 and θ_2 are holomorphic forms on Σ_1 and Σ_2, respectively. When Σ_1 and Σ_2 are complex lines, we can compute the holomorphic linking and compare it with the "Gauss linking number" of two real lines in $\mathbb{R}^3$. When two lines are not parallel to each other the integral formulas both in real and in complex cases can be easily identified as follows. We pick three vectors $\overrightarrow{e_1}, \overrightarrow{e_2}$ and $\overrightarrow{e_3}$ such that the first two determine the direction of two lines and the third vector has the initial point on the first line and the end point on the second line. Then the Gauss integral formula yields $\frac{1}{2}\mathrm{sign}(\det(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}))$ and it depends on the choice of the orientation of the first and the second line, but not on the order of the two lines. A similar computation in the complex case gives, up to a scalar multiple $\det(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})^{-1} < \overrightarrow{e_1}, \theta_1 >< \overrightarrow{e_2}, \theta_2 >$, where θ_i for $i = 1$ and 2 are elements of the dual space and can be viewed as 1-forms. Thus, the main information of the holomorphic linking is contained in the determinant. The latter degenerates when two lines cross each other or become parallel, and is also covariant with respect to the linear transformation. This elementary example indicates that the complex linking is the "measure" of closeness of two curves in a three-dimensional complex manifold and is the simplest possible invariant in the complex case. This supports our belief in the intrinsic nature of the new invariant.

It is well known that the Gauss linking number of two circles in S^3 has a simple geometric interpretation as an intersection number of one circle with a disk bounded by the second one. The analytic expression for the holomorphic linking number can also be written in a similar form. The new ingredients are the holomorphic forms θ_1 and θ_2 attached to the curves Σ_1 and Σ_2. Instead of a disk with prescribed boundary circle in the real class, one should consider a surface S_1 with a prescribed divisor Σ_1 in the complex case. Moreover, the holomorphic form θ_1 on Σ_1 is "lifted" to a meromorphic form ω_1 on S_1 such that $\mathrm{res}(\omega_1) = \theta_1$. See [11] and a further generalization in [20] and [21]. Then the formula for the holomorphic linking in M becomes the following

$$\#\left(\left(\Sigma_1, \theta_1\right), \left(\Sigma_2, \theta_2\right)\right) = \sum_{x \in S_1 \cap \Sigma_2} \frac{\omega_1(x) \wedge \theta_2(x)}{\eta(x)},$$

where $\omega_1(x) \wedge \theta_2(x) \in \wedge^3 \left(T^{1,0}_{x,\mathrm{M}}\right)^* \cong \Omega^3_{x,\mathrm{M}}$ and η is a holomorphic volume form on M. As in the real case, the invariant does not depend on the choice of the complex surface S_1 and the meromorphic form ω_1. It is easy to see directly that when Σ_1 and Σ_2 are complex lines and S_1 is a complex plane containing Σ_1, the above geometric formula for the holomorphic linking yields up to a scalar multiple the expression $\det(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})^{-1} \langle \overrightarrow{e_1}, \theta_1 \rangle \langle \overrightarrow{e_2}, \theta_2 \rangle$ discussed above.

As we have mentioned before, we have derived the analytic and geometric formulas for holomorphic linking studying complex counterpart of CSW with a path integral

$$\int_{\mathcal{A}^{0,1}/\mathcal{G}^{0,1}} \mathcal{D}A \exp\left(\sqrt{-1}h \int_{\mathrm{M}} A \wedge \overline{\partial} A \wedge \eta\right) \exp\left(\int_{\Sigma_1} A \wedge \theta_1\right) \exp\left(\int_{\Sigma_2} A \wedge \theta_2\right),$$

where h is a real parameter, $\mathcal{A}^{0,1}$ is the space of $(0,1)$ forms on M and $\mathcal{G}^{0,1}$ is the subspace of all $\overline{\partial}$-closed forms. Its rigorous meaning can be expressed in terms of the Green current for the pair $(M \times M, \Delta)$, where Δ is the diagonal. Its existence follows from the general theory of the Green currents established by Gillet and Soulé (see [28]) in the context of Arakelov geometry. For any complex subvariety Y in a projective manifold X they defined a Green current g_{Y} satisfying the equation

$$\partial\overline{\partial} g_Y + \delta_{\mathrm{Y}} = \omega_{\mathrm{Y}},$$

where δ_{Y} is the Dirac delta current corresponding to Y and ω_Y is a smooth closed form representing the Poincare dual of the homology class $[Y]$ of Y. We use the existence of a Green current for the pair $(X, Y) = (M \times M, \Delta)$ and then restrict it to $U \times U$, where U is an affine open set in M. The comparison of the restriction to the explicit formula for the Bochner–Martinelli kernel on $U \times U$ yields the above analogue of the Gauss integral formula in the complex case. Similarly a Green current for the pairs $(X, Y) = (M, \Sigma_1)$ or (M, Σ_2) can be used for an alternative analytic expression of the holomorphic linking.

The holomorphic linking also admits a certain canonical expression in the language of homological algebra. For a complex subvariety Y in a projective manifold X and a top holomorphic form θ on Y we introduce generalized notions of Grothendieck and Serre classes of a pair (Y, θ) denoted by $\mu(Y, \theta)$ and $\lambda(Y, \theta)$, respectively. By definition $\mu(Y, \theta)$ and $\lambda(Y, \theta)$ belong to $\mathrm{Ext}^d_{\mathcal{O}_X}(\mathcal{O}_Y, \Omega^n_X)$ and $\mathrm{Ext}^{d-1}_{\mathcal{O}_X}(I_Y, \Omega^n_X)$, where I_Y is the ideal sheaf of Y, Ω^n_X is the locally free sheaf of top holomorphic forms on X, $\mathcal{O}_X$ and $\mathcal{O}_Y$ are the structure sheaves on X and Y and d is the codimension of Y in X. This is a generalization of similar notions originally studied by Atiyah in [2] in the case of linking of two lines in a twistor space. From the definition it follows that the Grothendieck class always exists. The new point of our construction is the dependence of the Grothendieck and Serre classes on the holomorphic form θ, which is crucial for the existence of the Serre classes of pairs (Y, θ), where Y is a submanifold in a projective manifold. In particular, we show the existence of the Serre class of the pair (X, Y) where $X = M \times M$ and Y is the diagonal embedding of a CY manifold M with the holomorphic form η on it. Then the holomorphic linking of two curves in a CY 3-fold can be expressed via Yoneda pairing $\langle\, , \,\rangle$ of the corresponding Ext groups as follows:

$$\#\left(\left(\Sigma_1, \theta_1\right), \left(\Sigma_2, \theta_2\right)\right) = \left\langle \lambda\left(\Delta, \eta\right)|_{\Sigma_1 \times \Sigma_2}, \mu\left(\Sigma_1 \times \Sigma_2, \pi_1^*\left(\theta\right) \wedge \pi_2^*\left(\theta\right)\right)\right\rangle,$$

where π_1 and π_2 are the projections of $M \times M$ to the first and the second factor, respectively. The advantage of the latter interpretation of the holomorphic linking is that it makes sense over an arbitrary field and has universal algebro-geometric meaning. The relation with the analytic expression follows from the interpretation of the Serre class $\lambda(\Delta, \eta)$ as an element of $H^2(M \times M - \Delta, \mathcal{O}_{M \times M - \Delta})$, owing to the Grothendieck duality, and then from the identification of the local expression for $\lambda(\Delta, \eta)$ as a Bochner–Martinelli kernel. Similarly, we obtain an alternative expression for the holomorphic linking using the Grothendieck and Serre classes for the pairs $(X, Y) = (M, \Sigma_1)$ or (M, Σ_2).

We also compare our homological formula for the holomorphic linking with a similar expression found by Atiyah in [2] for the linking of the two lines in a twistor space. In order to obtain the direct relation between our holomorphic linking and Atiyah's, we extend our construction by considering complex curves with marked points on them and restrict it to the case of two spheres with two marked points.

The holomorphic linking studied in this paper has many predecessors. The Lagrangian of the complex counterpart of CSW theory was first proposed by Witten in [32]. He derived it as a low energy limit of an open string theory and argued that the theory is finite in spite of the fact that it is defined in a six-dimensional space. We rediscovered the same theory following the analogy with the real case, extensively studied in relation to representation theory of loop groups. Other approaches to holomorphic linking were previously considered by Atiyah [2], Penrose [26] and Gerasimov [12]. In particular, Gerasimov also suggested a path integral presentation of the Atiyah linking of two lines in a twistor space, which is similar to our path integral that arises in complexification of loop group theory [11]. We expect that the combination of the different motivations that lead to the holomorphic linking can be fruitful in the future developments in this rich new field.

The organization of the paper by sections is as follows:

In Section 2 we recall a path integral derivation of the Gauss linking number for CSW theory, previously obtained in [27] and [31]. The derivation yields, in particular, a well-known explicit integral formula for the Gauss invariant and relates it with the familiar geometric form of the Gauss linking number.

In Section 3 we repeat the path integral derivation in the complex case and obtain a definition of the holomorphic linking. To make it rigorous, we use the theory of currents on complex manifolds.

In Section 4 we use the existence of the Green current in Arakelov geometry [28] for the pair $M \times M$ and its diagonal Δ to recast the definition of the holomorphic linking in a more invariant form. As a result we also obtain a symmetric form of the holomorphic linking.

In Section 5 we give explicit formulas for the restriction of the Green kernel of the diagonal on an open affine set $U \times U$ in $M \times M$ by relating it to the Bochner–Martinelli kernel. This leads to an explicit analytic formula for the holomorphic linking which is the analogue of the classical Gauss formula.

In Section 6 we derive a geometric form of holomorphic linking. This is a direct complex generalization of the usual topological form of the Gauss linking number.

In Section 7 we introduce generalized notions of Grothendieck and Serre classes. We prove their existence for any embedded submanifold Y into a CY manifold M and the diagonal embedded in $M \times M$. Then we recast the analytic formulas for the holomorphic linking in the universal language of homological algebra.

In Section 8 we give a definition of the holomorphic linking of two Riemann surfaces with marked points on them. This allows us to define the holomorphic linking of spheres embedded in three-dimensional complex manifolds. As a consequence, we are able to explicitly relate our notion of holomorphic linking with the Atiyah holomorphic linking of rational curves in the twistor space of the four-dimensional sphere.

In conclusion, we would like to mention some future perspectives that result from our work. We have seen how different approaches to holomorphic linking lead to equivalent definitions. However, our approach via complex CSW theory has one important advantage, namely, it admits a non-abelian generalization at least at formal level. To make it rigorous, one can try the pertributive expansion of the complex CSW path integral. Donaldson and Thomas obtained the analogue of Casson's invariant for CY manifolds in [9, 29, 30], which should appear as the first non-trivial term in the pertributive expansion. In [29] one can find further ideas how to generalize the invariants obtained by Axelrod and Singer in [3, 4], coming from the expansion of the Chern–Simon functional in the complex case. Another possible approach is to find a combinatorial calculus for the complex CSW path integral and its generalizations. Interpretation of non-abelian generalizations of the holomorphic linking in the context of local cohomology can be an equally important application of complex CSW theory. Finally we would like to note that our notion of holomorphic linking can be easily generalized to submanifolds N_1 and N_2 in a n-dimensional CY manifold M, where $\dim_{\mathbb{C}} N_1 + \dim_{\mathbb{C}} N_2 = n - 1$. In this form it might be related to the height pairing for higher dimensional cycles that was first constructed by Bloch and Beilinson, see [5, 6, 8]. Other constructions related to the notion of holomorphic linking presented in this paper can be found in [20, 21, 29].

2 Abelian CSW theory and the Gauss linking number

Let C_1 and C_2 be two non-intersecting knots in a simply connected three-dimensional real manifold M. The simplest non-trivial invariant of the pair (C_1, C_2) is the Gauss linking number denoted by $\#(C_1, C_2)$. The abelian CSW theory gives a path integral presentation of this invariant via a correlation function of holonomy functionals attached to the knots. See [27, 31]. We will use this presentation as a starting point of our approach and then show that it leads to familiar analytic and geometric definitions of the

Gauss linking number. This point of view will allow us to produce complex counterparts of all classical formulas.

Let $\mathcal{A}$ be the space of 1-forms on M and let $\mathcal{G}$ be the subspace of exact 1-forms on M. The abelian Chern–Simons Lagrangian has the form

$$\mathcal{L}(A) = \int_M A \wedge dA,$$

where $A \in \mathcal{A}$ and it is invariant with respect to translation of an element from $\mathcal{G}$. One defines a holonomy functional on $\mathcal{A}$, called a Wilson loop, by

$$W_C(A) = \exp\left(\int_C A\right)$$

for any knot C in M. The Witten invariant of a link in M with components $C_1, \dots, C_n$ is given by the formal expression

$$\mathcal{Z}(\mathrm{M}; C_1, \dots, C_n) = \int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A \mathrm{e}^{\sqrt{-1}h\mathcal{L}(A)} W_{C_1}(A) \cdots W_{C_n}(A), \tag{2.1}$$

where h is a real parameter.

Though in the general case the rigorous definition of Witten's path integral remains a widely open publicized problem, the explicit meaning of the above definition can be described in the following way. Choose in $\mathcal{A}$ a linear complement $\mathcal{A}_0$ to $\mathcal{G}$. Then for any $\mathcal{G}$ invariant functional $\mathcal{F}$ on $\mathcal{A}$, one assumes the following "gauge fixing":

$$\int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A \mathrm{e}^{\sqrt{-1}h\mathcal{L}(A)} \mathcal{F}(A) = \int_{\mathcal{A}_0} \mathcal{D}A \mathrm{e}^{\sqrt{-1}h\mathcal{L}(A)} \mathcal{F}(A)$$

and the "quasi-invariance":

$$\int_{\mathcal{A}_0} \mathcal{D}A \mathrm{e}^{\sqrt{-1}h\mathcal{L}(A+A_1)} \mathcal{F}(A + A_1) = \int_{\mathcal{A}_0} \mathcal{D}A \mathrm{e}^{\sqrt{-1}h\mathcal{L}(A)} \mathcal{F}(A),$$

where $A_1 \in \mathcal{A}_0$.

The Gauss linking number of C_1 and C_2 appears if we "factor out" the self-linking. Namely, one has the following identity, which we will use as a definition.

Definition 2.1. The Gauss linking number $\#(C_1, C_2)$ is defined by the following formula:

$$\exp\left(\frac{\sqrt{-1}}{2h} \# (C_1, C_2)\right) = \frac{\mathcal{Z}(M; C_1, C_2)\mathcal{Z}(M)}{\mathcal{Z}(M; C_1)\mathcal{Z}(M; C_2)}. \tag{2.2}$$

We will show by formal application of the quasi-invariance that Definition 2.1 yields a certain well-defined functional on C_1 and C_2 which does not depend on the choice of the "gauge fixing". Let us choose two forms ω_1 and ω_2 such that

$$\int_M A \wedge \omega_i = \int_{C_i} A \quad \text{for } i = 1, 2.$$

In fact, the forms ω_1 and ω_2 are currents, i.e., linear functionals on the space of smooth 1-forms. However, by duality one can define a differential on the space of currents. (see [13] and Section 3.2.) Then it follows that ω_i are exact for $i = 1, 2$. We also obtain:

$$\begin{aligned}\mathcal{L}\left(A + \frac{\sqrt{-1}}{2h}A_1 + \frac{\sqrt{-1}}{2h}A_2\right) &= \mathcal{L}(A) + \frac{-1}{4h^2}\mathcal{L}(A_1) + \frac{-1}{4h^2}\mathcal{L}(A_2)\\ &\quad + \frac{\sqrt{-1}}{h}\left(\int_M A \wedge dA_1 + \int_M A \wedge dA_2\right)\\ &\quad - \frac{1}{4h^2}\int_M A_1 \wedge dA_2.\end{aligned}$$

Let us choose the real currents A_1 and A_2 satisfying the following relations:

$$dA_i = \omega_i \quad \text{for } i = 1, 2.$$

The currents A_i should be viewed as generalized 1-forms, and we choose them in a unique way by requiring that they belong to the completion of $\mathcal{A}_0$. We will also denote by

$$d^{-1}\omega_i = A_i$$

for $i = 1, 2$. With the above choice of A_1 and A_2, the integrand in $\mathcal{Z}(\mathrm{M};\mathrm{C}_1, C_2)$ becomes constant and the right-hand side of the expression in Definition 2.1 transforms to the following form:

$$\exp\left(\frac{\sqrt{-1}}{2h}\int_M A_1 \wedge dA_2\right) = \exp\left(\frac{\sqrt{-1}}{2h}\int_M d^{-1}\omega_1 \wedge \omega_2\right).$$

It is easy to check that the integral in the exponent does not depend on the choice of the complement $\mathcal{A}_0$ and is well defined. Thus, Definition 2.1 assumes the following form:

$$\#(C_1, C_2) = \int_M A_1 \wedge dA_2.$$

In order to relate it with the standard expression of the classical Gauss linking number, we will rewrite the above integral in a different form. Let us denote by δ_Δ a Dirac current concentrated on the diagonal $\Delta \subset \mathrm{M}\times\mathrm{M}$, namely for any smooth 3-form ξ on M one has

$$\int_{M\times M} \pi_i^*(\xi) \wedge \delta_\Delta = \int_\Delta \pi_i^*(\xi) = \int_M \xi, \quad i = 1, 2,$$

where π_1 and π_2 are the projections on the first and the second factor, respectively.

It is well known (see, e.g., [13, Chapter 3], that on a compact manifold M the cohomology theory defined by the currents coincides with the De Rham cohomology. We will recall the basic definitions related to currents in Section 3.2 below. Let us choose a closed smooth form ω_Δ which represents the same cohomology class as the Dirac current δ_Δ. Then there exists a current S such that

$$dS + \delta_\Delta = \omega_\Delta. \tag{2.3}$$

Equation (2.3) implies that the singular support of the current S is exactly the diagonal Δ, so the restriction of S to $M \times M - \Delta$ is a smooth form, which we denote by S_Δ.

Since the cohomology class of δ_Δ, and therefore of ω_Δ, is the Poincare dual class of Δ, the restriction of the closed smooth form ω_Δ to $M \times M - \Delta$ is an exact smooth form, i.e.,

$$\omega_\Delta|_{M\times M-\Delta} = d\psi_\Delta$$

for some smooth form ψ_Δ. Then we introduce a smooth 2-form on $M \times M - \Delta$

$$\phi_\Delta = \psi_\Delta - S_\Delta. \tag{2.4}$$

The cohomology class of ϕ_Δ on $M \times M - \Delta$ does not depend on the choices involved in the above construction.

Now we can rewrite

$$\begin{aligned}
\int_M d^{-1}\omega_1 \wedge \omega_2 &= \int_{M\times M} \pi_1^*\left(d^{-1}\omega_1\right) \wedge \pi_2^*\left(\omega_2\right) \wedge \delta_\Delta \\
&= \int_{M\times M} \pi_1^*\left(d^{-1}\omega_1\right) \wedge \pi_2^*\left(\omega_2\right) \wedge \left(\omega_\Delta - dS\right) \\
&= \int_{M\times M-\Delta} \pi_1^*\left(d^{-1}\omega_1\right) \wedge \pi_2^*\left(\omega_2\right) \wedge d\psi_\Delta \\
&\quad - \int_{M\times M} \pi_1^*\left(d^{-1}\omega_1\right) \wedge \pi_2^*\left(\omega_2\right) \wedge dS \\
&= \int_{M\times M-\Delta} \pi_1^*\left(\omega_1\right) \wedge \pi_2^*\left(\omega_2\right) \wedge \psi_\Delta \\
&\quad - \int_{M\times M-\Delta} \pi_1^*\left(\omega_1\right) \wedge \pi_2^*\left(\omega_2\right) \wedge S_\Delta \\
&= \int_{M\times M-\Delta} \pi_1^*\left(\omega_1\right) \wedge \pi_2^*\left(\omega_2\right) \wedge \phi_\Delta = \int_{C_1\times C_2} \phi_\Delta.
\end{aligned} \tag{2.5}$$

Let $M = S^3 = \mathbb{R}^3 \cup \infty$. The standard facts about Green currents imply that the form ϕ_Δ is given on $\mathbb{R}^3 \times \mathbb{R}^3 - \Delta$ by

$$\phi_\Delta = \frac{1}{4\pi} \frac{\varepsilon^{ijk}(x_i - y_i)}{|x-y|^3} dx_j \wedge dy_k, \tag{2.6}$$

where $x, y \in \mathbb{R}^3$, ε^{ijk} is the sign of permutation (i, j, k) and we assume the summation over all three indices. Thus (2.5) and (2.6) imply the classical Gauss formula for the linking number

$$\#\,(C_1, C_2) = \frac{1}{4\pi} \int_{C_1 \times C_2} \frac{\varepsilon^{ijk}(x_i - y_i)}{|x-y|^3} dx_j \wedge dy_k. \tag{2.7}$$

One can also derive a topological interpretation of the Gauss linking number by considering a disk D_1 in M with a boundary $\partial D_1 = C_1$ and "reversing" the steps in (2.5):

$$\begin{aligned} \int_{C_1 \times C_2} \phi_\Delta &= \int_{C_1 \times C_2} (\psi_\Delta - S_\Delta) = \int_{D_1 \times C_2 - \Delta} d\psi_\Delta - \int_{D_1 \times C_2} dS \\ &= \int_{D_1 \times C_2 - \Delta} \delta_\Delta \int_{D_1 \times C_2} dS = \int_{D_1 \times C_2} \delta_\Delta. \end{aligned} \tag{2.8}$$

The latter integral should be viewed as a pairing between the cohomology class represented by the current δ_Δ and the cycle $D_1 \times C_2$ (see Section 3.2. for details). It counts the number of the intersection points of D_1 and C_2 with a sign that depends on the orientation. This presentation provides a simple topological meaning of the Gauss linking number and implies its integrality properties.

Finally, we note that various forms and integrals that appeared in the above discussion of Gauss linking number admit a natural cohomological interpretation. The Gauss linking number can be obtained as a pairing of cohomology classes with the appropriate cycles.

In the rest of the paper, we will produce complex counterparts of various manifestation of the Gauss linking number.

3 Complex counterpart of the abelian CSW theory and holomorphic linking

The complexification of loop group theory developed in [10, 11] yields in particular a complex analogue of the abelian CSW Lagrangian and Wilson loop functional previously considered in the context of string theory [32]. This leads to a path integral definition of the holomorphic linking of Riemann surfaces in a CY 3-fold. We show that the formal calculation of the path integral gives rise to a rigorous mathematical notion of holomorphic linking which can be viewed as a complex counterpart of the Gauss linking number.

3.1 Path integral definition of the holomorphic linking

Let M be a Kähler manifold of complex dimension n. We recall that M is a CY manifold if it admits a metric with holonomy group $SU(n)$. It is a well-known fact that this definition of a CY manifold M implies that there exists a unique up to constant holomorphic n-form η without zeroes. See [7]. Let M be a CY 3-fold and η be a holomorphic 3-form on M. Until Section 8, we assume that M is a compact variety.

The Lagrangian of the complex abelian CSW theory is defined as follows. Let $\mathcal{A}^{0,1}$ denote the complex space of (0,1) forms on M and let $\mathcal{G}^{0,1}$ be the subspace of $\overline{\partial}$-exact forms. Then we set

$$\mathcal{L}(A) := \int_{\mathrm{M}} A \wedge \overline{\partial} A \wedge \eta, \quad A \in \mathcal{A}^{0,1}.$$

It follows immediately from Stokes' theorem that the Lagrangian is invariant with respect to $\mathcal{G}^{0,1}$, i.e.,

$$\mathcal{L}(A + \overline{\partial}\phi) = \mathcal{L}(A)$$

for any function ϕ.

For a Riemann surface Σ, together with a holomorphic 1-form θ, we define an analogue of the Wilson loop functional

$$W_{(\Sigma,\theta)}(A) := \exp\left(\int_{\Sigma} A \wedge \theta\right), \quad A \in \mathcal{A}^{0,1}. \tag{3.1}$$

The expression in (3.1) is also invariant with respect to $\mathcal{G}^{0,1}$. We will define the complex version of (2.1) as follows:

$$\begin{aligned}\mathcal{Z}(M; (\Sigma_1, \theta_1), \ldots, (\Sigma_n, \theta_n)) := \int_{\mathcal{A}^{0,1}/\mathcal{G}^{0,1}} & \mathcal{D}A \exp\left(\sqrt{-1} h \mathcal{L}(A)\right) \\ & \times \prod_{i=1}^{n} W_{(\Sigma_i,\theta_i)}(A),\end{aligned}$$

whereas in the real case, the path integral should satisfy the gauge fixing and quasi-invariance. Following the analogy with the real case we now define a complex counterpart of the Gauss linking number.

Definition 3.1. The holomorphic linking of two Riemann surfaces Σ_1 and Σ_2 with chosen holomorphic forms θ_1 and θ_2, respectively, is defined by the following formula:

$$\exp\left(\frac{\sqrt{-1}}{2h}\#\left((\Sigma_1,\theta_1),(\Sigma_2,\theta_2)\right)\right)=\frac{\mathcal{Z}(M;(\Sigma_1,\theta_1),(\Sigma_2,\theta_2))\mathcal{Z}(M)}{\mathcal{Z}(M;(\Sigma_1,\theta_1))\mathcal{Z}(M;(\Sigma_2,\theta_2))}. \tag{3.2}$$

Let $\mathcal{H}^1(\Sigma)$ denote the space of holomorphic differentials on Σ. We will show that for any two Riemann surfaces Σ_1 and Σ_2 the formal expression in Definition 3.1 gives rise to a well-defined linear map $\#$, which we will call the holomorphic linking:

$$\#\left((\Sigma_1,\circ),(\Sigma_2,\circ)\right):\mathcal{H}^1(\Sigma_1)\times\mathcal{H}^1(\Sigma_2)\longrightarrow\mathbb{C}.$$

3.2 Currents on complex manifolds

In order to transform the formal definition of the holomorphic linking to a rigorous form, we will need to recall some basic facts about currents on a complex manifold X. For more details see [13].

Let X be a compact Kähler complex manifold of complex dimension n. We will denote by $A^{p,q}(X)$ the vector space of C^∞ complex-valued forms of type (p,q). The space $A^m(X)$ of complex-valued C^∞ m-forms is given by

$$A^m(X)=\bigotimes_{p+q=m}A^{p,q}(X).$$

We denote by

$$\partial:A^{p,q}(X)\longrightarrow A^{p+1,q}(X)$$

and

$$\overline{\partial}:A^{p,q}(X)\longrightarrow A^{p,q+1}(X)$$

the Dolbeault differentials.

Let D_m denote the dual space of $A^m(X)$ with respect to the standard Frechet topology. We also denote by $D_{p,q}(X)$ the dual space of $A^{p,q}(X)$. We define $D^{p,q}(X)$ as $D_{n-p,n-q}(X)$. For any X there is a natural inclusion

$$A^{p,q}(X)\subset D^{p,q}(X).$$

In fact, to each form $\omega\in A^{p,q}(X)$ we can associate a continuous functional on $A^{n-p,n-q}(X)$ by the formula:

$$\langle\omega,\alpha\rangle=\omega(\alpha):=\int_X\omega\wedge\alpha$$

for any $\alpha\in A^{n-p,n-q}(X)$.

The differentials d, ∂ and $\overline{\partial}$ act on the spaces $D^m(X)$ and $D^{p,q}(X)$ in a natural way by extending the action on the subspaces $A^m(X)$ and $A^{p,q}(X)$, respectively. From Stokes' theorem we have

$$\langle \omega, d\alpha \rangle = \int_X \omega \wedge d\alpha = (-1)^{m-1} \int_X d\omega \wedge \alpha$$

for any $\alpha \in A^{2n-m}(X)$ and $\omega \in A^m(X)$. So we define $d\omega$ for any $\omega \in D^m(X)$ as follows:

$$\langle d\omega, \alpha \rangle := (-1)^{m-1} \langle \omega, d\alpha \rangle$$

for any $\alpha \in A^{2n-m-1}$. In the same way we start with the actions of ∂ and $\overline{\partial}$ on the everywhere-dense subspaces

$$A^{p,q}(X) \subset D^{p,q}(X),$$

and then we define $\partial\omega$ and $\overline{\partial}\omega$, respectively, for any $\omega \in D^{p-1,q}(X)$ or $\omega \in D^{p,q-1}(X)$

$$\langle \partial\omega, \alpha \rangle := (-1)^{p+q-1} \langle \omega, \partial\alpha \rangle$$

and

$$\left\langle \overline{\partial}\omega, \alpha \right\rangle := (-1)^{p+q-1} \left\langle \omega, \overline{\partial}\alpha \right\rangle$$

for any $\alpha \in A^{n-p,n-q}$. A current $\omega \in D^{p,q}(X)$ is $\overline{\partial}$-closed if and only if for any $\overline{\partial}$ exact form $\overline{\partial}\alpha \in A^{n-p,n-q}(X)$ one has

$$\left\langle \omega, \overline{\partial}\alpha \right\rangle = 0.$$

Similar characterization is valid for ∂-closed currents.

Definition 3.2. **(i)** Let Y be a complex subvariety in a projective variety X. We define a current δ_Y corresponding to Y via the integration pairing

$$\langle \delta_Y, \omega \rangle = \int_Y \omega|_Y,$$

where ω is a smooth form on X. In particular, the Dirac kernel δ_Δ of the diagonal embedding of M is a current of type (n, n) on $M \times M$, $\dim_{\mathbb{C}} M = n$, such that δ_Δ is zero on $M \times M - \Delta$ and

$$\langle \delta_\Delta, \alpha \rangle = \int_{\Delta \subset \mathrm{M} \times \mathrm{M}} \alpha|_\Delta$$

for any smooth form α of type (n, n) on $M \times M$.

(ii) Let X and Y be as above, such that X is CY, $\dim_{\mathbb{C}} X = n, \dim_{\mathbb{C}} Y = n - m$. We define the Dirac antiholomorphic current $\overline{\theta}_Y$ of type $(0, m)$

on X corresponding to the holomorphic $n-m$ form θ_Y on the subvariety Y as follows. Let β any smooth form of type $(n, n-m)$ on X, then

$$\langle\overline{\theta}_Y, \beta\rangle = \int_Y \theta \wedge \frac{\beta}{\eta}.$$

The holomorphic Dirac kernel θ_Y is defined in a similar manner. In particular, the antiholomorphic Dirac kernel $\overline{\eta}_\Delta$ is a current of type $(0, n)$ on $M \times M$ such that

$$\langle\overline{\eta}_\Delta, \beta\rangle = \int_\Delta \frac{\beta}{\pi_1^*(\eta)} = \int_\Delta \frac{\beta}{\pi_2^*(\eta)} \tag{3.3}$$

for any C^∞ form β of type $(2n, n)$ on $M \times M$, where π_1 and π_2 are the projections of $M \times M$ on the first and the second factor, respectively. Since the current $\overline{\eta}_\Delta$ is supported on the diagonal, its definition does not depend on the choice of the projection.

For any p-dimensional complex subvariety $Y \subset X$, we define the Green current $g_Y \in D^{p-1,p-1}(X)$ such that

$$\partial\overline{\partial} g_Y + \delta_Y = \omega_Y, \tag{3.4}$$

where ω_Y is some closed $C^\infty(p,p)$ form representing the Poincare dual class of $[Y]$. The existence of the Green current with only logarithmic singularities is one of the key results in Arakelov geometry (see [28]).

We will use two basic facts from the theory of currents. We can define both Dolbeault's and De Rham cohomology using currents instead of C^∞ forms. The basic theorem, due to Whitney, asserts that on compact smooth manifolds De Rham or, in the complex category, Dolbeault's cohomology are isomorphic to De Rham or Dolbeault's cohomology obtained from currents. (See [13, Chapter 5].) Also, according to Hörmander [19] we can define the exterior product $\alpha \wedge \beta$ of two currents as another current, if the singular supports of the currents α and β are disjoint as sets.

Since currents represent De Rham and Dolbeault cohomology classes, owing to the theorem of Whitney, we will sometimes denote their pairing with homology classes by the integral, e.g.,

$$\int_{Y'} \delta_Y$$

where $Y' \subset X$ is another subvariety in X of complementary complex dimension to Y. In particular, that is the precise meaning of formula (2.8) for the Gauss linking number.

Proposition 3.3. *For a pair (Σ, θ), where Σ is a Riemannian surface in the three-dimensional CY manifolds M, and θ is a holomorphic form on Σ, the antiholomorphic Dirac current $\overline{\theta}_\Sigma$ is $\overline{\partial}$ exact current of type $(0, 2)$ and*

$$\overline{\theta}_\Sigma = \overline{\partial} A, \tag{3.5}$$

for some current A of type $(0, 1)$ with a singular support on Σ.

Proof. Since M is a CY manifold, any closed $(0, 1)$-form β is an exact form. Let $\beta = \overline{\partial} f \wedge \eta$ on M. Then Stokes' theorem implies

$$\left\langle \overline{\theta}_\Sigma, \overline{\partial} f \wedge \eta \right\rangle = \int_\Sigma \theta \wedge \overline{\partial} f = \int_\Sigma d\,(f\theta) = 0.$$

Therefore the antiholomorphic Dirac current $\overline{\theta}_\Sigma$ is d and $\overline{\partial}$-closed. From the theorem of Whitney, and since $H^2(M, \Omega^3) = 0$ for a CY manifold, we obtain that the antiholomorphic Dirac current $\overline{\theta}_\Sigma$ is a $\overline{\partial}$ exact current of type $(0, 2)$ on M. Thus (3.5) holds for some current A of type (0,1). It follows from Definition 3.2 of the antiholomorphic Dirac current $\overline{\theta}_\Sigma$ and (3.5) that the singular supports of the currents A and $\overline{\partial} A$ are on Σ. Proposition 3.3 is proved. □

3.3 Definition and properties of holomorphic linking

We will use some formal path integral computations to find explicit formula for the expression in Definition 3.1 in the complex case following the exposition in Section 2. We define $\mathcal{A}^{0,1}$ as the space of (0,1) forms on M and $\mathcal{G}^{0,1}$ as the space of $\overline{\partial}$-closed (0,1) forms on M. As in the real case, we choose a complement to $\mathcal{G}^{0,1}$, which we call $\mathcal{A}_0^{0,1}$. For any invariant functional F on $\mathcal{A}^{0,1}$ with respect to $\mathcal{G}^{0,1}$ we assume that

a. $$\int_{\mathcal{A}^{0,1}/\mathcal{G}^{0,1}} \mathcal{D}A \exp\left(\sqrt{-1} h \mathcal{L}(A)\right) F(A) = \int_{\mathcal{A}_0^{0,1}} \mathcal{D}A \exp\left(\sqrt{-1} h \mathcal{L}(A)\right) F(A).$$

and for $A_1 \in \mathcal{A}_0^{0,1}$

b. $$\begin{aligned} &\int_{\mathcal{A}_0^{0,1}} \mathcal{D}A \exp\left(\sqrt{-1} h \mathcal{L}(A + A_1)\right) F(A + A_1) \\ &= \int_{\mathcal{A}_0^{0,1}} \mathcal{D}A \exp\left(\sqrt{-1} h \mathcal{L}(A)\right) F(A). \end{aligned}$$

The conditions a and b imply that we have the following expression for formula (3.2):

$$\frac{\mathcal{Z}(M,(\Sigma_1,\theta_1),(\Sigma_2,\theta_2))\mathcal{Z}(M)}{\mathcal{Z}(\mathrm{M},(\Sigma_1,\theta_1))\mathcal{Z}(M,(\Sigma_2,\theta_2))}=\exp\left(\frac{\sqrt{-1}}{2h}\int_M A_1\wedge\overline{\partial}A_2\wedge\eta\right),$$

where A_1 and A_2 are forms in $\mathcal{A}_0^{0,1}$ defined in Proposition 3.3 corresponding to Σ_1 and Σ_2, respectively. Now we can give a rigorous definition of the holomorphic linking.

Definition 3.4. The holomorphic linking between two curves Σ_1 and Σ_2 with two holomorphic 1-forms θ_1 and θ_2 on them is defined by the formula

$$\#\left((\Sigma_1,\theta_1),(\Sigma_2,\theta_2)\right)=\int_{\mathrm{M}} A_1\wedge\overline{\partial}A_2\wedge\eta, \tag{3.6}$$

where A_1 and A_2 are currents in $\mathcal{A}_0^{0,1}$ defined by (3.5) in Proposition 3.3 for the pairs (Σ_1,θ_1) and (Σ_2,θ_2), respectively.

The integral in (3.6) makes sense since A_1 and $\overline{\partial}A_2$ have disjoint supports. This follows from the definitions of A_1 and A_2 and since $\Sigma_1\cap\Sigma_2=\emptyset$ as in the real case. Substituting the expression for A_1 and $\overline{\partial}A_2$ from Proposition 3.3 and the Definition 3.2 of antiholomorphic currents $\left(\overline{\theta}_i\right)_{\Sigma_i}$, we obtain the first explicit expression for the holomorphic linking

$$\begin{aligned}\#\left((\Sigma_1,\theta_1),(\Sigma_2,\theta_2)\right)&=\int_M \overline{\partial}^{-1}\left(\left(\overline{\theta}_1\right)_{\Sigma_1}\right)\wedge\left(\overline{\theta}_2\right)_{\Sigma_2}\wedge\eta\\&=\int_{\Sigma_2}\left(\overline{\partial}^{-1}\left(\left(\overline{\theta}_1\right)_{\Sigma_1}\right)\right)|_{\Sigma_2}\wedge\theta_2.\end{aligned}\tag{3.7}$$

It is easy to check the following properties.

Proposition 3.5. (i) *The holomorphic linking does not depend on the choice of the complement* $\mathcal{A}_0^{0,1}$.

(ii) *The holomorphic linking is symmetric:*

$$\#\left((\Sigma_1,\theta_1),(\Sigma_2,\theta_2)\right)=\#\left((\Sigma_2,\theta_2),(\Sigma_1,\theta_1)\right).$$

(iii) $\#\left((\Sigma_1,\theta_1),(\Sigma_2,\theta_2)\right)$ *is linear in* θ_1 *and* θ_2.

4 Green kernel and symmetric form of holomorphic linking

Now that we have a rigorous definition of the holomorphic linking, we might try to express it in a more invariant form following the analogy with the real case. In Section 2, we derived the Gauss kernel for the linking number using the Green kernel for the exterior derivative operator. In order to obtain a similar formula for the holomorphic linking, we first show the existence of the analogue of the Green kernel of the operator $\overline{\partial}$, which can be viewed as a complexification of the exterior derivative in the real case.

4.1 The cohomology groups of Zariski open varieties

Let $X_0 = \bigcup_{i=1}^{N} C_i$ be a divisor with normal crossings in a projectve variety X. The groups of the cohomology of the variety $X - X_0$ can be computed as the cohomology groups of the de Rham log complex $\mathcal{A}^*(X, \log \langle X_0 \rangle)$.

Definition 4.1. (i) We will say that a form ω on one of the components C_{i_0} of X_0 has a logarithmic singularities if for each point

$$z \in C_{i_0} \cap \cdots \cap C_{i_k}$$

and some open neighbourhood $U \subset X$ of the point z we have

$$\omega|_U = \alpha \wedge \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_k}}{z_{i_k}}, \tag{4.1}$$

where α is a C^{∞} form in U and $z_{i_0} \times \cdots \times z_{i_k}\omega$ and $z_{i_0} \times \cdots \times z_{i_k} d\omega$ are smooth forms in U. $X_0 \cap U$ is defined by the equations

$$z_{i_1} \times \cdots \times z_{i_k} = 0$$

in U.

(ii) We define the de Rham log complex with the standard differential as follows:

$$\mathcal{A}^*(X_0, \log \langle X_0 \rangle) = \{\omega \in C^{\infty}(X - X_0, \Omega^*) \,|\omega \textit{ and } d\omega \textit{ are } C^{\infty} \textit{forms}$$
$$\textit{on } X - X_0 \textit{which have} \log \textit{singularities along} X_0\}. \tag{4.2}$$

By using the Poincare residue map Deligne proved the following.

Theorem 4.2. *The cohomology of $X - X_0$ is equal to the cohomology of the De Rham log complex $\mathcal{A}^*(X_0, \log \langle X_0 \rangle)$.*

For the proof of Theorem 4.2 see [14].

4.2 Basic definitions and the existence theorem

Let M be a CY of complex dimension n. We will normalize the holomorphic n-form η as follows:

$$(-1)^{(n(n-1)/2)} \left(\frac{\sqrt{-1}}{2}\right)^n \int_M \eta \wedge \overline{\eta} = 1. \tag{4.3}$$

According to the Theorem of Gillet and Soule stated in Section 3.2, for each closed form of type (n, n) that represents the Poincare dual of the homology class of the diagonal $\Delta \subset M \times M$ in $H_{2n}(M \times M, \mathbb{Z})$ there exists a Green current g_Δ of type $(n-1, n-1)$ with a logarithmic growth along the diagonal Δ in $M \times M$ such that

$$\partial\overline{\partial} g_\Delta + \delta_\Delta = \omega_\Delta, \tag{4.4}$$

where δ_Δ is a Dirac current of Δ and ω_Δ is a C^∞ form on $M \times M$ which represents the same cohomology class as the current δ_Δ. From the fact that the singular support of the Dirac kernel is Δ and equation (4.4), we can conclude that the restriction of the current g_Δ on $M \times M - \Delta$ is represented by a smooth C^∞ closed form of type $(n-1, n-1)$.

Proposition 4.3. *Let $\mathcal{U}$ be an affine open set in M. Then we have the following presentation for the restriction of ω_Δ on $\mathcal{U} \times \mathcal{U}$:*

$$\omega_\Delta|_{\mathcal{U}\times\mathcal{U}} = \overline{\partial}\psi_\Delta, \tag{4.5}$$

where ψ_Δ is a smooth form of type $(n, n-1)$ on $\mathcal{U} \times \mathcal{U}$. Moreover the form ψ_Δ has singularities of the type described by (4.1) *along the divisor, which is the complement of $\mathcal{U} \times \mathcal{U}$ in $M \times M$.*

Proof. It is a standard fact that for any coherent sheaf $\mathfrak{F}$ on the affine set $\mathcal{U} \times \mathcal{U}$, we have $H^k(\mathcal{U} \times \mathcal{U}, \mathfrak{F}) = 0$ for $k > 0$. Since ω_Δ is a closed form of type (n, n) representing the Poincare dual class of the diagonal Δ in $H^n(M \times M, \mathbb{Z})$, its restriction on $\mathcal{U} \times \mathcal{U}$ is a non-zero element of $H^n(\mathcal{U} \times \mathcal{U}, \mathbb{Z})$. On the other hand, ω_Δ is a closed form of type (n, n) and thus it is $\overline{\partial}$-closed. Thus its restriction on $\mathcal{U} \times \mathcal{U}$ is an element of $H^n(\mathcal{U} \times \mathcal{U}, \Omega^n)$. Since Ω^n is a coherent sheaf the restriction of ω_Δ on $\mathcal{U} \times \mathcal{U}$ is a $\overline{\partial}$ exact smooth form of type (n, n). Thus $\omega_\Delta|_{\mathcal{U}\times\mathcal{U}} = \overline{\partial}\psi_\Delta$, where ψ_Δ is a smooth form of type $(n, n-1)$ on $\mathcal{U} \times \mathcal{U}$. Since ω_Δ is a smooth form on $M \times M$ then according to Theorem 4.2 the restriction of ω_Δ on $\mathcal{U} \times \mathcal{U}$ can be represented as a cohomology class of the de Rham log complex (4.2). Thus we can choose ψ_Δ to have singularities of the type described by (4.1) along the divisor which is the complement of $\mathcal{U} \times \mathcal{U}$ in $M \times M$. Proposition 4.3 is proved. □

According to Proposition 4.3, the C^∞ form $((\psi_\Delta - \partial g_\Delta)/(\pi_1^*(\eta)))$ is of type $(0, n-1)$. It is non-zero and it is well defined on $\mathcal{U} \times \mathcal{U}$. We will show that the integration of any smooth $(2n, n+1)$ form β on $M \times M$ with this form defines a current on $M \times M$, which we denote by $\overline{\partial}^{-1}(\overline{\eta}_\Delta)$, namely

$$\left\langle \overline{\partial}^{-1}(\overline{\eta}_\Delta), \beta \right\rangle = \int_{\mathcal{U}\times\mathcal{U}-\Delta} \left(\frac{\psi_\Delta - \partial g_\Delta}{\pi_1^*(\eta)} \right) \wedge \beta. \tag{4.6}$$

Theorem 4.4. *The current $\overline{\partial}^{-1}(\overline{\eta}_\Delta)$ is well defined and it does not depend on the choice of ψ_Δ and $\mathcal{U}$.*

Proof. Let β be any $(2n, n+1)$ smooth form on $M \times M$. First we will show that the integral (4.6) does not depend on the choice of ψ_Δ when we fix the affine open set $\mathcal{U}$. To prove that the current $\overline{\partial}^{-1}(\overline{\eta}_\Delta)$ is well defined, it is enough to show that

$$\int_{\mathcal{U}\times\mathcal{U}-\Delta}\left(\frac{\psi_\Delta - \partial g_\Delta}{\pi_1^*(\eta)}\right)\wedge\beta \tag{4.7}$$

converges. So we need to prove that the integral (4.7) converges. In fact the integral

$$\int_{M\times M-\Delta}\frac{\partial g_\Delta}{\pi_1^*(\eta)}\wedge\beta \tag{4.8}$$

converges since the current g_Δ has a logarithmic growth along Δ. See [28]. The integral

$$\int_{M\times M-\Delta}\frac{\psi_\Delta}{\pi_1^*(\eta)}\wedge\beta$$

converges since according to Proposition 4.3, ψ_Δ is a smooth form on $\mathcal{U}\times\mathcal{U}$ and it has singularities along the complement of $\mathcal{U}\times\mathcal{U}$ in $M\times M$ described by (4.1). Thus we proved that $\overline{\partial}^{-1}(\overline{\eta}_\Delta)$ defines a current on $M \times M$.

Next we will verify that the current $\overline{\partial}^{-1}(\overline{\eta}_\Delta)$ does not depend on the choice of ψ_Δ. Indeed if we choose ψ'_Δ in $\mathcal{U}\times\mathcal{U}$ with logarithmic singularities along the complement of $\mathcal{U}\times\mathcal{U}$ in $M\times M$ such that

$$\overline{\partial}\psi'_\Delta = \overline{\partial}\psi_\Delta = \omega|_{\mathcal{U}\times\mathcal{U}},$$

then

$$\overline{\partial}\left(\psi'_\Delta - \psi_\Delta\right) = 0.$$

Thus the class of cohomology of $(\psi'_\Delta - \psi_\Delta)$ on $\mathcal{U}\times\mathcal{U}$ is zero. In particular, the singularities will cancel in $(\psi'_\Delta - \psi_\Delta)$. Thus $(\psi'_\Delta - \psi_\Delta) = \overline{\partial}\phi$, where ϕ is a smooth form defined on $M\times M$. So

$$\int_{\mathcal{U}\times\mathcal{U}}\left(\frac{\psi_\Delta}{\pi_1^*(\eta)} - \frac{\psi'_\Delta}{\pi_1^*(\eta)}\right)\wedge\beta = \int_{M\times M} d\left(\frac{\phi}{\pi_1^*(\eta)}\wedge\beta\right) = 0. \tag{4.9}$$

Suppose that $\mathcal{U}_1$ is another affine open set in M. Since $\mathcal{U}_1\cap\mathcal{U}$ is an affine set and the complement of $\mathcal{U}_1\cap\mathcal{U}$ in both of the affine open sets has measure zero, repeating the arguments that we used to prove (4.9) will show that

$$\int_{\mathcal{U}\times\mathcal{U}-\Delta}\left(\frac{\psi_\Delta - \partial g_\Delta}{\pi_1^*(\eta)}\right)\wedge\beta = \int_{\mathcal{U}_1\times\mathcal{U}_1-\Delta}\left(\frac{\psi_{1,\Delta} - \partial g_\Delta}{\pi_1^*(\eta)}\right)\wedge\beta.$$

Theorem 4.4 is proved. □

Corollary 4.5. *The form $(\psi_\Delta - \partial g_\Delta)/(\pi_1^*(\eta))$ is a $\overline{\partial}$-closed form of type $(0, n-1)$ on $M \times M - \Delta$ and it defines a class of cohomology in $H^{n-1}_{\overline{\partial}}(M \times M - \Delta, \mathcal{O}_{M\times M-\Delta})$.*

Remark 4.6. We would like to note that since the cohomology class of $\overline{\eta}_\Delta$ is non-zero, one cannot define $\overline{\partial}^{-1}(\overline{\eta}_\Delta)$ in a straightforward way. We hope that our notation would not lead to a possible confusion.

We will call the current $\overline{\partial}^{-1}(\overline{\eta}_\Delta)$ the Green kernel of the operator $\overline{\partial}$, and we will often use the same notation for defining C^∞ form in (4.6).

4.3 Symmetric form of holomorphic linking and complex linking number

From now on we will suppose that Σ_1 and Σ_2 are Riemann surfaces embedded in a CY 3-fold and $\Sigma_1 \cap \Sigma_2 = \emptyset$.

Theorem 4.7. *Let us consider the embedding $\Sigma_1 \times \Sigma_2 \subset M \times M$. Then the following formula holds:*

$$\#\left((\Sigma_1, \theta_1), (\Sigma_2, \theta_2)\right) = \int_{\Sigma_1\times\Sigma_2} \left(\overline{\partial}^{-1}(\overline{\eta}_\Delta)\right)|_{\Sigma_1\times\Sigma_2} \wedge \pi_1^*(\theta_1) \wedge \pi_2^*(\theta_2). \tag{4.10}$$

Proof. We can rewrite the holomorphic linking in the following form:

$$\begin{aligned}\#\left((\Sigma_1, \theta_1), (\Sigma_2, \theta_2)\right) &= \int_M \overline{\partial}^{-1}\left((\overline{\theta}_1)_{\Sigma_1}\right) \wedge (\overline{\theta}_2)_{\Sigma_2} \wedge \eta \\ &= \int_{M\times M} \delta_\Delta \wedge \pi_1^*\left(\overline{\partial}^{-1}\left((\overline{\theta}_1)_{\Sigma_1}\right)\right) \wedge \pi_2^*\left((\overline{\theta}_2)_{\Sigma_2} \wedge \eta\right).\end{aligned} \tag{4.11}$$

Substituting in (4.11) the expression for the current δ_Δ stated in (4.4),

$$\delta_\Delta = \pi_1^*(\eta) \wedge \left(\frac{\omega_\Delta - \overline{\partial}\partial g_\Delta}{\pi_1^*(\eta)}\right),$$

where

$$\omega_\Delta|_{M\times M-\Delta} = \overline{\partial}\psi_\Delta$$

and the ω_Δ is a form representing the Poincare dual class of Δ, we obtain that

$$
\begin{aligned}
\#\left(\left(\Sigma_1,\theta_1\right),\left(\Sigma_2,\theta_2\right)\right) = \int_{M\times M} &\left(\left(\frac{\omega_\Delta-\overline{\partial}\partial g_\Delta}{\pi_1^*(\eta)}\right)\right)\wedge \\
&\left(\overline{\partial}^{-1}\left(\pi_1^*\left(\left(\overline{\theta}_1\right)_{\Sigma_1}\wedge\eta\right)\right)\right)\wedge\pi_2^*\left(\left(\overline{\theta}_2\right)_{\Sigma_2}\wedge\eta\right) \\
= \int_{M\times M-\Delta} &\left(\overline{\partial}\left(\frac{\psi_\Delta}{\pi_1^*(\eta)}\right)\right)\wedge \\
&\left(\overline{\partial}^{-1}\left(\pi_1^*\left(\left(\overline{\theta}_1\right)_{\Sigma_1}\wedge\eta\right)\right)\right)\wedge\pi_2^*\left(\left(\overline{\theta}_2\right)_{\Sigma_2}\right) \\
-\int_{M\times M} &\left(\overline{\partial}\left(\frac{\partial g_\Delta}{\pi_1^*(\eta)}\right)\right)\wedge \\
&\left(\overline{\partial}^{-1}\left(\pi_1^*\left(\left(\overline{\theta}_1\right)_{\Sigma_1}\wedge\eta\right)\right)\right)\wedge\pi_2^*\left(\left(\overline{\theta}_2\right)_{\Sigma_2}\right). \qquad (4.12)
\end{aligned}
$$

From Stokes' Theorem, we obtain

$$
\begin{aligned}
\#\left(\left(\Sigma_1,\theta_1\right),\left(\Sigma_2,\theta_2\right)\right) = \int_{M\times M-\Delta} &\frac{\psi_\Delta}{\pi_1^*(\eta)}\wedge\left(\pi_1^*\left(\left(\overline{\theta}_1\right)_{\Sigma_1}\wedge\eta\right)\right)\wedge\pi_2^* \\
&\left(\left(\overline{\theta}_2\right)_{\Sigma_2}\wedge\eta\right) \\
-\int_{M\times M-\Delta} &\frac{\partial g_\Delta}{\pi_1^*(\eta)}\wedge\pi_1^*\left(\left(\overline{\theta}_1\right)_{\Sigma_1}\wedge\eta\right)\wedge\pi_2^* \\
&\left(\left(\overline{\theta}_2\right)_{\Sigma_2}\wedge\eta\right) \\
= \int_{M\times M-\Delta} &\left(\frac{\psi_\Delta-\partial g_\Delta}{\pi_1^*(\eta)}\right)\wedge\pi_1^*\left(\left(\overline{\theta}_1\right)_{\Sigma_1}\wedge\eta\right)\wedge\pi_2^* \\
&\left(\left(\overline{\theta}_2\right)_{\Sigma_2}\wedge\eta\right). \qquad (4.13)
\end{aligned}
$$

The antiholomorphic Dirac currents $(\overline{\theta}_1)_{\Sigma_1}$, $(\overline{\theta}_2)_{\Sigma_2}$ and $\overline{\eta}_\Delta$ have disjoint singularities sets. This fact guarantees that the current $(\pi_1^*((\overline{\theta}_1)_{\Sigma_1}))\wedge\pi_2^*$ $((\overline{\theta}_2)_{\Sigma_2})$ is well defined. Thus the convergence of all integrals is established.

Finally using formula (4.6) for $\overline{\partial}^{-1}(\overline{\eta}_\Delta)$ and the definition of the antiholomorphic Dirac currents $\pi_1^*((\overline{\theta}_1)_{\Sigma_1})$ and $\pi_2^*((\overline{\theta}_2)_{\Sigma_2})$, we deduce from (4.13) that

$$\begin{aligned}\#\left((\Sigma_1,\theta_1),(\Sigma_2,\theta_2)\right) &= \int_{M\times M-\Delta}\left(\overline{\partial}^{-1}(\overline{\eta}_\Delta)\right)\wedge\left(\pi_1^*\left((\overline{\theta}_1)_{\Sigma_1}\wedge\eta\right)\right)\wedge\pi_2^*\\ &\quad\left((\overline{\theta}_2)_{\Sigma_2}\wedge\eta\right)\\ &= \int_{\Sigma_1\times\Sigma_2}\left(\overline{\partial}^{-1}(\overline{\eta}_\Delta)\right)|_{\Sigma_1\times\Sigma_2}\wedge\pi_1^*(\theta_1)\wedge\pi_2^*(\theta_2).\end{aligned} \tag{4.14}$$

Thus we established formula (4.10). Theorem 4.7 is proved. □

Corollary 4.8. *The holomorphic linking does not depend on the choice of the representative of the current* $\overline{\partial}^{-1}(\overline{\eta}_\Delta)$.

Proof. The corollary follows directly from Stokes' Theorem. □

The expression of the holomorphic linking given by Definition 3.4 and Theorem 4.7 can be viewed as complex counterparts of the corresponding expressions for the Gauss linking number reviewed in Section 2. The essential difference between the two notions is that in the real case we obtain a topological invariant while in the complex case our linking map depends on the way the Riemannian surfaces are embedded in the CY manifold M and the choice of holomorphic forms on them but does not depend on the choice of the metric on CY manifold M.

The comparison of our holomorphic linking and the Gauss linking number suggests an alternative definition.

Definition 4.9. The complex linking number of two Riemann surfaces Σ_1 and Σ_2 embedded in a CY 3-fold such that $\Sigma_1\cap\Sigma_2=\emptyset$ is defined by the formula

$$\#(\Sigma_1,\Sigma_2)=\int_{\Sigma_1\times\Sigma_2}\left(\overline{\partial}^{-1}(\overline{\eta}_\Delta)\right)\wedge\left(\partial^{-1}(\eta_\Delta)\right),$$

where η is normalized by condition (4.3).

Simple examples show that the complex linking number contains less information than the holomorphic linking. It is an interesting question to find a relation between the two invariants.

5 Analytic expression of holomorphic linking

In this section we will give an explicit integral formula for the holomorphic linking. It can be viewed as a complex counterpart of the Gauss formula for the linking number. The key ingredient of our formula is an expression for the Green kernel $\overline{\partial}^{-1}(\overline{\eta}_\Delta)$ in terms of the Bochner–Martinelli form of type $(n, n-1)$. The letter form written in affine coordinates is precisely the classical Bochner–Martinelli form that yields a generalization of the Cauchy integral formula. It turns out that it finds another application in the integral formula for the holomorphic linking.

5.1 Bochner–Martinelli kernel

Let $\mathcal{U}$ be an affine open set in Zariski topology of the CY manifold M. Let $z^1, \ldots, z^n$ be local coordinates in $\mathcal{U}$ such that restriction of the holomorphic n-form η on $\mathcal{U}$ is expressed as

$$\eta\,|_{\mathcal{U}} = dz^1 \wedge \cdots \wedge dz^n.$$

Let

$$\Phi_j(z) := (-1)^{j-1} z^j dz^1 \wedge \cdots \wedge dz^{j-1} \wedge dz^{j+1} \wedge \cdots \wedge dz^n.$$

Following [13], Chapter 3 we will define the Bochner–Martinelli kernel on

$$\mathcal{U} \times \mathcal{U} - \Delta_{\mathcal{U}}, \quad \Delta_{\mathcal{U}} := \mathcal{U} \times \mathcal{U} \cap \Delta,$$

by the formula

$$\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}} = \frac{C_n \overline{\left(\sum_{r=1}^n \Phi_r(w-z)\right)} \wedge \left(dz^1 \wedge \cdots \wedge dz^n\right)}{\|w-z\|^{2n}}, \tag{5.1}$$

C_n is the volume of the unit $2n-1$ dimensional sphere, $\{w^k\}$ and $\{z^k\}$ are local coordinates in $\mathcal{U} \times \mathcal{U}$ such that

$$\pi_1^*(\eta)|_{\mathcal{U}\times\mathcal{U}} = dw^1 \wedge \cdots \wedge dw^n, \quad \pi_2^*(\eta)|_{\mathcal{U}\times\mathcal{U}} = dz^1 \wedge \cdots \wedge dz^n, \tag{5.2}$$

and

$$\|w-z\|^{2n} = \left(\sum_{k=1}^n \left|w^k - z^k\right|^2\right)^n.$$

It is proved in Chapter 3 of [13] that on $\mathcal{U} \times \mathcal{U} - \Delta_{\mathcal{U}}$ the forms $\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}}$ of type $(n, n-1)$ given by the expression (5.1) are d and so $\overline{\partial}$-closed, i.e.,

$$\overline{\partial}\left(\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}}\right)|_{\mathcal{U}\times\mathcal{U}-\Delta_{\mathcal{U}}} = d\left(\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}}\right)|_{\mathcal{U}\times\mathcal{U}-\Delta_{\mathcal{U}}} = 0. \tag{5.3}$$

Let $T_\varepsilon(\Delta)$ be a tubular neighbourhood of the diagonal Δ in $M \times M$. In [13], Chapter 5 it is proved that the limit

$$\lim_{\varepsilon \longrightarrow 0} \int_{\partial(T_\varepsilon(\Delta)) \cap \mathcal{U} \times \mathcal{U}} \mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}} \wedge \omega$$

exists. The form $\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}}$ defines a current on $M \times M$ as follows:

$$\left\langle \mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}}, \beta \right\rangle := \lim_{\varepsilon \longrightarrow 0} \int_{\mathcal{U}\times\mathcal{U}-(T_\varepsilon(\Delta))\cap\mathcal{U}\times\mathcal{U})} \mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}} \wedge \beta, \tag{5.4}$$

where β is any smooth form of type $(n, n+1)$ on $M \times M$. Based on Stokes' theorem and (5.4) we can compute the current

$$\overline{\partial}\left(\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}}\right) = d\left(\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}}\right)$$

on $M \times M$ as follows:

$$\left\langle \overline{\partial}\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}}, \omega \right\rangle = \left\langle d\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}}, \omega \right\rangle = \lim_{\varepsilon \longrightarrow 0} \int_{\partial(T_\varepsilon(\Delta))\cap\mathcal{U}\times\mathcal{U}} \mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}} \wedge \omega \tag{5.5}$$

for any smooth form ω of type (n, n) on $M \times M$.

Theorem 5.1. *Let M be a* CY *manifold. Then we have the following equality of currents*:

$$\overline{\partial}(\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}}) = \delta_\Delta|_{\mathcal{U}\times\mathcal{U}} \tag{5.6}$$

for any affine open set $\mathcal{U}$ in M.

Proof. In order to prove Theorem 5.1 we need to prove that for any smooth form ω of type (n, n) on M we have

$$\lim_{\varepsilon \longrightarrow 0} \int_{\partial(T_\varepsilon(\Delta))\cap\mathcal{U}\times\mathcal{U}} \mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}} \wedge \omega = \int_{\mathcal{U}_\Delta} \omega = \int_\Delta \omega. \tag{5.7}$$

Since $d(\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}}) = 0$ on $\mathcal{U} \times \mathcal{U} - (\mathcal{U}_\Delta)$, we have the following equation of currents:

$$\begin{aligned} d\left(\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}} \wedge \omega\right) &= d\left(\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}}\right) \wedge \omega + (-1)^{2n-1}\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}} \wedge \overline{\partial}\omega \\ &= -\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}} \wedge \overline{\partial}\omega. \end{aligned} \tag{5.8}$$

Let us denote $(\mathcal{U} \times \mathcal{U}) \cap T_\varepsilon(\Delta) = T_\epsilon(\Delta_\mathcal{U})$. Formula (5.8) and Stokes' Theorem imply

$$\begin{aligned} \int_{\mathcal{U}\times\mathcal{U}-T_\epsilon(\Delta_\mathcal{U})} d\left(\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}} \wedge \omega\right) &= -\int_{\mathcal{U}\times\mathcal{U}-T_\epsilon(\Delta_\mathcal{U})} \mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}} \wedge \overline{\partial}\omega \\ &= -\int_{\partial(T_\epsilon(\Delta_\mathcal{U}))} \mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}} \wedge \omega. \end{aligned} \tag{5.9}$$

We notice that $\partial(T_\varepsilon(\Delta) \cap \mathcal{U} \times \mathcal{U}$ is a fibration of $2n-1$ spheres over $\mathcal{U}_\Delta$. Since ω is a form of type (n, n), when we restrict it to the diagonal we will get

$$\omega|_\Delta = \phi(w)\,(\eta \wedge \overline{\eta})$$

for some C^∞ function $\phi(w)$. Now we may apply the Fubini theorem and rewrite the integral in (5.9) as follows:

$$\int_{\partial(T_\varepsilon(\Delta))\cap\mathcal{U}\times\mathcal{U}} \mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}} \wedge \omega = \int_{\mathcal{U}_\Delta} \left(\int_{\|z-w\|=\varepsilon,\omega\in\Delta} \phi(z)\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}} \right) \eta \wedge \overline{\eta}. \quad (5.10)$$

The expression in (5.10) makes sense since in [13, Chapter 3, paragraph 1], it is proved that if in the expression of the Bochner–Martinelli kernel given by (5.1) we substitute $z = w_0$ then

$$\int_{||z-w_0||=\varepsilon} \phi(z)\mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}} = \phi(w_0) \quad (5.11)$$

for any continuous function ϕ. Using the representation of the restriction of ω on Δ together with (5.10) and (5.11) we get formula (5.7). Theorem 5.1 is proved. □

We define

$$\mathcal{K}^{0,n-1}_{\mathcal{U}\times\mathcal{U}} = \frac{C_n\overline{(\sum_{r=1}^n \Phi_r(w-z)}}{\|w-z\|^{2n}} \quad (5.12)$$

and we call $\mathcal{K}^{0,n-1}_{\mathcal{U}\times\mathcal{U}}$ the holomorphic Bochner–Martinelli kernel. We have the following formula:

$$\mathcal{K}^{0,n-1}_{\mathcal{U}\times\mathcal{U}} \wedge \pi_1^*(\eta) = \mathcal{K}^{n,n-1}_{\mathcal{U}\times\mathcal{U}}. \quad (5.13)$$

The form $\mathcal{K}^{0,n-1}_{\mathcal{U}\times\mathcal{U}}$ defines a current of type $(0, n-1)$ on $M \times M$ as follows:

$$\left\langle \mathcal{K}^{0,n-1}_{\mathcal{U}\times\mathcal{U}}, \beta \right\rangle := \lim_{\varepsilon\longrightarrow 0} \int_{\mathcal{U}\times\mathcal{U}-(T_\varepsilon(\Delta)\cap\mathcal{U}\times\mathcal{U})} \mathcal{K}^{0,n-1}_{\mathcal{U}\times\mathcal{U}} \wedge \beta, \quad (5.14)$$

where β is any smooth form of type $(2n, n+1)$ on $M \times M$.

By using formulas (5.13), (5.14) and by repeating the arguments of the proof of Theorem 5.1 we obtain

Theorem 5.2. *Let M be a* CY *manifold. Then we have the following equality of currents:*

$$\overline{\partial}(\mathcal{K}^{0,n-1}_{\mathcal{U}\times\mathcal{U}}) = \overline{\eta}_\Delta|_{\mathcal{U}\times\mathcal{U}}, \quad (5.15)$$

where $\overline{\eta}_\Delta$ is the antiholomorphic Dirac kernel defined in Definition 3.2.

Remark 5.3. Instead of equation (3.4) we could consider the holomorphic analogue of (2.3):

$$\overline{\partial}\mathrm{S} + \overline{\eta}_\Delta = \omega_\Delta, \quad (5.16)$$

where ω_Δ is a smooth form on $M \times M$ which realizes the class of the current $\overline{\eta}_\Delta$. Then the holomorphic Bochner–Martinelli kernel represents a solution S of (5.16) on $M \times M - \Delta$.

5.2 The holomorphic analogue of the Gauss formula

Now we are ready to re-express the formula for the holomorphic linking established in Theorem 4.7 for the complex linking number introduced in Definition 4.9 using the Bochner–Martinelli kernel. This gives us the complex analogue of the Gauss integral formula for linking number.

Let Σ_1 and Σ_2 be two Riemann surfaces embedded in a CY 3-fold M such that

$$\Sigma_1 \cap \Sigma_2 = \emptyset.$$

Let $\mathcal{U}$ be an affine open set in M; then $\mathcal{U} \cap \Sigma_i$ are affine open sets in Σ_i, i.e., $\mathcal{U} \cap \Sigma_i$ are the Riemann surfaces Σ_i minus finite number of points. Suppose that θ_1 and θ_2 are two non-zero holomorphic forms on Σ_1 and Σ_2. We may also assume that

$$\theta_1 \,|_{\mathcal{U}\cap\Sigma_1} = f_1(w)dw \quad \text{and} \quad \theta_2 \,|_{\mathcal{U}\cap\Sigma_2} = f_2(z)dz. \tag{5.17}$$

Next we will derive an explicit integral form of the holomorphic linking.

Theorem 5.4. *The following formula holds:*

$$\#((\Sigma_1,\theta_1),(\Sigma_2,\theta_2)) = C_3 \int\limits_{(\mathcal{U}\cap\Sigma_1)\times(\mathcal{U}\cap\Sigma_2)} \mathcal{K}^{0,2}\left(f_1(w)dw\right)\wedge\left(f_2(z)dz\right) =$$

$$C_3 \int_{(\mathcal{U}\cap\Sigma_1)\times(\mathcal{U}\cap\Sigma_2)} \frac{\left(\overline{\sum_{i=1}^{3}\Phi_i(z-w)}\right)\wedge\left(f_1(w)dw\right)\wedge\left(f_2(z)dz\right)}{\left(\sum_{j=1}^{3}|z^j-w^j|^2\right)^3}, \tag{5.18}$$

where $\mathcal{U}$ is any open affine subset in M, C_3 is the volume of the unit sphere in $\mathbb{C}^3$ and the coordinates in $\mathcal{U}\times\mathcal{U}$ are chosen as in (5.2).

Proof. According to formula (4.13) we have

$$\#\left((\Sigma_1,\theta_1),(\Sigma_2,\theta_2)\right) = \int_{M\times M-\Delta}\left(\frac{\psi_\Delta-\partial g_\Delta}{\pi_1^*(\eta)}\right)\wedge\pi_1^*\left(\left(\overline{\theta}_1\right)_{\Sigma_1}\wedge\eta\right)\wedge\pi_2^*\left(\left(\overline{\theta}_2\right)_{\Sigma_2}\wedge\eta\right), \tag{5.19}$$

where ψ_Δ and g_Δ are defined as in Section 4.1. Since the complement to the affine set $\mathcal{U}$ in M has measure zero, we can rewrite formula (5.19) as follows:

$$\#\left(\left(\Sigma_1,\theta_1\right),\left(\Sigma_2,\theta_2\right)\right)=\int_{\mathcal{U}\times\mathcal{U}-\Delta_{\mathcal{U}}}\left(\frac{\psi_\Delta-\partial g_\Delta}{\pi_1^*(\eta)}\right)\wedge\pi_1^*\left(\left(\overline{\theta}_1\right)_{\Sigma_1}\wedge\eta\right)\wedge\pi_2^*\left(\left(\overline{\theta}_2\right)_{\Sigma_2}\wedge\eta\right). \tag{5.20}$$

Comparing equality (5.20) with equality (5.6) of Theorem 5.1 we can substitute the current $\psi_\Delta-\partial g_\Delta$ with the Bochner–Martinelli kernel $\mathcal{K}^{3,2}_{\mathcal{U}\times\mathcal{U}}$ since their derivatives restricted on $\mathcal{U}\times\mathcal{U}$ give the Dirac current δ_Δ of the diagonal restricted on $\mathcal{U}\times\mathcal{U}$ and we get

$$\begin{aligned}\#\left(\left(\Sigma_1,\theta_1\right),\left(\Sigma_2,\theta_2\right)\right)&=\int_{\mathcal{U}\times\mathcal{U}-\Delta_{\mathcal{U}}}\left(\frac{\mathcal{K}^{3,2}_{\mathcal{U}\times\mathcal{U}}}{\pi_1^*(\eta)}\right)\wedge\pi_1^*\left(\left(\overline{\theta}_1\right)_{\Sigma_1}\wedge\eta\right)\wedge\pi_2^*\\&\quad\left(\left(\overline{\theta}_2\right)_{\Sigma_2}\wedge\eta\right)\\&=\int_{\mathcal{U}\times\mathcal{U}-\Delta_{\mathcal{U}}}\mathcal{K}^{0,2}_{\mathcal{U}\times\mathcal{U}}\wedge\pi_1^*\left(\left(\overline{\theta}_1\right)_{\Sigma_1}\wedge\eta\right)\wedge\pi_2^*\\&\quad\left(\left(\overline{\theta}_2\right)_{\Sigma_2}\wedge\eta\right)\\&=\int_{(\mathcal{U}\cap\Sigma_1)\times(\mathcal{U}\cap\Sigma_2)}\mathcal{K}^{0,2}_{\mathcal{U}\times\mathcal{U}}\wedge\theta_1\wedge\theta_2.\end{aligned} \tag{5.21}$$

Substitituing the expression for the holomorphic Bochner–Martinelli kernel (5.12) and the local expressions for θ_1 and θ_2 in (5.17), we obtain

$$\#\left(\left(\Sigma_1,\theta_1\right),\left(\Sigma_2,\theta_2\right)\right)=C_3\int_{(\mathcal{U}\cap\Sigma_1)\times(\mathcal{U}\cap\Sigma_2)}\frac{\left(\overline{\sum_{i=1}^3\Phi_i(z-w)}\right)\wedge(f_1(w)dw)\wedge(f_2(z)dz)}{\left(\sum_{j=1}^3|z^j-w^j|^2\right)^3}.$$

Theorem 5.4 is proved. □

In the same way we prove

Corollary 5.5. *The following formula holds:*

$$\#(\Sigma_1,\Sigma_2)=C_3\int_{(\mathcal{U}\cap\Sigma_1)\times(\mathcal{U}\cap\Sigma_2)}\frac{\left(\sum_{j=1}^3\overline{\Phi_j(z-w)}\right)\wedge\left(\sum_{j=1}^3\Phi_j(z-w)\right)}{\left(\sum_{j=1}^3|z^j-w^j|^2\right)^3}.$$

The formulas of Theorem 5.4 and Corollary 5.5 suggest that we can define holomorphic linking for non-compact CY manifolds. In the case of $\mathbb{C}^3$, we can choose the holomorphic form η to be $dz^1\wedge dz^2\wedge dz^3$. In this case, the formula for the holomorphic linking becomes the complex version of the Gauss formula in $\mathbb{R}^3$ as it appears in Section 1.

6 Geometric interpretation of holomorphic linking

In this section, we will give a geometric interpretation of the holomorphic linking, which is a direct complex generalization of the usual topological definition of the Gauss linking number of two knots in $\mathbb{R}^3$. We will derive the geometric formula using the integral form of the holomorphic linking via the Green kernel. The derivation is parallel to the real case, but it also uses the Leray residue theory as in [11]. The explicit formula of Theorem 6.3 was communicated to us by B. Khesin.

6.1 Meromorphic forms with prescribed residues on Riemann surfaces

We will suppose that the Riemann surfaces Σ_1 and Σ_2 are of genus ≥ 1 embedded in a CY 3-fold M and

$$\Sigma_1 \cap \Sigma_2 = \emptyset.$$

Let us fix two non-zero holomorphic forms θ_i on each of Σ_i for $i = 1,\ 2$. According to [1], we can always find a very ample non-singular divisor S_1 containing Σ_1 since

$$\dim_{\mathbb{C}} \Sigma_1 < \frac{1}{2} \dim_{\mathbb{C}} M.$$

Proposition 6.1. *Let S_1 be a non-singular hypersurface section of $M \subset \mathbb{CP}^m$ which contains Σ_1, then there exist a holomorphic 2-form ω_1 on S_1 with a pole of order 1 along Σ_1 such that Leray's residues of ω_1 on Σ_1 is equal to θ_1.*

Proof. We have the following exact sequences:

$$0 \longrightarrow \Omega^2_{S_1} \longrightarrow \Omega^2_{S_1}(\Sigma_1) \xrightarrow{\text{res}} \Omega^1_{\Sigma_1} \longrightarrow 0$$

and

$$\begin{aligned} 0 \longrightarrow H^0\left(S_1, \Omega^2_{S_1}\right) &\longrightarrow H^0\left(S_1, \Omega^2_{S_1}(\Sigma_1)\right) \\ &\xrightarrow{\text{res}} H^0\left(\Sigma_1, \Omega^1_{\Sigma_1}\right) \longrightarrow H^1\left(S_1, \Omega^2_{S_1}\right) \longrightarrow \cdots, \end{aligned} \tag{6.1}$$

where $\Omega^2_{S_1}(\Sigma_1)$ is the locally free sheaf of holomorphic 2-forms on S_1 with a pole of order 1 on Σ_1. In order to deduce Proposition 6.1, we need to prove that $H^1\left(S_1, \Omega^2_{S_1}\right) = 0$. The definition of a CY manifold implies that $H_1(\mathrm{M}, \mathbb{C}) = 0$, and so by the Lefschetz Theorem (see [13]) we conclude that $H_1(S_1,\mathbb{C}) = 0$. Then the Poincare duality implies that $H^3(S_1,\mathbb{C}) = 0$. Hodge theory implies that $H^1\left(S_1,\Omega^2_{S_1}\right) = 0$. So the map

$$H^0\left(S_1,\Omega^2_{S_1}(\Sigma_1)\right) \xrightarrow{\text{res}} H^0\left(\Sigma_1,\Omega^1_{\Sigma_1}\right) \longrightarrow 0$$

is surjective. Proposition 6.1 is proved. □

Proposition 6.2. *Suppose that we can represent Σ_1 as*

$$S_1 \cap S_2 = \Sigma_1, \tag{6.2}$$

where S_1 and S_2 are non-singular hypersurface sections on M. Then there exists a meromorphic 3-form η_1 on M with poles of order 1 along S_1 and S_2 such that the double Leray residue of η_1 is equal to θ_1.

Proof. We have the following exact sequences:

$$0 \longrightarrow \Omega^3_M(S_2) \longrightarrow \Omega^3_M \otimes \mathcal{O}_M(S_1) \otimes \mathcal{O}_M(S_2) \xrightarrow{\text{res}} \Omega^2_{S_1}(S_2) \longrightarrow 0$$

and

$$\begin{aligned} 0 \longrightarrow H^0\left(M,\Omega^3_M(S_2)\right) &\longrightarrow H^0\left(M,\Omega^3_M \otimes \mathcal{O}_M(S_1) \otimes \mathcal{O}_M(S_2)\right) \\ &\xrightarrow{\text{res}} H^0\left(S_1,\Omega^2_{S_1}(S_2)\right) \longrightarrow H^1\left(M,\Omega^3_M(S_2)\right) \longrightarrow \cdots \end{aligned} \tag{6.3}$$

Since S_2 is a hypersurface section of CY manifold M, we can conclude from the Kodaira vanishing theorem that

$$H^1\left(M,\Omega^3_M(S_2)\right) = 0. \tag{6.4}$$

Combining (6.3) and (6.4), we deduce that the Leray's residue map

$$H^0\left(M,\Omega^3_M \otimes \mathcal{O}_M(S_1) \otimes \mathcal{O}_M(S_2)\right) \xrightarrow{\text{res}} H^0\left(S_1,\Omega^2_{S_1}(S_2)\right)$$

is surjective. This fact combined with Proposition 6.1 implies Proposition 6.2. □

From now on we will consider curves on CY 3-fold M which are represented by (6.2).

6.2 Geometric formula for holomorphic linking

Now we will express the holomorphic linking of (Σ_1,θ_1) and (Σ_2,θ_2) as a sum of residues of certain meromorphic 1-form over the intersection points of S_1 and Σ_2. In [20] and [21], expression (6.5) is interpreted via polar homologies.

Theorem 6.3. *Let ω_1 be a meromorphic 2-form defined as in Proposition 6.1 such that*

$$\mathrm{res}_{S_1}\omega_1 = \theta_1$$

Then the following formula is true:

$$\#\left((\Sigma_1,\theta_1),(\Sigma_2,\theta_2)\right) = \sum_{x\in S_1\cap\Sigma_2} \frac{\omega_1(x)\wedge\theta_2(x)}{\eta(x)}, \tag{6.5}$$

where $\omega_1(x)\wedge\theta_2(x) \in \wedge^3\left(T^{1,0}_{x,M}\right)^* \approxeq \Omega^3_{x,M}$.

Proof. Let $\mathrm{T}_\varepsilon(\Sigma_1)$ be a tubular neighbourhood of $\Sigma_1\times\Sigma_2$ in $S_1\times\Sigma_2$. From the definition of Leray's residue and formula (4.10), we deduce that

$$\begin{aligned}\#((\Sigma_1,\theta_1),(\Sigma_2,\theta_2)) &= \int_{\Sigma_1\times\Sigma_2}\left(\overline{\partial}^{-1}\left(\overline{\eta}_\Delta\right)\right)\wedge\pi_1^*(\theta_1)\wedge\pi_2^*(\theta_2)\\ &= \lim_{\varepsilon\longrightarrow 0}\int_{\partial T_\varepsilon(\Sigma_1)\times\Sigma_2}\left(\overline{\partial}^{-1}(\overline{\eta}_\Delta)\right)\wedge\pi_1^*(\omega_1)\wedge\pi_2^*(\theta_2),\end{aligned} \tag{6.6}$$

where $\partial T_\varepsilon(\Sigma_1)$ is the boundary of $T_\varepsilon(\Sigma_1)$ in S_1. Stokes' theorem implies that

$$\begin{aligned}&\lim_{\varepsilon\longrightarrow 0}\int_{\partial T_\varepsilon(\Sigma_1)\times\Sigma_2}\left(\overline{\partial}^{-1}(\overline{\eta}_\Delta)\right)\wedge\pi_1^*(\omega_1)\wedge\pi_2^*(\theta_2)\\ &\quad=\int_{S_1\times\Sigma_2-\Delta} d\left(\frac{\psi_\Delta}{\pi_1^*(\eta)}\right)\wedge\pi_1^*(\omega_1)\wedge\pi_2^*(\theta_2)\\ &\qquad-\int_{S_1\times\Sigma_2} d\left(\frac{\partial g_\Delta}{\pi_1^*(\eta)}\right)\wedge\pi_1^*(\omega_1)\wedge\pi_2^*(\theta_2)\\ &\quad=\int_{S_1\times\Sigma_2}\delta_\Delta\wedge\frac{\pi_1^*(\omega_1)\wedge\pi_2^*(\theta_2)}{\pi_1^*(\eta)} = \sum_{x\in S_1\cap\Sigma_2}\frac{\omega_1(x)\wedge\theta_2(x)}{\eta(x)}.\end{aligned}$$

Theorem 6.3 is proved. □

One can also deduce an alternative version of formula (6.5) using meromorphic 3-forms η_1 and η_2 with poles along S_1 and S_2 constructed in Proposition 6.2. Then the ratio (η_1/η) is a meromorphic section on M of the line bundle $\mathcal{O}_M(S_1)$ which we restrict to Σ_2 and multiply by the holomorphic form θ_2. The result is a meromorphic 1-form

$$\left(\frac{\eta_1}{\eta}\,|_{\Sigma_2}\right)\theta_2,$$

whose residues at $x \in S_1 \cap \Sigma_2$ are precisely the values

$$\frac{\omega_1(x) \wedge \theta_2(x)}{\eta(x)}.$$

This implies

Corollary 6.4. *Let* η_1 *be a meromorphic* 2*-form defined as in Proposition* 6.2*. Then the following formula holds*:

$$\#\left((\Sigma_1, \theta_1), (\Sigma_2, \theta_2)\right) = \sum_{x \in S_1 \cap \Sigma_2} \mathrm{res}_x \left(\frac{\eta_1}{\eta}\,|_{\Sigma_2}\right)\theta_2. \qquad (6.7)$$

We can explore the symmetry of the holomorphic linking and represent the other Riemann surface Σ_2 as an intersection of two surfaces. Then we obtain another version of formulas (6.5) and (6.7) using the meromorphic “lifts” of θ_2 instead of θ_1.

Theorem 6.3 and Corollary 6.4 are the complex analogues of the geometric form of the linking number of two knots in $\mathbb{R}^3$. In the real case, we took a disk whose boundary is one of the knots. We counted the number of points of intersection of the disk with the other knot, taking into account the orientation. The linking number of two knots is the intersection number just described. In the complex case, instead of a disk, we took a complex surface which contains one of the original curves. They summed again over points of intersection of the surface with the other curve weighted with the natural ratios of the forms associated to the geometric objects involved in the construction. Since we always recover the same holomorphic linking, the geometric realization in the complex case also does not depend on the choice of the complex surface and the meromorphic 2-form with prescribed residue.

7 Cohomological interpretation of holomorphic linking

In this section we will reformulate the analytic expressions for the holomorphic linking in the language of homological algebra. The homological interpretation of the analytic formulas for the holomorphic linking is based on the notions of generalized Grothendieck and Serre classes $\mu(Y, \theta)$ and $\lambda(Y, \theta)$ of subvariety Y in a projective variety X together with top holomorphic form θ on Y. Our definitions of Grothendieck and Serre classes are generalizations of the similar notions introduced by Atiyah in [2]. The

Grothendieck class $\mu(Y,\theta)$ always exists by definition. We will prove the existence of generalized Serre classes for the embedding of a submanifold Y in a CY manifold M and the diagonal embedding of M into $M \times M$. Then we derive homological expressions for the holomorphic linking using the Yoneda pairing. These expressions make sense over arbitrary field. They should be related to the height pairing in [5, 6, 8]. We will also illustrate why the original Atiyah's notion of Grothendieck and Serre classes cannot be used in our setting. We will start with the calculation of the local cohomology of the diagonal embedding and we will recall some basic facts from the general theory of local cohomology; a more detailed account of the theory can be found in [17, 15, 18].

7.1 Local cohomology of the diagonal

The key property of the local cohomology is the existence of the analogue of the Mayer–Vietoris exact sequence, namely, let $\mathcal{F}$ be a sheaf on a complex manifold X and Y be a closed submanifold, then the local cohomology groups $H^k_Y(X,\mathcal{F})$ will satisfy the following exact sequence:

$$\longrightarrow H^{k-1}(X,\mathcal{F}) \longrightarrow H^{k-1}(X-Y,\mathcal{F}) \xrightarrow{\delta_{k-1}} H^k_Y(X,\mathcal{F}) \longrightarrow H^k(X,\mathcal{F}) \longrightarrow$$

The boundary map δ in the exact sequence can be interpreted as an "explicit" construction and a generalization of distributions of "boundary values" of sections of the coherent sheaf $\mathcal{F}$.

One of the main results that was proved in [15] is that the local cohomology groups are related to the functor Ext. The computation of the local cohomology group $H^{k-1}_Y(X,\mathcal{F})$ is done by using the following spectral sequence.

First we define the sheaf of extensions in a standard way for any two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X [16]. We will denote this sheaf by $\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{G},\mathcal{F})$ and its global sections by $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{G},\mathcal{F})$. Then we define the sheaf $\underline{\mathcal{H}}^i_Y(\mathcal{F})$ as the following projective limit:

$$\underline{\mathcal{H}}^i_Y(\mathcal{F}) = \varinjlim_n \underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{O}_{n,Y},\mathcal{F}), \tag{7.1}$$

where $\mathcal{O}_{n,Y} = \mathcal{O}_X/I^n_Y$ and I_Y is the ideal sheaf in $\mathcal{O}_X$ that defines Y. According to [16, Exp. I, p. 9 and Exp. II, p. 2] spectral sequence with the initial term

$$E_2^{p,q} = H^p(X,\underline{\mathcal{H}}^q_Y(\mathcal{F}))$$

converges to $H^{p+q}_Y(X,\mathcal{F})$.

The following theorem can be found in [24].

Theorem 7.1. *Let M be a non-singular algebraic manifold. Let $\Delta \subset M\times M$ be the diagonal. Let I_Δ be the ideal sheaf of the diagonal, then I_Δ/I_Δ^2 is isomorphic to the cotangent sheaf Ω_M^1 and, moreover,*

$$I_\Delta^k/I_\Delta^{k+1} \backsimeq S^k(I_\Delta/I_\Delta^2)$$

is a locally free $\mathcal{O}_{M\times M}/I_\Delta$ module.

We will use this fact to establish the following.

Theorem 7.2. *Let M be a* CY *manifold and let T_M be the tangent bundle of M, then*

$$H^0(M, S^k(T_M)) = H^0(M, S^k(\Omega_M^1)) = 0$$

for all $k > 0$.

Proof. When $k = 1$, Theorem 7.2 follows from the isomorphism $T_M \overset{\lrcorner\eta_M}{\approxeq} \Omega_M^{n-1}$. Thus we have

$$H^0(M, T_M) = H^0(\mathrm{M}, \Omega_M^{n-1}).$$

The definition of a CY manifold implies that $H^0(M, \Omega_M^{n-1}) = 0$. Thus $H^0(M, T_M) = H^0(M, \Omega_M^1) = 0$.

Bochner's principle for the Ricci flat Kähler metric implies that if ϕ is any holomorphic tensor on a CY manifold, then it is parallel with respect to the Levi Cevita connection with respect to Yau's metric (see [7]). We also know from [7] that for a simply connected CY manifold the holonomy group of the CY metric is $\mathrm{SU}(n)$. These two facts imply that the globally defined holomorphic symmetric one forms are obtained from the $\mathrm{SU}(n)$ invariant k-symmetric tensors at one point by parallel transportation. So we have the following equality:

$$S^k(\mathbb{C}^n)^{\mathrm{SU}(n)} = H^0(M, S^k(T_M)) = H^0(M, S^k(\Omega_M^1)).$$

Since $S^k(\mathbb{C}^n)^{\mathrm{SU}(n)} = 0$, we conclude from Theorem 7.1 that

$$H^0(M, S^k(T_M)) = H^0(M, S^k(\Omega_M^1)) = H^0(M \times M, I_\Delta^k/I_\Delta^{k+1}) = 0$$

for $k \geq 1$. Theorem 7.2 is proved. □

Next, we will compute some local cohomology groups that will be needed in the construction and the cohomological interpretation of the Green current associated with subvarieties. By definition (7.1):

$$\underline{\mathcal{H}}^i_\Delta(\mathcal{O}_{M\times M}) = \varinjlim_k \underline{\mathrm{Ext}}^i_{\mathcal{O}_{M\times M}}(\mathcal{O}_{k,\Delta}, \mathcal{O}_{M\times M}),$$

where $\mathcal{O}_{k,\Delta} := \mathcal{O}_{M\times M}/I^k_\Delta$ and I_Δ is the ideal sheaf of $\Delta \subset M \times M$.

Theorem 7.3. $\underline{\mathcal{H}}^i_\Delta(\mathcal{O}_{M\times M}) = 0$ *for* $i \neq n$ *and* $\underline{\mathcal{H}}^n_\Delta(\mathcal{O}_{M\times M}) \neq 0$.

Proof. Let $\mathcal{U}$ be an open affine set in $M \times M$. We will prove by induction on k that we have

$$\underline{\mathrm{Ext}}^i_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/I^k_\Delta, \mathcal{O}_{M\times M}) = 0 \tag{7.2}$$

for $i \neq n$ and

$$\underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/I^k_\Delta, \mathcal{O}_{M\times M}) \neq 0. \tag{7.3}$$

The diagonal $\Delta \subset M \times M$ is a smooth algebraic variety. Therefore it is a local complete intersection in $M \times M$. Thus we have

$$\mathcal{O}_\Delta = \mathcal{O}_{M\times M}/(f_1, \ldots, f_n),$$

where Δ is locally defined by the regular sequence $f_1, \ldots, f_n$ of analytic functions in $\mathcal{O}_{M\times M}$ by

$$f_1 = \cdots = f_n = 0.$$

In Chapter 5 "Residues", paragraph "Kozul Complex and Its Applications" of [13] it is proved that for any regular sequence $(f_1, \ldots, f_p)$ in the local ring $\mathcal{O}_N$ of any complex manifold N we have

$$\underline{\mathrm{Ext}}^j_{\mathcal{O}_N}(\mathcal{O}_N/(f_1, \ldots, f_p), \mathcal{O}_N) = \begin{cases} \mathcal{O}_N/(f_1, \ldots, f_p) & j = p \\ 0 & j \neq p \end{cases}. \tag{7.4}$$

Thus we proved (7.2) and (7.3) for $k = 1$.

In order to proceed with the induction on k, we assume that (7.2) and (7.3) are true for $k = m$. The exact sequence

$$0 \longrightarrow I_\Delta^m/I_\Delta^{m+1} \longrightarrow \mathcal{O}_{M\times M}/I_\Delta^{m+1} \longrightarrow \mathcal{O}_{M\times M}/I_\Delta^m \longrightarrow 0$$

implies the long exact sequence

$$\begin{aligned}\cdots \longrightarrow \underline{\mathrm{Ext}}^i_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/I_\Delta^m, \mathcal{O}_{M\times M}) \longrightarrow \underline{\mathrm{Ext}}^i_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/I_\Delta^{m+1}, \mathcal{O}_{M\times M})\\ \longrightarrow \underline{\mathrm{Ext}}^i_{\mathcal{O}_{M\times M}}(I_\Delta^m/I_\Delta^{m+1}, \mathcal{O}_{M\times M}) \longrightarrow \cdots \qquad (7.5)\end{aligned}$$

Since by Theorem 7.1 $I_\Delta^m/I_\Delta^{m+1}$ is a free $\mathcal{O}_\Delta$ module, we obtain from

$$\underline{\mathrm{Ext}}^j_{\mathcal{O}_{M\times M}}(I_\Delta^m/I_\Delta^{m+1}, \mathcal{O}_{M\times M}) = \begin{cases} I_\Delta^m/I_\Delta^{m+1} & j = n \\ 0 & j \neq n \end{cases}. \qquad (7.6)$$

From the induction hypothesis, (7.6) and the long exact sequence (7.5), we can deduce (7.2) and (7.3) for any k. Thus Theorem 7.3 follows from the definition of the sheaves $\underline{\mathcal{H}}^i_\Delta(\mathcal{O}_{M\times M})$ given by (7.1). □

Corollary 7.4. *The formula*

$$\begin{aligned}H^0(M\times M, \underline{\mathrm{Ext}}^k_{\mathcal{O}_{M\times M}}(I_\Delta^m/I_\Delta^{m+1}, \mathcal{O}_{M\times M}))\\ = \mathrm{Ext}^k_{\mathcal{O}_{M\times M}}(I_\Delta^m/I_\Delta^{m+1}, \mathcal{O}_{M\times M}) = 0 \qquad (7.7)\end{aligned}$$

holds for $k \leq n$ and $m > 0$.

Proof. Formula (7.6) implies Corollary 7.4 for $k \neq n = \dim_{\mathbb{C}} M$. Thus (7.7) is proved for any $k \neq n$.

Suppose that $k = n$. Formula (7.6), Theorems 7.1 and 7.2 imply that for $m > 0$

$$\begin{aligned}\mathrm{Ext}^n_{\mathcal{O}_{M\times M}}(I_\Delta^m/I_\Delta^{m+1}, \mathcal{O}_{M\times M}) &= H^0(M\times M, \underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(I_\Delta^m/I_\Delta^{m+1}, \mathcal{O}_{M\times M}))\\ &= H^0(M\times M, I_\Delta^m/I_\Delta^{m+1}) = 0. \qquad (7.8)\end{aligned}$$

So Corollary 7.4 is proved. □

Theorem 7.5. $H^k_\Delta(M\times M, \mathcal{O}_{M\times M}) = 0$ *for $k \neq n$ and* $\dim_{\mathbb{C}} H^n_\Delta(M\times M, \mathcal{O}_{M\times M}) = 1$.

Proof. By Theorem 7.3 $\underline{\mathcal{H}}^q_\Delta(\mathcal{O}_{M\times M}) = 0$ for $q \neq 0$. Therefore $H^k_\Delta(M \times M, \mathcal{O}_{M\times M}) = 0$ for $k \neq n$.

According to [16], $H^n_\Delta(M\times M, \mathcal{O}_{M\times M})$ is obtained by the spectral sequence with a first term $E_2^{p,q} = H^p(M\times M, \underline{\mathcal{H}}^q_\Delta(\mathcal{O}_{\mathrm{M\times M}}))$. Thus we have for $p+q=n$

$$E_2^{p,q}=H^p(M\times M,\underline{\mathcal{H}}^q_\Delta(\mathcal{O}_{M\times M}))\Longrightarrow H^n_\Delta(M\times M,\mathcal{O}_{M\times M}).$$

From Theorem 7.3 and the definition of the cohomology group $H^n_\Delta(M\times M,\mathcal{O}_{M\times M})$ by the spectral sequence we get that

$$\begin{aligned}H^n_\Delta(M\times M,\mathcal{O}_{M\times M})&\backsimeq H^0(M\times M,\underline{\mathcal{H}}^n_\Delta(\mathcal{O}_{M\times M}))\\&\backsimeq H^0(M\times M,\varinjlim_k\underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/(I_\Delta)^k,\mathcal{O}_{M\times M})).\end{aligned}$$

Thus the nth local cohomology $H^n_\Delta(M\times M,\mathcal{O}_{M\times M})$ is the inductive limit of Cech cohomologies

$$\varinjlim_k H^0(M\times M,\underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/(I_\Delta)^k,\mathcal{O}_{M\times M})).$$

In [16] in Exp. II, it is proved that

$$\begin{aligned}&H^0\Big(M\times M,\varinjlim_k\underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}\Big(\mathcal{O}_{M\times M}/(I_\Delta)^k,\mathcal{O}_{M\times M}\Big)\Big)\\&\quad=\varinjlim_k H^0\big(M\times M,\underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/\big(I_\Delta\big)^k,\mathcal{O}_{M\times M})\big).\end{aligned}$$

Thus we have the following isomorphism:

$$\begin{aligned}&H^n_\Delta(M\times M,\mathcal{O}_{M\times M})\\&\quad=\varinjlim_k H^0\big(M\times M,\underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/\big(I_\Delta\big)^k,\mathcal{O}_{M\times M})\big).\end{aligned}\tag{7.9}$$

□

Lemma 7.6. *The natural restriction maps*

$$\begin{aligned}&H^0(M\times M,\underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/\big(I_\Delta\big)^{k+1},\mathcal{O}_{M\times M}))\\&\quad\longrightarrow H^0(M\times M,\underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/\big(I_\Delta\big)^k,\mathcal{O}_{M\times M}))\end{aligned}$$

are isomorphisms. Thus the following formula is true:

$$\dim_{\mathbb{C}}H^0(M\times M,\underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/\big(I_\Delta\big)^k,\mathcal{O}_{M\times M}))=1\quad for\ k\geq 0.$$

Proof. The proof of Lemma 7.6 is by induction. It is based on the following long exact sequence of sheaves:

$$\begin{aligned}0 \longrightarrow \underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/I^k_\Delta, \mathcal{O}_{M\times M}) &\longrightarrow \underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/I^{k+1}_\Delta, \mathcal{O}_{M\times M}) \\ \longrightarrow \underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(I^k_\Delta/I^{k+1}_\Delta, \mathcal{O}_{M\times M}) &\longrightarrow 0. \end{aligned} \tag{7.10}$$

Theorem 7.1 states that

$$\underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(I^k_\Delta/I^{k+1}_\Delta, \mathcal{O}_{M\times M}) \backsimeq I^k_\Delta/I^{k+1}_\Delta.$$

From (7.6) we have

$$\begin{aligned}&H^0(M\times M, \underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(I^k_\Delta/I^{k+1}_\Delta, \mathcal{O}_{M\times M})) \\ &\quad = H^0(M\times M, I^k_\Delta/I^{k+1}_\Delta) = H^0(\Delta, S^k(\Omega^1_\Delta)) = H^0(M, S^k(\Omega^1_M)) = 0.\end{aligned}$$

Combining this fact with the exact sequence (7.10) we obtain that

$$\begin{aligned}&H^0(M\times M, \underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/I^k_\Delta, \mathcal{O}_{M\times M}) \\ &\quad = H^0(M\times M, \underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/I^{k+1}_\Delta, \mathcal{O}_{M\times M}))\end{aligned} \tag{7.11}$$

for $k \geq 1$. The isomorphism

$$\underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}(\mathcal{U})/I_\Delta, \mathcal{O}_{M\times M}(\mathcal{U})) \backsimeq \mathcal{O}_{M\times M}(\mathcal{U})/I_\Delta,$$

implies that

$$\begin{aligned}&\dim_{\mathbb{C}} H^0\big(M\times M, \underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}\big(\mathcal{O}_{M\times M}/I_\Delta, \mathcal{O}_{M\times M}\big)\big) \\ &\quad = \dim_{\mathbb{C}} H^0\big(M\times M, \mathcal{O}_{M\times M}/I_\Delta\big) = \dim_{\mathbb{C}} H^0\big(\Delta, \mathcal{O}_\Delta\big) \\ &\quad = \dim_{\mathbb{C}} H^0\big(M, \mathcal{O}_M\big) = 1.\end{aligned} \tag{7.12}$$

Lemma 7.6 follows directly from (7.11) with (7.12). □

Theorem 7.5 follows from the isomorphism (7.9) and Lemma 7.6.

Corollary 7.7. $H^n_\Delta(M\times M, \mathcal{O}_{M\times M}) \approxeq \mathrm{Ext}^n_{\mathcal{O}_{M\times M}}\big(\mathcal{O}_{M\times M}/I_\Delta, \mathcal{O}_{M\times M}\big) \approxeq \mathbb{C}$.

Proof. Corollary 7.7 follows from the proof of Lemma 7.6 and the definition of $H^n_\Delta(M\times M, \mathcal{O}_{M\times M})$. □

From the Grothendieck duality it follows that $\mathrm{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/I_\Delta, \mathcal{O}_{M\times M})$ can be identified with the restriction of the space of antiholomorphic n forms on $M\times M$ on the diagonal Δ. Thus from the definition of the local cohomology it follows that $H^n_\Delta(M\times M, \mathcal{O}_{M\times M})$ is generated by $\overline{\eta}_\Delta$.

7.2 Definition of the Grothendieck and Serre classes

Let $E \to X$ be a vector bundle over a complex manifold X of a complex dimension n. Serre showed that the pairing

$$H^p(X,E) \times H^{n-p}(X,E^* \otimes \Omega_X^n) \longrightarrow \mathbb{C}$$

given by the integration over X of the corresponding pointwise pairing of the cohomology classes is nondegenerate.

Let $\mathcal{F},\mathcal{H}$ and $\mathcal{G}$ be three coherent sheaves on X. The Yoneda product

$$\mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \times \mathrm{Ext}^q_{\mathcal{O}_X}(\mathcal{G},\mathcal{H}) \longrightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{F},\mathcal{H})$$

is defined in a natural way by the composition of long exact sequences. It is a wellknown fact that if $\mathcal{F}$ is the sheaf of holomorphic function on X denoted by $\mathcal{O}_X$, then

$$\mathrm{Ext}^p(\mathcal{O}_X,\mathcal{F}) \approxeq H^p(X,\mathcal{F}).$$

Grothendieck proved that the map

$$H^p(X,\mathcal{F}) \times \mathrm{Ext}^{n-p}_{\mathcal{O}_X}(\mathcal{F},\Omega_X^n) \longrightarrow H^n(X,\Omega_X^n) \approxeq \mathbb{C} \tag{7.13}$$

given by the Yoneda pairing is non-degenerate. This pairing is called the Grothendieck duality.

If $\mathcal{E}$ is a locally free sheaf, i.e., $\mathcal{E}$ is the sheaf of sections of a vector bundle $E \to X$, then Grothendieck's duality implies Serre's duality by using the following isomorphism

$$\mathrm{Ext}^{n-p}_{\mathcal{O}_X}(\mathcal{E},\Omega_X^n) \approxeq H^{n-p}(X,\mathcal{E}^* \otimes \Omega_X^n). \tag{7.14}$$

Indeed by (7.13)

$$\mathrm{Ext}^{n-p}_{\mathcal{O}_X}(\mathcal{E},\Omega_X^n) \approxeq H^p(\mathrm{X},\mathcal{E})^*. \tag{7.15}$$

On the other hand we know that

$$H^p(X,\mathcal{E})^* \approxeq H^{n-p}(X,\mathcal{E}^* \otimes \Omega_X^n). \tag{7.16}$$

Combining (7.15) and (7.16) we get (7.14).

For any submanifold $\mathrm{Y} \subset X$ of codimension m, we will denote by I_Y the ideal sheaf consisting of functions vanishing on Y. Then the definition of the sheaf I_Y implies that the quotient sheaf $\mathcal{O}_X/I_Y$ is naturally identified with the structure sheaf $\mathcal{O}_Y$ extended by zero on $X - Y$.

Let (Y,θ) be a pair, where Y is a submanifold in X of codimension m and θ is a holomorphic $n - m = \dim_{\mathbb{C}} Y$ form on Y. By applying the Grothendieck duality twice one can deduce that

$$H^0(Y,\Omega_Y^{n-m}) \approxeq \left(\mathrm{Ext}^{n-m}_{\mathcal{O}_X}(\mathcal{O}_Y,\mathcal{O}_Y)\right)^* \approxeq \mathrm{Ext}^m_{\mathcal{O}_X}(\mathcal{O}_Y,\Omega^n_X). \tag{7.17}$$

Thus (7.17) defines a canonical isomorphism

$$\mu : H^0(Y,\Omega_Y^{n-m}) \approxeq \mathrm{Ext}^m_{\mathcal{O}_X}(\mathcal{O}_Y,\Omega^n_{\mathrm{X}}). \tag{7.18}$$

Definition 7.8. We define the Grothendieck class $\mu(Y,\theta)$ of the pair (Y,θ) in X as the image of

$$\theta \in H^0(Y,\Omega_Y^{n-m})$$

under the canonical map μ.

Proposition 7.9. *The Grothendieck class $\mu(\Delta,\pi_1^*(\eta)|_\Delta)$ can be canonically identified with the class of the antiholomorphic Dirac current $\overline{\eta}_\Delta$ given by Definition* 3.2.

Proof. From the proof of Lemma 7.6, it follows that there exist canonical identifications

$$\begin{aligned}\mathrm{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/I^k_\Delta,\mathcal{O}_{M\times M}) &\approxeq \mathrm{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}/I_\Delta,\mathcal{O}_{M\times M})\\ &\approxeq \mathrm{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_\Delta,\mathcal{O}_{M\times M})\end{aligned}$$

for $k>0$. According to Corollary 7.7, the following canonical identification

$$\mathrm{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_\Delta,\mathcal{O}_{M\times M}) = H^n_\Delta(M\times M,\mathcal{O}_{M\times M})$$

exists and by Theorem 7.5

$$\dim_{\mathbb{C}} \mathrm{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_\Delta,\mathcal{O}_{M\times M}) = \dim_{\mathbb{C}} H^n_\Delta(M\times M,\mathcal{O}_{M\times M}) = 1.$$

At the end of Section 7.1, we identified the generator of $H^n_\Delta(\mathrm{M}\times\mathrm{M},\mathcal{O}_{M\times M})$ with the class of the antiholomorphic Dirac current $\overline{\eta}_\Delta$. Thus the Grothendieck class $\mu(\Delta,\pi_1^*(\eta))$ can be interpreted as a local cohomology class, and in particular it can be identified with the class of the antiholomorphic Dirac current $\overline{\eta}_\Delta$. □

Since the Grothendieck class $\mu(\Delta,\pi_1^*(\eta)|_\Delta)$ does not depend on the pull-back of the holomorphic form η on $M\times M$ we will denoted it by $\mu(\Delta,\eta)$.

Remark 7.10. The definition of the Grothendieck class $\mu(\mathrm{Y})$ of a subvariety Y in an algebraic variety X given by Atiyah in [2] differs from ours. When we replace the holomorphic form $\theta \in H^0(\mathrm{Y},\Omega_{\mathrm{Y}}^{n-m})$ by $1 \in H^0(Y,\mathcal{O}_{\mathrm{Y}})$, we will get the Atiyah definition of the Grothendieck class

$$\mu(Y) \in \mathrm{Ext}^m_{\mathcal{O}_X}(\Omega_Y^{n-m},\Omega_X^n).$$

In particular, $\mu(\Delta)$ can be identified with the Dirac current δ_Δ.

Now we will define the Serre class $\lambda(Y,\theta)$ of a pair (Y,θ) in X. The definition of the Serre class $\lambda(Y,\theta)$ is possible under the assumption that the coboundary map

$$d_{m-1} : \mathrm{Ext}^{m-1}_{\mathcal{O}_X}(I_Y, \Omega^n_X) \longrightarrow \mathrm{Ext}^{m}_{\mathcal{O}_X}(\mathcal{O}_Y, \Omega^n_X) \tag{7.19}$$

resulting from the exact sequence

$$0 \longrightarrow I_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

is an isomorphism.

Definition 7.11. Suppose that Y is a non-singular subvariety in the projective smooth variety X and suppose that the map δ_{m-1} of (7.19) is an isomorphism. Then

$$\lambda(Y,\theta) := d^{-1}_{m-1}\big(\mu(Y,\theta)\big) \in \mathrm{Ext}^{m-1}(I_Y, \Omega^n_X).$$

is uniquely defined. We will call $\lambda(Y,\theta)$ the Serre class of the pair (Y,θ) in X.

We will prove the existence of the Serre class $\lambda(\Delta,\eta)$ of the diagonal in $M\times M$ for the CY manifold M as an element of $\mathrm{Ext}^{n-1}_{\mathcal{O}_{M\times M}}(I_\Delta, \mathcal{O}_{M\times M})$ by establishing the isomorphism (7.19) for $\Delta \subset M\times M$ for the sheaf $\Omega^{2n}_{M\times M} \approxeq \mathcal{O}_{M\times M}$.

Theorem 7.12. *Let M be a* CY *manifold of dimension n. Then we have the following canonical isomorphism:*

$$\delta_{n-1} : \mathrm{Ext}^{n-1}_{\mathcal{O}_{M\times M}}(I_\Delta, \mathcal{O}_{M\times M}) \longrightarrow \mathrm{Ext}^{n}_{\mathcal{O}_{M\times M}}(\mathcal{O}_\Delta, \mathcal{O}_{M\times M}). \tag{7.20}$$

Proof. The proof is based on the following long exact sequence

$$\begin{aligned}\cdots &\longrightarrow \mathrm{Ext}^{n-1}_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}, \mathcal{O}_{M\times M}) \longrightarrow \mathrm{Ext}^{n-1}_{\mathcal{O}_{M\times M}}(I_\Delta, \mathcal{O}_{M\times M})^{d_{n-1}}\\ &\longrightarrow \mathrm{Ext}^{n}_{\mathcal{O}_{M\times M}}(\mathcal{O}_\Delta, \mathcal{O}_{M\times M})\\ &\longrightarrow \mathrm{Ext}^{n}_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}, \mathcal{O}_{M\times M}) \longrightarrow \mathrm{Ext}^{n}_{\mathcal{O}_{M\times M}}(I_\Delta, \mathcal{O}_{M\times M})\\ &\longrightarrow \mathrm{Ext}^{n+1}_{\mathcal{O}_{M\times M}}(\mathcal{O}_\Delta, \mathcal{O}_{M\times M}) \longrightarrow \cdots\end{aligned} \tag{7.21}$$

From the Grothendieck duality, we obtain

$$\mathrm{Ext}^{k}_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}, \mathcal{O}_{M\times M}) = H^k\big(M\times M, \mathcal{O}_{M\times M}\big)$$

for $k \geq 0$. The definition of the CY manifold and the Kunneth formula imply that for $0 < k \neq n$, we have

$$\begin{aligned}\operatorname{Ext}^k_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}, \mathcal{O}_{M\times M}) &= H^k\big(M \times M, \mathcal{O}_{M\times M}\big)\\ &= \bigoplus_{p+q=k} H^p\big(\mathrm{M}, \mathcal{O}_M\big) \otimes H^q(M, \mathcal{O}_M) = 0.\end{aligned}$$

Thus from (7.21) we obtain

$$\begin{aligned}0 \longrightarrow \operatorname{Ext}^{n-1}_{\mathcal{O}_{M\times M}}(I_\Delta, \mathcal{O}_{M\times M}) &\xrightarrow{d_{n-1}} \operatorname{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_\Delta, \mathcal{O}_{M\times M})\\ \xrightarrow{i_n} \operatorname{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}, \mathcal{O}_{M\times M}) &\longrightarrow \cdots \end{aligned} \tag{7.22}$$

We will prove in Proposition 7.13 bellow that the map i_n in (7.22) is zero. Then Theorem 7.12, 7.13 follows immediately. □

Proposition 7.13. *The map*

$$i_n : \operatorname{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_\Delta, \mathcal{O}_{M\times M}) \longrightarrow \operatorname{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}, \mathcal{O}_{M\times M})$$

in the long exact sequence (7.22) *is the zero map.*

Proof. Let us consider the long exact sequence

$$\begin{aligned}\cdots \longrightarrow H^{n-1}(M \times M - \Delta, \mathcal{O}_{M\times \mathrm{M}-\Delta}) &\xrightarrow{\delta_{n-1}} H^n_\Delta(M \times M, \mathcal{O}_{M\times M}) \xrightarrow{\iota_n}\\ \xrightarrow{\iota_n} H^n(M \times M, \mathcal{O}_{M\times M}) &\xrightarrow{r_n} H^n(M \times M - \Delta, \mathcal{O}_{M\times \mathrm{M}-\Delta}) \longrightarrow \cdots\end{aligned} \tag{7.23}$$

From (7.23) we can conclude that we have

$$\begin{aligned}0 \longrightarrow H^{n-1}(M \times M - \Delta, \mathcal{O}_{M\times \text{M-}\Delta}) &\xrightarrow{\delta_{n-1}} H^n_\Delta(M \times M, \mathcal{O}_{M\times M})\\ \xrightarrow{\iota_n} H^n(M \times M, \mathcal{O}_{M\times M}) &\xrightarrow{r_n} H^n(M \times M - \Delta, \mathcal{O}_{M\times M-\Delta}) \longrightarrow 0\end{aligned} \tag{7.24}$$

We will need the following lemma. □

Lemma 7.14. *The map ι_n in the long exact sequence (*7.24*) is the zero map.*

Proof. It is easy to see that ι_n is the zero map if and only if r_n is an isomorphism. We will prove that r_n is an isomorphism by contradiction.

The map r_n is induced by the restriction map

$$r : M \times M \longrightarrow M \times M - \Delta.$$

According to Theorem 7.5

$$\dim_{\mathbb{C}} H^n_\Delta(M \times M, \mathcal{O}_{M\times M}) = 1. \tag{7.25}$$

From the definition of CY manifold and the Kunneth formula, we derive that

$$\dim_{\mathbb{C}} H^n(M \times M, \mathcal{O}_{M\times M}) = 2 \tag{7.26}$$

and $H^n(M \times M, \mathcal{O}_{M\times M})$ has a basis $\pi_1^*(\overline{\eta})$ and $\pi_2^*(\overline{\eta})$. If we assume that r_n is not an isomorphism, then (7.24), (7.25) and (7.26) will imply that

$$\dim_{\mathbb{C}} H^n(M \times M - \Delta, \mathcal{O}_{M\times \mathrm{M}-\Delta}) \leq 1. \tag{7.27}$$

Formula (7.27) means that on $M \times M - \Delta$, some linear combination $\alpha\pi_1^*(\overline{\eta}) + \beta\pi_2^*(\overline{\eta})$ for $\alpha, \beta \in \mathbb{C}$ will represent zero. So the holomorphic n-form $\alpha\pi_1^*(\eta) + \beta\pi_2^*(\eta)$ will be an exact form on $M \times M - \Delta$. So on $M \times M - \Delta$ there exists a holomorphic $(n-1)$-form $\omega(n-1,0)$ such that

$$\alpha\pi_1^*(\overline{\eta}) + \beta\pi_2^*(\overline{\eta}) = \overline{\partial}\left(\overline{\omega(n-1,0)}\right). \tag{7.28}$$

Let $\mathcal{U}$ be an affine open set in M. Thus the restriction of the holomorphic $(n-1)$-form $\omega(n-1,0)$ on $\mathcal{U} \times \mathcal{U} - (\mathcal{U} \times \mathcal{U} \cap \Delta)$ can be represented as follows:

$$\omega(n-1,0) = \sum_{I_p,J_q;p+q=n-1} f_{I_p,J_q}(z,w) dz^{I_[} \wedge dw^{J_q},$$

where $(z^1, \dots, z^n, w^1, \dots, w^n)$ are local coordinates in $\mathcal{U} \times \mathcal{U}$,

$$I_p = (i_1, \dots, i_p) \quad \text{for} \quad 0 < i_1 < \cdots < i_p \leq n,$$
$$J_q = (j_1, \dots, j_q) \quad \text{for} \quad 0 < j_1 < \cdots < j_q \leq n$$

and $f_{I_p,J_q}(z,w)$ are holomorphic functions in $\mathcal{U} \times \mathcal{U} - (\mathcal{U} \times \mathcal{U} \cap \Delta)$. Since Δ has a codimension $n \geq 3$ in $M \times M$, the Hartogs principle implies that the holomorphic functions $f_{I_p,J_q}(z,w)$ are well defined on $\mathcal{U} \times \mathcal{U}$. Thus $\omega(n-1,0)$ is a well defined holomorphic $(n-1)$-form on $M \times M$. This implies that $H^{n-1}(M \times M,\ \mathcal{O}_{M\times M}) \neq 0$. This fact contradicts the definition of a CY manifold. Thus the assumption that r_n is not an isomorphism leads to a contradiction. Lemma 7.14 is proved. □

The end of the proof of Proposition 7.13*:* Theorem 7.5 implies that there exists a canonical isomorphism

$$H^n_\Delta(M \times M, \mathcal{O}_{M\times M}) \approxeq \operatorname{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_\Delta, \mathcal{O}_{M\times M}).$$

The Grothendieck's duality implies that

$$\operatorname{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}, \mathcal{O}_{M\times M}) \approxeq H^n(M \times M, \mathcal{O}_{M\times M})$$

are canonically isomorphic. From the properties of the local cohomology as stated in [16] and Grothendieck duality, it follows that we have the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_\Delta, \mathcal{O}_{M\times M}) & \xrightarrow{i_n} & \operatorname{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}, \mathcal{O}_{M\times M}) \\ \| & & \| \\ H^n_\Delta(M\times M, \mathcal{O}_{M\times M}) & \xrightarrow{\iota_n} & H^n(M\times \mathrm{M}, \mathcal{O}_{M\times M}). \end{array} \tag{7.29}$$

The commutative diagram (7.29) and Lemma 7.14 imply Proposition 7.13. □

Corollary 7.15. *The Serre class $\lambda(\Delta,\eta)$ of the diagonal exists.*

Proposition 7.16. *The restriction of the Serre class $\lambda(\Delta,\eta)$ on an affine open set $\mathcal{U}\times\mathcal{U}$ in $M\times M$ can be identified with the holomorphic Bochner–Martinelli kernel $\mathcal{K}^{0,n-1}_{\mathcal{U}\times\mathcal{U}}$ defined by formula* (5.12).

Proof. The Grothendieck duality yields the commutative diagram

$$\begin{array}{ccc} \operatorname{Ext}^{n-1}_{\mathcal{O}_{M\times M}}(I_\Delta, \mathcal{O}_{M\times M}) & \xrightarrow{d_{n-1}} & \operatorname{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_\Delta, \mathcal{O}_{M\times M}) \\ \| & & \| \\ H^{n-1}(M\times M-\Delta, \mathcal{O}_{M\times M-\Delta}) & \xrightarrow{\delta_{n-1}} & H^n_\Delta(M\times M, \mathcal{O}_{M\times M}). \end{array} \tag{7.30}$$

Proposition 7.9 implies that we can identify the Grothendieck class $\mu(\Delta)$ with the antiholomorphic Dirac current $\overline{\eta}_\Delta$ by using the ismorphism bewteen $\operatorname{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_\Delta, \mathcal{O}_{M\times M})$ and $H^n_\Delta(M\times M,\ \mathcal{O}_{M\times M})$ in (7.30). Theorem 4.4 and Theorem 5.2 imply that the restriction $\delta^{-1}_{n-1}(\overline{\eta}_\Delta)$ on a Zariski open set $\mathcal{U}\times\mathcal{U}$ in $M\times M$ can be identified with the holomorphic Bochner–Martinelli kernel $\mathcal{K}^{0,n-1}_{\mathcal{U}\times\mathcal{U}}$. From the definition of the Serre class $\lambda(\Delta,\eta) = d^{-1}_{n-1}(\mu(\Delta))$ and the commutativity of the diagram (7.30), Proposition 7.16 follows directly. □

Remark 7.17. One can show that the Serre class corresponding to the Grothendieck class $\mu(\Delta)$ as defined in Remark 7.10 does not exist. In fact, it is not difficult to see that

$$\mu(\Delta)\in H^n_\Delta(M\times M, \Omega^n_{M\times M}).$$

We remarked that the Grothendieck class $\mu(\Delta)$ can be identified with the Dirac current δ_Δ. Thus it follows that in the exact sequence

$$\begin{aligned} 0 \longrightarrow H^{n-1}(M\times M-\Delta, \Omega^n_{M\times M}) &\xrightarrow{\delta_{n-1}} H^n_\Delta(M\times M, \Omega^n_{M\times M}) \\ &\xrightarrow{\iota_n} H^n(M\times M, \Omega^n_{M\times M}) \xrightarrow{r_n} H^n(M\times M-\Delta, \Omega^n_{M\times M}) \longrightarrow 0, \end{aligned}$$

the map

$$H^n_\Delta(M\times M, \Omega^n_{M\times M}) \xrightarrow{\iota_n} H^n(M\times M, \Omega^n_{M\times M})$$

is non-zero since $\iota_n(\delta_\Delta) = [\omega_\Delta]$, where $[\omega_\Delta]$ represents the Poincare dual class of the diagonal. Thus Theorem 7.5 implies that the map

$$H^{n-1}(M \times M - \Delta, \Omega^n_{M\times M}) \xrightarrow{\delta_{n-1}} H^n_\Delta(M \times M, \Omega^n_{M\times M})$$

is the zero map. This implies that the map

$$\mathrm{Ext}^{n-1}_{\mathcal{O}_{M\times M}}(I_\Delta, \Omega^n_{M\times M}) \xrightarrow{d_{n-1}} \mathrm{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_{M\times M}, \Omega^n_{M\times M})$$

is the zero map. Thus the Serre class $\lambda(\Delta)$ of the diagonal does not exist.

Next, we will show that the Serre class of the pair (Y, θ), where Y is a submanifold of codimension m embedded in an n-dimensional CY 3-fold M and θ is a holomorphic $(n-m)$-form on Y is well defined on M. From the definition of the Serre class of the pair (Y, θ) in a CY manifold, it follows that we need to check that the coboundary map δ_{m-1} in (7.19):

$$d_{m-1} : \mathrm{Ext}^{m-1}_{\mathcal{O}_M}(I_Y, \mathcal{O}_M) \longrightarrow \mathrm{Ext}^m_{\mathcal{O}_M}(\mathcal{O}_Y, \mathcal{O}_M) \tag{7.31}$$

is an isomorphism.

Proposition 7.18. *Let X be a compact Kähler manifold such that*

$$H^k(X, \mathcal{O}_X) = 0 \quad \textit{for } 0 < k < \dim_{\mathbb{C}} X = n. \tag{7.32}$$

Then δ_{m-1} defined by (7.31) *is an isomorphism.*

Proof. Proposition 7.18 follows from the long exact sequence

$$\begin{aligned}\cdots \longrightarrow \mathrm{Ext}^{m-1}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathrm{Ext}^{m-1}_{\mathcal{O}_X}(I_Y, \mathcal{O}_X) \longrightarrow \mathrm{Ext}^m_{\mathcal{O}_X}(\mathcal{O}_X/I_Y, \mathcal{O}_X)\\ \longrightarrow \mathrm{Ext}^m_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \cdots\end{aligned}$$

associated with the exact sequence

$$0 \longrightarrow I_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/I_Y = \mathcal{O}_Y \longrightarrow 0.$$

In fact the Grothendieck duality and condition (7.32) imply

$$\mathrm{Ext}^{m-1}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = H^{m-1}(X, \mathcal{O}_X) = \mathrm{Ext}^m_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = H^m(X\mathcal{O}_X) = 0$$

for $0 < m < n$. Proposition 7.18 is proved. $\square$

Thus Proposition 7.18 guarantees that the Serre class of the pair (Y, θ) is well defined on a CY manifold M since for CY manifolds condition (7.32) holds. In particular, we see that if (Σ, θ) is a pair of a Riemann surface embedded in a CY 3-fold and θ is a holomorphic form on Σ, then the Serre class $\lambda(\Sigma, \theta)$ and the Grothendieck class $\mu(\Sigma, \theta)$ are well defined on CY 3-fold M.

Suppose that Y_1 and Y_2 are two submanifolds of complex dimension m_1 and m_2 in a CY manifold M such that

$$\dim_{\mathbb{C}} Y_1 + \dim_{\mathbb{C}} Y_2 = \dim_{\mathbb{C}} M - 1$$

and $Y_1 \cap Y_2 = \varnothing$. We will define the restriction of the Serre class $\lambda(Y_1, \theta_1)|_{Y_2}$ as an element of $\mathrm{Ext}^{m_1-1}_{\mathcal{O}_M}(\mathcal{O}_{Y_2}, \mathcal{O}_{Y_2})$ as follows. Any element of

$$\alpha \in \mathrm{Ext}^{m_1-1}_{\mathcal{O}_M}(I_{Y_1}, \mathcal{O}_M)$$

corresponds to an exact sequence of length $m_1 - 1$ consisting of $\mathcal{O}_M$ modules with the first one being isomorphic to $\mathcal{O}_M$ and the last one to I_{Y_1}. The condition $Y_1 \cap Y_2 = \varnothing$ implies that the ideal sheaf I_{Y_1} when restricted on Y_2 will be the structure sheaf $\mathcal{O}_{Y_2}$ and thus we have two restriction maps

$$\rho_1 : I_{Y_1} \longrightarrow \mathcal{O}_{Y_2} \quad \text{and} \quad \rho_2 : \mathcal{O}_M \longrightarrow \mathcal{O}_{Y_2}.$$

Since the maps ρ_1 and ρ_2 are surjective, the exact sequence of length $m_1 - 1$ corresponds to the same exacts sequence tensored with $\mathcal{O}_{Y_2}$ and we will get a natural map

$$r_{m_1-1} : \mathrm{Ext}^{m_1-1}_{\mathcal{O}_M}(I_Y, \mathcal{O}_M) \longrightarrow \mathrm{Ext}^{m_1-1}_{\mathcal{O}_M}(\mathcal{O}_{Y_2}, \mathcal{O}_{Y_2}).$$

Then we define

$$\lambda(Y_1, \theta_1)|_{Y_2} = r_{m_1-1}\left(\lambda(Y_1, \theta_1)\right).$$

7.3 A homological interpretation of the holomorphic linking

Proposition 7.19. *Let Y be a smooth subvariety of codimension m in* CY *manifold M which is represented as the intersections of M with m hypersurfaces. Then*

$$H^m_Y(M,\mathcal{O}_M) \approxeq \mathrm{Ext}^m_{\mathcal{O}_M}(\mathcal{O}_Y, \mathcal{O}_M).$$

Proof. The proof of Proposition 7.19 is based on several lemmas. □

Lemma 7.20. *Let Y be a subvariety in a* CY *manifold M of dimension m. Then we have*

$$\underline{\mathrm{Ext}}^j_{\mathcal{O}_M}(I^k_Y / I^{k+1}_Y, \mathcal{O}_M) = 0$$

for $j < m$.

Proof. Since Y is a smooth variety in M then Y is a local complete intersection in M. Thus we can apply (7.4) to conclude

$$\underline{\mathrm{Ext}}^j_{\mathcal{O}_M}(I^k_Y / I^{k+1}_Y, \mathcal{O}_M) = 0$$

for $j < m$. Lemma 7.20 is proved. □

Lemma 7.21. *We have the following isomorphisms*

$$\underline{\mathrm{Ext}}^m_{\mathcal{O}_M}(\mathcal{O}_M/I_Y^{k+1},\mathcal{O}_M) \approxeq \underline{\mathrm{Ext}}^m_{\mathcal{O}_M}(\mathcal{O}_M/I_Y^k,\mathcal{O}_M) \tag{7.33}$$

for $k>0$.

Proof. From the exact sequence

$$0 \longrightarrow I_Y^k/I_Y^{k+1} \longrightarrow \mathcal{O}_M/I_Y^{k+1} \longrightarrow \mathcal{O}_M/I_Y^k \longrightarrow 0,$$

the associated long exact sequence

$$\begin{aligned}\cdots \longrightarrow \underline{\mathrm{Ext}}^j_{\mathcal{O}_M}(\mathcal{O}_M/I_Y^k,\mathcal{O}_M) &\longrightarrow \underline{\mathrm{Ext}}^j_{\mathcal{O}_M}(\mathcal{O}_M/I_Y^{k+1},\mathcal{O}_M)\\ \longrightarrow \underline{\mathrm{Ext}}^j_{\mathcal{O}_M}(I_Y^k/I_Y^{k+1},\mathcal{O}_M) &\longrightarrow \cdots\end{aligned} \tag{7.34}$$

and Lemma 7.20, (7.33) follows directly. Lemma 7.21 is proved. □

Corollary 7.22. $\underline{\mathrm{Ext}}^j_{\mathcal{O}_M}(\mathcal{O}_M/I_Y^k,\mathcal{O}_M)=0$ *for* $0<j<m$ *and* $k\geq 0$.

Lemma 7.23. $\underline{\mathcal{H}}^i_Y(\mathcal{O}_M)=0$ *for* $i\neq m$ *and*

$$\underline{\mathcal{H}}^m_Y(\mathcal{O}_M)=\mathcal{O}_M/(f_1,\ldots,f_{n-m}),$$

where Y *is locally defined by* $f_1=\cdots=f_{n-m}=0$.

Proof. The proof of Lemma 7.23 follows directly from the definition of the sheaves $\underline{\mathcal{H}}^i_Y(\mathcal{O}_M)$ given by (7.1), the result stated in [13] that

$$\underline{\mathrm{Ext}}^m_{\mathcal{O}_M}(\mathcal{O}_M/I_Y,\mathcal{O}_M)=\mathcal{O}_M/(f_1,\ldots,f_{n-m}),$$

Lemma 7.21 and Corollary 7.22. Lemma 7.23 is proved. □

Lemma 7.24. *We have* $H^j(Y,I_Y^k/I_Y^{k+1})=0$ *for* $0\leq j<m$ *and* $k\geq 0$.

Proof. The conditions that Y has a codimension m and Y can be represented as the intersections of m hyperplanes with M imply that the dual of the normal bundle is direct sum of line bundles of the type $\mathcal{O}_M(-n_i)$, where $n_i>0$. This fact combined with the fact I_Y^k/I_Y^{k+1} is the symmetric k-power of the conormal bundle of Y in M and Kodaira vanishing theorem imply Lemma 7.24. □

Proposition 7.19 follows directly from long exact sequence

$$\begin{aligned}\cdots \longrightarrow \mathrm{Ext}^j_{\mathcal{O}_M}(\mathcal{O}_M/I_Y^k,\mathcal{O}_M) &\longrightarrow \mathrm{Ext}^j_{\mathcal{O}_M}(\mathcal{O}_M/I_Y^{k+1},\mathcal{O}_M)\\ \longrightarrow \mathrm{Ext}^j_{\mathcal{O}_M}(I_Y^k/I_Y^{k+1},\mathcal{O}_M) &\longrightarrow \cdots,\end{aligned} \tag{7.35}$$

Lemmas 7.20, 7.21, 7.23, 7.24 and the definition of $H^m_Y(M,\mathcal{O}_M)$.

Proposition 7.19 implies that the Grothendieck class $\mu(Y,\theta)$ of (Y,θ) can be interpreted as a local cohomology class in $H^m_Y(M,\mathcal{O}_M)$.

Proposition 7.25. *Let Y be a submanifold of* CY *manifold M of codimension m. Let θ be a holomorphic $(n-m)$-form on Y. Then there exists a natural identification of the Grothendieck class $\mu(Y,\theta)$ with the Dirac anti-holomorphic current $\overline{\theta}_Y$.*

Proof. The definition of the Grothendieck class $\mu(\mathrm{Y},\theta)$ and Proposition 7.19 imply that

$$\mu(Y,\theta) \in \mathrm{Ext}^m_{\mathcal{O}_M}(\mathcal{O}_Y,\mathcal{O}_M) = H^m_Y(M,\mathcal{O}_M).$$

From here it follows that we can repeat the arguments in the proof of Proposition 7.9 to conclude Proposition 7.25. □

Proposition 7.26. *Let*

$$\lambda(Y,\theta) \in \mathrm{Ext}^{m-1}_{\mathcal{O}_M}(I_Y,\mathcal{O}_M)$$

be the Serre class of the pair (Y,θ), where Y is an intersection of M with d hypersurfaces and $m = \operatorname{codim}_{\mathbb{C}} Y$. Then we have the following isomorphism:

$$\mathrm{Ext}^{m-1}_{\mathcal{O}_M}(\mathcal{O}_{M-Y},\mathcal{O}_{M-Y}) \approxeq H^{m-1}(M-Y,\mathcal{O}_{M-Y}). \tag{7.36}$$

Let r be the restriction map composed of the isomorphism (7.36)

$$r : \mathrm{Ext}^{m-1}_{\mathcal{O}_M}(I_Y,\mathcal{O}_M) \longrightarrow \mathrm{Ext}^{m-1}_{\mathcal{O}_M}(\mathcal{O}_{M-Y},\mathcal{O}_{M-Y}) \approxeq H^{m-1}(M-Y,\mathcal{O}_{M-Y}).$$

Then we have the following expression for the Serre class $\lambda(Y,\theta)$:

$$r\left(\lambda(Y,\theta)\right) = \overline{\partial}^{-1}\left(\overline{\theta}_Y\right). \tag{7.37}$$

Proof. We will recall that according to the Grothendieck duality and since M is a CY manifold, we have

$$\mathrm{Ext}^p_{\mathcal{O}_M}(\mathcal{O}_M,\mathcal{O}_M) = H^p(M,\mathcal{O}_M) = 0 \tag{7.38}$$

for $0 < p < n$. From the long exact sequence (7.35) for $k = 1$ and (7.38) we can conclude that we have the following isomorphism:

$$\mathrm{Ext}^{m-1}_{\mathcal{O}_M}(I_Y,\mathcal{O}_M) \overset{\delta_{m-1}}{\longrightarrow} \mathrm{Ext}^m_{\mathcal{O}_M}(\mathcal{O}_M/I_Y,\mathcal{O}_M). \tag{7.39}$$

The long exact sequence

$$\cdots \longrightarrow H^k_Y(\mathrm{M},\mathcal{O}_M) \longrightarrow H^k(\mathrm{M},\mathcal{O}_M) \longrightarrow H^k(M-Y,\mathcal{O}_{M-Y}) \longrightarrow$$

combined with (7.38) give the following isomorphism:

$$H^{m-1}(M-Y,\mathcal{O}_M) \overset{\delta'_{m-1}}{\longrightarrow} H^m_Y(\mathrm{M},\mathcal{O}_M). \tag{7.40}$$

According to Proposition 7.19

$$\mathrm{Ext}^m_{\mathcal{O}_M}(\mathcal{O}_M/I_Y,\mathcal{O}_M) \approxeq H^m_Y(M,\mathcal{O}_M). \tag{7.41}$$

From (7.39), (7.40), (7.41) and the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^{m-1}_{\mathcal{O}_M}(I_Y,\mathcal{O}_M) & \stackrel{\delta_{m-1}}{\longrightarrow} & \mathrm{Ext}^{m}_{\mathcal{O}_M}(\mathcal{O}_M,\mathcal{O}_M) \\ \downarrow & & \downarrow \\ H^{m-1}(M-Y,\mathcal{O}_M) & \stackrel{\delta'_{m-1}}{\longrightarrow} & H^m_Y(\mathrm{M},\mathcal{O}_M) \end{array} \tag{7.42}$$

we can conclude that the map

$$r:\mathrm{Ext}^{m-1}_{\mathcal{O}_M}(I_Y,\mathcal{O}_M)\longrightarrow \mathrm{Ext}^{m-1}_{\mathcal{O}_M}(\mathcal{O}_{M-Y},\mathcal{O}_{M-Y})\approxeq H^{m-1}(M-Y,\mathcal{O}_{M-Y})$$

induced from the restriction map I_Y to $M-Y$ is an isomorphism. Formula (7.36) is proved. The definition of the Grothendieck class implies that

$$\mu(Y,\theta)\in \mathrm{Ext}^{m}_{\mathcal{O}_M}(\mathcal{O}_M/I_Y,\mathcal{O}_M)\approxeq H^m_Y(M,\mathcal{O}_M).$$

The definition of the Serre class $\lambda(Y,\theta)$ of a pair (Y,θ) and (7.42) imply that

$$\lambda(Y,\theta)\in \mathrm{Ext}^{m-1}_{\mathcal{O}_M}(I_Y,\mathcal{O}_M)\approxeq H^{m-1}(M-Y,\mathcal{O}_M).$$

Now from here and Proposition 49 we get formula (7.37). □

Proposition 7.27. *There exists a canonical pairing*

$$\langle\, ,\, \rangle:\mathrm{Ext}^{k}_{\mathcal{O}_M}(\mathcal{O}_Y,\mathcal{O}_Y)\times \mathrm{Ext}^{n-k}_{\mathcal{O}_M}(\mathcal{O}_Y,\mathcal{O}_M)\longrightarrow \mathbb{C}.$$

Proof. Yoneda product defines a pairing

$$\langle\, ,\rangle:\mathrm{Ext}^{k}_{\mathcal{O}_M}(\mathcal{O}_Y,\mathcal{O}_Y)\times \mathrm{Ext}^{n-k}_{\mathcal{O}_M}(\mathcal{O}_Y,\mathcal{O}_M)\longrightarrow \mathrm{Ext}^{n}_{\mathcal{O}_M}(\mathcal{O}_Y,\mathcal{O}_M). \tag{7.43}$$

By Grothendieck duality, $\mathrm{Ext}^{n}_{\mathcal{O}_M}(\mathcal{O}_Y,\mathcal{O}_M)$ is canonically isomorphic to the dual of

$$\mathrm{Ext}^{0}_{\mathcal{O}_M}(\mathcal{O}_Y,\mathcal{O}_M)=H^0(Y,\mathcal{O}_Y),$$

which is canonically isomorphic to $\mathbb{C}$. Proposition 7.27 is proved. □

Theorem 7.28. *Since M is a* CY *3-fold, we know that*

$$\lambda(\Sigma_1,\theta_1)|_{\Sigma_2}\in \mathrm{Ext}^{1}_{\mathcal{O}_M}(\mathcal{O}_{\Sigma_2},\mathcal{O}_{\Sigma_2}) \quad \textit{and} \quad \mu(\Sigma_2,\theta_2)\in \mathrm{Ext}^{2}_{\mathcal{O}_M}(\mathcal{O}_{\Sigma},\mathcal{O}_M).$$

Let

$$\langle\lambda(\Sigma_1,\theta_1)|_{\Sigma_2},\mu(\Sigma_2,\theta_2)\rangle$$

be the pairing defined by Proposition 7.27. *Then we have the following formula:*

$$\#\left((\Sigma_1,\theta_1),(\Sigma_2,\theta_2)\right)=\langle\lambda(\Sigma_1,\theta_1)|_{\Sigma_2},\mu(\Sigma_2,\theta_2)\rangle\,.$$

Proof. Since

$$\lambda(\Sigma_1,\theta_1)|_{\Sigma_2}\in \mathrm{Ext}^{1}_{\mathcal{O}_M}(\mathcal{O}_{\Sigma_2},\mathcal{O}_{\Sigma_2}) \quad \text{and} \quad \mu(\Sigma_2,\theta_2)\in \mathrm{Ext}^{2}_{\mathcal{O}_M}(\mathcal{O}_{\Sigma},\mathcal{O}_M)$$

by Propositions 7.25, 7.26 we can identify $\lambda(\Sigma_1,\theta_1)|_{\Sigma_2}$ and $\mu(\Sigma_2,\theta_2)$ with $\overline{\partial}^{-1}\left(\overline{\theta}_{\Sigma_1}\right)$ and $\mu\left(\overline{\theta}_{\Sigma_2}\right)$. According to Grothendieck, the Yoneda pairing is the same as the Serre pairing which is just integration (See [16]). The proof of Theorem 7.28 follows directly from the analytic formula (3.7) for the holomorphic linking. $\square$

The condition $\Sigma_1 \cap \Sigma_2 = \varnothing$ implies that $\Sigma_1 \times \Sigma_2 \cap \Delta = \varnothing$; thus we get that $I|_{\Sigma_1\times\Sigma_2} \approxeq \mathcal{O}_{\Sigma_1\times\Sigma_2}$. From here we conclude that for the Serre class

$$\lambda(\Delta,\eta)|_{\Sigma_1\times\Sigma_2} \in \mathrm{Ext}^2_{\mathcal{O}_{M\times M}}(\mathcal{O}_{\Sigma_1\times\Sigma_2},\mathcal{O}_{\Sigma_1\times\Sigma_2}). \tag{7.44}$$

The Grothendieck class $\mu(\Sigma_1\times\Sigma_2,\pi_1^*(\theta_1)\wedge\pi_2^*(\theta_2))$ of the pair

$$(\Sigma_1\times\Sigma_2,\pi_1^*(\theta_1)\wedge\pi_2^*(\theta_2))$$

is an element of $\mathrm{Ext}^4_{\mathcal{O}_{M\times M}}(\mathcal{O}_{\Sigma_1\times\Sigma_2},\mathcal{O}_{M\times M})$, i.e.,

$$\mu(\Sigma_1\times\Sigma_2,\pi_1^*(\theta_1)\wedge\pi_2^*(\theta_2)) \in \mathrm{Ext}^4_{\mathcal{O}_{M\times M}}(\mathcal{O}_{\Sigma_1\times\Sigma_2},\mathcal{O}_{M\times M}).$$

Yoneda pairing

$$\begin{aligned}\langle\,,\rangle : {}&\mathrm{Ext}^2_{\mathcal{O}_{M\times M}}(\mathcal{O}_{\Sigma_1\times\Sigma_2},\mathcal{O}_{\Sigma_1\times\Sigma_2}) \times \mathrm{Ext}^4_{\mathcal{O}_{M\times M}}(\mathcal{O}_{\Sigma_1\times\Sigma_2},\mathcal{O}_{M\times M})\\ &\longrightarrow Ext^6_{\mathcal{O}_{M\times M}}(\mathcal{O}_{\Sigma_1\times\Sigma_2},\mathcal{O}_{M\times M}).\end{aligned} \tag{7.45}$$

defines a non-degenerate pairing since by Grothendieck duality

$$\mathrm{Ext}^6_{\mathcal{O}_{M\times M}}(\mathcal{O}_{\Sigma_1\times\Sigma_2},\mathcal{O}_{M\times M}) \approxeq H^0(\Sigma_1\times\Sigma_2,\mathcal{O}_{\Sigma_1\times\Sigma_2}) = \mathbb{C}.$$

Next we will give a homological algebra interpretation of Theorem 4.7.

Theorem 7.29. *The following formula holds*:

$$\#\left((\Sigma_1,\theta_1),(\Sigma_2,\theta_2)\right) = \left\langle \lambda(\Delta,\eta)|_{\Sigma_1\times\Sigma_2}, \mu(\Sigma_1\times\Sigma_2,\pi_1^*(\theta_1)\wedge\pi_2^*(\theta_2))\right\rangle. \tag{7.46}$$

Proof. According to Proposition 7.16, the restriction of $\lambda(\Delta,\eta)$ on the product $\mathcal{U}\times\mathcal{U}$ of an affine set $\mathcal{U}$ in M can be identified with the holomorphic Bochner–Martinelli kernel $\mathcal{K}^{0,2}_{\mathcal{U}\times\mathcal{U}}$. Thus we can identify the restriction of the Serre class $\lambda(\Delta,\eta)$ on $(\Sigma_1\times\Sigma_2)\cap\mathcal{U}\times\mathcal{U}$ with the restriction of the holomorphic Bochner–Martinelli kernel on $(\Sigma_1\times\Sigma_2)\cap\mathcal{U}\times\mathcal{U}$. Proposition 7.25 implies that the Grothendieck class $\mu(\Sigma_1\times\Sigma_2,\pi_1^*(\theta_1)\wedge\pi_2^*(\theta_2))$ of the pair $(\Sigma_1\times\Sigma_2,\pi_1^*(\theta_1)\wedge\pi_2^*(\theta_2))$ can be identified with with the antiholomorphic Dirac current $\overline{(\pi_1^*(\theta_1)\wedge\pi_2^*(\theta_2))}_{\Sigma_1\times\Sigma_2}$. We pointed out above that the Yoneda pairing is the same as the Serre pairing which is just integration. Formula (7.46) follows directly from the above-described identifications of the restrictions of the Serre class and the Grothendieck class, the interpretation of the Yoneda pairing as integration and formula (5.18) in Theorem 5.4. Theorem 7.29 is proved. $\square$

One can generalize formulas (5.18) and (6.5) and Theorems 7.28, 7.29 for the pairs (Y_1, θ_1) and (Y_2, θ_2) of submanifolds Y_1 and Y_2 in a CY manifold M of any dimension such that

$$\dim_{\mathbb{C}} Y_1 + \dim_{\mathbb{C}} Y_2 = \dim_{\mathbb{C}} M - 1$$

and $Y_1 \cap Y_2 = \varnothing$ to define their holomorphic linking.

8 Generalizations of holomorphic linking

In this section, we would like to establish an explicit connection between our holomorphic linking and the one studied by Atiyah. Atiyah arrived at his formula for holomorphic linking by considering the twistor transform of the Green function of the Laplacian. Thus he only defined linking of spheres, while our definition makes sense for curves with genus greater than zero. Also the twistor space is never a CY space required in our approach. Thus in order to include the Atiyah holomorphic linking in our picture, we need to extend further our construction.

8.1 The holomorphic linking of Riemann surfaces with marked points

We would like to generalize the holomorphic linking of two Riemann surfaces in a CY manifold to the case of Riemann surfaces with punctures. Let

$$(\Sigma_i; p_1, \ldots, p_{m_i}; \theta_i), i = 1, 2$$

be two Riemann surfaces with m_i marked points on them embedded in a CY 3-fold M, and let θ_i be meromorphic forms on Σ_i with poles of order at most 1 at $p_1, \ldots, p_{m_i}$. We will assume as before that

$$\Sigma_1 \cap \Sigma_2 = \emptyset.$$

We can define the holomorphic linking of

$$(\Sigma_i; p_1, \ldots, p_{m_i}; \theta_i)$$

in any of the equivalent ways considered above with appropriate modifications. For example, the definition of the holomorphic linking via the Green kernel is defined as follows:

$$\#((\Sigma_1; p_1, \ldots, p_{m_1}; \theta_1), (\Sigma_2; p_1, \ldots, p_{m_2}; \theta_2))$$

$$= \lim_{\varepsilon \longrightarrow 0} \int_{\Sigma_1 - D_{1,\varepsilon} \times \Sigma_2 - D_{2,\varepsilon}} \left(\overline{\partial}^{-1} (\overline{\eta}_\Delta) |_{\Sigma_1 \times \Sigma_2} \right) \wedge \pi_1^*(\theta_1) \wedge \pi_2^*(\theta_2),$$

where $D_{i,\varepsilon}$ are the union of disks with radius ε around the marked points $p_1, \ldots, p_{m_i}$ with fixed local coordinates. We denote the limit as before by

$$\int_{\Sigma_1 \times \Sigma_2} \left(\overline{\partial}^{-1} (\overline{\eta}_\Delta) |_{\Sigma_1 \times \Sigma_2} \right) \wedge \pi_1^*(\theta_1) \wedge \pi_2^*(\theta_2).$$

Repeating the arguments in the proof of Theorem 6.3, one can also derive the geometric formula for the holomorphic linking of the punctured spheres

$$\#((\Sigma_1; p_1, \ldots, p_{m_1}; \theta_1), (\Sigma_2; p_1, \ldots, p_{m_2}; \theta_2)) = \sum_{x \in S_1 \cap \Sigma_2} \frac{\omega_1(x) \wedge \theta_2(x)}{\eta(x)},$$

where ω_1 is a meromorphic 2-form on the complex surface S_1 containing Σ_1, whose residue on Σ_1 is θ_1.

One can also generalize the notion of holomorphic linking to the case when M is not necessarily CY manifold. In the general case, a holomorphic form η does not exist and we have to fix a meromorphic 3-form. In certain cases, (for example, when M is a Fano variety) we can uniquely fix a meromorphic 3-form η by prescribing its poles along a given hypersurface $H \subset M$ whose homology class represents the first Chern class of M.

Another generalization of the holomorphic linking of Riemann surfaces in an arbitrary complex manifolds can be obtained by considering still holomorphic forms θ_1, θ_2 and η but coupled with a line bundle. The simplest example of this sort is the linking of two spheres in a three-dimensional projective space. In this example, one can consider unique up to a scale holomorphic forms θ_1, θ_2 and η with values in natural line bundles. Then the holomorphic linking becomes a well-defined number depending only on the embeddings of the Riemann surfaces.

In fact, the two generalizations of the holomorphic linking of Riemann surfaces in CY manifolds are closely related when the holomorphic forms are replaced by holomorphic forms with values in the line bundles on Σ_1, Σ_2 defined by the fixed points. Thus we can replace the holomorphic form η on M with a meromorphic form with a simple pole along some divisor and θ_i can be replaced by meromorphic forms with values in the trivial bundle. In particular, the example of the linking of two spheres in the $\mathbb{CP}^3$ with natural line bundles correspond to the holomorphic linking of two spheres with two marked points embedded in $\mathbb{CP}^3$ and the poles of meromorphic 3-form is determined by hypersurface consisting of four hyperplanes in general position. In the next subsection, we will consider this example in a more general context of twistor spaces first studied by Atiyah in [2].

8.2 Relations with Atiyah's results on linking of rational curves in twistor space

In [2], Atiyah discovered a version of holomorphic linking while studying the twistor transform of the Green function of the Laplacian. In order to express his result in an invariant form he used the Serre classes of the pairs

$$\Delta \subset M \times M$$

and

$$\Sigma_i \subset M,$$

$i = 1, 2$, where M (denoted by Z in his paper) is the twistor space of a compact four-dimensional manifold N with a self-dual metric, and Σ_i are rational curves that appear as the preimages of the points of N in M in the twistor transform. If N is a spin-manifold, there is a natural holomorphic line bundle L on M such that

$$L^{-4} \approxeq \Omega^3_M.$$

We denote by $\mathcal{O}_M(n)$ the sheaf of sections of L^n. If N is not a spin-manifold, then L does not exist but L^2 always exists yielding $\mathcal{O}_M(n)$ for even n. In particular, there is a canonical 3-form η^0 with coefficients in $\mathcal{O}_M(4)$, which plays a similar role for the twistor space as the holomorphic volume form η for the CY space. Since Σ_i are spheres, there are also canonical one forms θ_i^0 with coefficients in $\mathcal{O}_{\Sigma_i}(2)$. Thus the forms

$$\theta_i^0 \in H^0(\Sigma_i, \mathcal{O}_{\Sigma_i})$$

and

$$\eta^0 \in H^0(M, \mathcal{O}_M).$$

If we apply directly twice the Grothendieck duality to the pairs

$$(\Delta, \eta^0) \quad \text{and} \quad (\Sigma_1 \times \Sigma_2, \pi_1^*(\theta_1^0) \wedge \pi_1^*(\theta_1^0))$$

as we did in Section 7.2, we can define the analogues of Grothendieck classes

$$\mu(\Delta, \eta^0) \quad \text{and} \quad \mu(\Sigma_1 \times \Sigma_2, \pi_1^*(\theta_1^0) \wedge \pi_1^*(\theta_1^0))$$

and Serre classes

$$\lambda(\Delta, \pi_1^*(\eta^0)|_\Delta), \quad \lambda(\Sigma_1 \times \Sigma_2, \pi_1^*(\theta_1^0) \wedge \pi_1^*(\theta_1^0)).$$

Since

$$\mathcal{O}_{M\times M}(-2,-2)) \approxeq \Omega^6_{M\times M},$$

it is easy to see that

$$\lambda(\Delta, \eta^0) \in \operatorname{Ext}^2_{\mathcal{O}_{M\times M}}(I_\Delta, \mathcal{O}_{M\times M}(-2,-2)).$$

and

$$\mu(\Sigma_1 \times \Sigma_2, \pi_1^*(\theta_1^0) \wedge \pi_1^*(\theta_1^0)) \in \operatorname{Ext}^4_{\mathcal{O}_{M\times M}}(\mathcal{O}_{\Sigma_1\times\Sigma_2}, \mathcal{O}_{M\times M}(-2,-2)).$$

In the same manner, we can define the Grothendieck $\mu((\Sigma_i, \theta_i^0)$ and Serre classes $\lambda(\Sigma_i, \theta_i^0)$ of the pair (Σ_i, θ_i^0) and the definitions imply

$$\lambda(\Sigma_i, \theta_i^0) \in \operatorname{Ext}^1_{\mathcal{O}_M}(I_{\Sigma_i}, \mathcal{O}_M(-2))$$

and

$$\mu((\Sigma_i, \theta_i^0) \in \operatorname{Ext}^2_{\mathcal{O}_M}(\mathcal{O}_{\Sigma_i}, \mathcal{O}_M(-2)).$$

By repeating the arguments from Proposition 7.27 we can see that the Grothendieck duality defines the following two non degenerate pairings:

$$\operatorname{Ext}^1_{\mathcal{O}_M}(\mathcal{O}_{\Sigma_2}, \Omega^1_{\Sigma_2}) \times \operatorname{Ext}^2_{\mathcal{O}_M}(\Omega^1_{\Sigma_2}, \mathcal{O}_M(-4)) \longrightarrow \mathbb{C} \tag{8.1}$$

and

$$\operatorname{Ext}^2_{\mathcal{O}_M}(\mathcal{O}_{\Sigma_1\times\Sigma_2}, \Omega^2_{\Sigma_1\times\Sigma_2}) \times \operatorname{Ext}^4_{\mathcal{O}_{M\times M}}(\Omega^2_{\Sigma_1\times\Sigma_2}, \mathcal{O}_M(-4,-4)) \longrightarrow \mathbb{C}. \tag{8.2}$$

Let

$$\lambda(\Sigma_1, \theta_1)|_{\Sigma_2} \in \operatorname{Ext}^1_{\mathcal{O}_M}(\mathcal{O}_{\Sigma_2}, \Omega^1_{\Sigma_2}) \quad \text{and}$$
$$\lambda\left(\Delta, \eta^0\right)|_{\Sigma_1\times\Sigma_1} \in \operatorname{Ext}^2_{\mathcal{O}_M}(\mathcal{O}_{\Sigma_1\times\Sigma_2}, \Omega^2_{\Sigma_1\times\Sigma_2})$$

be the restrictions of the Serre classes of the pairs (Σ_1, θ_1) and $\left(\Delta, \pi_1^*(\eta^0)|_\Delta\right)$ on Σ_2. Then we can define the holomorphic linking of the pair (Σ_i, θ_i^0) by using (8.1) and (8.2) in two different ways according to Theorems 7.28 and 7.29:

$$\operatorname{At}(\Sigma_1, \Sigma_2) = \left\langle \lambda(\Sigma_1, \theta_1^0)|_{\Sigma_2}, \mu(\Sigma_2, \theta_2^0) \right\rangle \tag{8.3}$$

and

$$\operatorname{At}(\Sigma_1, \Sigma_2) = \left\langle \lambda(\Delta, \eta^0)|_{\Sigma_2}, \mu(\Sigma_1 \times \Sigma_2, \pi_1^*(\theta_1^0) \wedge \pi_2^*(\theta_2^0) \right\rangle. \tag{8.4}$$

Then one can prove that that formulas (8.3) and (8.4) are the same up to a constant with Atiyah's formulas for the linking of Σ_1 and Σ_2 expressed as the value of the Green function $G(x, y)$ of the conformally invariant Laplacian $\square$ on a self dual compact four-manifold N of the rational curves Σ_x and Σ_y in N corresponding to $x, y \in$ N.

We expect that the various forms of the holomorphic linking considered in this paper have appropriate analogues for the Atiyah linking number.

The possibility of the path integral presentation has been noted by Gerasimov [12]. The analytic expression for the Atiyah linking number should coincide with the Penrose integral formula for the Green function of the Laplacian. Finally, the geometric formula should have the following form similar to (6.7):

$$\mathrm{At}(\Sigma_1, \Sigma_2) = \sum_{x \in \mathrm{S}_1 \cap \Sigma_2} \mathrm{Res}_x \left(\frac{\eta_1^0}{\eta^0} |_{\Sigma_2} \right) \theta_2^0,$$

where η_1^0 is a twisted holomorphic 3-form on M whose double residue is the 1-form θ_2^0 on Σ_1.

In order to relate the Atiyah holomorphic linking with ours, we will pick out a pair of marked points for each rational curve Σ_i in M. In the homogeneous coordinates $(z_0 : z_1)$, the marked points can be represented by linear functionals, say $p_1(z)$, $p_2(z)$ for Σ_1 and $p_3(z)$, $p_4(z)$ for Σ_2. Then the meromorphic forms with simple poles at the marked points will have the forms

$$\theta_1 = \frac{\theta_1^0}{p_1(z)p_2(z)} \quad \text{and} \quad \theta_2 = \frac{\theta_2^0}{p_3(z)p_4(z)}.$$

Suppose that the twistor space is a projective variety and there is a section of the canonical bundle on M that yields a hypersurface $H \subset M$ that passes through the four marked points and $M - H$ is an open CY manifold. Let η be a meromorphic form M with a singularities along the hypersurface H. Then we define the holomorphic linking of a rational curves with two marked points as discussed in Section 8.1. We conjecture that the Atiyah linking coincides with ours, namely,

$$\mathrm{At}(\Sigma_1, \Sigma_2) = \#((\Sigma_1, \theta_1), (\Sigma_2, \theta_2)).$$

We will illustrate this relation in the case of the twistor space of the four-dimensional sphere S^4. In this case, $M = \mathbb{P}^3$ and its canonical bundle is isomorphic to $\mathcal{O}_M(-4)$. Thus $H^0(M, \Omega^3_M(4))$ is one-dimensional and its generator can be written in the projective coordinates as follows:

$$\eta^0 := \sum_{i=0}^{3} (-1)^i z_i dz_0 \wedge \cdots \wedge d\hat{z}_i \wedge \cdots \wedge dz_3.$$

Let us consider two non-intersecting lines Σ_1 and Σ_2 in $\mathbb{P}^3$ given by pairs of linear functionals

$$l_1(z) = l_2(z) = 0$$

and

$$l_3(z) = l_4(z) = 0.$$

We denote by S_i a hyperplane in M given by the equation $l_i(z) = 0$. We also consider

$$\theta_1^0 := \frac{\eta^0}{l_1(z)l_2(z)},$$

the meromorphic 3-form on M whose double residue is the 1-form η_1^0 on Σ_1. So we obtain

$$\mathrm{At}(\Sigma_1, \Sigma_2) = \sum_{x\in S_1\cap\Sigma_2} \mathrm{res}\left(\frac{\eta_1^0}{\eta^0}\theta_2^0\right) = \sum_{x\in S_1\cap\Sigma_2} \mathrm{res}\left(\frac{z_0dz_1 - z_1dz_0}{l_1(z)l_2(z)}\right). \tag{8.5}$$

On the other hand, we can compute the holomorphic linking of two non-intersecting lines $\Sigma_1$and Σ_2 with two marked points also using the geometric formula. Let $p_i(z) = 0$, $i = 1, 2, 3, 4$ defines four hyperplanes H_i, which are in general position and also intersect the hyperplanes defined by $l_i(z) = 0$, $i = 1, 2, 3, 4$ transversely. The pairs of marked points on Σ_1 and Σ_2 are precisely their intersection with the hyperplanes H_1, H_2 and H_3, H_4, respectively. We can consider $M - \{\cup_{i=1}^4 H_i\}$ as an open CY manifold with a holomorphic form

$$\eta := \frac{\eta^0}{p_1(z)p_2(z)p_3(z)p_4(z)}.$$

We will also define a meromorphic 3-form on $M - \{\cup_{i=1}^4 H_i\}$

$$\eta_1 := \frac{\eta^0}{p_1(z)p_2(z)l_1(z)l_2(z)}$$

whose double residue is the 1-form θ_1 on Σ_1. Then the geometric formula for the holomorphic linking yields

$$\#((\Sigma_1, \theta_1), (\Sigma_2, \theta_2))$$

$$\sum_{S_1\cap\Sigma_2} \mathrm{res}\left(\frac{\eta_1}{\eta}\theta_2\right) = \sum_{S_1\cap\Sigma_2} \mathrm{res}\left(\frac{\eta_1^0/(p_1(z)p_2(z))}{\eta^0/(p_1(z)p_2(z)p_3(z)p_4(z))}\frac{\theta_2^0}{p_3(z)p_4(z)}\right),$$

is the Atiyah holomorphic linking $\mathrm{At}(\Sigma_1, \Sigma_2)$.

We would like to conclude the discussion of the relation between the Atiyah holomorphic linking and the one arising from the complex CSW theory with a remark concerning the non-abelian case. Atiyah noted in his paper [2] that the conformal Laplacian has a natural covariant analogue for a unitary vector bundle E with a connection over X and if the connection is self-dual then E lifts to a holomorphic bundle $\widetilde{E}$ on M. In this case, the Green function of the conformally invariant Laplacian can be expressed using a non abelian version of an Atiyah holomorphic linking. In this case, the Serre class of the diagonal $\Delta \subset M \times M$ should be extended to endomorphisms of $\widetilde{E}$ and yields a non-abelian version of the holomorphic linking of two rational curves in M.

A variant of Atiyah construction for arbitrary pairs of non-intersecting curves in a CY manifold will also yield a non-abelian generalization of the holomorphic linking studied in this paper. We conjecture that it can also be derived from the complex CSW theory with a gauge group $\mathrm{GL}(n, \mathbb{C})$, where n is the rank of $\widetilde{E}$.

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