

Topological strings in generalized complex space

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Abstract

A two-dimensional topological sigma-model on a generalized Calabi–Yau target space X is defined. The model is constructed in a Batalin–Vilkovisky formalism using only a generalized complex structure \mathcal{J} and a pure spinor ρ on X . In the present construction, the algebra of Q -transformations automatically closes off-shell, the model transparently depends only on \mathcal{J} , the algebra of observables and correlation functions for topologically trivial maps in genus zero are easily defined. The extended moduli space appears naturally. The familiar action of the twisted $\mathcal{N} = 2$ conformal field theory (CFT) can be recovered after a gauge fixing. In the open case, we consider an example of generalized deformation of complex structure by a holomorphic Poisson bivector β and recover holomorphic noncommutative Kontsevich $*$ -product.

e-print archive: <http://lanl.arXiv.org/abs/hep-th/0603145>

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1 Introduction

The topological [1, 2] A/B-sigma models [3] were introduced by Witten as twists of the $\mathcal{N} = 2$ two-dimensional supersymmetric conformal field theory of maps from the worldsheet Σ into the target space X . Topological strings [4] are obtained by coupling topological sigma models to topological 2D gravity on Σ , so that in the path integral one integrates over the moduli space of complex structures on Σ . See [5] for references and an extensive review of the progress in the subject.

The topological A-model can be defined for any almost Kahler manifold X . An almost Kahler structure is a pair (ω, J) of a symplectic structure ω and a compatible almost complex structure J . In the Batalin–Vilkovisky approach, the almost complex structure J is used for the gauge fixing of the A-model, so the correlation functions depend only on the symplectic structure ω . The observables t of the A-model are identified with the de Rham cohomology classes $t \in H^*(X, \mathbb{C})$. The A-model free energy $\mathcal{F}_g(t)$ is the generating function of genus g Gromov–Witten invariants of X [6, 7].

On a Kahler manifold, the space of physical observables $\phi(x)_{i_1 \dots i_p \bar{i}_1 \dots \bar{i}_q}$ $\chi^{i_1} \dots \chi^{i_p} \chi^{\bar{i}_1} \dots \chi^{\bar{i}_q}$ of the A-model [3] is conveniently graded by a pair (p, q) according to the Hodge decomposition of the de Rham cohomology $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C})$. The cohomology class of symplectic structure ω can be deformed by an element of $H^2(X, \mathbb{C})$, which corresponds to an observable of degree $p + q = 2$. Physically, such observables give rise to ghost number zero deformations of the action [3] for the A-model. The deformations corresponding to the observables of degree $p + q = 2$ will be called *geometrical deformations*. The hermitian observables of type $(1, 1)$ are deformations of the Kahler structure. If we include the B-field, then the hermitian condition can be dropped. In other words, the real B-field and the real symplectic structure ω can be combined together into a complex two-form $\omega + iB$.

If $H^{2,0}(X, \mathbb{C})$ is trivial, then deformations of type $(1, 1)$ are the only geometrical deformations of the A-model. However, if $H^{2,0}(X, \mathbb{C})$ is not trivial, then there are also observables of type $(2, 0)$ and $(0, 2)$. Let us call the moduli space generated by the observables of degree $(1, 1)$ —*the ordinary geometric moduli space*, for the degrees $(2, 0), (1, 1), (0, 2)$ —*the geometric moduli space*, and for all (p, q) —*the extended moduli space* following [3].

Now consider the topological B-model on a Calabi–Yau manifold X . The B-model couples to a complex structure J on X . Its observables $\phi_{\bar{j}_1 \dots \bar{j}_q}^{i_1 \dots i_p}(x)$ $\theta^{i_1} \dots \theta^{i_p} \eta^{\bar{j}_1} \dots \eta^{\bar{j}_q}$ [3] of type $(-p, q)$ are identified with the Dolbeault

cohomology classes $H^q(\Lambda^p TX^{1,0})$.¹ The ordinary geometric moduli space of complex structures is generated by Beltrami differentials μ_j^i . They deform the Dolbeault differential $\bar{\partial}_j \rightarrow \bar{\partial}_j + \mu_j^i \partial_i$. The genus g free energy \mathcal{F}_g of the B-model does not have such a clear geometrical description as in the case of the A-model. For the recent mathematical progress in definition of the B-model, see [8–11]. In genus zero, the extended moduli space of the B-model was studied in [12–14].

Again, the observables of type $(-1, 1)$ correspond to ordinary geometrical deformations of complex structure generated by Beltrami differentials μ_j^i . If $H^0(\Lambda^2 TX^{1,0})$ or $H^{0,2}(X)$ are nontrivial, then there exist *generalized* deformations of complex structure generated by a holomorphic Poisson bivector β^{ij} or by a closed two-form $B_{\bar{i}\bar{j}}$.

$$\boxed{\begin{array}{l} \text{ordinary geometric moduli} \\ (p, q) = (1, 1) \end{array}} \subset \boxed{\begin{array}{l} \text{(Hitchin's generalized) geometric moduli} \\ p + q = 2 \end{array}} \subset \boxed{\begin{array}{l} \text{extended moduli} \\ \text{all } (p, q) \end{array}} \tag{1.1}$$

The geometrical meaning of these deformations was unclear in [3, 4]. The missing notions appeared in the studies of the homological mirror symmetry [15] and the generalized complex geometry [16].²

In the framework of the Kontsevich homological mirror symmetry conjecture [15], we need to consider not only closed strings, but also open strings together with all possible branes. Physically, branes are the boundary conditions for the open strings. Mathematically, branes are described by certain derived categories. For example, in the case of the B-model, branes could be holomorphic vector bundles supported on holomorphic submanifolds. The full category of B-branes is the bounded derived category $\mathcal{D}^b(X)$ of coherent sheaves on X . Physically, the worldsheet path integral defines the structure of algebra on the space of open string states \mathcal{A} . For example, the product $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ can be defined in terms of the open string three-point function. The worldsheet action in the path integral can be deformed by closed string observables. By definition, this deforms the open string algebra. Therefore, deformations of the open string algebra (modulo some symmetry) are described by the closed string states [8, 9, 17–21].

Mathematically, these deformations of the category of branes are computed by the Hochschild cohomology $\text{HH}(\mathcal{D}^b(X))$ [14, 20–25]. In [24], it was

¹For convenience, we will call sections of $\Lambda^p(TX^{1,0}) \otimes \Lambda^q(T^*X^{0,1})$ by $(-p, q)$ forms.

²Let us remark that these deformations were often neglected in the literature, since usually the target space X was taken to have $b_1(X) = 0$.

argued that computation of $\mathrm{HH}(\mathcal{D}^b(X))$ can be reduced to the computation of the Hochschild cohomology of the single space-filling brane. The algebra of observables on such a brane is just an algebra of holomorphic functions $\mathcal{O}(X)$. Thus, the hard computation of $\mathrm{HH}(\mathcal{D}^b(X))$ reduces to an easier computation of $\mathrm{HH}(\mathcal{O}(X))$ and gives the result [24, 26]

$$\mathrm{HH}(\mathcal{O}(X)) = \oplus H^q(\Lambda^p(TX^{1,0})).$$

This is precisely the moduli space of the closed B-model [3]. Given this observation, the geometrical nature of $(-2, 0)$ and $(0, 2)$ deformations of the closed B-model might be interpreted in terms of equivalent deformation of the open B-model algebra. It was claimed in [27, 28] that $(0, 2)$ deformations by a closed two-form $b_{\bar{i}\bar{j}}$ transform a sheaf structure on a brane into a more general gerby structure. On the other hand, $(-2, 0)$ deformations by a holomorphic bivector β^{ij} deform the ordinary product of functions on a brane into the noncommutative Kontsevich $*$ -product [21–23, 29, 30]. See [31] for a review of noncommutative field theories and [32–34] for the study of mirror symmetry in the context of generalized complex structures.

The notion of a generalized complex structure and generalized Calabi–Yau manifold was defined by Hitchin [16, 35] and then fully developed in Gualtieri’s thesis [36]. There are two ideas behind the notion of generalized complex structure. The first, coming from theory of constrained systems, is to generalize structures on the tangent bundle TX to structures on the direct sum of the tangent and the cotangent bundle $E = TX \oplus T^*X$ and consider Dirac structure on E [37]. The second, coming from string theory, is to extend the diffeomorphism group by action of B-field.

An *almost complex structure* is a section I of $\mathrm{End}(TX)$ such that $I^2 = -1$. Similarly, an *almost generalized complex structure* is a section \mathcal{J} of $\mathrm{End}(TX \oplus T^*X)$ such that $\mathcal{J}^* = -\mathcal{J}$, $\mathcal{J}^2 = -1$. An almost complex structure is integrable if $+i$ -eigenbundle of J (holomorphic subbundle of TX) is involutive under the Lie bracket on TX . Similarly, an almost generalized complex structure is integrable if $+i$ -eigenbundle L of \mathcal{J} (holomorphic subbundle of $TX \oplus T^*X$) is involutive under the Courant bracket [37].

The generalized complex geometry incorporates symplectic structures and ordinary complex structures as particular cases. Therefore, the topological A-model and the topological B-model could be particular cases of a certain generalized topological model [30, 38, 39]. This topological sigma-model of maps from Σ to X depends on a generalized complex structure \mathcal{J} , so let us call it *the topological \mathcal{J} -model*. See [40] for the analogue of the present construction in the real case and [41] for studies of the current algebra associated with \mathcal{J} .

If \mathcal{J} is an ordinary symplectic (complex) structure, let us call it to be of A (B)-type. Gualtieri [36] shows that deformations of a generalized complex structure at the symplectic point (A-type) are parameterized by $H^2(X, \mathbb{C})$, whereas deformations in the complex point (B-type) are parameterized by $\oplus_{p+q=2} H^q(\Lambda^p T X^{1,0})$. This is the space of $p + q = 2$ deformations of the topological A/B-model [3]. Thus it is very natural to suggest that the topological A/B-model is a particular case of the generalized \mathcal{J} -model.

Then one can also ask the following question. If the Hitchin’s *generalized* complex geometry corresponds to $p + q = 2$ deformations of the A/B-model, then what could be the *extended* complex geometry that corresponds to arbitrary (p, q) deformations? The answer is known under the name BV geometry. It first appeared in papers of Batalin and Vilkovisky [42, 43], who suggested a powerful generalization of Becchi-Rouet-Stora-Tyupin (BRST) quantization method for the case when gauge symmetries are reducible. A clear geometric interpretation of the BV formalism was given in [44–46]. A general method to construct a topological sigma-model for a BV target manifold was suggested in [47] by Alexandrov, Kontsevich, Schwarz and Zaboronsky (AKSZ). They illustrated the method by example of Chern–Simons theory and the topological A/B-model. Later AKSZ method was used by Kontsevich [22] to find the $*$ -product formula in the context of deformation quantization (see also [48–50]).

This is a summary of the steps to get the \mathcal{J} -model

$$\boxed{\text{generalized CY structure}} \xrightarrow{1} \boxed{\text{Lie bialgebroid}} \xrightarrow{2} \boxed{\text{BV geometry}} \xrightarrow{3} \boxed{\mathcal{J}\text{-model}}.$$

1. A generalized complex structure \mathcal{J} defines decomposition of the bundle $E = (TX \oplus T^*X) \otimes C$ into the $+i$ -eigenbundle L and its conjugate $L^* \simeq \bar{L}$, so $E = L \oplus \bar{L}$ [16, 36]. Each of the bundles L and \bar{L} has a structure of Lie algebroid [37, 51–53]. The Lie bracket $\{, \} : \Gamma(L) \otimes \Gamma(L) \rightarrow \Gamma(L)$ is equal to the restriction of the Courant bracket from $TX \oplus T^*X$ to L . The Lie bracket on L^* is also the restriction of the Courant bracket. The Lie algebroid differential $\bar{\partial} : \Gamma(\Lambda^k L^*) \rightarrow \Gamma(\Lambda^{k+1} L^*)$ is canonically defined by the Lie bracket on L . It satisfies $\bar{\partial}^2 = 0$. The pair (L, L^*) has the structure of Lie bialgebroid (L, L^*) [54], which means that the operator $\bar{\partial}$, defined by the Lie bracket on L , satisfies the Leibnitz’s rule for the Lie bracket on L^* .
2. A BV manifold (M, ω, ρ) is generally defined as a supermanifold M equipped with a nondegenerate symplectic structure ω and a measure ρ such that the corresponding odd Laplacian Δ squares to zero $\Delta^2 =$

0 [45, 46]. Here the Laplacian of a function f is the divergence $\Delta f := \operatorname{div} X_f$ of the vector field X_f generated by f , where the divergence of a vector field X is defined by the formula $\rho \operatorname{div} X := d(i_X \rho)$. The Poisson bracket $\{, \}$ and the solution S of the BV classical master equation $\{S, S\} = 0$ define the operator Q on functions on M as $Qf := \{S, f\}$. The operator Q is a derivation of the Poisson bracket $\{, \}$ and satisfies $Q^2 = 0$. The space of functions $\mathcal{A} := C^\infty(M)$ on such a manifold M is a differential BV algebra [49, 55–57]. This algebra has an odd Lie bracket $\{, \} : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$, canonically defined by the odd Poisson structure $\{f, g\} := \partial_i f \omega^{ij} \partial_j g$, and the differential $Q : \mathcal{A} \rightarrow \mathcal{A}$, defined by the solution of the classical master equation $Qf := \{S, f\}$. The Poisson bracket in \mathcal{A} is generated by the BV Laplacian Δ as $\{a, b\} = (-1)^{|a|}(\Delta(ab) - (\Delta a)b - (-1)^a a(\Delta b))$.

Now consider the supermanifold $N = \Pi L$, which is the total space of the bundle L with parity reversed on the fibers. In other words, coordinates in the fibers are odd (fermionic) variables. The space of functions $\mathcal{A} = C^\infty(N)$ is naturally identified with the space of sections of $\oplus_k \Lambda^k L^*$. The Lie algebroid differential $\bar{\partial} : \Gamma(\Lambda^k L^*) \rightarrow \Gamma(\Lambda^{k+1} L^*)$ is equivalent to the BV differential Q of degree 1 on \mathcal{A} . The Lie algebroid bracket $\{, \} : \Gamma(L^*) \otimes \Gamma(L^*) \rightarrow \Gamma(L^*)$ is equivalent to the BV bracket $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ of degree -1 . The generalized CY condition is equivalent to the existence of BV measure ρ . It allows us to define the BV Laplacian Δ . In terms of the Lie algebroid one has $\Delta = \bar{\partial}$, where $\bar{\partial}$ is the generalized $\bar{\partial}$ -divergence $\Gamma(\Lambda^k L^*) \rightarrow \Gamma(\Lambda^{k-1} L^*)$. Actually, the generators of the Poisson bracket on ΠL are in one-to-one correspondence with flat Lie algebroid L^* -connections on the top external power $\Lambda^{\operatorname{deg} L} L^*$ —generalized divergence operators [56, 58–60].

Thus for any generalized CY manifold, there is a corresponding BV supermanifold $N = \Pi L$ equipped with a homological differential Q . Assume that the odd Poisson structure on N is invertible, so N is a symplectic manifold. If it is not, then we will take symplectic realization M of N [61–63]. Since the odd vector field Q preserves the symplectic structure, the form $i_Q \omega$ is closed. If it is also exact, then $i_Q \omega = dS$. The function S is a Hamiltonian function for the vector field Q , so $Q = \{S, \cdot\}$.

3. Given the BV target manifold N and the Hamiltonian function S that satisfies $\{S, S\} = 0$, the topological \mathcal{J} -model is obtained along the lines of [47, 49]. The BV fields of the model are maps from $\Pi T\Sigma$ to N , where the supermanifold $\Pi T\Sigma$ is the total space of the tangent bundle of a Riemann surface Σ with parity reversed on the fibers. Let X^i and p_i be local coordinates on N , which are canonically conjugate with respect to the BV odd symplectic structure. Then the BV action of the model

is simply

$$S = \int_{\Pi T\Sigma} p_i dX^i + S(p_i, X^i). \tag{1.2}$$

The physical action of the sigma-model is obtained after the gauge fixing of this BV master action. See [40, 64, 65] on the Poisson sigma-model.

In Section 2, we consider in more details the algebraic and geometric structures mentioned above. In Section 3, we will construct the \mathcal{J} -model. In Section 4, we will compute the correlation functions on the boundary of the disk for the \mathcal{J} -model obtained by a finite deformation of the B-model by a holomorphic Poisson bivector β^{ij} and reproduce the Kontsevich $*$ -product formula. Section 5 concludes the paper.

2 A differential BV algebra \mathcal{A} of a generalized CY manifold

A *generalized complex structure* [16, 36] on a manifold X is a section of $\mathcal{J} \in \text{End}(TX \oplus T^*X)$ such that $\mathcal{J}^* = -\mathcal{J}$ and $\mathcal{J}^2 = -1$ with a certain integrability condition: the $+i$ -eigenbundle $L \subset (TX \oplus T^*X) \otimes \mathbb{C}$ is involutive with respect to Courant bracket [37]:

$$[X + \xi, Y + \eta] = [X, Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) \tag{2.1}$$

where $X, Y \in \Gamma(TX)$ and $\xi, \eta \in \Gamma(T^*X)$.

Then $(TX \oplus T^*X) \otimes \mathbb{C} = L \oplus \bar{L}$, where \bar{L} can be identified with L^* , since L is maximal isotropic³ and $L \cap \bar{L} = 0$.

The bundle L has a structure of a *Lie algebroid* [16, 36, 52, 53, 59]. The *Lie algebroid bracket* is the restriction of the *Courant bracket* on L . The *anchor map* $a : L \rightarrow TX$ is the restriction of the projection $TX \oplus T^*X \rightarrow TX$.

Generally speaking, the structure of Lie algebroid on a bundle L can be conveniently described by an odd 2-nilpotent vector field Q of degree 1 on

³With respect to the canonical metric on $(TX \oplus T^*X) \otimes \mathbb{C}$.

ΠL [47]. Let (x^μ, ψ^a) be coordinates in ΠL . Functions $f \in C^\infty(\Pi L)$ can be expanded in ψ^a

$$f(x, \psi) = \sum \frac{1}{k!} f_{a_1, \dots, a_k}(x) \psi^{a_1} \dots \psi^{a_k} \tag{2.2}$$

and the coefficients f_{a_1, \dots, a_k} are identified with sections of $\Lambda^k L^*$, so $C^\infty(\Pi L) = \Gamma(\Lambda^\bullet L^*)$.

The vector field Q defines the operator, which acts on the space of functions $C^\infty(\Pi L) = \Gamma(\Lambda^\bullet L^*)$ as Lie derivative. Note that any vector field Q of degree 1 can be represented as

$$Q = e_a^\mu \psi^a \frac{\partial}{\partial x^\mu} - \frac{1}{2} f_{bc}^a \psi^b \psi^c \frac{\partial}{\partial \psi^a} \tag{2.3}$$

and it corresponds to the Lie algebroid differential

$$\dots \xrightarrow{\bar{\partial}} \Lambda^k L^* \xrightarrow{\bar{\partial}} \Lambda^{k+1} L^* \xrightarrow{\bar{\partial}} \dots \tag{2.4}$$

The cohomology groups of this complex are called the Lie algebroid cohomology groups $H^k(L)$. The structure functions e_a^μ and f_{bc}^a define the anchor map $L \rightarrow TX$ and the Lie algebroid bracket. The notion of Lie algebroid generalizes the notion of Lie algebra and the tangent bundle. For a Lie algebra L , one can put $e_a^\mu = 0$ and f_{bc}^a to be the structure constants of L . For the tangent bundle, one can put e_a^μ to be the identity map and $f_{bc}^a = 0$.

For the case when L is the Lie algebroid that describes the $+i$ -eigenbundle of a generalized complex structure on X , we will use the supermanifold ΠL to construct the target space of the \mathcal{J} -model. (In the case of \mathcal{A} -model, ΠL is actually the target space.) The algebra \mathcal{A} of functions on ΠL is (super-) associative, commutative algebra with the differential Q .

The differential Q on ΠL is not enough to define a quantum field theory in the BV formalism. One also needs a BV bracket on the algebra $\mathcal{A} = C^\infty(\Pi L) = \Gamma(\Lambda^\bullet L^*)$.⁴ There is a natural notion to define simultaneously the compatible differential and the bracket on ΠL —a Lie bialgebroid [52, 53, 56, 58]. A Lie bialgebroid is a pair (L, L^*) of Lie algebroid L and its dual L^* , such that the differential $Q : \Lambda^k L^* \rightarrow \Lambda^{k+1} L^*$ of the Lie algebroid L satisfies the Leibnitz’s rule for the Lie algebroid bracket on L^* . This is the case for the pair (L, L^*) associated with generalized complex structure [36, 52, 53]. The Lie bracket $\{, \}_L$ on L can be extended to $\Lambda^\bullet L^*$. That equips the algebra $\mathcal{A} = C^\infty(\Pi L)$ with the odd Poisson bracket of degree -1 . This is the BV bracket. It satisfies the Leibnitz’s rule for the Lie algebroid differential $\bar{\partial}$. To summarize, a Lie bialgebroid structure (L, L^*)

⁴The differential $Q : \Lambda^k L^* \rightarrow \Lambda^{k+1} L^*$ is equivalent to the Lie algebroid structure on L , so Q defines a bracket on $\Lambda^\bullet L$, but not on $\Lambda^\bullet L^*$.

is equivalent to the structure on $\mathcal{A} = C^\infty(\Pi L)$ of a differential odd Poisson algebra⁵ $(\mathcal{A}, \bar{\partial}, [,]_{L^*})$ [56, 58, 59].

A (differential) *Batalin–Vilkovisky algebra* is a special case of a (differential) odd Poisson algebra. In a BV algebra [44–46], the Poisson bracket is generated by an odd 2-nilpotent differential operator of the second-order Δ , which is called *BV Laplacian* [56, 58]. It generates *BV bracket* on \mathcal{A} according to the formula

$$\{a, b\} = (-1)^{|a|} \Delta(a \wedge b) - \Delta a \wedge b - (-1)^{|a|} a \wedge \Delta b \quad \text{for } a, b \in \mathcal{A}. \quad (2.5)$$

Such an operator Δ can be naturally constructed for a generalized CY manifold as follows. The spinors of $\text{Spin}(TX, T^*X)$ are identified with all differential forms $\Lambda^\bullet(X) \equiv \Lambda^\bullet T^*X$. The canonical line bundle U_0 [16, 36] is a *pure spinor* line bundle U_0 defined as a subbundle of those spinors of $\text{Spin}(TX, T^*X)$, which are annihilated by all sections of L . In the Fock space spin representation, the U_0 is *the vacuum* line bundle, the sections of L are lowering operators and the sections of L^* are increasing operators. The canonical line bundle U_0 defines the alternative grading on $\Lambda^\bullet T^*X$ as follows:

$$\Lambda^\bullet T^*X = U_0 \oplus U_1 \oplus \cdots \oplus U_{2n},$$

where $U_k = \Lambda^k L^* \otimes U_0$. It is explained in [36] that integrability of L is equivalent to the condition $d(\Gamma(U_0)) \subset d(\Gamma(U_1))$. Then one can define generalized ∂ and $\bar{\partial}$ operators on $\Lambda^\bullet(T^*X)$ in such a way that $d = \partial + \bar{\partial}$ and $\bar{\partial} : \Gamma(U_k) \rightarrow \Gamma(U_{k+1})$ and $\partial : \Gamma(U_k) \rightarrow \Gamma(U_{k-1})$ [36]. A generalized CY manifold X is defined [16, 36] by the condition that on X exists a nowhere-vanishing closed section ρ of the canonical bundle U_0 —“*pure spinor*”. (In the case of CY manifold ρ is the holomorphic $(3, 0)$ form. In the case of a symplectic manifold ρ is $e^{i\omega}$, where ω is a symplectic structure.) Using ρ , the operators ∂ and $\bar{\partial}$ on the differential forms $\Lambda^\bullet T^*X$ can be mapped to the operators ∂ and $\bar{\partial}$ on sections of $\Lambda^\bullet L^*$ by the formula

$$\bar{\partial}(\mu \cdot \rho) = \bar{\partial}\mu \cdot \rho, \quad \partial(\mu \cdot \rho) = \partial\mu \cdot \rho \quad \text{for } \mu \in \Gamma(\Lambda^\bullet L^*). \quad (2.6)$$

To summarize, a generalized CY structure defines a differential BV algebra (\mathcal{A}, Q, Δ) where $\mathcal{A} = \Gamma(\Lambda^\bullet L^*)$, $Q = \bar{\partial}$, $\Delta = \partial$ [12, 36, 53, 55, 56, 58]. The BV bracket $\{, \}$ is defined in terms of $\Delta = \partial$ by (2.5). Since $\partial\bar{\partial} + \bar{\partial}\partial = 0$, the operator $\bar{\partial}$ is a derivation of the bracket $\{, \}$, and since $\partial^2 = 0$, the operator ∂ is a derivation of the bracket $\{, \}$.

⁵Sometimes, it is called Gerstenhaber algebra. See also [57, 66].

To describe quantum field theory in BV formalism, one also needs a measure on the space of fields \mathcal{A} . It is called *trace map* $\text{tr} : \mathcal{A} \rightarrow \mathbb{C}$ on the algebra \mathcal{A} . It satisfies

$$\text{tr}((\Delta a)b) = (-1)^a \text{tr}(a\Delta b), \tag{2.7}$$

$$\text{tr}((Qa)b) = (-1)^{a+1} \text{tr}(aQb). \tag{2.8}$$

For a generalized CY manifold, the trace map is defined by a section ρ of the canonical line bundle U_0 . Contracting with ρ , one can map an element μ of $\mathcal{A} = \Gamma(\Lambda^\bullet L^*)$ to a differential form in $\Omega^\bullet(X)$. This differential form $\mu \cdot \rho$ is also a spinor of $\text{Spin}(TX, T^*X)$. There is a natural $\text{Spin}(TX, T^*X)$ -invariant bilinear form on $\Omega^\bullet(X)$ [16, 36] with values in $\Omega^{\dim X}(X)$. It is given by the wedge product with a certain sign

$$(a, b) = \tilde{a} \wedge b, \tag{2.9}$$

where $\tilde{a} = a$ for $\deg a = 4k, 4k + 1$, and $\tilde{a} = -a$ for $\deg a = 4k + 2, 4k + 3$. This bilinear form is symmetric in dimension $4k, 4k + 1$ and antisymmetric otherwise.

Using the bilinear form $(,)$ and the canonical pure spinor ρ , we define

$$\text{tr } a \equiv \int_X (a \cdot \Omega, \Omega). \tag{2.10}$$

2.1 Examples

2.1.1 The complex case (the B-model)

A generalized complex structure \mathcal{J} that corresponds to an ordinary complex structure has the following matrix $\mathcal{J} \in \text{End}(TX \oplus T^*X)$:

$$\mathcal{J} = \begin{pmatrix} -I & 0 \\ 0 & I^\Gamma \end{pmatrix}, \tag{2.11}$$

with $+i$ -eigenbundle $L = TX^{01} \oplus T^*X^{10}$, so that $L^* = TX^{10} \oplus T^*X^{01}$. We consider the case of Calabi–Yau, then the canonical pure spinor is a nowhere vanishing holomorphic $(n, 0)$ form ρ . The algebra $\mathcal{A} = C^\infty(\Pi L) = \Gamma(\Lambda^\bullet L^*)$ is the familiar complex [3, 12, 55] of the observables of the B-model $\Lambda^\bullet L^* = \Lambda^\bullet(TX^{10} \oplus T^*X^{01}) = \Omega^\bullet(\Lambda^\bullet TX^{10})$.

The Lie algebroid differential $Q = \bar{\partial}$ is the standard Dolbeault differential $\bar{\partial} : \Omega^{-p,q} \rightarrow \Omega^{-p,q+1}$. The BV Laplacian Δ is the holomorphic divergence $\Delta = \partial \cdot : \Omega^{-p,q} \rightarrow \Omega^{-p+1,q}$. The BV bracket on \mathcal{A} is generated by $\Delta = \partial$ and can be viewed as the Lie bracket on holomorphic polyvector fields with coefficients in $(0, q)$ -forms. As explained above, the definition of Δ depends

on the existence of the *pure spinor*, or, equivalently, measure ρ on the BV manifold. For the B-model, we take the pure spinor to be the holomorphic $(n, 0)$ -form. The Lie algebroid cohomology is the Dolbeault cohomology $H_{\bar{\partial}}^q(\Lambda^p TX^{10})$.

Explicitly, let $(x^i, x^{\bar{i}})$ be complex coordinates on a CY manifold X and let $(\psi^i, \psi_{\bar{i}})$ be coordinates in the fibers of $L = TX^{10} \oplus T^*X^{01}$. Then the algebra $\mathcal{A} = C^\infty(\Pi L)$ is the algebra of functions $f(x^i, x^{\bar{i}}, \psi_i, \psi^{\bar{i}})$. If we take local coordinates, where the coefficients of the holomorphic $(n, 0)$ form are constant functions, then the BV operators Q and Δ have the following form:⁶

$$Q = \bar{\partial} = \psi^{\bar{i}} \frac{\partial}{\partial x^{\bar{i}}}, \tag{2.12}$$

$$\Delta = \partial = \frac{\partial}{\partial \psi_i} \frac{\partial}{\partial x^i}. \tag{2.13}$$

The trace map on \mathcal{A} is familiar from the standard path integral of the B-model [3] $\text{tr } a = \int_X (a \cdot \Omega) \wedge \Omega$. It can be written as a Berezian ρ on the superspace ΠL

$$\text{tr } a = \int_{\Pi L} \rho a(x^i, x^{\bar{j}}, \psi_i, \psi^{\bar{j}}), \tag{2.14}$$

where

$$\rho = \Omega_{i_1, \dots, i_n} \frac{\partial}{\partial \psi^{i_1}} \cdots \frac{\partial}{\partial \psi^{i_n}} \Omega_{j_1, \dots, j_n} dx^{j_1} \cdots dx^{j_n} dx^{\bar{k}_1} \cdots dx^{\bar{k}_n} \frac{\partial}{\partial \psi_{\bar{k}_1}} \frac{\partial}{\partial \psi_{\bar{k}_n}}. \tag{2.15}$$

2.1.2 The symplectic case (the A-model)

A generalized complex structure that corresponds to a symplectic structure ω has the following matrix $\mathcal{J} \in \text{End}(TX \oplus T^*X)$:

$$\mathcal{J} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{2.16}$$

The sections of the $+i$ -eigenbundle $L \subset TX \oplus T^*X$ are given by pairs $(X, -i\omega X)$ where $X \in \Gamma(TX)$ is an arbitrary vector field. The Lie algebroid L

⁶The expression for Q is the same in any coordinates. However, the expression for Δ involves holomorphic $(n, 0)$ -form ρ . For example, if $\rho := \rho dx^1 \wedge dx^2 \cdots \wedge dx^n$, then the divergence of the vector field μ^i is defined as $\Delta\mu = \partial_i \mu^i + \mu^i \partial_i \log \rho$. To simplify formulas, in the text below, we will always assume that the formula for Δ is written in the local coordinates where ρ is a constant. It is easy to recover the general formula, using the definition of Δ by means of the Dolbeault differential ∂ and the isomorphism map $\rho : \Lambda^\bullet(L^*) \simeq \Lambda^\bullet(TX^*)$.

of the A-model is isomorphic to the tangent bundle TX . The Lie algebroid bracket on L is mapped to the standard Lie bracket on vector fields: one can check that the restriction of the Courant bracket on L satisfies

$$[X - i\omega X, Y - i\omega Y] = [X, Y] - i\omega[X, Y] \tag{2.17}$$

for vector fields $X, Y \in \Gamma(TX)$. The Lie algebroid differential $Q: \Gamma(\Lambda^k L^*) \rightarrow \Gamma(\Lambda^{k+1} L^*)$ is mapped to the de Rham differential $d: \Omega^k(X) \rightarrow \Omega^{k+1}(X)$. The algebra $\mathcal{A} = C^\infty(\Pi L) = \Gamma(\Lambda^\bullet L^*)$ is isomorphic to the de Rham complex $\Omega^\bullet(X)$. The Lie algebroid cohomology groups $H^k(L)$ are the de Rham cohomology groups $H^k(L) = H^k_{DR}(X)$. The Lie algebroid bracket on $\Lambda^\bullet L^* \simeq \Lambda^\bullet(T^*X) = \Omega^\bullet(X)$ is the generalization of the Poisson bracket on functions to the space of differential forms $\Omega^\bullet(X)$. The BV Laplacian $\Delta: \Omega^k(X) \rightarrow \Omega^{k-1}(X)$ generates this bracket on $\Omega^\bullet(X)$ [67, 68]. Explicitly $\Delta = [\Lambda, d]$, where $\Lambda: \Omega^k(X) \rightarrow \Omega^{k-2}(X)$ is the operator of contraction with the Poisson structure ω^{-1} and $d: \Omega^k(X) \rightarrow \Omega^{k+1}(X)$ is the de Rham differential. The cohomology of Δ are called *canonical cohomology* in [67, 68].

Explicitly, let (x^i, ψ^i) be coordinates in $\Pi TX \simeq \Pi L$. Then

$$\bar{\partial} = Q = d = \psi^i \frac{\partial}{\partial x^i}, \tag{2.18}$$

$$\Lambda = \omega^{ij} \frac{\partial}{\partial \psi^i} \frac{\partial}{\partial \psi^j}, \tag{2.19}$$

and $\partial = \Delta = \delta = [\Lambda, d]$. In Darboux coordinates, where ω is a constant, one simply has $\Delta = \omega^{ij}(\partial/\partial x^i)\partial/\partial \psi^j$. In other words, Δ is the symplectic conjugate of d .

The trace map on the algebra $\mathcal{A} \simeq \Gamma(\Omega(X))$ is defined as the integral of the top degree component

$$\text{tr } a = \int_X a \quad \text{for } a \in \Omega^\bullet(X). \tag{2.20}$$

One can also take a dual point of view and consider the isomorphism $\Pi L \simeq \Pi T^*X$ induced by ω . Then in the coordinates (x^i, ψ_i) one has $\Delta = (\partial/\partial \psi_i)\partial/\partial x^i$ and $\bar{\partial} = Q = \{S, \cdot\}$ with $S = \omega^{ij}\psi_i\psi_j$.

Other types of generalized complex structure in the example of K3

Let X be a generalized CY manifold equipped with a generalized complex structure \mathcal{J} and a canonical pure spinor Ω [16, 36]. There is a notion of type of generalized complex structure \mathcal{J} [16, 36]. The type of \mathcal{J} is defined as the codimension of projection of the associated Lie algebroid L on $TX \otimes$

\mathbb{C} . The sections of L for an ordinary symplectic structure are represented by $(X^\mu, -i\omega_{\mu\nu}X^\nu) \in \Gamma(TX \oplus T^*X) \otimes \mathbb{C}$, so L is mapped on TX , so the codimension is 0, so the type is 0. The sections of L of an ordinary complex structure are represented by $(X^i, \xi_i) \in \Gamma(TX^{01}, T^*X^{10})$, so the codimension is $n = \dim_{\mathbb{C}} X$. The type 0 of an ordinary symplectic structure is the most general type, and the type n of an ordinary complex structure is the most singular type. Under deformation, the type of complex structure changes by even numbers. There is well defined notion of chirality of a given complex structure.

A canonical pure spinor of $Spin(TX, T^*X)$, which is a spinor annihilated by all sections of L , can be represented by a differential form of mixed degree in $\Omega^{odd/even}(X)$. The type is the degree of the lowest component of this differential form. For an ordinary symplectic structure $\Omega = e^{i\omega}$. For an ordinary complex structure Ω is a holomorphic $(n, 0)$ form.

Since the A-model generalized CY structure is of type 0, and the B-model generalized CY structure is of type $\dim_{\mathbb{C}} X$, we see that if $\dim_{\mathbb{C}} X$ is odd, the A-model and B-model are of different chiralities, therefore, a generalized \mathcal{J} -model in *odd* complex dimension *cannot* interpolate between the A-model and the B-model. Still it *might* interpolate if $\dim_{\mathbb{C}} X$ is even. For example, we can take the B-model and deform it by a holomorphic Poisson bivector β . This deformation on the other hand can be viewed as B-field deformation of the A-model.

For example, consider the complex dimension two, and take $\Omega_A = e^{b+i\omega}$ and $\Omega_B = \sigma$, where σ is the holomorphic $(2, 0)$ form. Now consider a deformation of B-model by the holomorphic bivector β^{ij} such that $\beta \cdot \sigma = 1$. Under this deformation $\Omega_B \rightarrow e^\beta \sigma = 1 + \sigma$. On the other hand take the symplectic structure of the A-model $\omega = \text{Im} \sigma$, and then consider deformation of this A-model with $\Omega_A = e^{i\omega}$ by the B-field $b = \text{Re} \sigma$. Under such a deformation $\Omega_A \rightarrow e^{b+i\omega} = e^\sigma = 1 + \sigma$, since $\sigma \wedge \sigma = 0$. Therefore, a holomorphic Poisson bivector β -field deformation of the B-model is the B-field deformation of the A-model⁷.

In complex dimension two, we can take, for example, X to be the $K3$ surface [71]. Then moduli space of generalized complex structures is of even

⁷In [69] it argued that the relation holds for D-branes. More precisely, let X is a complex manifold equipped with holomorphic $(2, 0)$ form $\sigma = b + i\omega$. Then the b -field transformation of the A-model (X, ω) is equivalent to the β -transformation by $\beta = \sigma^{-1}$ of the B-model. In terms of matrices of generalized complex structure for b -transformation of symplectic structure ω we have $\mathcal{J} = \begin{pmatrix} \omega^{-1}b & -\omega^{-1} \\ \omega + b\omega^{-1}b & -b\omega^{-1} \end{pmatrix}$. This is actually a β -transformation of the complex structure $I = \omega^{-1}b$. (One has $\omega + b\omega^{-1}b = 0$ as the condition for the existence of space-filling coisotropic branes in the A-model [70].)

type (which is the type of the A-model and the B-model in two dimensions), is parameterized by the Grassmanian $\text{Gr}_2(\mathbb{R}^{24})$ of real 2-planes in the real 24-dimensional space $H^\bullet(K3) = H^{\text{even}}(K3) \simeq \mathbb{R}^{24}$. A 2-plane is spanned by real and imaginary parts of the canonical pure spinor Ω . The physical $N = (2, 2)$ CFT with $K3$ -surface target space requires existence of *two* generalized orthogonal complex structures $(\mathcal{J}, \mathcal{J}')$ represented by *two* orthogonal pure spinors Ω_1 and Ω_2 . Each of these spinors spans a 2-plane in $H^\bullet(K3)$ and these planes must be orthogonal. Therefore, the moduli space of $N = (2, 2)$ CFT is locally parameterized by the Grassmanian $\text{Gr}_{2,2}(\mathbb{R}^{24})$ of two orthogonal 2-planes in \mathbb{R}^{24} . The moduli space of $N = (4, 4)$ CFTs is locally the space of 4-planes $\text{Gr}_4(\mathbb{R}^{24})$. The moduli space of $N = (2, 2)$ CFTs fibers over the moduli space of $N = (4, 4)$ CFTs. The fibers $S^2 \times S^2$ parameterize decomposition of a 4-plane into two orthogonal 2-planes [72, 73].

The topological twisting [3] makes theory to depend only on a half of this $\mathcal{N} = (2, 2)$ CFT structure. That is, the twisted theory couples only to \mathcal{J} and does not couple to \mathcal{J}' . Thus, we see that in the case of $K3$ surface, the present construction is in agreement with the traditional considerations [30, 72, 73].

3 The two-dimensional topological \mathcal{J} -model

The data (X, Q, Δ, ρ) associated with a generalized CY manifold X allow us to construct a topological \mathcal{J} -model using the AKSZ method [47].

3.1 The AKSZ construction of sigma-model for $\text{Maps}(\hat{\Sigma}, M)$ for a target space M with a PQ -structure

A PQ -manifold is a supermanifold equipped with an odd symplectic structure ω and a Hamiltonian vector field Q . In [47], a topological sigma-model was constructed for any such target space M .

Let us review the key properties of a PQ -target space. The symplectic structure defines the odd Poisson bracket $\{\cdot, \cdot\}$. Since ω is Q -invariant

$$L_Q \omega = (di_Q + i_Q d)\omega = d(i_Q \omega) = 0, \quad (3.1)$$

the one-form $i_Q \omega$ is closed. We consider the case when $i_Q \omega$ is exact, so there is a function S such that $i_Q \omega = dS$. Such a function on X is called *Hamiltonian function* for the vector field Q . For any function $f \in C^\infty(M)$, its Lie

derivative $L_Q f$ can be computed as a bracket with the Hamiltonian function S associated with Q

$$L_Q f = \{S, f\}. \tag{3.2}$$

The homological property $Q^2 = 0$ of the Q -structure can be written as the *BV classical master equation* [44–46]

$$\{S, S\} = 0. \tag{3.3}$$

A canonical example of a Q -manifold is a tangent bundle with parity reversed on the fibers $\Pi T X$. In coordinates (x^μ, ψ^μ) , one has $Q = \psi^\mu \partial_\mu$. As discussed in the previous section, the tangent bundle is an example of a Lie algebroid. The total space ΠL of any Lie algebroid is a particular case of Q -manifold with the vector field Q of degree 1.

A canonical example of a P -manifold is a cotangent bundle with parity reversed on the fibers $\Pi T^* X$. In coordinates (x^μ, π_μ) , the canonical symplectic form is $\omega(\delta x, \delta \pi) = \delta x^\mu \delta \pi_\mu$.

A PQ -manifold M can be constructed starting from any Q -manifold N as $M = \Pi T^* N$. If (x^μ, π_μ) are coordinates on $\Pi T^* N$, then for any vector field $Q = v^\mu (\partial/\partial x^\mu)$ on N , there is a Hamiltonian function $S = v^\mu \pi_\mu$ on M which generates Q on N .

Let Σ be a two-dimensional bosonic worldsheet. We extend it by fermionic directions and consider the Q -supermanifold $\hat{\Sigma} = \Pi T X$ with a canonical measure. That is, we take the total space $\Pi T \Sigma$ of the tangent bundle of Σ with parity reversed on the fibers. We equip $\hat{\Sigma} = \Pi T \Sigma$ with the standard Berezian measure. If (σ, θ) are coordinates on $\hat{\Sigma}$, the standard measure is $\rho = d\sigma^1 d\sigma^2 (\partial/\partial \theta^1) \partial/\partial \theta^2$. Functions on $\hat{\Sigma} = \Pi T \Sigma$ are differential forms $\Gamma(\Omega^\bullet(\Sigma))$. The product of functions on $\Pi T \Sigma$ is the wedge product of differential forms on Σ . For any function f on $\Pi T \Sigma$, which is a collection of differential forms $f^{(i)}$ of all degrees $f = f^{(0)} + f^{(1)} + f^{(2)}$, one can take the highest component $f^{(2)}$ and integrate over Σ , so we define the integral $\int f := \int_\Sigma f^{(2)}$. Equivalently, it is the integral over $\hat{\Sigma}$ with the standard Berezian measure

$$\int f := \int_{\hat{\Sigma}} \rho f. \tag{3.4}$$

In the following formulas, the standard Berezian measure $\rho = d\sigma^1 d\sigma^2 (\partial/\partial \theta^1) \partial/\partial \theta^2$ on the worldsheet will be omitted under the sign of integral. The Q -structure $Q_\Sigma = \theta^i \partial/\partial \sigma^i$ on $\hat{\Sigma}$ is the standard de Rham differential d on Σ .

The idea of [47] is to construct the BV structure on the space $\text{Maps}(\hat{\Sigma}, M)$ using the Q -structure Q_Σ on the worldsheet, the Q -structure Q_M on the

target space is generated by the Hamiltonian function S_M , the odd symplectic structure on the target space ω and the integral on the worldsheet $\int_{\hat{\Sigma}}$.

The space $\text{Maps}(\hat{\Sigma}, M)$ is the BV phase space of the model. The odd symplectic structure $\hat{\omega}$ on $\text{Maps}(\hat{\Sigma}, M)$ is defined by the integral of the pullback of the odd symplectic structure ω from the target space. For variations $\delta\phi_1$ and $\delta\phi_2$ of a map $\phi \in \text{Maps}(\hat{\Sigma}, M)$, we define the value of $\hat{\omega}$ on $\delta\phi_1$ and $\delta\phi_2$ to be the integral over $\hat{\Sigma}$

$$\tilde{\omega}(\delta\phi_1, \delta\phi_2) = \int_{\hat{\Sigma}} \omega(\delta\phi_1, \delta\phi_2). \tag{3.5}$$

As for the Q -structure on $\text{Maps}(\hat{\Sigma}, M)$, we can take

$$Q = Q_{\Sigma} + Q_M. \tag{3.6}$$

Any Q -structure generates diffeomorphism. There is a natural action of $\text{Diff}(\hat{\Sigma})$ and $\text{Diff}(M)$ on the space $\text{Maps}(\hat{\Sigma}, M)$. The Q -structure (3.6) corresponds to composition of two diffeomorphisms [47, 49]. For any such Q , there is a corresponding Hamiltonian function S . The function S is the *BV master action* [42, 43].

To write down S_{Σ} , it is convenient to represent the odd symplectic target space M as ΠT^*N for some manifold N . Let x^{μ} be coordinates on N and p_{μ} be coordinates on the fiber of ΠT^*N . Then x^{μ} and p_{μ} are canonically conjugate fields on $\hat{\Sigma}$. The Hamiltonian function S_{Σ} , which generates the de Rham differential $Q_{\Sigma} = d$, is

$$S_{\Sigma}[p, x] = \int_{\hat{\Sigma}} p_{\mu} dx^{\mu}. \tag{3.7}$$

The Hamiltonian function S_M for the structure Q_M is the integral over $\hat{\Sigma}$ of the pullback of S_M

$$S_M[p, x] = \int_{\hat{\Sigma}} S_M(p, x). \tag{3.8}$$

The total BV master action of the topological model is $S = S_{\Sigma} + S_Q$:

$$S[p, x] = \int_{\hat{\Sigma}} p_{\mu} dx^{\mu} + S_M(p, x). \tag{3.9}$$

Let us check that $S = S_{\Sigma} + S_M$ satisfies the BV classical master equation $\{S, S\} = 0$. That is equivalent to $\{S_{\Sigma}, S_{\Sigma}\} = 0$, $\{S_M, S_M\} = 0$ and

$\{S_\Sigma, S_M\} = 0$. The first two equations are satisfied because $Q_\Sigma^2 = 0$ and $Q_M^2 = 0$. The third equation

$$\{S_\Sigma, S_M\} = Q_\Sigma S_M = \int_{\hat{\Sigma}} dS_M = \oint_{\partial\Sigma} S_M$$

is automatically true for a closed surface. If Σ has a boundary, then in order to satisfy the BV classical master equation for S , we need to impose the boundary conditions $S_M|_{\partial\Sigma} = 0$. The model is a BV version of the Poisson sigma-model [40, 49, 64, 74–78]. The functional integral is supposed to be taken over a Lagrangian submanifold in the BV phase space [42, 43, 45, 46] of fields (p_μ, x^μ) on $\hat{\Sigma}$.

The space of functions on a PQ -manifold M with a measure is a differential BV algebra. More precisely [45, 56, 58], the algebra $C^\infty(M)$ for a general PQ -manifold is a differential odd Poisson algebra. However, it is not always a BV algebra. It is a BV algebra if M is equipped with a generator of the Poisson bracket—BV Laplacian Δ , which can be constructed by a measure. A BV Laplacian is an odd differential 2-nilpotent operator Δ of the second-order generating the Poisson bracket. The canonical example ΠT^*N of a P -manifold with coordinates (x^μ, p_μ) does have such an operator Δ . It has explicit form $\Delta = (\partial/\partial p_\mu)\partial/\partial x^\mu$.

3.2 Construction of target space with PQ -structure for a generalized CY manifold

We reviewed the AKSZ procedure [47] of constructing a topological sigma-model for a target space M with PQ -structure. Now we need to construct such a space M starting from a generalized CY manifold X .

As explained in Section 2, there is a differential BV algebra $(\mathcal{A}, Q, \Delta, \rho)$ of functions on $N = \Pi L$, which is the total space of the Lie algebroid L with parity reversed on the fibers. In other words, N is equipped with the odd Poisson bracket $\{, \}$ and the Q -structure $\bar{\partial}$. Recall that a PQ -manifold is a supermanifold equipped with an odd symplectic structure ω and a Hamiltonian vector field Q . That means that $N = \Pi L$ is a PQ -manifold if the Poisson structure is an inverse of some symplectic structure ω and Q is generated by some Hamiltonian function S .

In the AKSZ approach, the symplectic structure is required to construct the BV master action and its gauge fixed version. It is possible, however, that one could define amplitudes of the generalized topological sigma-model starting from any differential BV algebra $(\mathcal{A}, Q, \Delta, \rho)$, without assumption, that the Poisson structure generated by Δ is invertible. As we will see

later, this is indeed the case for tree level amplitudes in the large volume limit. Using only BV algebra $(\mathcal{A}, Q, \Delta, \rho)$, it is possible⁸ to define closed topological string field theory, which generalizes the Kodaira–Spencer theory of [4]. It definitely works in the genus zero and in the sector of topologically trivial maps, but it is not yet clear whether the BV algebra $(\mathcal{A}, Q, \Delta, \rho)$ completely defines the full theory, however there are some indications [10].

So far we will take more pragmatic approach and will try to reduce the problem to the AKSZ framework. The problem with the manifold $N = \Pi L$ is that the Poisson structure on it is not always invertible. For example, it is invertible in the case of the A-model, where $\Pi L \simeq \Pi T^*X$. However, it is *not* invertible in the case of the B-model, where $\Pi L = TX^{01} \oplus T^*X^{10}$. Indeed, in the coordinates $(x^i, x^{\bar{i}}, \psi^i, \psi_i)$ the odd Poisson bracket is generated by the BV Laplacian $\Delta = (\partial/\partial x^i)\partial/\partial \psi_i$. We can see that x^i and ψ_i are conjugate fields to each other, but $x^{\bar{i}}$ and $\psi^{\bar{i}}$ do not have conjugates. The Poisson structure degenerates on the fields $x^{\bar{i}}$ and $\psi^{\bar{i}}$. In this case, it is impossible to find a Hamiltonian function S , such that it generates the operator $Q = \psi^i \partial/\partial x^i$ via the Poisson bracket as $Q = \{S, \cdot\}$. The AKSZ construction cannot be used directly.

One way to solve this problem is to extend the set of fields ΠL by auxiliary fields and to make the Poisson structure nondegenerate.

To construct M , let us recall that any Poisson manifold N in the vicinity of a regular point is a symplectic fibration. Let K be the space of symplectic leaves. The Poisson bracket vanishes for functions that are constant along the leaves, i.e., for functions that depend only on K . Explicitly, near regular point of N , one can pick up the local coordinates (p_i, q^j, z^a) on N in such a way⁹ that the Poisson structure has the standard form: the only non-vanishing bracket is $\{p_i, q^j\} = \delta_i^j$. Here (p_i, q^j) are the local coordinates on the symplectic fibers and z^a are the local coordinates on the base K , i.e., $\{z^a\}$ are Casimir functions on N . Let us introduce the new fields z_a^* , which are conjugate to z^a , and then consider the manifold M with the local coordinates (p_i, q^j, z^a, z_a^*) . The manifold M is a symplectic manifold; moreover the projection $\pi : M \rightarrow N$ is a Poisson map.¹⁰ The manifold M is a symplectic realization of N [61–63], moreover $N \hookrightarrow M$ is a leaf-symplectic embedding [79]. This local construction might fail to work at non-regular points of N , where the rank of the Poisson structure is not constant, and there also could be global obstructions [79]. We leave these two important

⁸With some additional mild assumption similar to the $\bar{\partial}\partial$ -lemma, see [10].

⁹By the Darboux theorem for Poisson manifolds.

¹⁰A map $\pi : M \rightarrow N$ is called Poisson if $\pi^*(\{f, g\}_N) = \{\pi^*(f), \pi^*(g)\}_M$.

and interesting questions for the future study and assume in the present work that such symplectic realization M is well defined globally.

The vector field Q on N can be generated by a Hamiltonian function S on M , i.e., for functions on N one has $Q = \{S, \cdot\}$. The function S on M defines the vector field \tilde{Q} on M . The cohomology of \tilde{Q} on M is isomorphic to the cohomology of Q on N , so the physical states in the algebra $C^\infty(M)$ are the same as in the algebra $C^\infty(N)$.

3.2.1 Example: the B-model

Let X be a CY manifold, $\dim_{\mathbb{C}} X = n$. Consider the generalized complex structure \mathcal{J} corresponding to the complex structure on X . The Lie algebroid of the +i-eigenbundle \mathcal{J} is $L = TX^{01} \oplus T^*X^{10}$. The total space of L with fermionic fibers is the supermanifold $N = \Pi L$. The BV algebra of observables is the algebra of functions $\mathcal{A} = C^\infty(\Pi TX^{01} \oplus \Pi T^*X^{10})$, equivalently the algebra of sections of $\Lambda^k(L^*) = \bigoplus_{p+q=k} \Lambda^p(TX^{10}) \otimes \Lambda^q(T^*X^{01})$. We pick up local coordinates $(x^i, x^{\bar{i}}, \psi_i, \psi^{\bar{i}})$ on ΠL , then functions on ΠL are expanded as $f_{\bar{i}_1, \dots, \bar{i}_q}^{i_1, \dots, i_p} \psi^{\bar{i}_1} \dots \psi^{\bar{i}_q} \psi_{i_1} \dots \psi_{i_p}$. The Lie algebroid differential Q is the Dolbeault differential $\bar{\partial}$

$$Q \equiv \bar{\partial} = \psi^{\bar{i}} \frac{\partial}{\partial x^{\bar{i}}}. \tag{3.10}$$

The Q -cohomology of the algebra \mathcal{A} is the familiar space of physical observables of the B-model [3, 12].

The BV Laplacian Δ is the holomorphic divergence ∂ on holomorphic vector fields $\Lambda^p(TX^{10}) \otimes \Lambda^q(T^*X^{01})$ that can be mapped by the holomorphic $(n, 0)$ form to the Dolbeault differential ∂ on the differential forms $\Lambda^{n-p}(T^*X^{10}) \otimes \Lambda^q(T^*X^{01})$. In coordinates, where the holomorphic $(n, 0)$ form is constant, the operator Δ can be written as

$$\Delta \equiv \partial = \frac{\partial}{\partial \psi_i} \frac{\partial}{\partial x^{\bar{i}}}. \tag{3.11}$$

The Poisson bracket, generated as in (2.5) by Δ , is equivalent to the Lie bracket on holomorphic polyvector fields.

Looking at (3.11), we see that the Poisson structure is non-degenerate on the space (x^i, ψ_i) , but it is degenerate on the space $(x^{\bar{i}}, \psi^{\bar{i}})$. In other words, x^i and ψ_i are antifields to each other, but $x^{\bar{i}}$ and $\psi^{\bar{i}}$ are missing their antifields. So we extend the bundle ΠL by additional fields canonically conjugate to $x^{\bar{i}}$ and $\psi^{\bar{i}}$. Let us call them x_i^* and ψ_i^* . To summarize, M is a direct sum of the vector bundles $\Pi TX^{01} \oplus \Pi T^*X^{10} \oplus \Pi T^*X^{01} \oplus T^*X^{01}$ over the base X . If local coordinates on X are $(x^i, x^{\bar{i}})$, then the local coordinates

on the fibers of M are correspondingly $(\psi^{\bar{i}}, \psi_i, x_i^*, \psi_i^*)$. The total space of fields of the BV sigma-model is $(x^i, x^{\bar{i}}, \psi^{\bar{i}}, \psi_i, x_i^*, \psi_i^*)$. The present example of the construction of the B-model differs from the AKSZ [47],¹¹ but agrees with [21]. The odd symplectic structure ω on M has a local form

$$\omega = \delta x^i \delta \psi_i + \delta x^{\bar{i}} \delta x_i^* + \delta \psi^{\bar{i}} \delta \psi_i^*. \quad (3.12)$$

The Poisson bracket generated by (3.12) and restricted for functions on ΠL is the same as the Poisson bracket generated by (3.11). The Hamiltonian function S for the vector field Q (3.10) is quadratic in the coordinates of the fibers of M

$$S_M = \psi^{\bar{i}} x_i^*. \quad (3.13)$$

This function S_M gives us the master action of the B-model. More precisely, as explained above, the action of the B-model is obtained by pulling back the function S_M to a super Riemann surface $\hat{\Sigma}$ by means of $\text{Maps}(\hat{\Sigma}, M)$. One can also add the term S_Σ for the de Rham differential d on Σ . Finally, the BV master action of the B-model is

$$S_B = \int_{\hat{\Sigma}} \psi_i dx^i + \psi^{\bar{i}} x_i^*. \quad (3.14)$$

The fields x^i and $x^{\bar{i}}$ are treated as independent variables and the complex conjugate condition is relaxed. In the classical limit of the BV formalism, one usually considers deformations of the master action satisfying the condition $\{S, S\} = 0$ and keeps the same measure and the odd Poisson bracket on the space of antifields. In the language of the B-model, that means to deform the operator $Q = \bar{\partial}$, but to keep fixed the operator $\Delta = \partial$ and the holomorphic $(n, 0)$ -form of the base point. This issue is related to the holomorphic anomaly of the B-model [3, 4]. Given a base point in the moduli space of CY structures on X , the nearby complex structures can be parameterized as deformations of the operator $Q = \bar{\partial}$, and the corresponding observables can be identified with deformations of the master action S . However, in this approach, in the definition of the BV path integral, one keeps fixed the measure and the odd Poisson bracket defined by the base point. If the base point is changed, then the definition of the Laplacian, the odd Poisson bracket, and, thus, the measure in the functional integral in the BV phase space are changed, that is, the origin of the holomorphic anomaly [3, 4]. There is a dependence of the partition function on the base point, since the base point defines the measure in the BV path integral. It is possible that a background independent model could be formulated along the lines of [11, 16, 35, 80–82].

¹¹In their construction, the authors start from eight fields instead of six.

In the present formulation, it is convenient to study the generalized topological \mathcal{J} -model with respect to a simple base point on the moduli space of generalized complex structures. In some cases, it might be possible to take the ordinary B-model as a base point. Then topological \mathcal{J} -model is nothing else but a finite deformation of the B-model in generalized complex directions. At the classical level, it was studied in details in [12].

We will consider an explicit example of a generalized complex structure $\mathcal{J} = (I, \beta)$ described by means of an ordinary complex structure I and the holomorphic Poisson bivector β . The type of such generalized complex structure jumps where β vanishes: the description of the manifold $N = \Pi L$ is complicated. However, if we take the base point to be the ordinary complex structure I , then the full \mathcal{J} -model can be described by means of deformations of the master action S only. Explicitly $S_{\mathcal{J}} = S_I + \beta^{ij} \psi_i \psi_j$. That describes generalized deformation of the homological vector field $Q = \bar{\delta}$ in the frame where the odd Poisson bracket is fixed to be the same as defined by I . The target space and the measure of such \mathcal{J} -model is the same as of the ordinary B-model. The difference is only in the master action.

We will leave the study of the dependence of the \mathcal{J} -model on the base point for the future work. That should bring a generalization of the holomorphic anomaly equation. Physically, the holomorphic anomaly equation was explained in [83] by a change of holomorphic polarization in $H^3(X, \mathbb{C})$ with a change of the base point. In the BV formulation that corresponds to a change of the definition of the BV Laplacian Δ and the BV bracket, if one changes the base point.

3.2.2 Example: the A-model

Consider a generalized complex structure \mathcal{J} defined by a non-degenerate symplectic structure ω . The symplectic structure provides the isomorphism between the tangent TX and cotangent T^*X bundle of X . The Lie algebroid L is isomorphic to each of them, and each of TX or T^*X can be used as a model for the target space of the A-model. Let us take the $\Pi L \simeq T^*X$ and consider the canonical coordinates on it (x^i, p_i) . The odd symplectic structure is simply $\omega = \delta x^i \delta p_i$. The Q -structure is generated by the Hamiltonian function $S_M = \omega^{ij} p_i p_j$. For the A-model the BV Poisson structure is nondegenerate on $N = \Pi L$, therefore no completion of N is required. The target space of \mathcal{J} -model is simply $M = N$. The BV action of the A-model is the same as of the Poisson sigma-model [47, 75, 76, 84]

$$S = \int_{\Sigma} p_i dx^i + \omega^{ij} p_i p_j. \tag{3.15}$$

3.3 The BV gauge fixing

The quantization in the BV formalism [42–45] is done by taking path integral over a Lagrangian submanifold L in the BV phase space of the theory

$$Z = \int_L e^{-(1/\hbar)S}. \quad (3.16)$$

The partition function does not depend on a deformation of the Lagrangian submanifold L . Indeed, let us represent M as T^*L with the coordinates (x^i, p_i) . The Lagrangian manifold L is the zero section of T^*L . A Lagrangian deformation is defined by a function Ψ on L as $p_i = (\partial/\partial x^i)\Psi$. A deformation of the integral $\int_L f$ of any harmonic function $\Delta f = 0$ vanishes

$$\int_{L+\delta L} f - \int_L f = \int_L \frac{\partial f}{\partial p_i} p_i = \int_L \frac{\partial f}{\partial p_i} \frac{\partial}{\partial x^i} \Psi = \int_L \Psi \frac{\partial}{\partial x^i} \frac{\partial f}{\partial p_i} = \int_L \Psi \Delta f = 0. \quad (3.17)$$

A restriction of the BV master action S to a Lagrangian submanifold is called *gauge fixing*. Locally, a deformation of a Lagrangian submanifold L can be described by a function Ψ on L which is called the *gauge fixing fermion*.

To show the idea, let us consider the BV gauge fixing of the ordinary B-model defined on a Kahler manifold X . The Kahler metric of X will be used to conveniently describe a Lagrangian submanifold in the BV phase space of fields on $\hat{\Sigma}$. In a general case, the \mathcal{J} -model might be gauge fixed using another generalized complex structure \mathcal{J}' which commutes with \mathcal{J} . In the case of the B-model, such a generalized complex structure \mathcal{J}' is a compatible symplectic structure. In the case of A-model, such a generalized complex structure \mathcal{J}' is a compatible complex structure. The geometry of $(\mathcal{J}, \mathcal{J}')$ is a generalized Kahler geometry [36]. In [30, 38, 39], it was suggested to make a generalized topological B-model by twisting a certain sigma-model. This sigma-model *a priori* depends on both $(\mathcal{J}, \mathcal{J}')$, but dependence on \mathcal{J}' disappears after a twist. As Gualtieri showed [36] the generalized Kahler geometry $(\mathcal{I}, \mathcal{J})$ is equivalent to the data (g, b, I_+, I_-) with certain compatibility conditions, where g is an ordinary metric, B is a two-form and I_+ and I_- are the ordinary complex structures. This geometry was encountered in a study of $\mathcal{N} = (2, 2)$ CFT in [85], and recently studied in works [86–99].

From the example of the A-model (Gromov–Witten invariants) which exists on any almost Kahler manifold, where \mathcal{J} is a symplectic structure and \mathcal{J}' is a almost complex structure, we know that \mathcal{J}' does not have to be integrable. Therefore, an existence of integrable \mathcal{J}' that together with \mathcal{J}

makes a generalized Kahler geometry seems not to be generally required for a definition of the topological model. However, in the case when integrable \mathcal{J}' is used for the gauge fixing of the topological \mathcal{J} -model, one should recover a twisted version of sigma-model with generalized Kahler space like in [97].

For an illustration, let us consider the ordinary B-model. The BV master action is written as

$$S_B = \int_{\hat{\Sigma}} \mu(\psi_i dx^i + \psi^{\bar{i}} x_i^*) \tag{3.18}$$

in terms of the fields $(x^i, x^{\bar{i}}, \psi^{\bar{i}}, \psi_i, x_i^*, \psi_i^*)$ in the phase space $\text{Maps}(\hat{\Sigma}, M)$, where M was described above. A field on $\hat{\Sigma}$ is a collection of differential forms of all degrees on Σ . The symplectic pairing is given by the wedge product and integral over Σ in each of the pairs $(x^i, \psi_i), (x^{\bar{i}}, x_i^*), (\psi^{\bar{i}}, \psi_i^*)$. As in [47], we will choose a Lagrangian submanifold in two steps. First, let us algebraically choose one-half of the fields in such a way that the symplectic form vanishes on them. This set of fields will be called antifields Φ^* . The submanifold of the phase space, where all antifields vanish, is a Lagrangian submanifold $L : \Phi^* = 0$. Such naive choice of L gives a degenerate physical action. To get a nondegenerate physical action, we can deform L by a gauge fixing fermion to get

$$L_{\Psi} : \Phi^* = \frac{\partial \Psi}{\partial \Phi}. \tag{3.19}$$

For illustration purposes, let us write the usual derivatives instead of covariant, which one has to use in a non-flat case.

The full set of fields represented by differential forms on Σ is written in this diagram.¹²

$$\begin{array}{cccc}
 x^i & x_z^i & x_{\bar{z}}^i & x_{z\bar{z}}^i \\
 \psi_i & \psi_{iz} & \psi_{i\bar{z}} & \psi_{iz\bar{z}} \\
 x^{\bar{i}} & x_z^{\bar{i}} & x_{\bar{z}}^{\bar{i}} & x_{z\bar{z}}^{\bar{i}} \\
 x_{\bar{i}}^* & x_{iz}^* & x_{i\bar{z}}^* & x_{iz\bar{z}}^* \\
 \psi^{\bar{i}} & \psi_z^{\bar{i}} & \psi_{\bar{z}}^{\bar{i}} & \psi_{z\bar{z}}^{\bar{i}} \\
 \psi_i^* & \psi_{\bar{i}}^* & \psi_{i\bar{z}}^* & \psi_{i\bar{z}\bar{z}}^*
 \end{array} \tag{3.20}$$

The canonically conjugate variables are written as three pairs of rows, and in each pair of rows the conjugate fields have an opposite horizontal position. (If we enumerate the rows from 0 to 5 and the columns from 0 to 3 in this diagram, then the fields in positions $(2i, j)$ and $(2i + 1, 3 - j)$ are

¹²Hopefully, the mixing of notation x^i for a field on $\hat{\Sigma}$ and its zero degree component on Σ will not cause confusion, but will be clear from the context. If we need to distinguish a field ϕ on $\hat{\Sigma}$ from its zero component, then we will use the notation $\hat{\phi}$ for a field on $\hat{\Sigma}$.

canonically conjugate pairs of a field and antifield. As usual, they have opposite statistics.) The statistics of the zero forms $x^i, x^{\bar{i}}, \psi_i^*$ is even (bosonic). The statistics of the remaining zero forms $\psi_i, x_i^*, \psi^{\bar{i}}$ is odd (fermionic). It alternates with the degree of the differential form on Σ in the each row.

Let us make the first step and choose the physical fields Φ by a box around their symbols

$$\begin{array}{cccc}
 \boxed{x^i} & \boxed{x_z^i} & \boxed{x_{\bar{z}}^i} & x_{z\bar{z}}^i \\
 \boxed{\psi_i} & \psi_{iz} & \psi_{i\bar{z}} & \psi_{iz\bar{z}} \\
 \boxed{x^{\bar{i}}} & \boxed{x_z^{\bar{i}}} & \boxed{x_{\bar{z}}^{\bar{i}}} & x_{z\bar{z}}^{\bar{i}} \\
 \boxed{x_i^*} & \boxed{x_{iz}^*} & \boxed{x_{i\bar{z}}^*} & x_{iz\bar{z}}^* \\
 \boxed{\psi^{\bar{i}}} & \boxed{\psi_z^{\bar{i}}} & \boxed{\psi_{\bar{z}}^{\bar{i}}} & \boxed{\psi_{z\bar{z}}^{\bar{i}}} \\
 \psi_i^* & \psi_{iz}^* & \psi_{i\bar{z}}^* & \psi_{iz\bar{z}}^*
 \end{array} \tag{3.21}$$

In each pair of conjugate fields, only one field is boxed, therefore the boxed fields make a Lagrangian submanifold in the full BV phase space. The boxed fields Φ are left in the theory after the gauge fixing. At the second step, the unboxed fields Φ^* are expressed in terms of Φ as in (3.19) via a suitable gauge fixing function $\Psi(\Phi)$.

Let us take $\Psi(\Phi)$ similar to the ordinary B-model [3]

$$\Psi = g_{i\bar{j}}(x_z^i \partial_{\bar{z}} x^{\bar{j}} + x_{\bar{z}}^{\bar{i}} \partial_z x^i). \tag{3.22}$$

That gives

$$x_{i\bar{z}\bar{z}}^* = g_{i\bar{i}}(\partial_z x_{\bar{z}}^i + \partial_{\bar{z}} x_z^i), \tag{3.23}$$

$$\psi_z^i = g_{i\bar{i}} \partial_{\bar{z}} x^{\bar{i}}, \quad \psi_z^{\bar{i}} = g_{i\bar{i}} \partial_z x^i. \tag{3.24}$$

The remaining unboxed fields are zero on the Lagrangian L_Ψ . We get the physical Lagrangian of the B-model¹³

$$(\hat{\psi}_i d\hat{x}^i + \hat{x}_i^* \hat{\psi}^i)|_{L_\Psi} = [g_{i\bar{i}}(\partial_{\bar{z}} x^{\bar{i}} \partial_z x^i + \partial_z x^{\bar{i}} \partial_{\bar{z}} x^i) + \psi_i(\partial_z x_{\bar{z}}^i - \partial_{\bar{z}} x_z^i) \psi^{\bar{i}}] \tag{3.25}$$

$$+ [g_{i\bar{i}} \psi^{\bar{i}}(\partial_z x_{\bar{z}}^i + \partial_{\bar{z}} x_z^i) + x_i^* \psi_{z\bar{z}}^{\bar{i}} + x_{i\bar{z}}^* \psi_z^{\bar{i}} + x_{i\bar{z}}^* \psi_z^{\bar{i}}] \tag{3.26}$$

The last three quadratic terms can be physically interpreted as auxiliary. In the remaining action, we recognize that of [3] with a change of notations $x_z^i \rightarrow \rho_z^i, x_{\bar{z}}^{\bar{i}} \rightarrow \rho_{\bar{z}}^{\bar{i}}, \psi^{\bar{i}} \rightarrow \eta^{\bar{i}}, \psi_i \rightarrow \theta_i$. The algebra of physical observables is

¹³The present construction of the B-model differs from [47]. In [47], the BV action was written in terms of $8n$ fields on $\hat{\Sigma}$ corresponding to the coordinates in the target space $\Pi T^* \Pi T X$. The present construction takes the target space M to be a certain extension of $N = \Pi L$ that has a nondegenerate symplectic structure. The dimension of the target space M in this work is $6n$ compared to $8n$ in [47].

given by the Q -cohomology of $C^\infty(M)$, which is the same as Q -cohomology of $C^\infty(N)$, which is the Q -cohomology of the algebra of functions of $(x^i, x^{\bar{i}}, \psi^{\bar{i}}, \psi_i)$. This is the same space as [3] where the primary observables $\mathcal{O}^{(0)}$ were identified with the Q -cohomology of functions of $(x^i, x^{\bar{i}}, \eta^{\bar{i}}, \theta_i)$.

One can also see that the deformation of the master action of the B-model by a holomorphic function $f(x)$ corresponds to the deformation of the operator Q to the topological Landau–Ginzburg model with the superpotential $f(x)$. The physical observables are identified with cohomologies of $Q = \bar{\psi}^i \partial / \partial x^{\bar{i}} + (\partial f / \partial x^i) \partial / \partial \psi_i$. For polynomial $f(x)$ on $X = \mathbb{C}^n$, the cohomology of Q in degree 0 is the polynomial ring $\mathbb{C}[x^i]$ factorized over the ideal generated by $\partial_i f(x)$. The ring $\mathbb{C}[x^i] / df(x^i)$ is the familiar ring of observables of the topological Landau–Ginzburg model [100].

3.4 Observables and deformations

What are the observables of the model? In the path integral formulation of a quantum field theory, observables can be associated with deformations of the action. If the theory is gauge-invariant, then the deformed action must be again a gauge-invariant functional.

Let us see what happens concretely in the BV approach [45, 47]. After the gauge fixing, the partition function is the integral over a Lagrangian submanifold in the BV phase space

$$Z = \int_L e^{-(1/\hbar)S[\phi]}. \tag{3.27}$$

The gauge invariance of the theory means that Z does not change with a change of the gauge fixing condition. Thus, the partition function Z has to be invariant under deformations of the Lagrangian submanifold L in the space of fields. For any function f , the integral $\int_L f$ is invariant under such deformations if $\Delta f = 0$. Taking $f = e^{-(1/\hbar)S}$, one obtains BV *quantum master equation* $\Delta e^{-(1/\hbar)S} = 0$, or

$$-\hbar \Delta S + \frac{1}{2} \{S, S\} = 0. \tag{3.28}$$

Let us assume that S_0 satisfy (3.28) and let us consider a deformation $S_0 \rightarrow S_0 + \delta S$. We obtain

$$-\hbar \Delta(S_0 + \delta S) + \frac{1}{2} \{S_0 + \delta S, S_0 + \delta S\} = 0, \tag{3.29}$$

so

$$-\hbar \Delta \delta S + Q \delta S + \frac{1}{2} \{\delta S, \delta S\} = 0, \tag{3.30}$$

where we used the definition of the operator $Q \cdot = \{S_0, \cdot\}$. In the limit $\hbar = 0$, the BV quantum master equation becomes *the BV classical master equation*

$$Q\delta S + \frac{1}{2}\{\delta S, \delta S\} = 0, \quad (3.31)$$

which is also well known under the name Maurer–Cartan equation in deformation theory. Deformation of S corresponds to the deformation of Q according to the definition $Q = \{S, \cdot\}$. The classical BV equation is the homological property $Q^2 = 0$. Not all deformations of S lead to a new theory. Consider deformation of S by a diffeomorphism that preserves the symplectic structure, that is, by a Hamiltonian vector field. The vector field, generated by a function δS , acts on S as $S \rightarrow S + \{\delta S, S\}$. We see that such diffeomorphisms corresponds to deformations S of the form $Q\delta S$. The theory is invariant under Q -exact deformations $S \rightarrow S + Q\delta S$. From (3.31), infinitesimal deformations have to be Q -closed $Q\delta S = 0$. Therefore, the physical space of non-equivalent gauge-invariant infinitesimal deformations (=infinitesimal observables) is the cohomology group of the operator Q . On the moduli space of physical non-equivalent theories, infinitesimal observables can be viewed as vector fields.

Example. Consider the point in the moduli space of generalized complex structures corresponding to the ordinary complex structure—the B-model. The space of functions $C^\infty(\Pi L)$ is the space of $\Omega^{-p,q}$ forms $\mu_{\bar{i}_1 \dots \bar{i}_q}^{j_1 \dots j_p}$, the operator Q is $\bar{\partial}$, the Laplacian Δ is the holomorphic divergence ∂ and the bracket $\{, \}$ is the Lie bracket on the holomorphic polyvector fields. The deformations S are functions on $C^\infty(\Pi L) = \Gamma(\Lambda^p T X^{10} \otimes \Lambda^q T^* X^{01}) \cong \Omega^{-p,q}(X)$.

The case $(p, q) = (-1, 1)$ corresponds to an ordinary deformation of the complex structure by a Beltrami differential $\bar{\partial} \rightarrow \bar{\partial} + \mu \cdot \partial$. Equation (3.31) is the requirement that the new Dolbeault operator $\bar{\partial}$ squares to zero. It is called the Kodaira–Spencer equation.

The other deformations of degree 2, the types $(-2, 0)$ and $(0, 2)$, are geometrical deformations of the underlying Lie bialgebroid structure (L, L^*) . Indeed, functions S on ΠL of degree 2 by the relation $Q = \{\mu, \cdot\}$ correspond to operators Q of degree 1. Such an operator defines a Lie algebroid structure on the vector bundle L . Deformations S of other degrees are not described by a Lie algebroid structure and do not correspond to a generalized complex structure. The moduli space of generalized complex structures is locally generated by deformations (observables) of degree 2. The moduli space of deformations of arbitrary degrees is the *extended* moduli space [3, 12].

Consider a $(-2, 0)$ deformation by a bivector field β^{ij} . Contrary to the $(-1, 1)$ case, the equation $\bar{\partial}\beta + 1/2\{\beta, \beta\} = 0$ is now equivalent to two separate equations $\bar{\partial}\beta = 0$ and $\{\beta, \beta\} = 0$, since the first term has the grade $(-2, 1)$, whereas the second term has the grade $(-3, 0)$. The equation tells us that β^{ij} is the holomorphic bivector. The second condition tells us that β^{ij} is holomorphic Poisson. Therefore, the space of $(-2, 0)$ deformations is the space of holomorphic Poisson bivector fields. In Section 4, we consider the open B-model deformed by β^{ij} .

The space of $(0, 2)$ deformations is the space of $\bar{\partial}$ -closed $(0, 2)$ forms $b_{\bar{i}\bar{j}}$.

3.5 The extended moduli space

Here we will review the geometrical structure of the extended moduli space.

Witten [3] introduced the notion of the *extended moduli space*. Baranikov and Kontsevich [12] showed that the extended moduli space \mathcal{M} has a structure of *Frobenius manifold*: there is a potential Φ and metric g_{ij} on \mathcal{M} such that the structure functions $C^i_{jk} = g^{il}C_{ljk}$, obtained from the third-derivative $C_{ijk} = \partial_{ijk}\Phi$, define an associative product on the space of vector fields. In [101], the same property was shown for the moduli space of generalized complex structure. Manin [55] showed that the Frobenius structure naturally appears on the moduli space of deformations of any differential BV algebra.¹⁴ Since any generalized CY manifold is associated with a differential BV algebra, the moduli space of generalized complex structures is also Frobenius manifold. Physically, the structure function C_{ijk} is the three-point function of the topological \mathcal{J} -model in genus zero.

We consider a quantum theory in the BV formalism $\int_L e^{-(1/\hbar)S}$. The space of BV functionals is differential BV algebra $(\mathcal{A}, Q, \Delta, \text{tr})$, where Q and Δ are the corresponding BV operators as defined above. The trace map is defined in terms of the path integral \int_L over some Lagrangian submanifold L in the BV phase space.

Consider a deformation $S_0 \rightarrow S_0 + \delta S$. We observed above that δS must satisfy the quantum master equation

$$-\hbar\Delta\delta S + Q\delta S + \frac{1}{2}\{\delta S, \delta S\} = 0. \tag{3.32}$$

Looking at this equation, one can keep in mind the familiar case of the ordinary B-model, where $Q = \bar{\partial}$ and $\Delta = \partial$. All arguments are parallel.

¹⁴With an additional technical requirement similar to the $\bar{\partial}\partial$ -lemma.

Let us require that δS solves both the quantum and the classical BV equations, so $\Delta\delta S = 0$.¹⁵ Note that since Δ is of degree -1 , Q is of degree 1 , and $\{, \}$ is of degree -1 , then for geometrical deformations δS of degree 2 the equations $Q\delta S + (1/2)\{\delta S, \delta S\} = 0$ and $\Delta\delta S = 0$ follow automatically from the BV quantum master equation. Let us denote $\delta S = \mu$, $Q = \bar{\partial}$ and $\Delta = \partial$. We still remember that $(\mathcal{A}, \bar{\partial}, \partial, \text{tr})$ is an arbitrary differential BV algebra corresponding to an arbitrary generalized complex structure. The equations on $\mu \in \mathcal{A}$ are written as

$$\bar{\partial}\mu + \frac{1}{2}\{\mu, \mu\} = 0, \tag{3.33}$$

$$\partial\mu = 0. \tag{3.34}$$

Infinitesimal deformations μ are represented by cohomology classes of $\bar{\partial}$ restricted to the kernel of ∂ . What about finite deformations? Let $\mu = x + a$, where x is a harmonic representative of a $\bar{\partial}$ -cohomology and a is a ∂ -exact correction that we have to find. The equation on a is

$$\bar{\partial}a + \frac{1}{2}\{x + a, x + a\} = 0. \tag{3.35}$$

Let us use the definition of the BV bracket (2.5) in terms of the BV Laplacian ∂ . Since x is harmonic and a is ∂ -exact, one has $\partial(x + a) = 0$, therefore

$$\bar{\partial}a + \frac{1}{2}\partial((x + a) \cdot (x + a)) = 0. \tag{3.36}$$

Then we can get a formula to recursively solve for a order in order in x :

$$a = -\frac{1}{2}\bar{\partial}^{-1}\partial((x + a) \cdot (x + a)) \tag{3.37}$$

This Kodaira–Spencer method was used in [4] in the context of deformations of the B-model. It works exactly in the same way for any differential BV algebra [55]. The harmonic representatives x of the cohomology classes physically correspond to the external background. The recursive solution of (3.37) can be drawn in terms of tree level Feynman diagrams [4].

The solution $\mu(x)$ exists if we assume that the analogue of $\partial\bar{\partial}$ -lemma holds. The solution defines a map from the $\bar{\partial}$ -cohomology $H_{\bar{\partial}}^{\bullet}(\mathcal{A})$ to elements of the BV algebra which satisfy the classical and the quantum BV master equation. The master action $S_0 + \mu$ solves simultaneously the classical and quantum BV master equations, and thus corresponds to a topological model.

¹⁵This restriction appeared in [4] in the target-space formulation of the B-model (Kodaira–Spencer theory) from the string field theory point of view. The target space fields $\mu \in \Gamma(\Omega^{-p,q})$ in [4] were restricted by the condition $\partial A = 0$, which was interpreted as a condition for a string field to be in b_0 cohomology.

In other words, the solution $\mu(x)$ provides us with linear coordinates $H_{\bar{\partial}}^{\bullet}(\mathcal{A}) \simeq TM_{S_0}$ on the extended moduli space. The definition of this linear structure depends on the base point $S_0 \in \mathcal{M}$. The linear coordinates on the classical geometrical moduli are called *canonical coordinates* in the context of the B-model in [4].

Example. Let us consider $X = T^2$. A generalized complex structure on T^2 could be only of the ordinary or symplectic type. As a reference point, let us take the complex structure with a period τ . The linear coordinates in \mathcal{M} are bosonic for $H^{0,0}$ and $H^{-1,1}$ and fermionic for $H^{-1,0}$ and $H^{0,1}$. The space $H^{-1,1}(X)$ describes ordinary deformations of complex structure by Beltrami differentials. The space $H^{0,0}$ describes constant terms that could be added to the BV master action. Such constant terms correspond to a dilatation of the BV measure.

A deformation of a complex structure of a CY manifold $\partial_{\bar{j}} \rightarrow \partial_{\bar{j}} + \mu_{\bar{j}}^i \partial_i$ corresponds to a deformation of the holomorphic $(n, 0)$ -form $\Omega \rightarrow e^{-\mu} \Omega$.¹⁶ For T^2 let (x, y) be real coordinates on the standard unit square. The complex structure τ_0 corresponds to the complex coordinate $z = x + \tau_0 y$ and the holomorphic $(1, 0)$ form $\Omega_0 = dz = dx + \tau_0 dy$. The periods are $a_0 = \int_A dz = 1$ and $b_0 = \int_B dz \tau_0$. Let $\alpha \equiv \mu^{0,0}$ and $\beta \equiv \mu^{-1,1}$. After the deformation $\Omega = e^{-\mu} \Omega_0 = e^{-\alpha} (dz - \beta d\bar{z})$. The new periods are $a = e^{-\alpha} (1 - \beta)$ and $b = e^{-\alpha} (\tau_0 - \beta \bar{\tau}_0)$. The new complex structure is $\tau = b/a = (\tau_0 - \beta \bar{\tau}_0)/(1 - \beta)$. If we assume that the definition domain of τ is the upper half plane (neglecting discrete $SL(2, \mathbb{Z})$ transformations), then the domain for $\beta = (\tau - \tau_0)/(\tau - \bar{\tau}_0)$ is the unit disk $D : |\beta| < 1$. The boundary $|\beta| = 1$ is the degenerate complex structure $\text{Im } \tau = 0$. The scale is parameterized by a complex plane \mathbb{C} . The total bosonic moduli space is $D \times \mathbb{C}$.

In this example, deformations of the complex structure were mapped to deformations of the canonical holomorphic form $\Omega_0 \rightarrow e^{-\mu} \Omega_0$. The same can be done in the extended moduli space. The coordinates are the *periods* $X^i(\mu) = \int_{A^i} \Omega$, where A^i is a basis of cycles $H_{\bullet}(X)$. The formula $X^i(x|\Omega_0) = \int_{A^i} e^{-\mu(x)} \Omega_0$ gives a *period map* $H^{\bullet}(L) \rightarrow H^{\bullet}(X, \mathbb{C})$. The dimension of $H^{\bullet}(L)$ and $H^{\bullet}(X, \mathbb{C})$ is the same (recall that multiplication map by the canonical pure spinor Ω_0 in the generalized CY case provides isomorphism

¹⁶A generalized complex structure \mathcal{J} defines +i-eigenbundle $L = TX^{01} \oplus T^*X^{10}$. The term TX^{01} corresponds to the \bar{z} -direction in the tangent space. It is described by the vector field $\partial_{\bar{i}}$. The deformation by $\mu_{\bar{j}}^i$ is a deformation of the \bar{z} -direction in the tangent bundle. A new \bar{z} -direction $\bar{\partial}_{\bar{i}}$ in the old basis is represented by the vector field $\partial_{\bar{j}} + \mu_{\bar{j}}^i \partial_i$. The new holomorphic one-forms dz must annihilate antiholomorphic vector fields $\partial_{\bar{j}} + \mu_{\bar{j}}^i \partial_i$. Therefore, they can be locally written in the old basis as $dz^i - \mu_{\bar{j}}^i dz^{\bar{j}}$. Therefore, a new holomorphic $(n, 0)$ form Ω is $e^{-\mu} \Omega_0$ up to a scale.

between $\Lambda^\bullet(L^*)$ and $\Omega^\bullet(X, \mathbb{C})$). The formula $\Omega_0 \rightarrow e^{-\mu}\Omega_0$ can be interpreted as the action in the spinor representation of the group element $e^{-\mu}$ that corresponds to the algebra element μ . The bosonic part of \mathcal{M} is mapped to $H^{\text{odd/even}}(X)$ if Ω_0 is of the odd/even type.

The dimension counting in the T^2 example is as follows. There are two complex bosonic directions in \mathcal{M} : deformations β of the complex structure and rescaling α of the canonical pure spinor Ω_0 . They are mapped into $H^1(X)$ and parameterized by two periods a and b .

Usually the B-model is considered in $\dim_{\mathbb{C}} X = 3$. What is special in the geometry of the extended moduli space if $\dim_{\mathbb{C}} X = 3$? Let the reference point be an ordinary CY manifold X with a holomorphic $(3, 0)$ -form Ω_0 . The geometrical deformations of the generalized complex structure (which are of degree 2) and dilatations (which are of degree 0) are parameterized by $H^{-2,0} \oplus H^{-1,1} \oplus H^{0,2}$ and $H^{0,0}$. Using the map by Ω_0 , this space is isomorphic to $H^{1,0} \oplus H^{2,1} \oplus H^{3,2}$ and $H^{3,0}$. If $\dim_{\mathbb{C}} X = 3$, the space of geometrical deformations of generalized CY structure is the half of all bosonic deformations $H^{\text{odd}}(X)$. Therefore, precisely in dimension 3, the *geometrical* moduli space of generalized CY structures can be parameterized by half of all odd periods $X^i = \int_{A_i} \Omega$, where A_i is a basis in $H_{\text{odd}}(X)$.

In the case of *ordinary* CY structures in dimension 3, this is a very well-known parameterization by *half of all 3-cycles* A_i that make basis in $H_3(X)$

$$X^i = \int_{A_i} \Omega_0. \tag{3.38}$$

The complementary periods F_i are defined for the dual¹⁷ 3-cycles B^i

$$F_i = \int_{B^i} \Omega_0. \tag{3.39}$$

Here F_i are functions of X^i .

In the case of *generalized* CY structures in dimension 3, one needs to consider the *half of all odd cycles*. We can choose a basis $A_i \circ B^j = \delta_i^j$, $A_i \circ A_j = B^i \circ B_j = 0$, where $A_i, B^j \in H_1 \oplus H_3 \oplus H_5$ and write

$$X^i = \int_{A^i} \Omega, \quad F_i = \int_{B^i} \Omega, \tag{3.40}$$

where Ω_0 is the canonical pure spinor on X [16, 36]. X^i can be taken as coordinates on the generalized geometrical moduli space of \mathcal{J} -model. As usual, one can get rid of scale and consider the projective coordinates $(X^0 : X^1 : \dots : X^{h_{1,0}+h_{2,1}+h_{3,2}})$. Alternatively, instead of X^i , one can use

¹⁷We use the basis $A_i \circ A_j = B_i \circ B_j = 0$, $A_i \circ B^j = \delta_i^j$.

$(\text{Re } X^i, \text{Re } F_i)$. It was shown by Hitchin [16] that this parameterization of generalized CY structures by $H^{\text{odd}}(X, \mathbb{R})$ is not degenerate. He also constructed a certain explicit map¹⁸ that takes a real 1+3+5-form ρ and gives a real 1+3+5-form $\hat{\rho}$, such that the spinor $\Omega = \rho + i\hat{\rho}$ is pure and closed. Therefore, Ω defines a generalized complex structure.

Moreover, Hitchin [16] showed that for $\dim_{\mathbb{C}} X = 3$ the moduli space of *generalized CY structures* is a special Kahler manifold. A special Kahler geometry [102, 103] is a familiar attribute of $\mathcal{N} = 2$ theories [104, 105] and integrability [11, 106].

Intrinsically, a *special Kahler* [102, 103, 105, 107] structure¹⁹ on a Kahler manifold (M, J, ω) is defined as a flat torsion-free connection ∇ on TM such that (i) the symplectic structure is covariantly constant $\nabla\omega = 0$ and (ii) the complex structure is d_{∇} -closed $d_{\nabla}J = 0$. This definition is related with the commonly used extrinsic definition of a special Kahler geometry by means of a holomorphic prepotential \mathcal{F} as follows. Consider a $2n$ -complex dimensional space $V = \mathbb{C}^{2n}$ with a canonical holomorphic symplectic form

$$\Omega = dX^i \wedge dF_i, \tag{3.41}$$

where (X^i, F_i) are coordinates on \mathbb{C}^{2n} . Consider a Hermitian form $h(u, v) = -i\Omega(u, \bar{v})$ for $u, v \in V$. Then any special Kahler manifold M locally is realized as a submanifold $\phi : M \rightarrow \mathbb{C}^{2n}$ such that (i) it is holomorphic, (ii) the restriction ϕ^*h of the Hermitian form on M is nondegenerate and (iii) it is holomorphic Lagrangian $\phi^*\Omega = 0$. The conditions (i) and (ii) give a Kahler structure ϕ^*h on M and the condition (iii) gives the *special* property of the Kahler structure. In real coordinates (x^i, y^i, u^i, v^i) , where $X^i = x^i + iy^i$ and $F_i = u_i + iv_i$, we have the Kahler form $\omega = -\text{Im } \phi^*h = dx^i \wedge du_i + dy^i \wedge dv_i$. Since M is Lagrangian, $\phi^*\Omega = 0$, and since $\text{Re } \Omega = dx^i \wedge du_i - dy^i \wedge dv_i$, the Kahler form on M is

$$\omega = 2dx^i \wedge du_i. \tag{3.42}$$

Since $du_i = \text{Re } dF_i = \text{Re}(F_{ij}dz^j) = \text{Re } F_{ij}dx^j - \text{Im } F_{ij}dy^j$, we get

$$\omega = -2 \text{Im } F_{ij}dx^i \wedge dy^j = -i \text{Im } F_{ij}dz^i \wedge \overline{dz^j}. \tag{3.43}$$

¹⁸It is a critical point of a certain functional. See [82] for a one-loop study of that functional.

¹⁹A special Kahler geometry is sometimes called “rigid special” Kahler geometry (global $\mathcal{N} = 2$), whereas a projective special Kahler geometry is called “local special” Kahler geometry (local $\mathcal{N} = 2$). A projective special Kahler can be obtained from a special Kahler that admit a certain C^* -action.

The symplectic form ω on M can also be expressed in terms of the Kahler potential $\omega = i\partial\bar{\partial}K$, where

$$K = -\text{Im}(F_i\overline{X^i}). \tag{3.44}$$

The real part of the Kahler form $\text{Re } \phi^*h$ is the Kahler metric on M . In real flat coordinates ($x = \text{Re } X, u = \text{Re } F$), the metric g_{ij} is given by the second derivative $g_{ij} = \partial_i\partial_j K$.

The real coordinates $x^i = \text{Re } X^i, u_i = \text{Im } F_i$ define a flat symplectic torsion-free connection ∇ on M by the condition $\nabla dx^i = 0, \nabla du_i = 0$. This connection can be extended to the complexified tangent bundle $TM \otimes \mathbb{C}$, and then $(0, 1)$ part of ∇ is the $\bar{\partial}$ -operator $\nabla = \bar{\partial}$, and Christoffel symbol of $(1, 0)$ part of ∇ is $\partial_k\tau_{ij} = \partial_{kij}\mathcal{F}$, so this is the familiar connection on the moduli space of an ordinary topological B-model [4]. The moduli space of generalized CY structures for $\dim_{\mathbb{C}} X = 3$ has exactly the same geometry.

Locally, a holomorphic Lagrangian submanifold $M \subset \mathbb{C}^{2n}$ is defined by a holomorphic function $\mathcal{F}(X_i)$ and equation $F_i = \partial\mathcal{F}/\partial X_i$. The X^i are holomorphic coordinates on M . They are called special coordinates [4].

To summarize, in complex dimension 3, the *geometrical moduli space* of the generalized \mathcal{J} -model is precisely half-dimensional subspace of the bosonic part of the *extended moduli space*. This half-dimensional subspace is realized as a holomorphic Lagrangian submanifold with respect to the canonical holomorphic structure in $H^{\text{odd/even}}(X, \mathbb{C})$. It is Lagrangian, because in complex dimension 3, the canonical symplectic form²⁰ on \mathcal{M} vanishes for geometrical deformations.

Note that the definition of *special* and *canonical* coordinates is very different. The canonical coordinates x are defined in tangent space to the extended moduli space $T\mathcal{M}_{\Omega_0}$ at the reference point Ω_0 . A finite deformation is given by solution $\mu(x)$ of the BV-Maurer–Cartan equation (3.33). As we will see, the holomorphic metric $\text{tr } \mu_a\mu_b$ in the extended moduli space is a constant matrix in canonical coordinates.

On the other hand, the *special* coordinates are defined only for $\dim_{\mathbb{C}} X = 3$ and only for *geometrical* deformations of generalized CY structures. In this case, the geometrical moduli space is the half-dimensional holomorphic

²⁰ $\omega(\mu_1, \mu_2) = \text{tr } \mu_1\mu_2$ for $\mu_1, \mu_2 \in \mathcal{A}$. It is non-zero only if $\deg \mu_1 + \deg \mu_2 = 2n$, where $\dim_{\mathbb{C}} X = n$. The geometrical deformations by definition are of degree 0 and 2. That is a half of all bosonic deformations of degrees 0, 2, 4, 6 if $\dim_{\mathbb{R}} X = 6$. And that is a Lagrangian submanifold since the degrees 0 and 2 cannot make 6, thus the symplectic structure on \mathcal{M} vanishes.

Lagrangian submanifold of the extended moduli space. The special coordinates are convenient to compare the generalized \mathcal{J} -model and the $\mathcal{N} = 2$ four-dimensional supergravity in the type IIA/B string compactifications on X [95, 108–112].

There are other related reasons why the six real dimensions is a special case. Nekrasov [81] conjectures Z -theory (a topological analogue of M -theory), which should provide a non-perturbative completion of topological strings. It is possible that this theory should be formulated in terms of some G_2 theory on a real seven-dimensional manifold X_7 . The six-dimensional theory on X_6 arises after compactification of one dimension for $X_7 = S^1 \times X_6$ (see also [11, 80]).

Geometrically, infinitesimal observables of the \mathcal{J} -model are vector fields $\Gamma(T\mathcal{M})$. Using the trace map of the BV algebra \mathcal{A} , we can define n -point function of observables. Let $x^i \in H_{\bar{0}}^\bullet(\mathcal{A}) \simeq T\mathcal{M}_{\Omega_0}$ be local coordinates on the moduli space \mathcal{M} near the point $\Omega_0 \in \mathcal{M}$. The n -point function of vector fields v_1^i, \dots, v_n^i is defined in terms of the trace map tr and the map $\mu(x)$

$$\langle v_1, \dots, v_n \rangle = v_1^{i_1} \cdots v_n^{i_n} \text{tr} \partial_{i_1} \mu(x) \partial_{i_2} \mu(x) \cdots \partial_{i_n} \mu(x). \quad (3.45)$$

The one-point function defines the identity vector field, the two-point function defines the metric on \mathcal{M} and the three-point function defines the Frobenius structure [55]. In the *canonical* coordinates x , the metric is a constant matrix

$$g_{ij} = \text{tr} \partial_i \mu(x) \partial_j \mu(x) = \text{tr} \partial_i (x + a(x)) \partial_j (x + a(x)) = \text{tr} \partial_i x \partial_j x. \quad (3.46)$$

The terms with $a(x)$ vanish since they contain a product of ∂ -exact and ∂ -closed terms, but $\text{tr}(\partial a)b = \pm \text{tr} a(\partial b)$. The three-point function is not generally constant.

The n -point function is the n -point function of the toy version of the \mathcal{J} -model on the zero-dimensional worldsheet—point. The space of maps is simply the target space M and the BV algebra of the model is the BV algebra \mathcal{A} of functions on M .

Of course, the \mathcal{J} -model on a two-dimensional worldsheet is more complicated. Instead of M , one has to consider the space $\text{Maps}(\tilde{\Sigma}, M)$.

We begin with the study of topologically trivial maps in genus zero. By topologically trivial we mean maps which are homotopic to constant maps. The three-point function is given by the same formula (3.45), since the moduli space of Riemann surfaces of genus zero with three marked points is a point, and the BV path integral is reduced to the integral over M . Let us consider the simple case in more details.

3.6 The topologically trivial maps in genus zero

This problem was essentially solved already in [4] where the Kodaira–Spencer action, evaluated in a critical point, was suggested as a free energy of B-model in genus zero. Geometrically, the moduli space of the \mathcal{J} -model is a Frobenius manifold. The construction was generalized for the extended moduli space of B-model in [12] and for the moduli space of any differential BV algebra in [55]. Generalized CY structure is associated with a differential BV algebra, so by the construction of [55] the moduli space of generalized CY structures is also Frobenius. Thus, it is automatically equipped with a certain potential function Φ . The derivative Φ_{ijk} is the three-point function, which was defined above in terms of the trace map tr and $\mu(x)$ -map. Li [101] directly studied the moduli space of generalized CY structures.

Let us review the generalized Kodaira–Spencer construction. Let Ω_0 in \mathcal{M} be the reference point and $x \in T\mathcal{M}_{\Omega_0} = H^\bullet(L_{\Omega_0})$ be the canonical coordinates. The three-point function C_{ijk} is defined as

$$C_{ijk}(x|\Omega_0) = \text{tr}_{\Omega_0} \mu_i \mu_j \mu_k, \tag{3.47}$$

where $\mu_i \equiv \partial\mu(x)/\partial x^i$. The index tr_{Ω_0} is written to remember that the trace map (2.10) is defined in terms of the canonical pure spinor Ω_0 of the *reference* point in \mathcal{M} .

Is there a potential function $\Phi(x|\Omega_0)$ such that the three-point function is the third derivative? That is

$$C_{ijk}(x|\Omega_0) = \partial_i \partial_j \partial_k \Phi(x|\Omega_0). \tag{3.48}$$

The answer is very natural from the BV point of view [44].

Namely, let us consider the classical BV master equation

$$\bar{\partial}\mu + \frac{1}{2}\{\mu, \mu\} = 0 \tag{3.49}$$

for deformations of the master action $S_0 \rightarrow S_0 + \mu$. We can think about this equation in analogy with the flat curvature equation $dA + A \wedge A = 0$ for a connection A . What is the functional on a space of connections that has the equation of motion $dA + A \wedge A = 0$? This is the Chern–Simons [113, 114] functional

$$S[A] = \int \frac{1}{2} \text{tr} A \, dA + \frac{1}{6} \text{tr} A[A, A]. \tag{3.50}$$

For any special differential BV algebra, we can consider Chern–Simons like functional,²¹

$$S_{\text{CS}}[\mu|\Omega_0] = \frac{1}{2} \text{tr } \mu \bar{\partial} \mu + \frac{1}{6} \text{tr } \mu \{ \mu, \mu \} \tag{3.51}$$

whose equation of motion is the BV master equation [44]. Let us impose the constraint $\partial \mu = 0$ (3.33), which makes $1/\partial$ to be well defined. Then $\{ \mu, \mu \} = \partial(\mu \cdot \mu)$. We can pull out ∂ and get the action

$$S_{\text{CS}}[\mu|\Omega_0] = \frac{1}{2} \text{tr } \mu \frac{\bar{\partial}}{\partial} \mu + \frac{1}{6} \text{tr } \mu \mu \mu. \tag{3.52}$$

The equation of motion of (3.52) is the BV classical master equation (3.49). This action in the context of an ordinary B-model was written in [4] and was called Kodaira–Spencer target space action. Here, as in [55], it is written in the context of any differential BV algebra $(\mathcal{A}, \bar{\partial}, \partial, \text{tr})$. The case of generalized CY structures is a particular case of the general construction. The abstract generalized Chern–Simons theory was studied in [115].

Let us rewrite (3.52) more carefully. Let us assume that the BV algebra \mathcal{A} satisfies the analogue of the $\partial\bar{\partial}$ -lemma [116]. Let $x \in H^\bullet(\mathcal{A})$ be harmonic, so $\bar{\partial}x = 0, \partial x = 0$. Let us represent the ∂ -closed element $\mu \in \mathcal{A}$ as $\mu = x + a$, where a is ∂ -exact. Using the relation $\{ \mu, \mu \} = \partial(\mu \cdot \mu)$ for ∂ -closed μ , we get

$$S_{\text{CS}}[a|x, \Omega_0] = \frac{1}{2} \text{tr } a \frac{\bar{\partial}}{\partial} a + \frac{1}{6} \text{tr}(x + a) \cdot (x + a) \cdot (x + a). \tag{3.53}$$

As explained in the previous section, the solution $\mu(x) = x + a(x)$ of the equation (3.49) for a critical point of (3.53) can be found perturbatively. By the standard field theoretical argument, the solution is a generating function of tree Feynmann diagrams of the cubic theory (3.53). The vertex in these diagrams is $\text{tr } \mu \mu \mu$, the propagator is $\bar{\partial}^{-1} \partial$. The three-point function C_{ijk} is the third derivative of the action (3.53) evaluated at the critical point $a(x)$

$$C_{ijk}(x|\Omega_0) = \partial_i \partial_j \partial_k \Phi(x|\Omega_0), \tag{3.54}$$

$$\Phi(x|\Omega_0) = S_{\text{CS}}[a(x)|x, \Omega_0]. \tag{3.55}$$

In different, but similar contexts, this fact was shown in [4, 12, 55]. Let us briefly remind the computation. Let $\dot{f}(x) = (d/d\tau)f(x)$ be the derivative in an arbitrary constant direction $\partial_\tau = v^i \partial_i$ in $H^\bullet(\mathcal{A})$. We need to

²¹The functional on space of BV actions!

show that

$$\ddot{\Phi}(x) = \text{tr } \dot{\mu}^3(x) \tag{3.56}$$

We have

$$\ddot{\Phi} = \left(\frac{1}{2} \text{tr } a \bar{\partial} \partial^{-1} a + \frac{1}{6} \text{tr } \mu \mu \mu \right) \cdots \tag{3.57}$$

Let us compute derivatives

$$\frac{1}{6} (\text{tr } \mu^3) \cdots = \text{tr } \dot{\mu}^3 + 3 \text{tr } \ddot{\mu} \dot{\mu} \mu + \frac{1}{2} \text{tr } \ddot{\mu} \mu^2, \tag{3.58}$$

$$\frac{1}{2} (\text{tr } a \bar{\partial} \partial^{-1} a) \cdots = 3 \text{tr } \ddot{a} \bar{\partial} \partial^{-1} a + \text{tr } \ddot{a} \bar{\partial} \partial^{-1} a. \tag{3.59}$$

Since $\mu = x + a$, we have $\dot{\mu} = \dot{a}$ and $\ddot{\mu} = \ddot{a}$, therefore

$$\ddot{\Phi}(x) = \text{tr } \dot{\mu}^3 + 3 \text{tr } \ddot{\mu} (\bar{\partial} \partial^{-1} a + \mu \dot{\mu}) + \text{tr } \ddot{\mu} \left(\bar{\partial} \partial^{-1} a + \frac{1}{2} \mu^2 \right). \tag{3.60}$$

The factors in the brackets of $\ddot{\mu}$ and $\ddot{\mu}$ terms vanish at the critical point (3.49). Indeed, starting from (3.49), we have $\bar{\partial} \partial^{-1} a + 1/2 \mu \cdot \mu = 0$, and taking the derivative ∂_τ we get $\bar{\partial} \partial^{-1} \dot{a} + \mu \dot{\mu} = 0$.

Thus, for any special differential BV algebra, in particular, for any that comes from a generalized CY manifold, there is a secondly quantized theory of the Chern–Simons type. This theory was studied in more details in [10].

There is one interesting remark about loop computations in (3.52). Counting degrees in the cubic vertices and propagators, one can see that there is a selection rule for nonzero correlation functions

$$\frac{1}{2} \sum_{i=1}^n d_i = (\dim_{\mathbb{C}} X - 3)(1 - g) + n. \tag{3.61}$$

Proof. Let $D = \dim_{\mathbb{R}} X$, the degree of $\mu_i \in \Lambda^k(L^*)$ be k . Let us mark degrees at the incoming line of each vertex. Each propagator has two marks on its ends. In each vertex, we have $d_{j_1} + d_{j_2} + d_{j_3} = 2D$, where j_1, j_2, j_3 enumerate the incoming lines to that vertex. In each propagator we have $d_{i_1} + d_{i_2} + 2 = 2D$, where d_{i_1}, d_{i_2} are degrees at the ends of a given propagator i . This condition is due to the form $\text{tr } a \partial^{-1} \bar{\partial} a$ of the kinetic term, where $\partial^{-1} \bar{\partial}$ has degree 2. Let d_i be degrees of fields in the external lines entering the diagram, n be the number of external lines, V be the number of vertexes, E be the number of propagators, g be the number of loops. The total sum of degrees over ends of all lines can be computed as sum over vertexes or the sum over lines. Since the result is the same we have the equation $2VD = 2E(D - 1) + \sum d_i$. Plugging relations for a trivalent graph $V = 2(g - 1) + n$ and $E = n - 3 + 3g$ we get (3.61).

This is precisely the condition on degrees of observables in the A-model with non-vanishing Gromov–Witten invariants. In Gromov–Witten theory $(\dim_{\mathbb{C}} X - 3)(1 - g) + n + \int_{\beta} c_1(TX)$ is the dimension of the fundamental class in the moduli space of Riemann surfaces $\mathcal{M}_{g,n}$ of genus g and degree β .

For maps of degree 0, the Chern–Simons like theory (3.52) gives the same condition. Moreover, that condition holds in arbitrary $\dim_{\mathbb{C}} X$ and for an arbitrary generalized CY structure. It is not specific to the A-model. That tells us that $\dim_{\mathbb{C}} X = 3$ is not crucial for the existence of the target space theory of the Chern–Simons form (3.52). Indeed, we never used condition $\dim_{\mathbb{C}} X = 3$ to write the action (see also [115]). The peculiarity of $\dim_{\mathbb{C}} X = 3$ is that this CS theory is nontrivial in the sector of purely *geometric* deformations of generalized CY structure.

In the case when \mathcal{J} -model is the B-model, the theory (3.52) reduces to the Kodaira–Spencer theory of gravity of [4], in the case when \mathcal{J} -model is the A-model, it reduces to the Kahler gravity of [117].

The Chern–Simons theory for BV actions again satisfies the BV master equation [4]. In [4], the action was written initially only for ordinary geometric deformations given by Beltrami differentials, then by arguments from string field theory all other degrees were introduced that had interpretation of BV ghosts and antifields. Here we obtain the complete Chern–Simons functional with all degrees included from the very beginning.

In a spirit similar to [82], one can also compute the 1-loop partition function in (3.53) and find a certain product of generalized Ray–Singer torsions in terms of determinants of generalized $\bar{\partial}_L, \partial_L$ operators of the Lie algebroid [118].

3.7 The $\dim_{\mathbb{C}} X = 3$ case

In case $\dim_{\mathbb{C}} X = 3$ there exists additional structure on the moduli space of \mathcal{J} -model. The logic below closely follows the standard considerations of the B-model [4]. The difference is that a holomorphic $(3, 0)$ form is replaced by a canonical pure spinor (a mixed differential form). The familiar basis of all *3-cycles* is replaced by the basis of all *odd (even)* cycles.

It was explained in Section 3.5 that the moduli space of geometrical deformations is a holomorphic Lagrangian half-dimensional submanifold L in the bosonic slice of the total extended moduli space. We can parameterize the bosonic extended moduli space $H^{\text{odd/even}}(X)$ by periods of the canonical

pure spinor

$$X^I = \int_{A_I} \Omega, \quad F_I = \int_{B^I} \Omega, \tag{3.62}$$

where A_I and B^I satisfying $A_I \circ A_I = B^I \circ B^I = 0, A_I \circ B^I = \delta_I^J$ is a basis of cycles in $H_{\text{odd/even}}(X)$. The number of X^i is the half of $\dim H^{\text{odd/even}}$, so we can consider X to be coordinates on the geometric moduli space. Let us consider a geometric deformation $\mu \cdot \Omega$ of the canonical pure spinor. By degree counting, we see that

$$\int_X (\mu\Omega, \Omega) = 0, \tag{3.63}$$

so²² $\dot{X}^I F_I - X^I \dot{F}_I = 0$ for any geometrical deformation \dot{X} . A variation along X^I gives

$$F_I = X^J \partial_I F_J. \tag{3.64}$$

Let us consider the function

$$\mathcal{F}(X) = \frac{1}{2} X^I F_I, \tag{3.65}$$

and compute $\partial_J \mathcal{F}$

$$\partial_J \mathcal{F}(X) = \frac{1}{2} (\partial_J F_I) X^I + \frac{1}{2} F_J. \tag{3.66}$$

Using equation (3.64), we see that

$$\partial_J \mathcal{F}(X) = F_J, \tag{3.67}$$

so $\mathcal{F}(X)$ is the generating function of the Lagrangian submanifold L embedded into $H^{\text{odd/even}}$ with symplectic coordinates (X^I, F_I) .

We see that the geometrical moduli space of the generalized \mathcal{J} -model for $\dim_{\mathbb{C}} X = 3$ has the familiar special Kahler structure. It is equipped by the holomorphic potential $F(X)$ and the special coordinates X^I .

Let us also note that the third derivative $C_{IJK} = \partial_I \partial_J \partial_K \mathcal{F}$ defines the three-point function that agrees with (3.47). Indeed, consider the third-derivative $\ddot{\mathcal{F}}(X)$ along some constant vector field in X^I coordinates. Using

²²Actually we adjust signs of cycles such that bilinear pairing defined on pure spinors gives $\int(\Omega_1, \Omega_2) = \int_{A_I} \Omega_1 \int_{B^I} \Omega_2 - \int_{B^I} \Omega_2 \int_{A^I} \Omega_1$ for two closed pure spinors Ω_1, Ω_2 .

the relation $\mathcal{F} = 1/2X^I F_I$ and $F_I = \partial_I \mathcal{F}$, we get

$$\dot{\mathcal{F}} = \dot{F}_I X^I = F_I \dot{X}^I, \tag{3.68}$$

$$\ddot{\mathcal{F}} = \ddot{F}_I X^I + \dot{F}_I \dot{X}^I = \dot{F}_I \dot{X}^I, \quad \ddot{F}_I X^I = 0, \quad \ddot{F}_I X^I + \ddot{F} \dot{X} = 0, \tag{3.69}$$

$$\ddot{\mathcal{F}} = \ddot{F}_I X^I + 2\dot{F}_I \dot{X}^I = -\ddot{F}_I X^I. \tag{3.70}$$

From the last line, we obtain

$$\ddot{\mathcal{F}} = \int (\ddot{\Omega}, \Omega). \tag{3.71}$$

To make the relation with the three-point function, defined in terms of the $\mu(x)$ -map in the canonical coordinates, let us represent $\Omega = e^{-\mu} \Omega_0$. Then let us compute $\ddot{\Omega}$

$$(e^{-\mu})^{\cdots} = (-\ddot{\mu} + 3\dot{\mu}\dot{\mu} - \dot{\mu}^3)e^{-\mu} \tag{3.72}$$

Therefore,

$$\ddot{\mathcal{F}} = -\text{tr}_\Omega(\dot{\mu}^3 + 3\dot{\mu}\dot{\mu} - \dot{\mu}^3) = -\text{tr}_\Omega(\dot{\mu}^3), \tag{3.73}$$

since the second and the third terms in the bracket vanish for geometrical deformations by a simple degree counting. Explicitly, we can write

$$C_{IJK} = \partial_I \partial_J \partial_K \mathcal{F} = - \int (\mu_I \mu_J \mu_K \cdot \Omega, \Omega) \tag{3.74}$$

where μ_I represents the basis for deformations of Ω , that is, $\delta X^I \delta_I \Omega = \mu_I \Omega$. Up to a sign redefinition, we nearly recovered the standard formula for (3.47)

$$\partial_i \partial_j \partial_k \Phi = C_{ijk} = \text{tr}_{\Omega_0} \mu_i \mu_j \mu_k = \int (\mu_i \mu_j \mu_k \Omega_0, \Omega_0). \tag{3.75}$$

The apparent difference is that (3.74) uses the measure Ω for the trace map of that point x where C_{IJK} is computed, but (3.75) uses the measure Ω_0 of some reference point. However, if $\Omega = e^{-\mu(x)} \Omega_0$ where $\mu(x)$ is variation of the generalized complex structure by means of $\mu \in \Lambda^2 L^*$, without a change of the scale of the canonical pure spinor, then

$$\text{tr}_{\Omega_0} \mu_i \mu_j \mu_k = \text{tr}_\Omega \mu_i \mu_j \mu_k. \tag{3.76}$$

This is easily to seen using $\dim_{\mathbb{C}} X = 3$ and counting degrees.

So we showed that the formalism of the previous section, with the Chern–Simons type formulas (3.52) and (3.54) for the partition function of BV actions, which is valid in any dimension and for all types of deformations, reduces for $\dim_{\mathbb{C}} X = 3$ and geometrical deformations $\mu(x) \in \Lambda^2 L^*$ to the familiar formulas of the special Kahler geometry (3.65) and (3.75).

It is known that the topological string on a CY threefold X computes the effective prepotential for vector multiplets of the $\mathcal{N} = 2$ four-dimensional supergravity arising after compactification of type II string theory on X [4, 119].

It is also possible to consider compactification of type II string theory on a generalized CY threefold [95, 108, 109, 111, 120]. One can expect that the relation between the physical and topological string also holds for such compactifications. A generalization of the Ooguri-Strominger-Vafa (OSV) conjecture [121] on the relation between black hole entropy in the type II string compactifications and the topological strings is considered in [112].

We did not discuss the question of the holomorphic anomaly [4, 83] in the generalized complex case. However, as it was demonstrated by Gerasimov and Shatashvili [11], the holomorphic anomaly equation can be defined on an arbitrary moduli space that has a structure of a special Kahler manifold M . Since the geometric moduli space of generalized complex structures for $\dim_{\mathbb{C}} X = 3$ is special Kahler, the holomorphic anomaly is defined in the same way. Presumably, the specifics of $\dim_{\mathbb{C}} X = 3$ and special Kahler geometry is not important for the holomorphic anomaly equation. The key point of the holomorphic anomaly equation is a parallel transport of observables on the moduli space. Given a flat connection on the extended moduli space for generalized \mathcal{J} -model, one should be able to get again the holomorphic anomaly equation. It would be interesting to see its simple derivation from the target space perspective of the action (3.52).

4 The Kontsevich $*$ -product as a $(-2,0)$ deformation of the open topological B-model by a holomorphic Poisson bivector

In this section, we consider an example of the topological \mathcal{J} -model, which is not equivalent to the ordinary A-model or B-model. More specifically, we will consider \mathcal{J} to be a finite deformation of the ordinary B-model by a holomorphic Poisson bivector β^{ij} . We will take the B-model as a reference point and consider perturbation theory over β^{ij} . We expect to get the Kontsevich $*$ -product formula [22, 23] as a deformation of the algebra of the open B-model in accordance with [20, 29, 30].

The β -perturbed B-model has the BV master action

$$S_B = \int_{\Sigma} \mu(\psi_i dx^i + \bar{\psi}^{\bar{i}} x_{\bar{i}}^* + \beta^{ij} \psi_i \psi_j). \quad (4.1)$$

Let us stress that β is not supposed to be infinitesimally small. Now we want to consider the action (4.1) on a disk. The observables are holomorphic functions on the boundary: we consider the space-filling B-brane.

Let us quantize (4.1). We need to choose a Lagrangian submanifold. In the conjugate pairs of fields $(x^{\bar{i}}, x_{\bar{i}}^*)$ and $(\psi^{\bar{i}}, \psi_{\bar{i}}^*)$, we set to zero all components of $x^{\bar{i}}$ and $\psi_{\bar{i}}^*$:

$$\begin{array}{cccc}
 x^{\bar{i}} & x_z^{\bar{i}} & x_{\bar{z}}^{\bar{i}} & x_{z\bar{z}}^{\bar{i}} \\
 \boxed{x_i^*} & \boxed{x_{iz}^*} & \boxed{x_{i\bar{z}}^*} & \boxed{x_{iz\bar{z}}^*} \\
 \boxed{\psi^{\bar{i}}} & \boxed{\psi_z^{\bar{i}}} & \boxed{\psi_{\bar{z}}^{\bar{i}}} & \boxed{\psi_{z\bar{z}}^{\bar{i}}} \\
 \psi_i^* & \psi_{i\bar{z}}^* & \psi_{i\bar{z}}^* & \psi_{iz\bar{z}}^*
 \end{array} \tag{4.2}$$

The action for the boxed fields is a decoupled quadratic free action corresponding to the second term in (4.1). Since it decouples, we will forget about it.

We are left with the pair of conjugate fields (x^i, ψ_i) and purely holomorphic BV action for them

$$S_B = \int_{\hat{\Sigma}} \mu(\psi_i dx^i + \beta^{ij}(x)\psi_i\psi_j). \tag{4.3}$$

This is consistent, because β^{ij} is the holomorphic bivector, so it depends only on x^i , and the observables are holomorphic functions on a boundary, so they also depend only on x^i . In the BV holomorphic symplectic structure, x^i and ψ_i are conjugate variables.

In the real case precisely the action (4.3) was studied by Cattaneo and Felder [48, 49] in the context of the deformation quantization. In that case context x^i are real coordinates on a real manifold X and ψ_i are coordinate in the fibers of T^*X . There is the canonical symplectic form for (x^i, ψ_i) . In [48, 49], it is shown that the 2D perturbative diagrams for the 3-point boundary correlation function $\langle h(x(\infty))f(x(0))g(x(1)) \rangle$ precisely reproduce the terms in the Kontsevich *-product formula [22]

$$f(x) \star g(x) = f(x)g(x) + \frac{i\hbar}{2} \partial_i f(x) \partial_j g(x) \beta^{ij}(x) + \dots \tag{4.4}$$

The complete formula [22, 48, 49]

$$f \star g = fg + \sum_{i=1}^{\infty} \left(\frac{i\hbar}{2} \right)^i \sum_{\Gamma} w_{\Gamma} D_{\Gamma}(f \otimes g) \tag{4.5}$$

is the perturbative series in \hbar . Each term is given by a certain bi-differential operator D_Γ acting on f and g and associated with a certain Feynman diagram. The weight w_Γ is the amplitude of the diagram Γ . The derivation, following Cattaneo and Felder [48, 49], goes as follows.

The sum in the formula runs over diagrams Γ . A diagram of order n has $n + 2$ vertices. There are n vertices labeled $1, \dots, n$ and corresponding to an insertion of $\beta^{ij}(x)$ in the interior of the disk Σ . There are also two distinguished vertices labeled L and R for insertion of $f(x)$ at two points 0 and 1 at the boundary.

The vertices are connected by oriented propagators. Precisely two propagators start at each vertex and end somewhere else. To such a diagram corresponds a bi-differential operator Δ_Γ that acts on f and g . This operator is made of β^{ij} and its derivatives $\partial_{i_1}\partial_{i_2}\dots\beta^{i_1j_2}$. The arrows indicate contraction of indices of β^{ij} and derivatives ∂_i . The beginning of an arrow stands for an upper index i of β^{ij} , and the end of an arrow points to another $\beta(x)$ or $f(x)$ that should be acted by ∂_i . The coefficient w_Γ is given by a certain integral over position of n points z_i on the upper half plane (or disk) Σ . The integrand is a product of $2n$ functions $\phi(z_i, z_j)$ for each connecting arrow in the diagram.

That is just a Feynman diagram of the perturbation theory in β for the 2D theory on the disk (4.3). The free action is $\int \psi_i dx^i$. The propagator connects ψ^i and x^i . Each insertion of $\beta^{ij}\psi_i\psi_j$ gives two ψ 's. The ψ 's should be contracted with x 's, so we expand $\beta^{ij}(x)$ and $f(x)$ in x and contract them with ψ 's. Pulling out one x from $\beta^{ij}(x)$ or $f(x)$ and contracting it with ψ is the same as taking a derivative of $\beta^{ij}(x)$ or $f(x)$.

The coefficient of each diagram is given by a certain integral over position of n insertion points over Σ . The propagator $\phi(z, w)$ is the appropriately gauged fixed operator d^{-1} . It has a structure $\phi(z, w) = \log(z - w)(z - \bar{w})/(\bar{z} - \bar{w})(\bar{z} - w)$. The integrals over positions of points z_1, \dots, z_n of products of these propagators are computable [22]. They are of a topological nature, since propagator d^{-1} is represented by a closed differential form.

Exactly the same perturbation theory holds for (4.3). Indeed, the fields of (4.3), the BV symplectic structure, the Poisson bivector and the BV measure in the functional integral are precisely holomorphic analogue of the Cattaneo and Felder data. The $(-2, 0)$ finite perturbation of the B-model formally reproduces in all orders the Kontsevich $*$ -product formula in the holomorphic context.

See [122] for discussion of subtleties related with quantization of the Poisson sigma-model.

5 Conclusion

In this paper, we presented the topological sigma-model that depends only on a generalized complex structure \mathcal{J} on the target space. We employed the AKSZ formalism [47] for construction of the \mathcal{J} -model.

The closed sector was studied at the tree level without instanton corrections, reproducing [101]. The relevance of the Chern–Simons like functionals for *closed* string field theory was discussed. The observables and correlation functions are defined in agreement with Barannikov and Kontsevich [12] and Li [101]. The special properties of the case $\dim_{\mathbb{C}} X = 3$ are studied from the viewpoint of the general framework.

In the open sector, we considered generalized complex structure represented by an ordinary complex structure and holomorphic Poisson bivector. The product in the algebra of open strings is deformed into the noncommutative holomorphic Kontsevich $*$ -product with [20, 29, 30]. The computation is completely parallel to the real case of [48, 49].

There were some related developments recently. In [30, 38, 39], it was suggested to construct the generalized topological \mathcal{J} -model by means of the generalized Kahler geometry, which is described by a pair of commuting generalized complex structures $(\mathcal{J}, \mathcal{J}')$. As Gualtieri [36] showed the generalized Kahler geometry $(\mathcal{J}, \mathcal{J}')$ is equivalent to the data (g, b, I_+, I_-) with certain compatibility conditions, where g is an ordinary metric, B is a two-form, I_+ and I_- are ordinary complex structures. (This is the geometry discovered in studies of $\mathcal{N} = (2, 2)$ CFT's in [85], see also recent works [86–93, 97, 123].)

The difference of our approach is that from the very beginning, we require existence of only one generalized complex structure (and a canonical pure spinor, i.e., a weak generalized CY structure). A second generalized complex structure is used only to gauge fix the model. In general, it does not have to be integrable. It would be very interesting to compare explicitly the gauge fixed version of \mathcal{J} -model with constructions of [93, 97, 99]. It is also possible that one could find other physically interesting gauge fixing conditions. For example, the gauge that was used in Section 4 to study β^{ij} deformations of the B-model, is different. It is not yet clear, what is the space \mathcal{L} of physically non-equivalent gauges—Lagrangian submanifolds in the BV phase space of \mathcal{J} -model.

The interesting questions about the possibility of global symplectic realization [61–63, 79] of HLL , as well as about the behavior of the model at the non-regular points, were left out for the future study. The topological sigma-model on a generalized complex target space might be useful in the

context of the field theoretical study of geometrical Langlands program [124] and generalized type II string compactifications [109].

In [46], a generalization of BV formalism was suggested. There is a way to define amplitudes for cycles in the moduli space \mathcal{L} of Lagrangian submanifolds. The original BV construction is a particular case of zero cycles. The higher genus amplitudes of the \mathcal{J} -model coupled with the 2D gravity correspond to the integrals of closed forms over cycles in \mathcal{L} . One needs to study these higher genus contributions. There are some indications [10, 115] that the generalized CS theory of BV actions might define higher genus amplitudes in the no instanton approximation. The relation [4] with the closed string field theory needs to be studied in more details. In particular, one has to find out what are the descendants from the viewpoint of the generalized CS theory.

The generalized CS theory (3.52) on the space of BV functionals exists for any theory formulated in the BV way. The classical solutions of this CS theory correspond to the solutions of the BV classical master equation. What is the meaning of the generalized CS action for BV functionals at the quantum level?

Acknowledgements

I thank E. Witten for coordination of the project, many interesting discussions, important comments and suggestions. I am very grateful to N. Nekrasov for fruitful discussions, ideas and numerous answers to my questions. I thank M. Grana, F. Denef, A. Kapustin, A. Losev, A. Neitzke, S. Shatashvili, D. Shih and A. Tomassielo for interesting discussions and remarks. Part of this research was done during my visits to Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette, France, and the third Simons Workshop in Mathematics and Physics at Stony Brook University, NY. I thank these institutions for their kind hospitality. The work was supported in part by grant RFBR 04-02-16880 and grant NSF 245-6530.

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