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Topological sigma-models with *H*-flux and twisted generalized complex manifolds

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Abstract

We study the topological sector of $N = 2$ sigma-models with H-flux. It has been known for a long time that the target-space geometry of these theories is not Kähler and can be described in terms of a pair of complex structures, which do not commute, in general, and are parallel with respect to two different connections with torsion. Recently an alternative description of this geometry was found, which involves a pair of commuting twisted generalized complex structures on the target space. In this paper, we define and study the analogs of A and B-models for $N = 2$ sigma-models with H-flux and show that the results are naturally expressed in the language of twisted generalized complex geometry. For example, the space of topological observables is given by the cohomology of a Lie algebroid associated to one of the two twisted generalized complex structures. We determine the topological scalar product, which endows the algebra of observables with the structure of a Frobenius algebra. We also discuss mirror symmetry for twisted generalized Calabi–Yau manifolds.

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1 Introduction

It was pointed out by Witten [19] in 1988 that one can construct interesting examples of topological field theories by "twisting" supersymmetric field theories. This observation turned out to be very important for quantum field theory and string theory, since observables in topologically twisted theories are effectively computable on one hand and can be interpreted in terms of the untwisted theory on the other. In other words, supersymmetric field theories tend to have large integrable sectors.

From the string theory viewpoint, the most important class of supersymmetric field theories admitting a topological twist are sigma-models with $(2,2)$ supersymmetry. Usually one considers the case when the B-field is a closed 2-form, in which case $(2,2)$ supersymmetry requires the target M to be a Kähler manifold. In this case, the theory admits two different twists, which give rise to two different topological field theories, known as the A and B-models.[1] In any topological field theory, observables form a supercommutative ring. For the A-model, this ring turns out to be a deformation of the complex de Rham cohomology ring of M, which is known as the quantum cohomology ring. It depends on the symplectic (Kähler) form on M, but not on its complex structure. For the B-model, the ring of observables turns out to be isomorphic to

$$\oplus_{p,q} H^p(\Lambda^q T^{1,0} M),$$

which obviously depends only on the complex structure of M. Furthermore, it turns out that all correlators in the A-model are symplectic invariants of M, while all correlators in the B-model are invariants of the complex structure on M [20].

In this paper, we analyze more general topological sigma-models for which $H = dB$ is not necessarily zero. It is well known that for $H \neq 0$ $(2,2)$ supersymmetry requires the target manifold M to be "Kähler with torsion" [2]. This means that we have two different complex structures $I_\pm$ for right movers and left movers, such that the Riemannian metric g is Hermitian with respect to either one of them, and I_+ and I_- are parallel with respect to two different connections with torsion. The torsion is proportional to $\pm H$. The presence of torsion implies that the geometry is not Kähler (the forms $\omega_\pm = gI_\pm$ are not closed). Upon topological twisting, one obtains a topological field

[1] More precisely, the B-model makes sense on the quantum level if and only if M is a Calabi–Yau manifold. For the A-model, the Calabi–Yau condition is unnecessary.

theory, and one would like to describe its correlators in terms of geometric data on M. As in the Kähler case, there are two different twists (A and B), and by analogy one expects that the correlators of either model depend only on "half" of the available geometric data. Furthermore, it is plausible that there exist pairs of $(2, 2)$ sigma-models with H-flux for which the A-model are B-model are "exchanged." This would provide an interesting generalization of mirror symmetry to non-Kähler manifolds.

The main result of the paper is that topological observables can be described in terms of a (twisted) generalized complex structure on M. This notion was introduced by Hitchin [6] and studied in detail by Gualtieri [5]; we review it below. One can show that the geometric data H, g, I_+, I_- can be repackaged as a pair of commuting twisted generalized complex structures on M [5]. We show in this paper that on the classical level the ring of topological observables and the topological metric on this ring depend only on one of the two twisted generalized complex structures. This strongly suggests that all the correlators of either A or B-model (encoded by an appropriate Frobenius manifold) are invariants of only one twisted generalized complex structure. Therefore, if M and M' are related by mirror symmetry (i.e., if the A-model of M is isomorphic to the B-model of M' and vice versa), then the appropriate moduli spaces of twisted generalized complex structures on M and M' will be isomorphic.

To state our results more precisely, we need to recall the definition of the (twisted) generalized complex structure (TGC-structure for short). Let M be a smooth even-dimensional manifold, and let H be a closed 3-form on M. The bundle $TM \oplus TM^*$ has an interesting binary operation, called the twisted Dorfman bracket. It is defined, for arbitrary $X, Y \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(TM^*)$, as

$$(X \oplus \xi) \circ (Y \oplus \eta) = [X, Y] \oplus (\mathcal{L}_X \eta - i_Y d\xi + \iota_Y \iota_X H).$$

It is not skew-symmetric, but satisfies a kind of Jacobi identity. Its skew-symmetrization is called the twisted Courant bracket and does not satisfy the Jacobi identity. The bundle $TM \oplus TM^*$ also has an obvious pseudo-Euclidean metric of signature (n, n), which we call q. For a detailed discussion of the Dorfman and Courant brackets and their geometric meaning, see Ref. [16].

A TGC-structure on M is a bundle map $\mathcal{J}$ from $TM \oplus TM^*$ to itself which satisfies the following three requirements.

- $\mathcal{J}^2 = -1$.
- $\mathcal{J}$ preserves q, i.e., $q(\mathcal{J}u, \mathcal{J}v) = q(u, v)$ for any $u, v \in TM \oplus TM^*$.

- The eigenbundle of $\mathcal{J}$ with eigenvalue i is closed with respect to the twisted Dorfman bracket. (One may replace the Dorfman bracket with the Courant bracket without any harm.)

In the special case $H = 0$, the adjective "twisted" is dropped everywhere, and one gets the notion of a generalized complex structure (GC-structure).

To any TGC-structure on M, one can canonically associate a complex Lie algebroid E (which is, roughly, a complex vector bundle with a Lie bracket which has properties similar to that of a complexification of the tangent bundle of M). From any complex, Lie algebroid E, one can construct a differential complex whose underlying vector space is the space of sections of $\Lambda^\bullet(E^*)$ (which generalizes the complexified de Rham complex of M). We will show that the space of topological observables is isomorphic to the cohomology of this differential complex. We will also write down a formula for the metric on the cohomology, which makes it into a supercommutative Frobenius algebra. Both the ring structure and the topological metric depend only on one of the two TGC-structures available.

Even if $H = 0$, one can consider the situation when $I_+ \neq I_-$. This is possible, for example, when M is hyper-Kähler, and there is a family of complex structure parametrized by S^2 and compatible with a fixed Riemannian metric g. This case was considered in Ref. [10], where the relation with GC-structures was first noted. In this paper, we extend the results of Ref. [10] to the case $H \neq 0$. The relation of twisted generalized complex geometry with $N = 2$ supersymmetric sigma-models has also been studied in Refs. [11, 12]. This subject may also be relevant to flux compactifications of superstring theories [4].

The organization of the paper is as follows. In Section 2, we give a brief review of $(2, 2)$ supersymmetric sigma-models. Our emphasis is on their relation to generalized complex geometry. In Section 3, we construct the topological theories by "twisting" $(2, 2)$ supersymmetric sigma-models with H-flux. In particular, we discuss the relevance of the twisted generalized Calabi–Yau condition for our construction. The Ramond–Ramond ground states of the $(2, 2)$ theory are studied in Section 4. The results of this section are used to prove that the space of topological observables is given by an associated Lie algebroid cohomology. In Section 5, we discuss the topological metric on the space of observables in the topologically twisted theory and write down a formula for tree-level topological correlators, neglecting quantum corrections. In Section 6, we discuss possible quantum corrections due to worldsheet instantons. In Section 7, we discuss the implications of our results, including a possible generalization of mirror symmetry to non-Kähler manifolds with H-flux.

2 The geometry of (2,2) supersymmetric sigma-models

We start by reviewing $(2,2)$ supersymmetric sigma-models and setting up the notation. It is well known that the bosonic sigma model in $1+1$ dimensions with any Riemannian target manifold M admits a $(1,1)$ supersymmetric extension. The action of the $(1,1)$-extended theory has a superfield formulation:

$$S = \frac{1}{2}\int \mathrm{d}^2\sigma \mathrm{d}^2\theta \big(g_{ab}(\Phi) + B_{ab}(\Phi)\big) D_+\Phi^a D_-\Phi^b. \tag{2.1}$$

Here $\Phi^a = \phi^a + \theta^+\psi_+^a + \theta^-\psi_-^a + \theta^-\theta^+F^a$ are $(1,1)$ superfields (components of a supermap from a $(1,1)$ superworldsheet to the bosonic target M), g is a Riemannian metric on M, and $H = dB$ is a real closed 3-form on M. Note that if H defines a non-trivial class in $H^3(M)$, then B is only locally well defined. The super covariant derivatives $D_\pm$ are defined by

$$D_+ = \frac{\partial}{\partial\theta^+} + i\theta^+\partial_+, \;\; D_- = \frac{\partial}{\partial\theta^-} + i\theta^-\partial_-, \quad \partial_\pm \equiv \partial_0 \pm \partial_1.$$

The action is invariant under the standard supersymmetry generated by

$$Q_+ = \frac{\partial}{\partial\theta^+} - i\theta^+\partial_+, \quad Q_- = \frac{\partial}{\partial\theta^-} - i\theta^-\partial_-.$$

When the target manifold M possesses additional structure, the theory may possess a larger supersymmetry. For example, it is well known that when (M,g) is Kähler and $H=0$, the theory has $(2,2)$ supersymmetry. A natural question is whether $(2,2)$ supersymmetry implies Kähler geometry. This has been answered in the negative by Gates *et al.* [2]. It is shown there that the general form of a second supersymmetry (as opposed to the standard one generated by $Q_\pm$ above) is

$$\tilde{\delta}\Phi^a = (\tilde{\epsilon}^+\tilde{Q}_+ + \tilde{\epsilon}^-\tilde{Q}_-)\Phi^a = \tilde{\epsilon}^+ I_+(\Phi)^a{}_b D_+\Phi^b + \tilde{\epsilon}^- I_-(\Phi)^a{}_b D_+\Phi^b,$$

where the tensors $I^a_{\pm b}$ satisfy the following constraints. First, the condition that the above transformation generates a separate (on-shell) $(1,1)$ supersymmetry, which commutes with the standard one, requires I_+, I_- to be a pair of integrable almost complex structures on TM. Second, the invariance of the action (2.1) requires that the metric g be Hermitian with respect to both I_+ and I_-, and that the tensors $I_\pm$ are covariantly constant with respect to certain affine connections with torsion. More explicitly, one must have

$$\nabla_a^{(\pm)} I_{\pm}{}^b_c = \partial_a I_{\pm}{}^b_c + \Gamma_{\pm}{}^b_{ad} I_{\pm}{}^d_c - \Gamma_{\pm}{}^d_{ac} I_{\pm}{}^b_d = 0,$$

with the affine connections defined by

$$\Gamma_{\pm}{}^a_{bc} = \Gamma^a_{bc} \pm \frac{1}{2} g^{ad} H_{dbc}.$$

Here Γ^a_{bc} is the Levi–Civita connection:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}).$$

Note that in general (g, I_+) and (g, I_-) do not define Kähler structures (i.e., the 2-forms $\omega_\pm = gI_\pm$ are not closed) due to the presence of torsion in $\Gamma_\pm$.

An interesting special case is when $[I_+, I_-] = 0$. In this case, one can simultaneously diagonalize $I_\pm$ and one can decompose $TM = \ker(I_+ - I_-) \oplus \ker(I_+ + I_-)$. It was shown in [2] that $\ker(I_+ - I_-)$ is integrable to $N = 2$ chiral superfields whilst $\ker(I_+ + I_-)$ is integrable to twisted chiral superfields. Such a manifold is said to admit a product structure defined by $P = I_+ I_-$. Locally, it is a product of two Kähler manifolds, but globally it is not Kähler in general. Another interesting class of examples is provided by hyper-Kähler manifolds, which admit a family of complex structures parametrized by $\vec{x} \in S^2$. One may take I_+ and I_- to be any two complex structures parametrized by two points $\vec{x}_\pm \in S^2$. We refer to Refs. [1, 2, 8, 13, 15] for more details on these and related issues.

Remarkably, the geometric data required by generic $(2, 2)$ supersymmetry are equivalent to those which define the so-called twisted generalized Kähler structure [5]. Here we briefly recall the definitions which are needed later. Let M be a real manifold of dimension $2n$. As already mentioned in the introduction, the bundle $TM \oplus TM^*$ has a natural pseudo-Euclidean scalar product of signature $(2n, 2n)$ which we will denote q. A twisted generalized complex structure on M is a section $\mathcal{I}$ of $\mathrm{End}(TM \oplus TM^*)$ which preserves q, satisfies $\mathcal{I}^2 = -1$, and such that its i-eigenbundle is closed with respect to the twisted Courant bracket. A twisted generalized Kähler structure on M is a pair of commuting twisted generalized complex structures, $(\mathcal{J}_1, \mathcal{J}_2)$, such that $\mathcal{G} = -q\mathcal{J}_1\mathcal{J}_2$ defines a positive-definite metric on $TM \oplus TM^*$. It is shown in Ref. [5] that specifying a twisted generalized Kähler structure is equivalent to specifying g, H, I_+, and I_- satisfying the constraints of $(2, 2)$ supersymmetry. Explicitly, the two commuting twisted generalized complex structures can be taken as

$$\mathcal{J}_1 = \begin{pmatrix} \tilde{I} & -\alpha \\ \delta\omega & -\tilde{I}^t \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} \delta I & -\beta \\ \tilde{\omega} & -\delta I^t \end{pmatrix} \tag{2.2}$$

with

$$\begin{aligned} \tilde{I} &= \frac{I_+ + I_-}{2}, \quad \delta I = \frac{I_+ - I_-}{2} \\ \beta &= \frac{\omega_+^{-1} + \omega_-^{-1}}{2}, \quad \alpha = \frac{\omega_+^{-1} - \omega_-^{-1}}{2} \\ \tilde{\omega} &= \frac{\omega_+ + \omega_-}{2}, \quad \delta\omega = \frac{\omega_+ - \omega_-}{2}. \end{aligned}$$

It can be shown that both α and β are (possibly degenerate) Poisson bivectors [13].

3 Construction of topological theories

3.1 Twisting

In this section, we construct topologically twisted versions of $(2,2)$ sigma-models without assuming $I_+ = I_-$. In fact, the case $H = 0, I_+ \neq I_-$ has already been analyzed in [10]. It is shown there that the space of local observables of the topologically twisted theory can be identified with the cohomology of a certain Lie algebroid associated to generalized complex structure. We will analyze the general case.

We follow the approach pioneered in Ref. [20], which dealt with the case when M is a Kähler manifold, with vanishing H-field, and $I_+ = I_-$. If there is a non-anomalous $U(1)$ R-symmetry satisfying an integrality constraint (for any state the sum of spin and one-half the R-charge must be integral), then one may shift the spin of all fields by one-half of their R-charges and obtain a topological field theory. In the case when the target space is a Kähler manifold, there are two classical $U(1)$ R-symmetries: the vector R-symmetry $U(1)_V$ and the axial R-symmetry $U(1)_A$. At the quantum level, $U(1)_V$ remains a good symmetry, while $U(1)_A$ suffers from an anomaly unless M satisfies the condition $c_1(TM) = 0$. Twisting by the vector R-symmetry yields the so-called A-model, while twisting by the axial R-symmetry (if it is not anomalous) yields the B-model.

It is not difficult to see that the construction in Ref. [20] can be applied to the more general case at hand. The two complex structures, $I_\pm$, induce two different decompositions of the complexified tangent bundle

$$TM_{\mathbb{C}} \simeq T_+^{1,0} \oplus T_+^{0,1} \simeq T_-^{1,0} \oplus T_-^{0,1}.$$

Under such decompositions, the fermionic fields $\psi_\pm$ splits accordingly into the holomorphic and anti-holomorphic components:

$$\begin{aligned}
\psi_+ &= \frac{1}{2}(1 - iI_+)\psi_+ + \frac{1}{2}(1 + iI_+)\psi_+,\\
\psi_- &= \frac{1}{2}(1 - iI_-)\psi_- + \frac{1}{2}(1 + iI_-)\psi_-.
\end{aligned}$$

At the classical level, there are two inequivalent ways to assign $U(1)$ R-charges to fermions (the bosons having zero charge):

$$U(1)_V\colon\ q_V\left(\frac{1}{2}(1 - iI_+)\psi_+\right) = -1, \quad q_V\left(\frac{1}{2}(1 - iI_-)\psi_-\right) = -1$$

$$U(1)_A\colon\ q_A\left(\frac{1}{2}(1 - iI_+)\psi_+\right) = -1, \quad q_A\left(\frac{1}{2}(1 - iI_-)\psi_-\right) = 1$$

The topological twisting is achieved by shifting the spin of fermions either by $q_V/2$ or $q_A/2$. We will call the corresponding topological theories generalized A and B-models. Note that flipping the sign of I_- exchanges them.

So far our analysis has been at the classical level. For the generalized A and B-models to make sense as quantum field theories, one must require that the $U(1)$ symmetry used in the twisting be anomaly-free. The anomalies are easily computed by the Atiyah–Singer index theorem, and the resulting conditions are

$$\begin{aligned}
&U(1)_V\colon\ c_1(T_-^{1,0}) - c_1(T_+^{1,0}) = 0\\
&U(1)_A\colon\ c_1(T_-^{1,0}) + c_1(T_+^{1,0}) = 0
\end{aligned} \tag{3.1}$$

It is possible to express the anomaly conditions in terms of twisted generalized complex structures. Recall that $(2,2)$ supersymmetry requires M to be a twisted generalized Kähler manifold, with two commuting twisted generalized complex structures $(\mathcal{J}_1, \mathcal{J}_2)$ and positive definite metric $\mathcal{G} = -q\mathcal{J}_1\mathcal{J}_2$ on $TM \oplus TM^*$. Let $C_\pm$ be the ± 1 eigenbundles of $\mathcal{G}$. The natural projection from $TM \oplus TM^*$ to TM induces bundle isomorphisms $\pi_\pm\colon C_\pm \simeq TM$. The twisted generalized complex structure $\mathcal{J}_1$ induces two complex structures on TM, one from $\pi_+\colon C_+ \to TM$ and the other from $\pi_-\colon C_- \to TM$. These are the two complex structures $I_\pm$ which appeared above. Let E_1 and E_2 denote the i-eigenbundles of $\mathcal{J}_1$ and $\mathcal{J}_2$, respectively. Since $\mathcal{J}_1$ and $\mathcal{J}_2$ commute, one has the further decompositions $E_1 = E_1^+ \oplus E_1^-$ and

$E_2 = E_2^+ \oplus E_2^-$, where the superscripts $\pm$ label the eigenvalues $\pm i$ of the other twisted generalized complex structure. It follows that

$$C_\pm \otimes \mathbb{C} = E_1^\pm \oplus (E_1^\pm)^* = E_2^\pm \oplus (E_2^\pm)^*.$$

Now we can rewrite the conditions Equation (3.1) in terms of bundles E_1 and E_2:

$$\begin{aligned} &U(1)_V\colon c_1(E_2) = 0 \\ &U(1)_A\colon c_1(E_1) = 0. \end{aligned} \tag{3.2}$$

It seems natural to call either of these conditions a twisted generalized Calabi–Yau condition (for $\mathcal{J}_1$ and $\mathcal{J}_2$, respectively). However, this name is already reserved for a somewhat stronger condition introduced by Hitchin [6] and Gualtieri [5]. The Hitchin–Gualtieri condition on $\mathcal{J}$ implies the vanishing of $c_1(E)$, but the converse is not true in general. Physically, the vanishing of $c_1(E)$ is also not sufficient for the topological twist to make sense. We will see in Sections 4 and 5 that the topological twist makes sense if and only if the Hitchin–Gualtieri condition is satisfied.

As already mentioned, flipping the relative sign of $I_\pm$ exchanges the generalized A and B-models. This is consistent with the anomaly-cancellation condition, since changing the relative sign of $I_\pm$ is equivalent to exchanging $\mathcal{J}_1$ and $\mathcal{J}_2$.

3.2 BRST cohomology of operators

Next we describe the BRST operator and BRST-invariant observables. We shall focus mainly on the generalized B-model, since the A-model can be obtained from it by flipping the sign of I_-. As we just discussed, $(M, \mathcal{J}_1)$ must be a twisted generalized Calabi–Yau manifold. After the topological twist, $(1 + iI_+)\psi_+$ and $(1 + iI_-)\psi_-$ become sections of $T_+^{0,1}$ and $T_-^{0,1}$, respectively.[2] On the other hand, $(1 - iI_+)\psi_+$ and $(1 - iI_-)\psi_-$ become worldsheet spin-1 fields and they should not appear in the BRST variation of ϕ. We obtain two scalar nilpotent operators from the original supersymmetry generators: $Q_L = (Q_+ + i\tilde{Q}_+)/2$ and $Q_R = i(Q_- + i\tilde{Q}_-)/2$. The overall factor of i in the expression of Q_R is for later convenience. The $N = 2$ supersymmetry algebra implies that $Q_L^2 = 0$, $Q_R^2 = 0$, and $\{Q_L, Q_R\} = 0$. We take the BRST operator of the generalized B-model to be $Q_{\mathrm{BRST}} = Q_L + Q_R$.

[2]Actually, these are sections of the *pullbacks* of $T_+^{0,1}$ and $T_-^{0,1}$. However, we shall slightly abuse the notation by not spelling out explicitly the word "pullback" in the following.

To simplify notation, let us define:

$$\chi \equiv \frac{1}{2}(1 + iI_+)\psi_+, \quad \lambda \equiv \frac{i}{2}(1 + iI_-)\psi_-.$$

The scalar fields transform under Q_L and Q_R as follows:

$$\begin{aligned}
\{Q_L, \phi^a\} &= \chi^a \\
\{Q_L, \chi^a\} &= 0 \\
\{Q_L, \lambda^a\} &= -\Gamma_-{}^a{}_{bc}\chi^b\lambda^c \\
\{Q_R, \phi^a\} &= \lambda^a \\
\{Q_R, \lambda^a\} &= 0 \\
\{Q_R, \chi^a\} &= -\Gamma_+{}^a{}_{bc}\lambda^b\chi^c.
\end{aligned} \tag{3.3}$$

Local observables of the topological theory must take the following form

$$\mathcal{O}_f = f_{a_1\cdots a_p;b_1\cdots b_q}\chi^{a_1}\cdots\chi^{a_p}\lambda^{b_1}\cdots\lambda^{b_q},$$

where f is totally anti-symmetric in a's as well as in b's. Recall that $\chi \in \Gamma(T_+^{0,1}), \lambda \in \Gamma(T_-^{0,1})$, so one can regard f as a section of $\Omega_+^{0,p}(M) \otimes \Omega_-^{0,q}(M)$. Here the subscripts $\pm$ remind us with respect to which complex structure the differential forms are decomposed. Next we must require that $\mathcal{O}_f$ be annihilated by the BRST operator $Q_L + Q_R$. To write down the action of Q_L on $\mathcal{O}_f$, it is convenient to regard f as a $(0,p)$ form for the complex structure I_+, with values in $\Omega_-^{0,q}(M)$. A straightforward calculation gives

$$\{Q_L, \mathcal{O}_f\} = \mathcal{O}_{\bar{D}_{(+)}f}.$$

Here $\bar{D}_{(+)}$ is a covariantization of the ordinary Dolbeault operator $\bar{\partial}$ corresponding to I_+. The covariantization uses the connection on $\Omega_-^{0,q}(M)$ coming from the connection Γ_- on TM. On the other hand, one can regard f as a $(0,q)$ form for I_-, taking values in $\Omega_+^{0,p}(M)$. One gets

$$\{Q_R, \mathcal{O}_f\} = \mathcal{O}_{\bar{D}_{(-)}f},$$

where $\bar{D}_{(-)}$ now stands for a covariantization of Dolbeault operator $\bar{\partial}$ for I_- using the connection Γ_+ on TM.

The space of local observables has a natural bigrading by the left and right moving R-charges. With respect to it, Q_L has grade $(1,0)$, and Q_R

has grade $(0,1)$. The local observables fit into the following bicomplex:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & \mathcal{O}^{p-1,q+1} \arrow[r] \arrow[u] & \mathcal{O}^{p,q+1} \arrow[r] \arrow[u, "Q_R"] & \mathcal{O}^{p+1,q+1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r, "Q_L"] & \mathcal{O}^{p-1,q} \arrow[r, "Q_L"] \arrow[u] & \mathcal{O}^{p,q} \arrow[r, "Q_L"] \arrow[u, "Q_R"] & \mathcal{O}^{p+1,q} \arrow[r, "Q_L"] \arrow[u] & \cdots \\
\cdots \arrow[r] & \mathcal{O}^{p-1,q-1} \arrow[r] \arrow[u] & \mathcal{O}^{p,q-1} \arrow[r] \arrow[u, "Q_R"] & \mathcal{O}^{p+1,q-1} \arrow[r] \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u, "Q_R"] & \vdots \arrow[u] &
\end{tikzcd}$$

The total cohomology of this bicomplex is the space of "physical" observables in our topological theory. As usual, this means that there are two spectral sequences which converge to the BRST cohomology $H^*_{Q_{\mathrm{BRST}}}$. In practice, the computation is usually quite involved, unless the spectral sequences degenerate at a very early stage.

3.3 Relation with twisted generalized complex structures

In the special case $H = 0$, it has been argued in [10] that the BRST complex coincides with the cohomology of the Lie algebroid E_1 associated with the generalized complex structure $\mathcal{J}_1$. We will show that the statement is true for arbitrary H.

First let us recall the necessary definitions. A Lie algebroid, by definition, is a real vector bundle E over a manifold M equipped with two structures: a Lie bracket $[\cdot,\cdot]$ on the space of smooth sections of E, and a bundle morphism $a\colon E \to TM$, called the anchor map. These data satisfy two compatibility conditions:

(i) $a([s_1, s_2]) = [a(s_1), a(s_2)]\ \forall s_1, s_2 \in \Gamma(E)$, i.e., a is a homomorphism of Lie algebras.

(ii) $[f \cdot s_1, s_2] = f \cdot [s_1, s_2] - a(s_2)(f) \cdot s_1, \quad \forall f \in C^\infty(M), \forall s_1, s_2 \in \Gamma(E)$.

If we take $E = TM$, $a = id$, and the bracket to be the ordinary commutator of vector fields, then both conditions are obviously satisfied. Thus a Lie algebroid over M should be thought of as a "generalized tangent bundle."

A complex Lie algebroid is defined similarly, except that E is a complex vector bundle, and TM is replaced with its complexification $TM_{\mathbb{C}}$.

There is an alternative, and perhaps more intuitive, way to think about Lie algebroids. For any vector bundle E, we may consider a graded supermanifold ΠE, i.e., the total space of the bundle E with the fiber directions regarded as odd and having degree 1. It turns out that there is a one-to-one correspondence between Lie-algebroid structures on E and degree 1 odd vector fields Q on ΠE satisfying $\{Q, Q\} = 2Q^2 = 0$ [18]. The correspondence goes as follows. Let (x^b, ξ^μ) be local coordinates on ΠE, where x^b are local coordinates on M, and ξ^μ are linear coordinates on the fiber. Any degree 1 odd vector field on ΠE has the form

$$Q = a^b_\mu \xi^\mu \frac{\partial}{\partial x^b} + c^\mu_{\nu\rho} \xi^\nu \xi^\rho \frac{\partial}{\partial \xi^\mu},$$

where a^b_μ and $c^\mu_{\nu\rho}$ are locally defined functions on M. Let e_μ be the basis of sections of E dual to the coordinates ξ^μ. Define a map a: $E \to TM$ by

$$a(e_\mu) = a^b_\mu \frac{\partial}{\partial x^b}$$

and a bracket by

$$[e_\nu, e_\rho] = c^\mu_{\nu\rho} e_\mu.$$

One can show that these data define on E the structure of a Lie algebroid if and only if $Q^2 = 0$.

Identifying functions on ΠE with sections of the graded bundle $\Lambda^\bullet E^*$, we may regard Q as a differential on the space of sections of this bundle. We will call the resulting complex the canonical complex of the Lie algebroid, and its cohomology will be called the Lie algebroid cohomology. Let us take, for example, $E = TM$, with a the identity map, and the standard Lie bracket. Then the canonical complex is the complex of differential forms on M, with Q being the usual de Rham differential and the Lie algebroid cohomology is simply the de Rham cohomology of M.

To every twisted generalized complex manifold $(M, H, \mathcal{J})$, one can associate a complex Lie algebroid by letting E be the eigenbundle of $\mathcal{J}$ with eigenvalue $-i$. The bracket on E is induced by the Courant bracket on $TM \oplus TM^*$, and the anchor map is the projection to $TM_{\mathbb{C}}$. The associated complex controls the deformations of the twisted generalized complex structure on M (with H fixed). We claim that the BRST complex discussed above is isomorphic to the Lie algebroid complex associated to the twisted generalized complex manifold $(M, H, \mathcal{J} = \mathcal{J}_1)$.

To see the relation between the two complexes, let us define new fermionic coordinates:

$$\psi^a = \frac{1}{\sqrt{2}}(\psi_+^a + i\psi_-^a), \quad \rho_a = \frac{1}{\sqrt{2}} g_{ab}(\psi_+^b - i\psi_-^b). \tag{3.4}$$

One may regard ψ and ρ as fermion fields taking values in the pullback of $TM_{\mathbb{C}}$ and $TM_{\mathbb{C}}^*$, respectively. Their anticommutation relations are

$$\{\psi^a, \psi^b\} = \{\rho_a, \rho_b\} = 0, \quad \{\psi^a(\sigma), \rho_b(\sigma')\} = \delta_b^a \delta(\sigma - \sigma'). \tag{3.5}$$

It is convenient to introduce a fermion field

$$\Psi = \begin{pmatrix} \psi \\ \rho \end{pmatrix}$$

taking values in $TM_{\mathbb{C}} \oplus TM_{\mathbb{C}}^*$. In terms of Ψ, the anticommutation relations read

$$\{\Psi(\sigma)^\alpha, \Psi(\sigma')^\beta\} = \left(q^{-1}\right)^{\alpha\beta} \delta(\sigma - \sigma'),$$

where q is the canonical scalar product on $TM_{\mathbb{C}} \oplus TM_{\mathbb{C}}^*$ which plays such a fundamental role in generalized complex geometry.

It is easy to check that

$$\begin{pmatrix} \chi + \lambda \\ g(\chi - \lambda) \end{pmatrix} = (1 + i\mathcal{J}_1)\Psi.$$

Therefore any function of the bosonic coordinates ϕ^i and fermionic scalars χ, λ can be rewritten as a function on ΠE, where E is the eigenbundle of $\mathcal{J}_1$ with eigenvalue $-i$. Thus the graded vector spaces underlying the two complexes are naturally isomorphic. It remains to show that the BRST differential Q_{BRST} coincides with the Lie algebroid differential Q. One could check this by a direct computation, but there is a more efficient route to this goal, with the added advantage of making the isomorphism of the two complexes more obvious. To explain this indirect proof, we must first discuss ground states in the Ramond–Ramond sector.

4 Ramond–Ramond ground states

4.1 Cohomology of states and differential forms

So far we have been discussing the BRST cohomology of operators in the twisted theory. One may also consider the BRST cohomology of states. In a topological field theory, there is a state-operator isomorphism, so the two cohomologies are identical. In the physical (untwisted) SCFT, the cohomology of operators is reinterpreted as the chiral ring, while the cohomology of states is reinterpreted as the space of zero-energy states in the Ramond–Ramond sector. The isomorphism between these two spaces is given by the spectral flow.

In this section, we compute the space of ground states in the RR sector from scratch. There are several reasons to do this. First, it may be interesting to consider $N = 2$ sigma-models with H-flux when the $U(1)_A$ charge is anomalous, i.e., the condition Equation (3.1) is not fulfilled, or more generally, when the twisted generalized Calabi–Yau condition is not fulfilled. Such theories cannot be topologically twisted, but both the chiral ring and the space of RR ground states are perfectly well defined and in general non-isomorphic. In the Kähler case ($H = 0$), this is a familiar situation: the chiral ring is given by $H^\bullet(\Lambda^\bullet TX)$, while the space of RR ground states is $H^\bullet(\Omega^\bullet_X)$. Only in the Calabi–Yau case are the two spaces naturally isomorphic. Second, if we use the point-particle approximation and replace the 2d sigma-model with supersymmetric quantum mechanics (this approximation can be shown to be exact as far as RR ground states are concerned), then the Hilbert space of the theory can be naturally identified with the space of differential forms on X (of all degrees). The supercharge becomes a differential operator on forms, and can be easily computed. We will see that this operator is exactly the differential associated to the twisted generalized complex structure $\mathcal{J}_1$ in Ref. [5]. From this, one can infer without any computations that the chiral ring coincides with the Lie algebroid cohomology associated to $\mathcal{J}_1$. This is the result claimed in the end of the previous section, except that we do not need to assume the existence of the topological twist.

Let us start by writing down the Noether charges associated with Q_+ and Q_- in the point-particle approximation:[3]

$$\begin{aligned} Q_+ &= \psi^a_+ g_{ab}\dot{\phi}^b - \frac{i}{6}H_{abc}\psi^a_+\psi^b_+\psi^c_+ \\ Q_- &= \psi^a_- g_{ab}\dot{\phi}^b + \frac{i}{6}H_{abc}\psi^a_-\psi^b_-\psi^c_-. \end{aligned} \tag{4.1}$$

Let $Q = Q_+ + iQ_-$ and $Q^* = Q_+ - iQ_-$. The supersymmetry algebra implies that

$$Q^2 = Q^{*2} = 0, \quad \{Q, Q^*\} = 4\mathcal{H}$$

where $\mathcal{H}$ is the Hamiltonian of the supersymmetric quantum mechanics:

$$\mathcal{H} = \frac{1}{2}g_{ab}\dot{\phi}^a\dot{\phi}^b - \frac{1}{4}R^{(+)}_{abcd}\psi^a_+\psi^b_+\psi^c_-\psi^d_-.$$

By the standard Hodge–de Rham argument, the supersymmetric ground states are in one-to-one correspondence with the elements of Q-cohomology.

[3] We use the same symbols for the generators and their associated charges.

The charge Q can be thought of as an operator on differential forms via canonical quantization. The classical phase space of the supersymmetric quantum mechanical system is $TM \oplus \Pi(TM \oplus TM)$, where $\Pi(TM \oplus TM)$ is the parity reversal of $TM \oplus TM$. The two "fermionic" copies of TM come from ψ_+ and ψ_-. The symplectic form on TM is the standard one, while the symplectic form in the odd directions (which is actually symmetric) is given by the Riemannian metric g:

$$\{\psi^a_\pm, \psi^b_\pm\}_{P.B.} = -ig^{ab}, \quad \{\psi^a_\pm, \psi^b_\mp\}_{P.B.} = 0.$$

Canonical quantization identifies the Hilbert space with $L^2(S)$, the space of square-integrable sections of the spin bundle $S = S(TM \oplus TM)$. In the case at hand, $TM \oplus TM$ has a natural complex polarization, using which the spin bundle S can be identified with $\wedge^\bullet(TM^*)$. In other words, instead of $\psi_\pm$, we use the coordinates ψ and ρ, which can be quantized by letting ψ^a be a wedge product with dx^a, and letting ρ_b be a contraction with the vector field $\frac{\partial}{\partial x^b}$.

Now we discuss how $N = 1$ supercharges Q and Q^* act on the Hilbert space. Let us first consider the case $H = 0$. Following the standard quantization procedure, one can easily show that $Q = -i\sqrt{2}\psi^a\nabla_a$, with ∇ being the covariant derivative on the sections of the spin bundle $S(TM \oplus TM)$ that is induced from the Levi–Civita connection on TM, and with ψ^a acting as a Clifford multiplication. Under the isomorphism $S(TM \oplus TM) \simeq \wedge^\bullet(TM^*)$, Q is identified with the de Rham differential d, up to a factor $-i\sqrt{2}$. This is the familiar statement that the space of ground states in an $N = 1$ supersymmetric quantum mechanics is isomorphic to the de Rham cohomology of the target space. Now let us consider the case $H \neq 0$. Using (3.4) and (4.1) one can show that

$$Q = -\sqrt{2}i\psi^a\nabla_a + \frac{\sqrt{2}i}{6}H_{abc}\psi^a\psi^b\psi^c$$
$$Q^* = -\sqrt{2}ig^{ab}\rho_b\nabla_a + \frac{\sqrt{2}i}{6}H^{abc}\rho_a\rho_b\rho_c$$

Up to a numerical factor $-\sqrt{2}i$, Q is identified with a twisted de Rham operator

$$d_H = d - H\wedge$$

while Q^* is identified with its adjoint. Therefore the supersymmetric ground states are in one-to-one correspondence with the d_H-cohomology. This statement is also well known [17].

It remains to identify the BRST operator, Q_{BRST}, in this context. The R-current is given by

$$J = -\frac{i}{2}(\omega_+(\psi_+, \psi_+) + \omega_-(\psi_-, \psi_-)),$$

under which $(1 + iI_+)\psi_+$ and $(1 + iI_-)\psi_-$ have charge $+1$ by canonical anticommutation relations. For our purpose, it will be more convenient to express J in terms of the fermions ψ, ρ:

$$J = -\frac{i}{2}(\delta\omega(\psi, \psi) - \alpha(\rho, \rho) - 2\langle \tilde{I}\psi, \rho\rangle).$$

As discussed above, quantization amounts to substitutions:

$$\psi^a \longleftrightarrow dx^a \wedge, \quad \rho_a \longleftrightarrow \iota_{\partial/\partial x^a} \equiv \iota_a.$$

Then the R-current is identified with the following operator on differential forms:

$$J = -i(\delta\omega \wedge - \iota_\alpha - \iota_{\tilde{I}})$$

where ι_α is the contraction with the Poisson bivector α, and $\iota_{\tilde{I}}$ is defined in a local coordinate basis as

$$\iota_{\tilde{I}} = \tilde{I}^a_b(dx^b \wedge) \circ \iota_a.$$

Note that $\delta\omega, \alpha$, and $\tilde{I}$ can be read off $\mathcal{J}_1$ (Equation (2.2)), and therefore the operator J depends only on the TGC-structure $\mathcal{J}_1$.

The BRST operator is given by

$$Q_{\mathrm{BRST}} = \frac{1}{2}(Q + [J, Q]).$$

Since $Q = d_H$, it is clear that Q_{BRST} depends only on the 3-form H and the twisted generalized complex structure $\mathcal{J}_1$. In the following two subsections, we will relate Q_{BRST} to a differential operator on forms defined in Ref. [5] as well as to the canonical complex of the Lie algebroid E.

4.2 Differential forms on a twisted generalized complex manifold

To proceed further, we need to discuss some properties of differential forms on a twisted generalized complex manifold. Recall that on an ordinary complex manifold, the space of differential forms is graded by a pair of integers, the first integer being the ordinary degree of a form, and the second integer being the difference between the number of holomorphic and antiholomorphic indices. If we think of a form as a function on a supermanifold ΠTM, then the first integer is the eigenvalue of a differential operator

$$\deg = \theta^a \frac{\partial}{\partial \theta^a},$$

while the second integer is the eigenvalue of

$$-i \cdot \iota_I = -i I^a_b \theta^b \frac{\partial}{\partial \theta^a},$$

where I^a_b is the complex structure tensor. The existence of the second grading allows us to decompose the de Rham differential d into a holomorphic exterior differential ∂ and its antiholomorphic twin $\bar{\partial}$. Now, following Gualtieri [5], we will define analogs of $-i \cdot \iota_I, \partial$, and $\bar{\partial}$ for twisted generalized complex manifolds.

Recall that $TM \oplus TM^*$ acts on $\Omega^\bullet(M)$ via the spinor representation. Consider the subbundle E of $TM_{\mathbb{C}} \oplus TM^*_{\mathbb{C}}$ defined as the eigenbundle of the TGC-structure $\mathcal{J}$ with eigenvalue $-i$, and its complex-conjugate $\bar{E}$. Since E is isotropic with respect to the form q, one may regard elements of E as annihilation operators, and elements of $\bar{E}$ as creation operators, which act on the fermionic Fock space $\Omega^\bullet(M)$. In other words, the complex structure $\mathcal{J}$ on the vector bundle $TM \oplus TM^*$ allows one to identify the Clifford algebra generated by $TM_{\mathbb{C}} \oplus TM^*_{\mathbb{C}}$ with the fermionic creation–annihilation algebra generated by $E \oplus \bar{E}$. In each fiber of $\Omega^\bullet(M)$, we thus have a vacuum vector, defined up to a factor by the condition that E annihilates it. These vacuum vectors fit into a complex line bundle U_0 over M, which is obviously a subbundle of $\Omega^\bullet(M)$. We will call it the canonical line bundle of the TGC-manifold $(M, \mathcal{J})$. If $\mathcal{J}$ arises from an ordinary complex structure on M, then U_0 is the bundle of top-degree holomorphic forms on the corresponding complex manifold. In general, U_0 does not have a definite degree (i.e., its sections are inhomogeneous forms).

More generally, we can decompose $\Omega^\bullet(M)$ into a direct sum

$$U_0 \oplus U_1 \oplus \cdots \oplus U_{2n}$$

where U_0 is the canonical line bundle for the TGC structure $\mathcal{J}$, and $U_k = \wedge^k \bar{E} \cdot U_0$. Fiberwise, this is simply a decomposition of the fermionic Fock space into subspaces with a definite fermion number. Thus we obtained a grading of the space of differential forms $\Omega^\bullet(M)$ by a non-negative integer k. In the case when $\mathcal{J}$ arises from an ordinary complex structure I, this grading reduces to $i \cdot \iota_I + n$, where n is the complex dimension of M.

The analogue of the de Rham operator d is the "twisted" de Rham operator operator $d_H = d - H$. The analogs of the Dolbeault operators ∂ and $\bar{\partial}$

are defined as follows:

$$\bar{\partial}_H = \pi_{k+1} \circ d_H \colon \Gamma(U_k) \longrightarrow \Gamma(U_{k+1})$$
$$\partial_H = \pi_{k-1} \circ d_H \colon \Gamma(U_k) \longrightarrow \Gamma(U_{k-1}).$$

Note that all these constructions make sense even when $\mathcal{J}$ fails to be integrable with respect to the twisted Courant bracket. It turns out that $\mathcal{J}$ is integrable with respect to the twisted Courant bracket if and only if $d_H = \partial_H + \bar{\partial}_H$. This was proved in Ref. [5] in the case $H = 0$, but one can easily modify the argument so that it applies in general. For the sake of completeness, we provide a proof in the appendix. Similarly, the Dolbeault operators can be defined for almost complex manifolds, but the identity $d = \partial + \bar{\partial}$ holds if and only if the almost complex structure is integrable.

The twisted generalized Calabi–Yau condition that we mentioned above can be formulated in terms of the canonical line bundle U_0 [5,6]. Namely, a TGC-manifold is called a TG-Calabi–Yau manifold if there exists a nowhere vanishing section Ω of U_0 which is d_H-closed. In view of the definition of U_0, this is the same as requiring

$$\bar{\partial}_H \Omega = 0.$$

The topological part of the TG-Calabi–Yau condition (i.e., topological triviality of the line bundle U_0) is equivalent to $c_1(E) = 0$. We have seen above that this condition ensures that the R-current necessary for twisting is non-anomalous. The second part (the existence of a "twisted-holomorphic" section Ω) is also quite important from the physical viewpoint: we will see that it ensures the absence of BRST anomaly.

It is interesting to ask if the second part of the TG-Calabi–Yau condition follows from the first one. The answer turns out to be negative, and this can be seen already for ordinary Calabi–Yau manifolds.[4] Namely, if $c_1(M) = 0$ and M is not simply connected, it may happen that the canonical line bundle is not trivial as a holomorphic line bundle (even though it is trivial topologically). In this case, there are no nowhere-vanishing holomorphic sections of the canonical line bundle.

4.3 Cohomology of states and twisted generalized complex structures

We are going to show that the BRST-cohomology of states is isomorphic to the cohomology of $\bar{\partial}_H$ on differential forms on the TGC-manifold $(M, \mathcal{J}_1)$.

[4]We are grateful to Misha Verbitsky for explaining this to us.

As a preliminary step, let us obtain a convenient explicit formula for the grading operator on $\Omega^\bullet(M)$, defined in the previous subsection, in terms of the twisted generalized complex structure $\mathcal{J}_1$. Let $A = X + \xi \in \Gamma(TM_{\mathbb{C}} \oplus TM^*_{\mathbb{C}})$. It can be regarded as an endomorphism of $\wedge^\bullet TM^*_{\mathbb{C}}$:

$$A \cdot \rho = \iota_X \rho + \xi \wedge \rho. \tag{4.2}$$

On the other hand, $\mathcal{J}_1$ is an endomorphism of $TM_{\mathbb{C}} \oplus TM^*_{\mathbb{C}}$ with eigenvalues i and $-i$. By definition, the grading operator $R(\mathcal{J}_1)$ must satisfy

$$[R(\mathcal{J}_1), A] = -i\mathcal{J}_1 A, \quad \forall A \in \Gamma(TM_{\mathbb{C}} \oplus TM^*_{\mathbb{C}}).$$

Obviously, this condition determines $R(\mathcal{J}_1)$ up to a constant. Using the explicit matrix form (2.2) of $\mathcal{J}_1$, one gets

$$\mathcal{J}_1 A \cdot \rho = \iota_{\tilde{I}X} + \iota_{\alpha(\xi)} - \iota_X \delta\omega \wedge - \tilde{I}^t(\xi) \wedge .$$

Then it is easy to check that the following is a solution to the above equation:

$$R(\mathcal{J}_1) = -i(\delta\omega \wedge - \iota_\alpha - \iota_{\tilde{I}}).$$

The general solution may differ from this only by a constant.

From the result of the last section, one immediately sees that under the identification of $Q \leftrightarrow d_H$ and the identification of the Hilbert space as the space of differential forms, the R-current generator J is identified as

$$J \longleftrightarrow R(\mathcal{J}_1) + \text{const}.$$

The BRST operator Q_{BRST} then becomes

$$\begin{aligned} Q_{\text{BRST}} &= \frac{1}{2}\big(Q + [J, Q]\big) \\ &= \frac{1}{2}\big(d_H + [R(\mathcal{J}_1), d_H]\big) \\ &= \bar{\partial}_H \end{aligned}$$

This is the desired result.

Now we can show that the BRST-cohomology of operators is isomorphic to the Lie algebroid cohomology of E. Recall that the $\bar{\partial}_H$ complex is a differential graded module over the Lie algebroid complex $(\Lambda^\bullet \bar{E}, d_L)$, i.e., the following identity holds for any section s of $\Lambda^\bullet \bar{E}$ and any differential form ρ:

$$\bar{\partial}_H(s \cdot \rho) = (d_L s) \cdot \rho + (-1)^{|s|} s \cdot \bar{\partial}_H \rho.$$

This identity is proved in Ref. [5] for the case $H = 0$, but the proof is valid more generally. Since we identified the space of sections of $(\Lambda^\bullet \bar{E}, d_L)$ with the space of operators, the space of forms with the Hilbert space of the

SUSY quantum mechanics, and $\bar{\partial}_H$ with the representation of the BRST charge on the Hilbert space, it follows that

$$[Q_{\mathrm{BRST}}, s] = d_L s.$$

This implies that the cohomology of d_L is isomorphic to the BRST cohomology of operators, as claimed.

5 Topological correlators and the Frobenius structure

For any $N = 2$ $d = 2$ field theory, we may consider the chiral ring, as well as the cohomology of the supercharge $Q_L + Q_R$ on the states in the Ramond sector. The latter is a module over the former. The two spaces are not isomorphic in general. But if the theory admits a topological B-twist, the two spaces are always isomorphic, by virtue of the state-operator correspondence in a topological field theory. More precisely, the space of states of a 2d TFT is an algebra with a nondegenerate scalar product $(\cdot, \cdot)$ such that

$$(a, bc) = (ab, c).$$

Such algebras are called Frobenius. All topological correlators can be expressed in terms of the Frobenius structure on the space of states. For example, genus zero correlators are given by

$$\langle a_1 \cdots a_n \rangle_{g=0} = (a_1, a_2, \ldots, a_n).$$

Consider now an $N = 2$ sigma-model for which condition (2.1) is satisfied, and the $U(1)_A$ R-charge is non-anomalous. One expects that the theory admits a topological B-twist, and therefore the chiral ring, which is known to be isomorphic to the Lie algebroid cohomology of E_1, is a supercommutative Frobenius algebra. In fact, we will see that in order for a BRST-invariant measure in the path-integral to exist, the target manifold must be a TG-Calabi–Yau manifold, which is stronger than Equation (2.1).

Note that the Frobenius scalar product $(\cdot, \cdot)$ can be recovered from the "trace" function:

$$\mathrm{Tr}(a) = (1, a)$$

by letting $(a, b) = \mathrm{Tr}(ab)$. The name "trace" is used because Tr vanishes on commutators (in the graded case, on graded commutators). Let Ω be a $\bar{\partial}_H$-closed differential form which sits in the component U_0. For a twisted generalized Calabi–Yau such a form exists and is unique up to a constant factor. Note that Ω is also d_H-closed. Consider now a bundle automorphism $p : TM \oplus TM^*$ which looks as follows:

$$p : (v, \xi) \longmapsto (v, -\xi), \ \ \forall v \in \Gamma(TM), \quad \forall \xi \in \Gamma(TM^*).$$

This automorphism takes the form q to $-q$ and maps the Courant bracket twisted by H to the Courant bracket twisted by $-H$. It follows from this that for any twisted generalized complex structure $\mathcal{J}$ the bundle map $\mathcal{J}' = p^{-1}\mathcal{J}p$ is also a twisted generalized complex structure, for the opposite H-field $H' = -H$. Moreover, it is easy to see that $(M, -H, \mathcal{J}')$ is a twisted generalized Calabi–Yau if and only if $(M, H, \mathcal{J})$ is one. (From the physical viewpoint, p corresponds to worldsheet parity transformation, and the above facts are obvious.) In particular, we have a decomposition

$$\wedge^\bullet TM^* \otimes \mathbb{C} = U_0' \oplus U_1' \oplus \cdots \oplus U_{2n}'$$

Let Ω' be the $\bar{\partial}_{H'}$-closed differential form which sits in the component U_0'. We claim that the trace function on the Lie algebroid cohomology is given by

$$\mathrm{Tr}(\alpha) \sim \int_M \Omega' \wedge \alpha \cdot \Omega,$$

where α is a d_L-closed section of $\Lambda^\bullet(E_1^*)$.

To derive this formula, we recall that the Frobenius trace is computed by the path-integral on a Riemann sphere with an insertion of the operator corresponding to α. Since we are dealing with a topological theory, we must also turn on a $U(1)$ gauge field coupled to the R-current participating in the twisting. This gauge field must be equal to the spin connection, which means that the total flux through the sphere is 2π. Let us stretch the sphere into a long and thin cigar, so that the insertion point of α is somewhere in the middle portion. The value of the path-integral does not change, of course, but it may now be evaluated more easily by reducing the computation to the supersymmetric quantum mechanics. The path-integral on each hemisphere gives a state in the Ramond–Ramond sector, which may be approximated in the point-particle limit by a function of the zero-modes. Bosonic zero-modes are simply coordinates on M, while fermionic zero-modes are ψ^i, taking values in $TM_{\mathbb{C}}$. Thus the Ramond–Ramond state is represented by a function on $\Pi TM_{\mathbb{C}}$, i.e., by a (complex-valued) differential form. We have described above how α acts on differential forms.

It remains to identify the particular RR states arising from performing the path-integral over each hemisphere, and then integrate over the zero-modes. Since we are not inserting any operators on the hemispheres, the RR ground state in question is the spectral flow of the unique vacuum state in the NS sector, and therefore in the point-particle approximation is represented by the form Ω defined above.[5] However, there is a subtlety related to the choice of orientation. This subtlety arises because our identification of RR states with differential forms depends on orientation: exchanging left-moving and right-moving fermions is equivalent to performing a Hodge duality on forms. In the physical language, Hodge duality is simply the Fourier transform of fermionic zero-modes. If we induce the orientations of both hemispheres from a global orientation of the Riemann sphere, then the wave-function coming from one hemisphere will be given by a function of the fermionic zero-modes ψ^i, while the wave-function from the other hemisphere will be a function of the Fourier-dual zero-modes. In order to evaluate the path-integral one first has to Fourier transform the second state, and only then multiply the wave functions and integrate over the zero-modes. Alternatively, we can choose the opposite orientation for the second hemisphere, so that there is no need for Fourier transform. This also requires flipping the sign of H, since the worldsheet theory is not parity-invariant. We conclude that the wave function from the second hemisphere is given by Ω', and the path-integral in question is given by

$$\mathrm{Tr}(\alpha) \sim \int_M \Omega' \wedge \alpha \cdot \Omega.$$

Let us check that this formula is BRST-invariant, i.e., that it vanishes if $\alpha = d_L\beta$ for some β. Indeed, we have

$$\mathrm{Tr}(d_L\beta) = \int_M \Omega' \wedge \bar{\partial}_H(\beta \cdot \Omega) = \int_M \Omega' \wedge (d_H + [R(\mathcal{J}_1), d_H])(\beta \cdot \Omega).$$

Next we have to use the following two identities valid for any two forms γ, η:

$$\int_M \gamma \wedge d_H\eta = -(-1)^{|\gamma|} \int_M (d_{H'}\gamma) \wedge \eta \tag{5.1}$$

$$\int_M \gamma \wedge R(\mathcal{J}_1)\eta = -\int_M (R(\mathcal{J}_1')\gamma) \wedge \eta, \tag{5.2}$$

where $H' = -H$ and $\mathcal{J}' = p^{-1}\mathcal{J}p$. Then we get

$$\begin{aligned}\mathrm{Tr}(d_L\beta) &= -(-1)^{|\Omega'|} \int_M ((d_{H'} + [R(\mathcal{J}_1'), d_{H'}])\Omega') \wedge \beta \cdot \Omega \\ &= -(-1)^{|\Omega'|} \int_M \bar{\partial}_{H'}\Omega' \wedge \beta \cdot \Omega = 0.\end{aligned} \tag{5.3}$$

[5] If a d_H-closed section of the line bundle U_0 does not exist, then supersymmetry is spontaneously broken, i.e., there are no RR states with zero energy. From the point of view of the topological theory, this means that the measure in the path-integral fails to be BRST invariant, i.e., there is a BRST anomaly.

Let us also check that this formula reduces to the known expressions in the case of the ordinary A and B-models with $H = 0$ and $I_+ = I_-$. For the ordinary B-model, $\mathcal{J}_1' = \mathcal{J}_1$, $\Omega' = \Omega$, and the form Ω is simply the top holomorphic form on M. It is obvious that our formula for the trace function reduces to the standard formula for the B-model [20]. For the A-model, the situation is more interesting. The relevant generalized complex structure is $\mathcal{J}_2$, and we have $\mathcal{J}_2' = -\mathcal{J}_2$. The forms Ω and Ω' are given by

$$\Omega = e^{i\omega}, \quad \Omega' = e^{-i\omega}.$$

The complex Lie algebroid for the A-model is isomorphic to $TM_{\mathbb{C}}$, thus the Lie algebroid cohomology is isomorphic to the complex de Rham cohomology. The usual formula for the Frobenius trace on $H^\bullet(M)$ is

$$\mathrm{Tr}(\beta) = \int_M \beta, \ \ \beta \in \Omega^\bullet(M), d\beta = 0.$$

This does not seem to agree with our formula. But one should keep in mind that the identification between the Lie algebroid cohomology and de Rham cohomology is non-trivial, and as a result, although the bundle $\Lambda^\bullet E_2^*$ is isomorphic to $\Omega^\bullet(M)$, the action of $\Omega^\bullet(M)$ on itself coming from the action of $\Lambda^\bullet E_2^*$ on $\Omega^\bullet(M)$ is not given by the wedge product. To describe this action, let us identify the space of sections of $\Omega^\bullet(M)$ with the graded supermanifold ΠTM. Let $\alpha \in \Omega^k(M)$ be given by

$$\alpha = \frac{1}{k!}\alpha_{a_1\cdots a_k} dx^{a_1} \wedge \cdots \wedge dx^{a_k}.$$

The action we are after is obtained by associating to α the following differential operator on ΠTM:

$$\frac{1}{k!}\alpha_{a_1\cdots a_k} D^{a_1} \cdots D^{a_k},$$

where

$$D^a = \theta^a + i(\omega^{-1})^{ab}\frac{\partial}{\partial\theta^b}.$$

The operators D^a anticommute, so this is well defined. On the other hand, in the usual description of the A-model, the action of $\Omega^\bullet(M)$ on itself is given by the ordinary wedge product (plus quantum corrections, which we neglect in this paper).

The difference between our description of the A-model and the usual one is due to a different identification of the fermionic fields with operators on forms. While we identified ψ^a with "creation" operators dx^a and ρ_a with "annihilation" operators, the usual identification is different:

$$\psi_+^{\bar{i}} \longmapsto dx^{\bar{i}}, \quad \psi_-^{i} \longmapsto dx^{i}, \quad g_{\bar{i}j}\psi_+^{j} \longmapsto \iota_{\frac{\partial}{\partial x^{\bar{i}}}}, \quad g_{j\bar{i}}\psi_+^{\bar{i}} \longmapsto \iota_{\frac{\partial}{\partial x^{\bar{j}}}}.$$

This choice is related to ours by a Bogolyubov transformation. In the usual description, the vacuum state with the lowest R-charge $J_L - J_R$ is given by the constant 0-form on M. It is easy to see that the Bogolyubov transformation maps it to the inhomogeneous form $e^{i\omega}$. The same transformation also maps the ordinary degree of a differential form to the nonstandard grading on $\Omega^\bullet(M)$ defined in Ref. [5] and explained above. Thus our formula agrees with the standard one after a Bogolyubov transformation (and if one neglects quantum corrections).

6 Towards the twisted generalized quantum cohomology ring

So far we have only discussed the classical ring structure on the space of topological observables. In general, the actual ring structure is deformed by quantum effects. A well-known example is the ordinary A-model, whose ring of BRST-invariant observables (the quantum cohomology ring) is a deformation of the de Rham cohomology ring $H^\bullet(M, \mathbb{C})$ induced by worldsheet instantons. In this section, we carry out the analysis for generic twisted topological sigma-model with H-flux and identify worldsheet instantons which can contribute to the deformation of the ring structure. This section is essentially an extension of the analysis of Section 8.2 of Ref. [10] to the case $H \neq 0$.

As is well known, the path integral of a (cohomological) TFT can be localized around the Q_{BRST} invariant field configurations [20]. For our generalized B-model, the BRST variations of some of the fields are already given in Equation (3.4). We will also need the following BRST transformations for $(1 - iI_\pm)\psi_\pm$:

$$\left\{Q_{\mathrm{BRST}}, \frac{1}{2}(1 - iI_+)\psi_+\right\} = \frac{i}{2}(1 - iI_+)\partial_+\phi + \cdots$$
$$\left\{Q_{\mathrm{BRST}}, \frac{1}{2}(1 - iI_-)\psi_-\right\} = -\frac{1}{2}(1 - iI_+)\partial_-\phi + \cdots$$

where the dots involve fermion bilinear terms. The BRST invariant configurations are given by setting the fermionic fields $\psi_\pm$ to zero and demanding

$$\frac{1}{2}(1 - iI_+)\partial_+\phi = 0, \quad \frac{1}{2}(1 - iI_-)\partial_-\phi = 0.$$

In terms of the generalized complex structure, the above equation is equivalent to

$$\frac{1}{2}(1 - i\mathcal{J}_1)\begin{pmatrix}\partial_1\phi \\ g\partial_0\phi\end{pmatrix} = 0. \tag{6.1}$$

This is the same instanton equation as obtained in Ref. [10] in the case of $H = 0$, and the results of [10] carry over to our case. For reader's convenience, we summarize them below.

To find Euclidean instantons, we Wick-rotate $\partial_0 \to \sqrt{-1}\partial_2$ as in Ref. [10]. Equation (6.1) then leads to the following equations

$$\begin{aligned} \tilde{\omega}\partial_1\phi = 0, \quad \partial_1\phi = -\delta I\partial_2\phi \\ \tilde{\omega}\partial_2\phi = 0, \quad \partial_2\phi = \delta I\partial_1\phi. \end{aligned} \tag{6.2}$$

It is not difficult to see that the solutions to Equations (6.2) are "twisted generalized holomorphic maps" with respect to $\mathcal{J}_2$. The precise meaning of this is as follows. The differential $d\phi$ composed with the natural embedding $j : TM \to TM \oplus TM^*$ defines a map

$$j \circ d\phi\colon T\Sigma \longrightarrow TM \oplus TM^*.$$

Equation (6.2) then says that $j \circ d\phi$ intertwines the TGC structure $\mathcal{J}_2$ and the complex structure on the worldsheet I_Σ. That is,

$$\mathcal{J}_2(j \circ d\phi) = (j \circ d\phi)I_\Sigma$$

Solutions of this equation generalize both the holomorphic maps of the ordinary A-model and the constant maps of the ordinary B-model.

On general grounds, the ring structure of topological observables may admit non-trivial quantum corrections coming from these worldsheet instantons. We will not try to describe these corrections more precisely here. But note that for a generic TGC-structure $\mathcal{J}_2$ the TG-holomorphic instanton equation is much more restrictive than the ordinary holomorphic instanton equation. Indeed, it requires the image of $T\Sigma$ under $d\phi$ to lie in the kernel of the map $\tilde{\omega}$. For a generic $\mathcal{J}_2$ and at a generic point of M, the 2-form $\tilde{\omega}$ is non-degenerate, and so this condition does not allow non-constant instantons. In other words, all non-trivial instantons must be contained in the subvariety, where $\tilde{\omega}$ is degenerate. The extreme cases are the ordinary B-model, where $\tilde{\omega}$ is a symplectic form and there are no nontrivial instantons, and the ordinary A-model, where $\tilde{\omega}$ vanishes identically.

7 Discussion

In this paper, we have studied the topological sector of $(2,2)$ sigma-models with H-flux. We found that the results are very conveniently formulated in terms of twisted generalized complex structures. For example, the chiral ring is isomorphic (on the classical level) to the cohomology of a certain Lie algebroid which controls the deformation theory of a twisted generalized complex structure. On the quantum level, the two rings are isomorphic as vector spaces, but the ring structures may be different due to worldsheet instantons. It would be interesting to further study these quantum corrections. In particular, we expect that the quantum ring structure depends only on one of the two twisted generalized complex structures present. (This is the analog of the statement that the quantum cohomology ring is independent of the choice of the complex structure [20].) To prove this, one has to show that varying the TGC-structure $\mathcal{J}_2$ changes the action of the sigma-model by BRST-exact terms.

It is expected on general grounds that the moduli space of $N = 2$ SCFTs is a product of two spaces, corresponding to deformations by elements of the (c,c) and (a,c) rings. It follows from our work that for $N = 2$ sigma-models with H-flux these two moduli spaces are identified with the moduli spaces of two independent twisted generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$. From the mathematical viewpoint, this means that the deformation theory of twisted generalized Calabi–Yau manifolds is unobstructed. It would be very interesting to prove this rigorously.

Recall that the well-known Kähler identities

$$\partial\partial^* + \partial^*\partial = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} = \frac{1}{2}(dd^* + d^*d)$$

are interpreted physically as $N = 2$ supersymmetry relations. It follows from the results of this paper that for twisted generalized Kähler manifolds analogous relations hold true:

$$\partial_H\partial_H^* + \partial_H^*\partial_H = \bar{\partial}_H\bar{\partial}_H^* + \bar{\partial}_H^*\bar{\partial}_H = \frac{1}{2}(d_H d_H^* + d_H^* d_H).$$

It should be clear that all the notions pertaining to ordinary $N = 2$ sigma-models make sense when one allows for the possibility of H-flux. For example, a pair of twisted generalized Calabi–Yau manifolds M and M' are called mirror if the generalized A-model of M is isomorphic to the generalized B-model of M', and vice versa. In mathematical terms, this means that the Frobenius manifold corresponding to deformations of the twisted generalized complex structure $\mathcal{J}_1$ on M is isomorphic to the Frobenius manifold corresponding to deformations of $\mathcal{J}_2$ on M', and vice versa. Further, it is possible to define the categories of generalized A and B-branes, and one expects that mirror symmetry exchanges them. The geometry of generalized A and B-branes deserved further study; initial steps in this direction have been made in Refs. [10, 21].

It would be very interesting to find examples of mirror pairs of twisted generalized Calabi–Yau manifolds. There is a slight problem here though: we do not expect any *compact* examples of twisted generalized Calabi–Yau manifolds with $H \neq 0$ to exist. If such an example existed, it would give rise to a superconformal $N = 2$ sigma-model with integral central charge. By rescaling the metric and the H-field (so that the volume of the manifold is large), we would get a metric and an H-field on M which satisfy supergravity equations of motion. But there are well-known theorems that force all such smooth supergravity solutions to have zero H-field [3, 9, 14].

Note in this connection that the simplest non-trivial example of a TG Kähler manifold $M = S^3 \times S^1$ [5] is not a TG-Calabi–Yau manifold. Even though the topological condition $c_1(E) = 0$ is trivially satisfied (M has no cohomology in degree 2), the line bundle U_0 does not have a section which is d_H-closed.[6]

Thus to study mirror symmetry for twisted generalized Kähler manifolds, we either have to work with non-compact manifolds or drop the Calabi–Yau condition. Both possibilities are interesting. What we are lacking at present is a generalization of the Kähler quotient (or toric geometry) construction. This would provide us with a large supply of TG Kähler manifolds and perhaps would also enable us to find their mirrors (cf. [7]). We plan to return to this subject in the future.

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Appendix A

A twisted generalized almost complex structure is defined just like a twisted generalized complex structure, except that the last condition (integrability of $E = \ker(\mathcal{J} + i)$ with respect to the Dorfman bracket) is dropped. Given a TG almost complex structure $\mathcal{J}$ on (M, H), one can define operators ∂_H and $\bar{\partial}_H$ on differential forms on M (see Section 4.2). In this appendix, we prove the following integrability criterion for $\mathcal{J}$.

Theorem A.1. *The twisted generalized almost complex structure $\mathcal{J}$ is integrable if and only if $d_H = \partial_H + \bar{\partial}_H$.*

We shall outline a proof of the theorem following Gualtieri, who proved it in the special case $H = 0$ [5]. Let ρ be an arbitrary differential form, and let $A = X + \xi$, $B = Y + \eta$ be arbitrary sections of E. It is straightforward to show (using Equation (4.2) and the Cartan identities $\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X$, $\iota_{[X,Y]} = [\mathcal{L}_X, \iota_Y]$) that

$$A \cdot B \cdot d\rho = d(BA\rho) + B \cdot d(A\rho) - A \cdot d(B\rho) + [A, B]\rho \tag{A.1}$$

$$\begin{aligned} A \cdot B \cdot (H \wedge \rho) = &-\iota_Y \iota_X H \wedge \rho + \iota_Y H \wedge (A\rho) \\ &- \iota_X H \wedge (B\rho) + H \wedge (AB\rho) \end{aligned} \tag{A.2}$$

Subtracting (A.2) from (A.1), one obtains

$$A \cdot B \cdot d_H\rho = d_H(BA\rho) + B \cdot d_H(A\rho) - A \cdot d_H(B\rho) + [A, B]_H \cdot \rho. \tag{A.3}$$

The rest of the proof now follows exactly as in Ref. [5]. First let us assume that $\mathcal{J}$ is integrable. For $\rho \in \Gamma(U_0)$, (A.3) reduces to $AB \cdot d_H\rho = [A, B]_H \cdot \rho = 0$. Since $d_H\rho$ has no component in U_0, it follows that $d(\Gamma(U_0)) \subset \Gamma(U_1)$ and thus $d_H = \partial_H + \bar{\partial}_H$ holds for $\rho \in \Gamma(U_0)$. Now assume $d_H = \partial_H + \bar{\partial}_H$ holds for all $U_k, 0 \leq k < i$, and let $\rho \in \Gamma(U_i)$ and $A, B \in \Gamma(E)$ as before. Equation (A.3) now shows that $AB \cdot d_H\rho \in \Gamma(U_{i-3} \oplus U_{i-1})$, which in turn implies $d_H\rho \in \Gamma(U_{i-1} \oplus U_{i+1})$. By induction, one concludes that $d_H = \partial_H + \bar{\partial}_H$ on $\wedge^\bullet TM^* \otimes \mathbb{C}$. The converse is also true by similar argument.

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