

# Two-dimensional models with (0, 2) supersymmetry: perturbative aspects

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## Abstract

Certain perturbative aspects of two-dimensional sigma models with  $(0, 2)$  supersymmetry are investigated. The main goal is to understand in physical terms how the mathematical theory of “chiral differential operators” is related to sigma models. In the process, we obtain, for example, an understanding of the one-loop beta function in terms of holomorphic data. A companion paper will study nonperturbative behavior of these theories.

## 1 Introduction

Two-dimensional sigma models with  $(0, 2)$  supersymmetry have attracted much interest over the years, largely because, in the conformally invariant case, they can be used to construct supersymmetric compactifications of heterotic string theory. These models have interesting nonperturbative effects that can spoil conformal invariance even when it

holds in perturbation theory [1]; much attention has focused on determining conditions for exact conformal invariance (for example, see [2–5]). The present paper, however, will pursue a rather different direction.

By taking the cohomology of one of the supercharges of a  $(0,2)$  model, one can construct a half-twisted model [6] that is characterized by a chiral algebra. This chiral algebra contains much information about the dynamics of the underlying model, but this has not yet been fully exploited. See [7] for an example of determination of such a chiral algebra. The structure is further enriched because the elliptic genus, a familiar invariant of two-dimensional supersymmetric models, can be expressed in terms of supersymmetric physical states which furnish a module for the chiral algebra. (The elliptic genus is defined using  $(0,1)$  supersymmetry [8] and can be interpreted in terms of a module for a chiral algebra when one has  $(0,2)$  supersymmetry.) Correlation functions of the half-twisted model are related in certain situations to Yukawa couplings in heterotic string compactifications. They have been studied from many points of view; for recent discussion, see [9].

In the present paper, we will study the half-twisted model *in perturbation theory*. We will interpret the *perturbative* approximation to the chiral algebra of the half-twisted model in terms of the mathematical theory of “Chiral Differential Operators” or CDO’s [10–12]. This theory was also developed independently in Section 3.9 of [13]; the CDO of a flag manifold had in essence been introduced earlier [14]. Some of these developments have been motivated by potential applications to the geometric Langlands program [15]. Borrowing the mathematical results, we acquire a few novel insights about the physics; for example, we interpret the one-loop beta function in terms of holomorphic data. Perhaps our results will also be of interest for mathematical applications of CDO’s.

Beyond perturbation theory, the chiral algebras of  $(0,2)$  models are no longer related to CDO’s. What happens nonperturbatively will be the subject of a separate paper, in which we will see that instanton effects often change the picture radically, triggering the spontaneous breaking of supersymmetry and making the chiral algebra trivial. This can only happen when the elliptic genus vanishes, since a nonzero elliptic genus is an obstruction to supersymmetry breaking.

If one drops the requirement of unitarity, theories obtained by half-twisting a  $(0,2)$  model are a special case of a larger class of sigma models that lead to chiral algebras. We describe this larger class in Section 2. In Section 3, we characterize the chiral algebra that arises in perturbation theory from such a model in terms of a sheaf of CDO’s. In Section 4, we

incorporate unitarity and specialize to twisted versions of models with  $(0, 2)$  supersymmetry. In Section 5, we perform explicit calculations illustrating how the standard one-loop anomalies of  $(0, 2)$  models — the beta function and the chiral anomaly — arise from the CDO point of view. These examples are inspired by and generalize an example considered in Section 5.6 of [10] as well as the detailed analysis of anomalies in [12].

The question of how to interpret sheaves of CDO's in terms of physics was originally raised by F. Malikov at the Caltech-USC Center for Theoretical Physics in 1999–2000 and by E. Frenkel at a conference on the geometric Langlands program held at the IAS in the spring of 2004. These questions were the genesis of the present paper. In addition, E. Frenkel generously explained a number of relevant mathematical constructions, including chiral algebras without stress tensors and Čech description of operators and anomalies, some of which would have been quite difficult to understand from the literature. I am grateful for his assistance. Finally, I would like to thank Meng-Chwan Tan for a careful reading of the paper and spotting of many imprecisions.

## 2 Chiral algebras from sigma models

### 2.1 Classical action

In the present section, we consider the minimal structure of a two-dimensional sigma model that enables one to define a chiral algebra. We consider a sigma model of maps  $\Phi$  from a Riemann surface  $\Sigma$  to a complex manifold  $X$ . We pick a local complex coordinate  $z$  on  $\Sigma$  to write formulas. The only bosonic field in the model is  $\Phi$  itself, which we describe locally via fields  $\phi^i(z, \bar{z})$  (which are pullbacks to  $\Sigma$ , via  $\Phi$ , of local complex coordinates on  $X$ ). There are also the following fermionic fields:  $\rho^i$  is a  $(0, 1)$ -form on  $\Sigma$  with values in  $\Phi^*(TX)$ ; and  $\alpha^{\bar{i}}$  is a zero-form on  $\Sigma$  with values in  $\Phi^*(\overline{TX})$ . Here  $\Phi^*(TX)$  and  $\Phi^*(\overline{TX})$  are the pullback to  $\Sigma$ , via  $\Phi$ , of the holomorphic and antiholomorphic tangent bundles  $TX$  and  $\overline{TX}$  of  $X$ .

We postulate a fermionic symmetry with

$$\begin{aligned}\delta\phi^i &= \delta\alpha^{\bar{i}} = 0 \\ \delta\bar{\phi}^{\bar{i}} &= \alpha^{\bar{i}} \\ \delta\rho_{\bar{z}}^i &= -\partial_{\bar{z}}\phi^i.\end{aligned}\tag{2.1}$$

We call the generator of this symmetry  $Q$ . We introduce a  $U(1)$  symmetry  $R$  (which nonperturbatively may be violated by instantons) under which  $\alpha$  has charge 1,  $\rho$  has charge  $-1$ , and  $\phi$  is invariant. So  $Q$  has  $R$ -charge 1.

Clearly,  $Q^2 = 0$  classically (and also quantum mechanically, as we discuss later), and so any action of the form  $I = \int |d^2z| \{Q, V\}$  is  $Q$ -invariant, for any  $V$ . (Here  $|d^2z| = idz \wedge d\bar{z}$ .) Choosing on  $X$  a hermitian (not necessarily Kahler) metric  $ds^2 = g_{i\bar{j}} d\phi^i d\bar{\phi}^{\bar{j}}$ , we take  $V = -g_{i\bar{j}} \rho^i \partial_z \bar{\phi}^{\bar{j}}$ . This leads to the action that we use to define our most basic sigma model:

$$I = \int |d^2z| \left( g_{i\bar{j}} \partial_z \phi^i \partial_z \bar{\phi}^{\bar{j}} + g_{i\bar{j}} \rho^i \partial_z \alpha^{\bar{j}} - g_{i\bar{j}, \bar{k}} \alpha^{\bar{k}} \rho^i \partial_z \bar{\phi}^{\bar{j}} \right). \quad (2.2)$$

It is very natural to extend this model to include additional fermionic fields valued in  $\Phi^*(E)$ , where  $E$  is a holomorphic vector bundle over  $X$ . The generalization is important in heterotic string compactification and hence is extensively analyzed in the literature on  $(0, 2)$  models (for recent discussion, see [9]); this more general possibility has also been considered in the mathematical literature on CDO's. A special case is that in which  $E = TX$ ; the resulting model is then a half-twisted version of the usual sigma model with  $(2, 2)$  supersymmetry, and the associated CDO has been called mathematically the chiral de Rham complex (CDR). For brevity, we will omit these generalizations. Other generalizations that have been studied in the mathematical literature [16, 17] could plausibly be interpreted physically in terms of the perturbative approximation to orbifolds and to gauged sigma models, respectively, with  $(2, 2)$  supersymmetry.

Our goal in this section is to study, in perturbation theory, the cohomology of  $Q$  acting on local operators of this sigma model. We will examine this more closely after describing some properties of the quantum model, but first we make a few observations about the classical theory. Classically, the model is conformally invariant; the trace  $T_{z\bar{z}}$  of the stress tensor vanishes. The nonzero components of the stress tensor are  $T_{zz} = g_{i\bar{j}} \partial_z \phi^i \partial_z \bar{\phi}^{\bar{j}}$  and  $T_{\bar{z}\bar{z}} = g_{i\bar{j}} \left( \partial_z \phi^i \partial_{\bar{z}} \bar{\phi}^{\bar{j}} + \rho^i D_{\bar{z}} \alpha^{\bar{j}} \right)$ . All components of the stress tensor are  $Q$ -invariant, but crucially,  $T_{\bar{z}\bar{z}}$  is trivial in cohomology, being  $\{Q, -g_{i\bar{j}} \rho^i \partial_z \bar{\phi}^{\bar{j}}\}$ .

We say that a local operator  $\mathcal{O}$  inserted at the origin has dimension  $(n, m)$  if under a rescaling  $z \rightarrow \lambda z$ ,  $\bar{z} \rightarrow \bar{\lambda} z$  (which is a symmetry of the classical theory), it transforms as  $\partial^{n+m} / \partial z^n \partial \bar{z}^m$ , that is, as  $\lambda^{-n} \bar{\lambda}^{-m}$ . Classical local operators have dimensions  $(n, m)$  where  $n$  and  $m$  are nonnegative

integers.<sup>1</sup> But the cohomology of  $Q$  vanishes except for  $m = 0$ . The reason for the last statement is that the rescaling of  $\bar{z}$  is generated by  $\bar{L}_0 = \oint d\bar{z} \bar{z} T_{\bar{z}\bar{z}}$ . As we noted in the last paragraph,  $T_{\bar{z}\bar{z}}$  is of the form  $\{Q, \dots\}$ , so  $\bar{L}_0 = \{Q, W_0\}$  for some  $W_0$ . Hence, if  $\{Q, \mathcal{O}\} = 0$ , then  $[\bar{L}_0, \mathcal{O}] = \{Q, [W_0, \mathcal{O}]\}$ . If  $[\bar{L}_0, \mathcal{O}] = m\mathcal{O}$  for  $m \neq 0$ , it follows that  $\mathcal{O}$  is trivial in cohomology.

By an argument similar to the above, if  $\mathcal{O}$  is annihilated by  $Q$ , then as an element of the cohomology,  $\mathcal{O}(z)$  varies holomorphically with  $z$ . Indeed,  $\partial_{\bar{z}}\mathcal{O}$  is the commutator with  $\mathcal{O}$  of  $\bar{L}_{-1} = \oint d\bar{z} T_{\bar{z}\bar{z}}$ . We have  $\bar{L}_{-1} = \{Q, W_{-1}\}$  for some  $W_{-1}$ , and hence  $\partial_{\bar{z}}(\mathcal{O}) = \{Q, [W_{-1}, \mathcal{O}]\}$ .

We conclude with a few comments on the quantum theory. In Section 2.3, we will show that quantum mechanically, just like classically,  $T_{\bar{z}\bar{z}}$  and  $T_{zz}$  are of the form  $\{Q, \dots\}$ . Hence the same is true for the momentum operator (called  $\bar{L}_{-1}$  in the last paragraph) that generates  $\partial/\partial\bar{z}$ . This implies that operators  $\mathcal{O}(z)$  that represent cohomology classes vary holomorphically with  $z$ , just as in the classical theory. As for the assertion that the cohomology of  $Q$  on operators of dimension  $(n, m)$  is trivial for  $m \neq 0$ , this statement is automatically true quantum mechanically (at least in perturbation theory or when quantum effects are small enough) if it is true classically, since a vanishing cohomology continues to vanish after any small perturbation.

Now, let  $\mathcal{O}(z)$  and  $\tilde{\mathcal{O}}(z')$  be two operators that commute with  $Q$ , so that their product does so likewise. Consider the operator product expansion or OPE of this product:

$$\mathcal{O}(z)\tilde{\mathcal{O}}(z') \sim \sum_k f_k(z - z')\mathcal{O}_k(z'). \quad (2.3)$$

In general the coefficient functions  $f_k$  are not holomorphic. But if we pass to the cohomology and drop operators on the right hand side that are  $\{Q, \dots\}$ , then the surviving coefficient functions are holomorphic. In fact,  $\partial/\partial\bar{z}$  acting on the left hand side of (2.3) gives terms that are cohomologically trivial,<sup>2</sup> so the  $f_k$ 's that are not annihilated by  $\partial/\partial\bar{z}$  multiply operators  $\mathcal{O}_k$  that are likewise cohomologically trivial. We have established, roughly speaking, that the cohomology of  $Q$  has a natural structure of a holomorphic chiral algebra, which we will call  $\mathcal{A}$ .

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<sup>1</sup>Quantum mechanically, anomalous dimensions shift the values of  $n$  and  $m$ , but the difference  $n - m$  is unchanged.

<sup>2</sup>Here we use the fact that  $\{Q, \mathcal{O}\} = 0$ . So  $\partial_{\bar{z}}\mathcal{O} = \{Q, S(z)\}$  for some  $S(z)$ , as we argued before. Hence  $\partial_{\bar{z}}\mathcal{O}(z) \cdot \mathcal{O}(z') = \{Q, S(z)\tilde{\mathcal{O}}(z')\}$ , where we use also the fact that  $[Q, \tilde{\mathcal{O}}] = 0$ .

We must warn the reader here of the following. As in the mathematical literature on this subject, the notion we use here of a chiral algebra does not quite coincide with the usual physical notion. In fact, reparameterization invariance on the worldsheet  $\Sigma$  is not one of the axioms. The sigma model (2.2) is generically not invariant at the quantum level under holomorphic changes of coordinate on  $\Sigma$  (because it is not invariant under conformal rescalings of the metric of  $\Sigma$ , which such changes of coordinate induce). As we see later from various points of view, such invariance is not necessarily recovered at the level of the chiral algebra. Our operators  $\mathcal{O}(z)$  vary holomorphically in  $z$ , and have operator product expansions that obey the usual relations of holomorphy, associativity, and invariance under translation and rescaling of  $z$ , but not necessarily invariance under arbitrary holomorphic reparameterization of  $z$ . Our chiral algebras are in general only defined locally, requiring a choice of complex parameter  $z$  up to translations and scaling, or alternatively, requiring a flat metric up to scaling on the Riemann surface  $\Sigma$ . This is enough to define a chiral algebra on a surface of genus one, but to define the chiral algebra on a Riemann surface of higher genus requires more analysis, and is potentially obstructed by an anomaly involving  $c_1(\Sigma)$  and  $c_1(X)$  that we will meet in Sections 2.3 and 3.5.

### 2.1.1 Relation to the elliptic genus

Though in this paper we focus on operators, it is also possible to construct states by canonical quantization of the theory on  $\mathbb{R} \times \mathbf{S}^1$ . The  $Q$ -cohomology of such states furnishes a module  $\mathcal{V}$  for the chiral algebra  $\mathcal{A}$  that we obtained from the  $Q$ -cohomology of operators. In case  $X$  is a Calabi-Yau manifold, the usual operator-state correspondence gives a natural isomorphism from operators to states. In that case, therefore,  $\mathcal{V}$  is isomorphic to  $\mathcal{A}$  itself. That is not so in general.

By counting bosonic and fermionic states in  $\mathcal{V}$  of energy  $n$  one can form a modular function called the elliptic genus which has no quantum corrections,<sup>3</sup> making it effectively computable. Explicitly, the elliptic genus is  $V(q) = q^{-d/12} \sum_{n=0}^{\infty} q^n \text{Tr}_{\mathcal{V}_n} (-1)^F$ , where  $\mathcal{V}_n$  is the space of supersymmetric states of energy  $n$  and  $\text{Tr}_{\mathcal{V}_n} (-1)^F$  is its “Euler characteristic” (difference of bosonic and fermionic dimensions). When the elliptic genus is nonzero,  $\mathcal{V}$  is nonempty and hence supersymmetry is not spontaneously broken. One can form an analogous generating series for  $\mathcal{A}$ , with the operators graded by dimension, and at least in perturbation theory this function appears to

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<sup>3</sup>Absence of quantum corrections is proved using the fact that both the energy and the operators  $(-1)^F$  that distinguishes bosonic and fermionic states are exactly conserved quantum mechanically.

have modular properties (though it is not clear that this statement has a natural path integral proof). Explicitly, this series, considered at the perturbative level in Section 5.6 of [10], is  $A(q) = q^{-d/12} \sum_{n=0}^{\infty} q^n \text{Tr}_{\mathcal{A}_n} (-1)^F$ , with  $\mathcal{A}_n$  being the space of operators of dimension  $n$ . However, this function, though constant in perturbation theory, *does* have nonperturbative quantum corrections (except on Calabi-Yau manifolds), because instanton corrections do not preserve the grading by dimension of an operator. So even when  $A(q)$  is nonzero in perturbation theory, it may vanish nonperturbatively. In fact,  $\mathbb{CP}^1$  gives an example, as we will discuss elsewhere.

## 2.2 Moduli of the chiral algebra

Here, we will consider the moduli of the chiral algebra found in the last section.

There are a few obvious considerations. The chiral algebra does not depend on the hermitian metric  $g_{i\bar{j}}$  that was used in writing the classical action, since this metric appears in the action entirely inside a term of the form  $\{Q, \dots\}$ .

The chiral algebra does depend on the complex structure of  $X$ , because this enters in the definition of the fields and the fermionic symmetry. In fact, the chiral algebra varies holomorphically with the complex structure of  $X$ . It is possible to show this by showing that if  $J$  denotes the complex structure of  $X$ , then an antiholomorphic derivative  $\partial/\partial\bar{J}$  changes the action by terms of the form  $\{Q, \dots\}$ .

If  $B$  is a closed two-form on  $X$ , then we can add to the action a topological invariant

$$I_B = \int_{\Sigma} \Phi^*(B). \quad (2.4)$$

Being a topological invariant,  $I_B$  is invariant under any local deformation of the fields and in particular under  $Q$ . Including this term in the action has the effect of introducing a factor  $\exp(-I_B)$  in the path integral. In perturbation theory, we consider only degree zero maps  $\Phi : \Sigma \rightarrow B$ , so  $I_B = 0$  and this factor equals 1. Hence, the interaction  $I_B$  is really not relevant for the present paper. Nonperturbatively,  $I_B$  affects the weights assigned to instantons and can affect the chiral algebra.

Our focus here is on a more subtle possibility involving a topologically nontrivial  $B$ -field which is not closed and so does affect perturbation theory.

First, we describe the situation locally. Let  $T = \frac{1}{2}T_{ij}d\phi^i \wedge d\phi^j$  be any two-form on  $X$  that is of type  $(2, 0)$ . Let

$$I_T = - \int |d^2z| \{Q, T_{ij}\rho^i \partial_z \phi^j\} \quad (2.5)$$

or in more detail

$$I_T = \int |d^2z| \left( T_{ij} \partial_{\bar{z}} \phi^i \partial_z \phi^j - T_{ij, \bar{k}} \alpha^{\bar{k}} \rho^i \partial_z \phi^j \right). \quad (2.6)$$

Here  $T_{ij, \bar{k}} = \partial T_{ij} / \partial \bar{\phi}^{\bar{k}}$ . As will become clear,  $T$  is best understood as a two-form gauge field, like the object  $B$  considered in the last paragraph, except that  $B$  was constrained to be closed but not necessarily of any particular Hodge type, while  $T$  is of type  $(2, 0)$  but not necessarily closed.

As written,  $I_T$  is  $Q$ -trivial and depends on the choice of a  $(2, 0)$ -form  $T$ . But in fact, with some mild restrictions, the definition of  $I_T$  depends only on the three-form  $\mathcal{H} = dT$ , and makes sense even if  $T$  is not globally defined as a two-form (and must be interpreted as a two-form gauge field, in the sense of string theory, or in terms of mathematical theories such as connections on gerbes or Cheeger-Simons differential characters).

In brief, we will find that  $I_T$  can be defined for any closed form  $\mathcal{H}$  that is of type  $(3, 0) \oplus (2, 1)$ , but the formula (2.5) that expresses  $I_T$  as  $\{Q, \dots\}$  is only valid globally if it is true globally that  $\mathcal{H} = dT$  for some  $T$  of type  $(2, 0)$ . Hence, the chiral algebra  $\mathcal{A}$  depends on the cohomology class (in a certain sense that will be clarified later) of  $\mathcal{H}$ .

In fact, as  $T$  is of type  $(2, 0)$ ,  $\mathcal{H} = dT$  is a sum of terms of types  $(3, 0)$  and  $(2, 1)$ . The second term in (2.6) is already written in terms of  $\mathcal{H}$ , since  $T_{ij, \bar{k}}$  is simply the  $(2, 1)$  part of  $\mathcal{H}$ . The first term in (2.6) can likewise be expressed in terms of  $\mathcal{H}$ . Recalling that  $|d^2z| = idz \wedge d\bar{z}$ , we write that first term (which is a generalization of the Wess-Zumino anomaly functional [18]) as

$$I_T^{(1)} = -\frac{i}{2} \int_{\Sigma} T_{ij} d\phi^i \wedge d\phi^j = -i \int_{\Sigma} \Phi^*(T). \quad (2.7)$$

Suppose now that  $C$  is a three-manifold whose boundary is  $\Sigma$  and over which the map  $\Phi : \Sigma \rightarrow X$  extends. Then, if  $T$  is globally defined as a  $(2, 0)$ -form, the relation  $\mathcal{H} = dT$  implies, via Stokes' theorem, that

$$I_T^{(1)} = -i \int_C \Phi^*(\mathcal{H}), \quad (2.8)$$

a formula which expresses  $I_T^{(1)}$ , and hence  $I_T$ , just in terms of  $\mathcal{H}$ .



We do not actually want to limit ourselves to the case that  $T$  is globally defined as a two-form. In general, the right hand side of (2.8) makes sense as long as  $\mathcal{H}$  is globally defined. But it depends on the choice of  $C$  and of the extension of  $\Phi$ . To make this explicit, we denote it as  $I_T^{(1)}(C)$ . If  $C$  and  $C'$  are two three-manifolds with boundary  $\Sigma$  and chosen extensions of  $\Phi$ , they glue together (after reversing the orientation of  $C'$ ) to a closed three-manifold  $D$  with a map  $\Phi : D \rightarrow X$ . Then  $I_T^{(1)}(C) - I_T^{(2)}(C') = -i \int_D \Phi^*(\mathcal{H})$ . In quantum theory, a shift in the (Euclidean) action by an integral multiple of  $2\pi i$  is irrelevant; the indeterminacy in  $I_T^{(1)}$  is of this nature if  $\mathcal{H}/2\pi$  has integral periods. Nonperturbatively, for the quantum theory associated with the classical actions considered here to be well defined,  $\mathcal{H}$  must obey this condition, and in particular, the integrality means that continuous moduli of  $\mathcal{H}$  that are present in perturbation theory may be absent in the non-perturbative theory. In the present paper, as we consider only perturbation theory, we will not see this effect.

In writing  $I_T^{(1)}$  in terms of  $\mathcal{H}$ , we have assumed that  $\Phi$  extends over *some* three-manifold  $C$  of boundary  $\Sigma$ . This assumption is certainly valid in perturbation theory, in which one considers topologically trivial maps  $\Phi$ ; such maps extend over any chosen  $C$ . Nonperturbatively, this framework for defining  $I_T^{(1)}$  is not adequate, as  $C$  may not exist. To define  $I_T^{(1)}$  in general, one must interpret  $T$  as a two-form gauge field (or in terms of gerbes or differential characters); in that framework, if  $\mathcal{H}/2\pi$  is associated with an integral cohomology class,  $I_T^{(1)}$  is always naturally defined mod  $2\pi i$ . In general,  $T$  is not completely determined as a two-form gauge field by its curvature  $\mathcal{H} = dT$ , as one may add a flat two-form gauge field to  $T$ . Hence, the functional  $I_T^{(1)}$  does depend in general on global information not contained in  $\mathcal{H}$ , but this dependence does not affect perturbation theory, which only depends on  $\mathcal{H}$ . This is why the sheaf of CDO's, as defined in the mathematical literature [10,13], and as interpreted in this paper from a physical viewpoint, only depends on  $\mathcal{H}$ .

### 2.2.1 Moduli

So far, we have locally a two-form gauge field  $T$  that is of type  $(2,0)$ , and whose curvature  $\mathcal{H} = dT$  is hence of type  $(3,0) \oplus (2,1)$ . Actually, given any closed form  $\mathcal{H}$  of type  $(3,0) \oplus (2,1)$ , we can always find locally a  $(2,0)$ -form  $T$  with  $\mathcal{H} = dT$ . To see this, we first select *any* local two-form  $W$  with  $\mathcal{H} = dW$ .  $W$  exists because of the Poincaré lemma. A priori,  $W$  is a sum of terms  $W^{(2,0)} + W^{(1,1)} + W^{(0,2)}$  of the indicated types. Now as  $\bar{\partial}W^{(0,2)} = 0$  (since  $\mathcal{H}$  has no component of type  $(0,3)$ ), the  $\bar{\partial}$  version of the Poincaré lemma asserts that locally  $W^{(0,2)} = \bar{\partial}\lambda$  for some  $(0,1)$ -form  $\lambda$ .

Upon replacing  $W$  by  $\widetilde{W} = W - d\lambda$ , we have a new two-form  $\widetilde{W}$  with  $\mathcal{H} = d\widetilde{W}$ , and such that  $\widetilde{W}$  has a decomposition  $\widetilde{W} = \widetilde{W}^{(2,0)} + \widetilde{W}^{(1,1)}$ . Now again as  $\overline{\partial}\widetilde{W}^{(1,1)} = 0$ , we have locally  $\widetilde{W}^{(1,1)} = \overline{\partial}\widetilde{\lambda}$ . Finally,  $T = \widetilde{W} - d\widetilde{\lambda}$  is the desired  $(2,0)$ -form with  $\mathcal{H} = dT$ .

Hence, given any closed form  $\mathcal{H}$  of type  $(3,0) \oplus (2,1)$ , we can locally represent it as  $\mathcal{H} = dT$  for some  $(2,0)$ -form  $T$  and write the  $Q$ -invariant functional  $I_T$ . The definition of  $I_T$  essentially depends only on  $\mathcal{H}$  (modulo terms that do not affect perturbation theory), but the formula (2.5), which locally shows that  $I_T$  is  $Q$ -trivial, makes sense globally only if  $T$  is globally defined as a two-form. So the moduli, in perturbation theory, of the chiral algebra derived from the sigma model with target a complex manifold  $X$  are parameterized by the closed form  $\mathcal{H}$  of type  $(3,0) \oplus (2,1)$ , modulo forms that can be written globally as  $dT$  for  $T$  of type  $(2,0)$ . Nonperturbatively, and thus beyond the scope of this paper, the situation is somewhat different, as  $\mathcal{H}$  must be an integral class, and the flat  $B$ -fields will also enter.

### 2.2.2 Interpretation Via $H^1(X, \Omega^{2,\text{cl}})$

Now we want to describe more precisely what sort of cohomology class  $\mathcal{H}$  represents.

Let  $U_a$ ,  $a = 1, \dots, s$  be a collection of small open sets giving a good cover of  $X$ . (This means that the individual  $U_a$  and all of their intersections are open balls.)

Suppose that (as in our problem) we are given a closed form  $\mathcal{H}$  on  $X$  that is of type  $(3,0) \oplus (2,1)$ . As we have seen, in each  $U_a$ , there is a  $(2,0)$ -form  $T_a$  with  $\mathcal{H} = dT_a$ . On each intersection  $U_{ab} = U_a \cap U_b$ , let  $T_{ab} = T_a - T_b$ . Clearly,

$$T_{ab} = -T_{ba} \tag{2.9}$$

for each  $a, b$ , and

$$T_{ab} + T_{bc} + T_{ca} = 0 \tag{2.10}$$

for each  $a, b, c$ . Moreover,  $dT_{ab} = 0$  and hence (as  $T_{ab}$  is of definite type  $(2,0)$ )  $\partial T_{ab} = \overline{\partial} T_{ab} = 0$ . The  $T_{ab}$  are not uniquely determined by  $\mathcal{H}$ . The definition of  $T_a$  would allow us to shift  $T_a \rightarrow T_a + S_a$ , where  $S_a$  is a  $(2,0)$ -form on  $U_a$  obeying  $dS_a = 0$  (and hence  $\partial S_a = \overline{\partial} S_a = 0$ ). This change in the  $T_a$  induces the equivalence

$$T_{ab} \longrightarrow T_{ab} + S_a - S_b. \tag{2.11}$$

One defines  $\Omega^2(X)$  as the sheaf of  $(2, 0)$ -forms on  $X$ , and  $\Omega^{2,\text{cl}}(X)$  as the sheaf of such forms that are annihilated by  $\partial$ . (The label “cl” is short for “closed” and refers to forms that are closed in the sense of being annihilated by  $\partial$ .) A section of  $\Omega^{2,\text{cl}}(X)$  that is holomorphic in a given set  $U \subset X$  is a  $(2, 0)$ -form in  $U$  that is annihilated by both  $\bar{\partial}$  and  $\partial$ . Similarly,  $\Omega^{n,\text{cl}}$  is the sheaf whose sections are  $(n, 0)$  forms that are annihilated by  $\partial$ ; its holomorphic sections are also annihilated by  $\bar{\partial}$ . In the last paragraph, we found in each double intersection  $U_a \cap U_b$  a holomorphic section  $T_{ab}$  of  $\Omega^{2,\text{cl}}(X)$ . The identities (2.9) and (2.10) and equivalence (2.11) means that these fit together into an element of the Čech cohomology group  $H^1(X, \Omega^{2,\text{cl}}(X))$ .

If  $\mathcal{H}$  is globally of the form  $dT$  for some  $(2, 0)$ -form  $T$ , then we can take all  $T_a$  to equal  $T$ , whereupon all  $T_{ab}$  vanish. So we have obtained a map from the space of closed forms  $\mathcal{H}$  of type  $(3, 0) \oplus (2, 1)$ , modulo forms that are globally  $dT$  for  $T$  of type  $(2, 0)$ , to  $H^1(X, \Omega^{2,\text{cl}})$ .

Conversely, suppose that we are given an element of this sheaf cohomology group, represented by such a family  $T_{ab}$ . Let  $f_a$  be a partition of unity subordinate to the cover  $U_a$  of  $X$ . This means that the  $f_a$  are continuous functions on  $X$  that vanish outside  $U_a$  and obey  $\sum_a f_a = 1$ . Let  $\mathcal{H}_a$  be the three-form defined in  $U_a$  by  $\mathcal{H}_a = \sum_c df_c \wedge T_{ac}$ .  $\mathcal{H}_a$  is well defined throughout  $U_a$ , since in  $U_a$ ,  $df_c$  vanishes wherever  $T_{ac}$  is not defined. Obviously,  $\mathcal{H}_a$  is of type  $(3, 0) \oplus (2, 1)$ , since the  $T_{ac}$  are of type  $(2, 0)$ , and moreover  $d\mathcal{H}_a = 0$ . For any  $a$  and  $b$ , we have  $\mathcal{H}_a - \mathcal{H}_b = \sum_c df_c \wedge (T_{ac} - T_{bc})$ . Using (2.10), this is  $\sum_c df_c \wedge T_{ab} = d(\sum_c f_c) \wedge T_{ab}$ . This vanishes, as  $\sum_c f_c = 1$ . So for all  $a, b$ ,  $\mathcal{H}_a = \mathcal{H}_b$  on  $U_a \cap U_b$ . The  $\mathcal{H}_a$  thus fit together to a closed three-form  $\mathcal{H}$  that is of type  $(3, 0) \oplus (2, 1)$ . Thus, we have found a map from the Čech cohomology group  $H^1(X, \Omega^{2,\text{cl}}(X))$  to closed forms  $\mathcal{H}$  of type  $(3, 0) \oplus (2, 1)$ , modulo those that are globally  $dT$  for  $T$  of type  $(2, 0)$ . We leave it to the reader to verify that the two maps we have described are inverses.

The conclusion of this analysis is that the sigma models considered here, and therefore the chiral algebras derived from them, can be “shifted” in a natural way by an interaction  $I_T$  determined by an element  $\mathcal{H}$  of  $H^1(X, \Omega^{2,\text{cl}}(X))$ .

Such an  $\mathcal{H}$ , being a closed three-form, also determines a class in the de Rham cohomology  $H^3(X, \mathbb{R})$ , but  $\mathcal{H}$  may vanish in de Rham cohomology even if it is nonzero in  $H^1(X, \Omega^{2,\text{cl}})$ . This occurs if  $\mathcal{H}$  can be globally written as  $dT$  for some two-form  $T$ , but  $T$  cannot be globally chosen to be of type  $(2, 0)$ . For an example, let  $X = \mathbb{C}^2 \times E$ , where  $\mathbb{C}^2$  has complex coordinates  $x, y$  and the elliptic curve  $E$  is the quotient of the complex  $z$  plane by a lattice. Let  $\mathcal{H} = dx \wedge dy \wedge d\bar{z}$ . Then  $\mathcal{H} = d(x dy \wedge d\bar{z})$ , but  $\mathcal{H}$  cannot be written as  $dT$  with  $T$  of type  $(2, 0)$ .

### 2.3 Anomalies

Before investigating the quantum properties of the model, the most basic question to consider is whether it exists at all — whether the Lagrangian that we have written leads to some kind of quantum theory.

A failure of the model to exist, even in perturbation theory, would come from an anomaly in the path integral of the world-volume fermions  $\alpha^{\bar{i}}$  and  $\rho^j$ . (For such sigma model anomalies, see [19].) In this discussion, we can omit the interaction  $I_T$ , as anomalies do not depend on continuously variable couplings such as this one. Considering only the basic action (2.2), the kinetic energy for the fermions is  $(\rho, D\alpha) = \int |d^2z| g_{\bar{i}i} \rho^i D\alpha^{\bar{i}}$ , where  $D$  is the  $\partial$  operator on  $\Sigma$  acting on sections of  $\Phi^*(\overline{TX})$ . Equivalently, if we pick a spin structure on  $\Sigma$ ,  $D$  can be interpreted as a Dirac operator on  $\Sigma$  acting on sections of  $V = \overline{K}^{-1/2} \otimes \Phi^*(\overline{TX})$ , with  $K$  the canonical bundle of  $\Sigma$  and  $\overline{K}$  its complex conjugate.

We consider a family of maps  $\Phi : \Sigma \rightarrow X$ , parameterized by a base  $B$ . In fact, in the path integral we want to consider the universal family of all maps of  $\Sigma$  to  $X$ . These maps fit together into a map  $\Phi : \Sigma \times B \rightarrow X$ . To make sense of the quantum path integral, we must be able to interpret the determinant of  $D$  as a function on  $B$ , but mathematically, it is interpreted in general as a section of a determinant line bundle  $\mathcal{L}$ . The quantum theory can only exist if  $\mathcal{L}$  is trivial. Conversely, the quantum theory will exist if  $\mathcal{L}$  is trivial and can be trivialized by a local formula similar to the Green-Schwarz anomaly cancellation mechanism.

The theory of determinant line bundles is usually expressed in terms of a family of  $\overline{D}$  operators, while here we have a family of  $D$  operators. The  $D$  operators would be converted into  $\overline{D}$  operators if we reverse the complex structure on  $\Sigma$ , but in most of this paper, the formulas look much more natural with the complex structure as we have chosen it. (If we were to reverse the complex structure, the  $D$  operator of the fermions would become a  $\overline{D}$  operator, but our chiral algebra would be antiholomorphic.) At any rate, the theory of determinants of  $D$  operators is isomorphic to the theory of determinants of  $\overline{D}$  operators, so we can borrow the usual results.

The basic obstruction to triviality of  $\mathcal{L}$  is its first Chern class. By the family index theorem, applied to anomalies in [20, 21], the first Chern class of  $\mathcal{L}$  is  $\pi_*(\text{ch}_4(V))$ , where  $\text{ch}_4$  is the dimension four part of the Chern character, and  $\pi : \Sigma \times B \rightarrow B$  is the projection to the second factor. This

vanishes if  $\text{ch}_4(V)$  vanishes before being pushed down to  $B$ .<sup>4</sup> To evaluate this, we note that  $\text{ch}_4(\overline{TX}) = p_1(X)/2$  and that tensoring with  $\overline{K}^{-1/2}$  adds an additional term  $c_1(\Sigma)c_1(X)/2$ . Here  $p_1(X)$  is the first Pontryagin class of the ordinary, real tangent bundle of  $X$ ; there is a natural way to divide it by 2 to get an integral characteristic class. The condition for vanishing is thus that

$$0 = \frac{1}{2}c_1(\Sigma)c_1(X) = \frac{1}{2}p_1(X). \quad (2.12)$$

The first condition means at the level of differential forms that either  $c_1(X) = 0$  and  $\Sigma$  is arbitrary, or  $c_1(X) \neq 0$ , and we must restrict ourselves to Riemann surfaces  $\Sigma$  with  $c_1(\Sigma) = 0$ .

The characteristic class  $p_1(X)/2$  can be interpreted as an element of  $H^2(X, \Omega^{2, \text{cl}})$ .<sup>5</sup> We will meet it in this guise in Sections 3.5 and 5.2. Likewise, as explained in the footnote,  $c_1(X)$  corresponds to an element of  $H^1(X, \Omega^{1, \text{cl}})$  and  $c_1(\Sigma)$  to a class in  $H^1(\Sigma, \Omega^{1, \text{cl}})$ . These likewise will make a later appearance.

In perturbation theory, it suffices for the conditions (2.12) to hold at the level of differential forms. Nonperturbatively (and thus beyond the scope of the present paper), these conditions must hold in integral cohomology. For a brief elucidation of this (and an implementation of the Green-Schwarz anomaly cancellation by which one defines the fermion path integral once conditions like (2.12) are imposed), see Section 2.2 of [22].

Both of these anomalies are familiar in closely related models. The  $p_1(X)$  anomaly appears equally with  $(0, 1)$  or  $(0, 2)$  supersymmetry and is quite important in the context of the heterotic string. The  $c_1(\Sigma)c_1(X)$  anomaly appears in sigma models with  $(2, 2)$  supersymmetry twisted to get the

<sup>4</sup>If  $\text{ch}_4(V) \neq 0$  but  $\pi_*(\text{ch}_4(V)) = 0$ , then  $\mathcal{L}$  is trivial but cannot be trivialized by a local Green-Schwarz mechanism, so the quantum sigma model does not exist.

<sup>5</sup>This statement is possibly most familiar for Kahler manifolds, where  $p_1(X)$  is represented by a form of type  $(2, 2)$ , annihilated by both  $\overline{\partial}$  and  $\partial$  and thus representing an element of  $H^2(X, \Omega^{2, \text{cl}})$ . However, on any complex manifold,  $p_1(X)$  can be represented by a closed form of type  $(2, 2) \oplus (3, 1) \oplus (4, 0)$ . To see this, pick any connection on the holomorphic tangent bundle  $TX$  whose  $(0, 1)$  part is the natural  $\overline{\partial}$  operator of this bundle. Since  $\overline{\partial}^2 = 0$ , the curvature of such a connection is of type  $(2, 0) \oplus (1, 1)$ , as a result of which, for every  $k \geq 0$ ,  $c_k(TX)$  (which is a polynomial of degree  $k$  in the curvature, and is usually abbreviated as  $c_k(X)$ ) is described by a form of type  $(k, k) \oplus (k+1, k-1) \oplus \dots \oplus (2k, 0)$ , and represents an element of  $H^k(X, \Omega^{k, \text{cl}}(X))$ . In particular,  $c_1(X)$  represents an element of  $H^1(X, \Omega^{1, \text{cl}}(X))$ , and  $p_1 = 2c_2(X) - c_1^2(X)$  represents an element of  $H^2(X, \Omega^{2, \text{cl}}(X))$ .

topological  $B$ -model and is the reason that the  $B$ -model (except in genus 1) is only consistent on Calabi-Yau manifolds. The  $c_1(\Sigma)c_1(X)$  anomaly, in models with  $(0, 2)$  or  $(2, 2)$  supersymmetry, is generated by the topological twist [6], while the  $p_1(X)$  anomaly is present in the underlying physical model with  $(0, 1)$  or  $(0, 2)$  supersymmetry, regardless of any topological twisting.

### 2.3.1 Other questions involving anomalies

Finally, we will just briefly hint at a few other questions involving anomalies that are important for a more complete study of the model.

One basic issue is to show that  $Q$  is conserved at the quantum level and that  $Q^2 = 0$ .

Assuming that  $Q$  is conserved, the fact that  $Q^2 = 0$  follows from the fact that for a generic hermitian metric on  $X$ , there are no locally defined conserved quantities in the model of charge  $q = 2$  with respect to the  $U(1)$  symmetry  $R$  introduced in Section 2.1. It suffices to show this classically (as small quantum corrections can only destroy conservation laws, not create them). So it suffices to show that for generic metric there are no nontrivial local conserved currents of  $q = 2$ . (A trivial conserved current is  $J = *dC$ , where  $C$  is a local operator of dimension  $(0, 0)$ ; the associated conserved charge vanishes.) Indeed, it is possible to show that such currents exist if and only if there are suitable covariantly constant tensors on  $X$  beyond the metric tensor.

This can be contrasted with what happens for the bosonic string outside the critical dimension. The BRST operator  $Q$  is conserved, but its square is nonzero and is a multiple of the conserved quantity  $\oint dz c \partial^3 c$ .

To show that  $Q$  is conserved, one approach is to note that  $Q$ -invariant Pauli-Villars regulator terms  $\int d^2x \{Q, V\}$  are possible (where  $V$  is a suitable higher derivative expression of  $R$ -charge  $-1$ , such as  $g_{i\bar{j}} \rho_{\bar{z}}^i \partial_{\bar{z}} \partial_z^2 \bar{\phi}^{\bar{j}}$ ). Upon adding such terms, all Feynman diagrams are regularized beyond one-loop order, so only one-loop anomalies are possible. To show that there is no one-loop anomaly in  $Q$  requires some more direct argument. One approach is to classify, in terms of the local differential geometry, the possible anomalies that might appear at one-loop order, and show that there are none. (Here one uses the fact that in general, sigma model perturbation theory is local on  $X$ , and the one-loop approximation only involves derivatives of the metric

of  $X$  to very low order. A similar argument can actually be carried out to all orders, to show that  $Q$  is conserved without using the fact that a regularization exists beyond one-loop order.)

The proof of conservation of  $Q$  can be carried out in the presence of an arbitrary metric  $f_{ab}$  on  $\Sigma$ , not just the flat metric that we have used in writing many formulas. Since the effective action  $\Gamma$  is thus  $Q$ -invariant for any metric, it follows that the stress tensor, whose expectation value is  $\langle T_{ab} \rangle = \partial\Gamma/\partial f^{ab}$ , is likewise  $Q$ -invariant. The components  $T_{\bar{z}\bar{z}}$  and  $T_{z\bar{z}}$  of the stress tensor have dimensions  $(0, 2)$  and  $(1, 1)$ . On a flat Riemann surface  $\Sigma$ , the  $Q$ -cohomology vanishes for operators of dimension  $(n, m)$  with  $m \neq 0$ , as we discussed in section 2.1. Hence, on a flat Riemann surface, we have  $T_{\bar{z}\bar{z}} = \{Q, \dots\}$  and  $T_{z\bar{z}} = \{Q, \dots\}$ . On a curved Riemann surface  $\Sigma$ , we have to allow operators that depend on the Ricci scalar  $R$  of  $\Sigma$  (which we consider to have dimension  $(1, 1)$  because of the way it scales under rescaling of the metric  $f$ ) as well as its derivatives, in addition to the usual quantum fields. In particular, in this enlarged sense, the  $Q$ -cohomology of operators of dimension  $(1, 1)$  is one-dimensional, being generated by  $R$  itself. So the general result is

$$T_{z\bar{z}} = \frac{c}{24\pi}R + \{Q, \dots\}, \quad (2.13)$$

where  $c$  is a constant. In particular, though classically  $T_{z\bar{z}} = 0$ , reflecting conformal invariance, quantum mechanically there may be an anomaly. The anomaly is the sum of a multiple of  $R$ , corresponding to the usual conformal anomaly, and a  $Q$ -trivial term that, being  $Q$ -trivial, does not affect correlation functions of operators in the chiral algebra, that is, operators annihilated by  $Q$ . The  $c$ -number anomaly can be considered to affect only the partition function, not the normalized correlation functions. Combining the statements in this paragraph, correlation functions of operators in the  $Q$ -cohomology are holomorphic and depend on  $\Sigma$  only via its complex structure, as is familiar for chiral algebras.

### 3 Sheaf of perturbative observables

In this section, we analyze the  $Q$ -cohomology in perturbation theory. Nonperturbatively, and beyond the scope of the present paper, instanton corrections can change the picture radically.

### 3.1 General considerations

A local operator is represented by an operator  $F$  that in general is a function of the fields  $\phi$ ,  $\bar{\phi}$ ,  $\rho_{\bar{z}}$ ,  $\alpha$ , and their derivatives with respect to  $z$  and  $\bar{z}$ .<sup>6</sup> However, as we saw in Section 2.1, the  $Q$ -cohomology vanishes for operators of dimension  $(n, m)$  with  $m \neq 0$ . Since  $\rho_{\bar{z}}$  and the derivative  $\partial_{\bar{z}}$  both have  $m = 1$  (and no ingredient in constructing a local operator has negative  $m$ ),  $Q$ -cohomology classes can be constructed from just  $\phi$ ,  $\bar{\phi}$ ,  $\alpha$ , and their derivatives with respect to  $z$ . The equation of motion for  $\alpha$  is  $D_z \alpha = 0$ , so we can ignore the  $z$ -derivatives of  $\alpha$ . A  $Q$ -cohomology class can thus be represented in general by an operator

$$F(\phi, \partial_z \phi, \partial_z^2 \phi, \dots; \bar{\phi}, \partial_z \bar{\phi}, \partial_z^2 \bar{\phi}, \dots; \alpha), \quad (3.1)$$

where we have tried to indicate that  $F$  might depend on  $z$  derivatives of  $\phi$  and  $\bar{\phi}$  of arbitrarily high order, though not on derivatives of  $\alpha$ . If  $F$  has bounded dimension, it depends only on derivatives up to some finite order and is polynomial of bounded degree in those.  $F$  is also polynomial in  $\alpha$ , simply because  $\alpha$  is fermionic and only has finitely many components. However, the dependence of  $F$  on  $\phi$  and  $\bar{\phi}$  (as opposed to their derivatives) is not restricted to have any simple form. Recalling the definition of the  $R$ -charge in section 2.1, we see that if  $F$  is homogeneous of degree  $k$  in  $\alpha$ , then it has  $R$ -charge  $q = k$ .

A general  $q = k$  operator  $F(\phi, \partial_z \phi, \dots; \bar{\phi}, \partial_z \bar{\phi}, \dots; \alpha)$  can be interpreted as a  $(0, k)$ -form on  $X$  with values in a certain holomorphic vector bundle. We will make this explicit for operators of dimension  $(0, 0)$  and  $(1, 0)$ , hoping that this will make the general idea clear. For dimension  $(0, 0)$ , the most general  $q = k$  operator is of the form  $F(\phi, \bar{\phi}; \alpha) = f_{\bar{j}_1, \dots, \bar{j}_k}(\phi, \bar{\phi}) \alpha^{\bar{j}_1} \dots \alpha^{\bar{j}_k}$ ; thus,  $F$  may depend on  $\phi$  and  $\bar{\phi}$  but not on their derivatives, and is  $k$ th order in  $\alpha$ . Mapping  $\alpha^{\bar{j}}$  to  $d\bar{\phi}^{\bar{j}}$ , such an operator corresponds to an ordinary  $(0, k)$ -form  $f_{\bar{j}_1, \dots, \bar{j}_k}(\phi, \bar{\phi}) d\bar{\phi}^{\bar{j}_1} \dots d\bar{\phi}^{\bar{j}_k}$  on  $X$ . For dimension  $(1, 0)$ , there are two cases. A dimension  $(1, 0)$  operator  $F(\phi, \partial_z \phi, \bar{\phi}; \alpha) = f_{i, \bar{j}_1, \dots, \bar{j}_k}(\phi, \bar{\phi}) \partial_z \phi^i \alpha^{\bar{j}_1} \dots \alpha^{\bar{j}_k}$  that is linear in  $\partial_z \phi$  and does not depend on any other derivatives is a  $(0, k)$ -form on  $X$  with values in  $T^*X$  (the holomorphic cotangent bundle of  $X$ ); alternatively, it is a  $(1, k)$ -form on  $X$ . Similarly, a dimension  $(1, 0)$  operator  $F(\phi, \bar{\phi}, \partial_z \bar{\phi}; \alpha) = f_{\bar{j}_1, \dots, \bar{j}_k}^i(\phi, \bar{\phi}) g_{i\bar{s}} \partial_z \bar{\phi}^{\bar{s}} \alpha^{\bar{j}_1} \dots \alpha^{\bar{j}_k}$  that is linear in  $\partial_z \bar{\phi}$  and does not depend on any other derivatives is a  $(0, k)$ -form on  $X$

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<sup>6</sup>In contrast to Section 2.3, here we work locally on a flat Riemann surface with local parameter  $z$ , so we need not include in our operators dependence on the scalar curvature of  $\Sigma$ .



with values in  $TX$ , the holomorphic tangent bundle of  $X$ . In a like fashion, for any integer  $n > 0$ , the operators of dimension  $(n, 0)$  and charge  $k$  can be interpreted as  $(0, k)$ -forms with values in a certain holomorphic vector bundle over  $X$ . This structure persists in quantum perturbation theory, but there may be perturbative corrections to the complex structure of this bundle.

The action of  $Q$  on such operators is easy to describe at the classical level. If we interpret  $\alpha^{\bar{i}}$  as  $d\bar{\phi}^{\bar{i}}$ , then  $Q$  acting on a function of  $\phi$  and  $\bar{\phi}$  is simply the  $\bar{\partial}$  operator. This follows from the transformation laws  $\delta\bar{\phi}^{\bar{i}} = \alpha^{\bar{i}}$ ,  $\delta\phi^i = 0$ . Classically, the interpretation of  $Q$  as the  $\bar{\partial}$  operator remains valid when  $Q$  acts on a more general operator  $F(\phi, \partial_z\phi, \dots; \bar{\phi}, \partial_z\bar{\phi}, \dots; \alpha)$  that does depend on derivatives of  $\phi$  and  $\bar{\phi}$ . The reason for this is that, because of the equation of motion  $D_z\alpha = 0$ , one can neglect the action of  $Q$  on derivatives  $\partial_z^m\bar{\phi}$  with  $m > 0$ . One is thus left classically only with the action of  $Q$  on  $\bar{\phi}$ , as opposed to its derivatives; this is interpreted as the  $\bar{\partial}$  operator.

Perturbatively, there definitely are corrections to the action of  $Q$ . The most famous such correction is associated with the one-loop beta function. Classically, the dimension  $(2, 0)$  part of the stress tensor is  $T_{zz} = g_{i\bar{j}}\partial_z\phi^i\partial_z\bar{\phi}^{\bar{j}}$ . Classically,  $\{Q, T_{zz}\} = 0$ , but at the one-loop order,

$$\{Q, T_{zz}\} = \partial_z(R_{i\bar{j}}\partial_z\phi^i\alpha^{\bar{j}}), \quad (3.2)$$

where  $R_{i\bar{j}}$  is the Ricci tensor. If  $X$  is Calabi-Yau (and thus  $R_{i\bar{j}} = \partial_i\partial_{\bar{j}}\Lambda$  for some function  $\Lambda(\phi, \bar{\phi})$ ), it is possible to modify  $T_{zz}$  (subtracting  $\partial_z(\partial_i\Lambda\partial_z\phi^i)$ ) so as to be annihilated by  $Q$ . But if  $c_1(X) \neq 0$ , the one-loop correction to  $Q$  is essential and a  $Q$ -invariant modification of  $T_{zz}$  does not exist. In Section 5, we will examine more closely, from a different point of view, the one-loop correction to the cohomology of  $Q$  that is associated with the beta function.

Gradually, we will obtain a fairly clear picture of the nature of perturbative quantum corrections to  $Q$ . For now, we make a few simple observations. Let  $Q_{\text{cl}} = \bar{\partial}$  denote the classical approximation to  $Q$ . Perturbative corrections to  $Q$  are local on  $X$ ; they modify the classical formula

$$Q_{\text{cl}} = \bar{\partial} = \sum_i d\bar{\phi}^{\bar{i}} \frac{\partial}{\partial\phi^i} = \sum_i \alpha^{\bar{i}} \frac{\partial}{\partial\phi^i} \quad (3.3)$$

by terms that, order by order in perturbation theory, are differential operators whose possible degree grows with the order of perturbation theory. This is so because, more generally, sigma model perturbation theory is local

on  $X$  (and to a given order, sigma model perturbation theory depends on an expansion of fields such as the metric tensor of  $X$  in a Taylor series up to a given order). Instanton corrections are not at all local on  $X$ , so they can change the picture radically.

The locality highly constrains the possible perturbative modifications of  $Q$ . Let us try to perturb the classical expression (3.3) to a more general operator  $Q = Q_{\text{cl}} + \epsilon Q' + \mathcal{O}(\epsilon^2)$ , where  $\epsilon$  is a small parameter that controls the magnitude of perturbative quantum corrections. To ensure that  $Q^2 = 0$ , we need  $\{Q_{\text{cl}}, Q'\} = 0$ ; moreover, if  $Q' = \{Q_{\text{cl}}, \Lambda\}$  for some  $\Lambda$ , then the deformation by  $Q'$  can be removed by conjugation with  $\exp(-\epsilon\Lambda)$ . So  $Q'$  represents a  $Q_{\text{cl}}$  or  $\bar{\partial}$  cohomology class. Similarly, the same is true of any essentially new correction to  $Q$  (not determined by lower order terms) that appears at any order in  $\epsilon$ . Moreover, if  $Q'$  is to be generated in sigma model perturbation theory, it must be possible to construct it locally from the fields appearing in the sigma model action. (This assertion has no analog for nonperturbative instanton corrections.) For example, the Ricci tensor is constructed locally from the metric of  $X$ , which appears in the action, and represents an element of the  $\bar{\partial}$  cohomology group  $H^1(X, T^*X)$ , so it obeys these conditions. Moreover, we saw in Section 2 that it is possible to perturb the action by an element of  $H^1(X, \Omega^{2,\text{cl}})$ ; once such an element appears in the action, we certainly might then expect it to appear in a correction to  $Q$  – we will see more about this later. But these classes are apparently unique as one-dimensional  $\bar{\partial}$  cohomology classes on  $X$  that can be constructed locally from fields appearing in the action, and it may be that in some sense they completely determine the perturbative corrections to  $Q$ .

### 3.2 A sheaf of chiral algebras

In general, as we have seen in Section 2, the  $Q$ -cohomology has the structure of a chiral algebra with holomorphic operator product expansions. In this context, the  $Q$ -cohomology of dimension zero plays a special role. If  $f(z)$  and  $g(z)$  are local operators of dimension zero representing  $Q$ -cohomology classes, then singularities in the operator product  $f(z)g(z')$ , by holomorphy, must be proportional to  $(z - z')^{-s}h(z')$  for some positive integer  $s$  and operator  $h$  of dimension  $-s$ . As there are no operators of negative dimension in sigma model perturbation theory, no such operator  $h$  exists, and hence the operator product  $f(z)g(z')$  is completely nonsingular as  $z \rightarrow z'$ . It follows that, for dimension zero, we can naively set  $z = z'$  and multiply  $Q$ -cohomology classes to get an ordinary ring (more precisely, as some of the operators may be fermionic, this is a  $\mathbb{Z}_2$ -graded ring with commutators and anticommutators). We might call this the chiral ring of the theory,

as opposed to its chiral algebra.<sup>7</sup> In perturbation theory, the chiral ring is actually  $\mathbb{Z}$ -graded by the  $R$ -charge; this grading reduces mod two to the  $\mathbb{Z}_2$ -grading just mentioned. (Instantons in general reduce the  $\mathbb{Z}$  grading to a  $\mathbb{Z}_{2k}$  grading, where  $2k$  is the greatest divisor of  $c_1(X)$ .)

If we can assume that  $Q$  coincides with  $Q_{\text{cl}} = \bar{\partial}$ , then this ring is just the graded ring  $H^{0,*}(X)$ . This is certainly an interesting ring, but it may be “small.” For example, it is finite-dimensional if  $X$  is compact.

We can do much better if we realize that in perturbation theory, because the local operators of the sigma model and the fermionic symmetry  $Q$  can be described locally along  $X$ , it makes sense to consider operators that are well defined not throughout  $X$ , but only in a given open set  $U \subset X$ . Concretely, we get such an operator if we allow the function  $F(\phi, \partial\phi, \dots; \alpha)$  considered earlier to be defined only for  $\phi \in U$ .  $Q$ -cohomology classes of operators defined in an open set  $U$  have sensible operator product expansions (in perturbation theory) involving operators that are also defined in  $U$ , and they can be restricted in a natural fashion to smaller open sets and glued together in a natural way on unions and intersections of open sets. So we get what is known mathematically as a “sheaf of chiral algebras,” associating a chiral algebra and a chiral ring to every open set  $U \subset X$ . We call this sheaf  $\hat{\mathcal{A}}$ . Nonperturbatively, this structure will break down since, with instanton effects, neither the local operators (that is, the operators that are local on the Riemann surface  $\Sigma$ ) nor their operator product expansions can be defined locally on  $X$ .

The operators that are of dimension zero and of  $R$ -charge  $q = 0$  in a given open set  $U \subset X$  are of special interest. If  $Q = Q_{\text{cl}} = \bar{\partial}$ , they are simply the holomorphic functions on  $U$ , with the obvious commutative ring structure. (This ring, however, does not act on arbitrary sections of the sheaf  $\hat{\mathcal{A}}$  over  $U$ , since a dimension zero operator  $f(\phi)$  may have short distance singularities with a general chiral operator  $F(\phi, \partial\phi, \dots; \bar{\phi}, \bar{\partial}\bar{\phi}, \dots; \alpha)$ . As a result, the sheaf  $\hat{\mathcal{A}}$  does not have the natural structure of a “sheaf of  $\mathcal{O}$ -modules.”)

Starting with this observation, we can show in a variety of ways that there are no perturbative quantum corrections to  $Q$  for dimension zero and charge zero. For the kernel of a differential operator  $Q$  (the solutions  $f$  of  $Qf = 0$ ) to have the structure of a sheaf of commutative rings,  $Q$  must be

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<sup>7</sup>The term chiral ring is most commonly used for a closely related notion in two-dimensional sigma models with  $(2, 2)$  supersymmetry, and in related models in other dimensions.

a homogeneous first order differential operator.<sup>8</sup> Though we could deform  $Q_{\text{cl}} = d\bar{\phi}^{\bar{j}}\partial/\partial\bar{\phi}^{\bar{j}}$  by adding a new term  $\epsilon d\bar{\phi}^{\bar{j}}h_{\bar{j}}^i\partial/\partial\phi^i$ , where  $h$  represents an element of  $H^1(X, TX)$ , this just amounts to deforming the complex structure of  $X$ . Both classically and quantum mechanically, we want to study the model with arbitrary complex structure on  $X$ , and we may as well parameterize the quantum theory by the same complex structure that we use classically.<sup>9</sup> So, with the right parameterization of operators and theories, we can assume that for dimension and  $R$ -charge zero, there is no perturbative quantum correction to  $Q$ .

A more abstract version of this argument is to assert that, since the kernel of  $Q$  for dimension and charge zero gives a sheaf of commutative rings, we can define a complex manifold  $X'$  as the “spectrum” of this ring. If  $Q$  is obtained by perturbative quantum corrections from  $Q_{\text{cl}}$ ,  $X'$  is a deformation of  $X$ ; after possibly reparameterizing the family of quantum theories that depends on the complex structure of  $X$ , we can assume that  $X' = X$ .

We can reach the same conclusion by showing that there is no locally constructible cohomology class with the right properties to describe a deformation of  $Q$  for operators of dimension and charge zero. A correction to  $Q$  acting on functions or operators of dimension and charge zero would have leading term  $d\bar{\phi}^{\bar{j}}h_{\bar{j}}^{i_1\dots i_s}\partial^s/\partial\phi^{i_1}\dots\partial\phi^{i_s}$ , for some  $s > 0$ . Here  $h$  represents an element of  $H^1(X, \text{Sym}^s TX)$ , where  $\text{Sym}^s TX$  is the  $s$ -fold symmetric tensor product of  $TX$ . No element of  $H^1(X, \text{Sym}^s TX)$  can be constructed locally, so  $Q$  is undeformed in acting on functions.

Finally, perhaps the most illuminating proof that  $Q$  is undeformed in its action on functions follows from the description of the sheaf of chiral algebras that we give in Section 3.3.

### 3.2.1 Description by Čech cohomology

We can alternatively describe the perturbative sheaf of  $Q$ -cohomology classes by a sort of Čech cohomology. This will bring us to the mathematical point of view on this subject [10]. In fact, we will show that the chiral algebra  $\mathcal{A}$  of the  $Q$ -cohomology of the sigma model with target space

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<sup>8</sup>Or conjugate to one by  $\exp(\epsilon D)$  for some differential operator  $D$ . Such conjugacy is inessential and would be removed by change of basis in the space of local operators.

<sup>9</sup>In general, a family of classical field theories with appropriate properties leads to a family of quantum theories depending on the same number of parameters, but there is no natural pointwise map between the two families.

$X$  can be computed in perturbation theory as the Čech cohomology of the sheaf  $\widehat{\mathcal{A}}$  of locally defined chiral (or  $Q$ -invariant) operators. The relation between  $\bar{\partial}$  and Čech cohomology is of course standard in ordinary differential geometry, but here we are working in quantum field theory, and  $Q$  does not, in general coincide with the  $\bar{\partial}$  operator. (As we noted above, there are, in general, nontrivial quantum corrections involving the Ricci tensor, and perhaps others.) Nevertheless, thinking of  $Q$ -cohomology as a generalization of  $\bar{\partial}$  cohomology, it can be related to Čech cohomology by following the standard arguments.

Consider an open set  $U \subset X$  that is isomorphic to an open ball in  $\mathbb{C}^n$ , where  $n = \dim_{\mathbb{C}}(X)$ . Any holomorphic vector bundle  $W \rightarrow U$  is trivial and the higher cohomology  $H^q(U, W) = 0$  for all  $q > 0$ . Hence, in particular, in the classical limit, the  $Q$ -cohomology of the sheaf of local operators over  $U$  vanishes for  $q > 0$ ; and since small quantum corrections can only annihilate cohomology classes, not create them, it follows perturbatively that the  $Q$ -cohomology of local operators over  $U$  likewise vanishes in positive degree.

Now consider a good cover of  $X$  by open sets  $U_a$ . Then the  $U_a$  and all of their intersections have the property just described:  $\bar{\partial}$  cohomology and hence  $Q$ -cohomology vanishes in positive degree.

Let  $F$  be a  $Q$ -cohomology class of  $q = 1$ . We can precisely imitate the usual arguments about  $\bar{\partial}$  cohomology. When restricted to  $U_a$ ,  $F$  must be trivial, so  $F = \{Q, C_a\}$  where  $C_a$  is an operator of  $q = 0$  that is well defined in  $U_a$ .  $C_a$  may depend on  $a$ , although of course  $F$  does not.

Now in the intersection  $U_a \cap U_b$ , we have  $F = \{Q, C_a\} = \{Q, C_b\}$ , so  $\{Q, C_a - C_b\} = 0$ . Let  $C_{ab} = C_a - C_b$ . For each  $a$  and  $b$ ,  $C_{ab}$  is defined in  $U_a \cap U_b$ . Clearly, for all  $a, b, c$ , we have

$$C_{ab} = -C_{ba}, \quad C_{ab} + C_{bc} + C_{ca} = 0. \quad (3.4)$$

The sheaf  $\widehat{\mathcal{A}}$  of chiral operators has for its local sections the  $\alpha$ -independent local operators  $F(\phi, \partial_z \phi, \dots; \bar{\phi}, \partial_z \bar{\phi}, \dots)$  that are annihilated by  $Q$ . Each  $C_{ab}$  is a section of  $\widehat{\mathcal{A}}$  over the intersection  $U_a \cap U_b$ . The properties found in the last paragraph means that it is natural to think of the collection  $C_{ab}$  as defining an element of the first Čech cohomology group  $H_{\check{C}ech}^1(X, \widehat{\mathcal{A}})$ .

Just as in the usual case of relating  $\bar{\partial}$  and Čech cohomology, we can run all this backwards. If we are given a family  $C_{ab}$  of elements of  $H^0(U_a \cap U_b, \widehat{\mathcal{A}})$  obeying (3.4), we proceed as follows. Let  $f_a$  be a partition of unity subordinate to the open cover of  $X$  given by the  $U_a$ . (We recall

that this means that  $f_a$  is nonzero only inside  $U_a$ , and  $\sum_a f_a = 1$ .) Let  $F_a = \sum_c [Q, f_c] C_{ac}$ .<sup>10</sup> Then in  $U_a \cap U_b$ ,  $F_a = F_b$ , since  $F_a - F_b = \sum_c [Q, f_c] (C_{ac} - C_{bc}) = \sum_c [Q, f_c] C_{ab} = 0$  (we used (3.4) and the fact that  $\sum_c f_c = 1$ ). So the  $F_a$  are equal to each other and hence to a  $q = 1$  operator  $F$  that obeys  $\{Q, F\} = 0$  and is globally defined throughout  $X$ .

The above argument should seem familiar from Section 2.2 (or from any description of the relation between Čech and  $\bar{\partial}$  cohomology). What we have done is simply to copy the standard argument relating  $\bar{\partial}$  and Čech cohomology to show that, for  $q = 1$ , the  $Q$ -cohomology coincides with the Čech cohomology of the sheaf  $\hat{\mathcal{A}}$ . Nothing is special here about  $q = 1$ , and imitating the standard argument, we learn that this is true for all  $q$ . So the chiral algebra  $\mathcal{A}$  is, as a vector space,  $\oplus_q H_{\text{Čech}}^q(X, \hat{\mathcal{A}})$ . Henceforth, we generally omit the label “Čech” in denoting the cohomology of  $\hat{\mathcal{A}}$ .

In effect, the physical description via a Lagrangian and a  $Q$  operator gives a  $\bar{\partial}$ -like description of a sheaf  $\hat{\mathcal{A}}$  of chiral algebras and its cohomology. In the mathematical literature, this sheaf is studied from the Čech point of view. Here, the field  $\alpha$  is omitted and locally one considers operators constructed only from  $\phi, \bar{\phi}$ , and their derivatives. Cohomology classes of positive  $q$  are constructed as Čech  $q$ -cocycles. Instead, in the physical approach, the sheaf appears in a  $\bar{\partial}$ -like language, using the differential  $Q$ , and classes of degree  $q$  are represented by operators that are  $q$ th order in the field  $\alpha$ .

In Section 3.3, we express in a physical language a few key points that are made [10] in the mathematical literature starting from the Čech viewpoint.

### 3.3 Relation to a free $\beta\gamma$ system

To begin, we will give a convenient description of the local structure of the sheaf  $\hat{\mathcal{A}}$ . That is, we will describe in a new way the  $Q$ -cohomology of operators that are regular in a small open set  $U \subset X$ . We assume that  $U$  is isomorphic to an open ball in  $\mathbb{C}^n$ .

The hermitian metric on  $X$  only enters the action in terms of the form  $\{Q, \dots\}$  and so does not affect the  $Q$ -cohomology. Hence, to describe the local structure, we can pick a hermitian metric that is flat when restricted to  $U$ . The action, in general, also contains terms derived from an element of  $H^1(X, \Omega^{2, \text{cl}}(X))$ , as we explained in Section 2.2. These terms are also

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<sup>10</sup>Some regularization of the operator product of  $[Q, f_c(\phi, \bar{\phi})]$  with  $C_{ac}$  is needed, for example by normal ordering.

$Q$ -exact locally, and so can be discarded in analyzing the local structure in  $U$ . We can pick coordinates in  $U$  such that the action derived from the flat hermitian metric in  $U$  is

$$I = \frac{1}{2\pi} \int_{\Sigma} |d^2z| \sum_{i,\bar{j}} \delta_{i,\bar{j}} \left( \partial_z \phi^i \partial_z \bar{\phi}^{\bar{j}} + \rho^i \partial_z \alpha^{\bar{j}} \right). \quad (3.5)$$

Now let us describe the  $Q$ -cohomology classes of operators regular in  $U$ . As explained above, these can be represented by operators of dimension  $(n, 0)$  that are independent of  $\alpha$ . Such operators in general are of the form  $F(\phi, \partial_z \phi, \dots; \bar{\phi}, \partial_z \bar{\phi}, \dots)$ . On this class of operators,  $Q$  acts as  $\alpha^{\bar{j}} \partial / \partial \bar{\phi}^{\bar{j}}$ , and the condition that  $F$  is annihilated by  $Q$  is precisely that, as a function of  $\phi$ ,  $\bar{\phi}$ , and their derivatives, it is independent of  $\bar{\phi}$  (as opposed to its derivatives), and depends only on the other variables, namely  $\phi$  and the derivatives of  $\phi$  and  $\bar{\phi}$ .<sup>11</sup> Thus the  $Q$ -invariant operators are of the form  $F(\phi, \partial_z \phi, \dots; \partial_z \bar{\phi}, \partial_z^2 \bar{\phi}, \dots)$ . Differently put, these operators have a general dependence on the  $z$ -derivatives of  $\phi$  and  $\bar{\phi}$ , but in their dependence on the center of mass coordinate of the string, they are holomorphic.

If we set  $\beta_i = \delta_{i\bar{j}} \partial_z \bar{\phi}^{\bar{j}}$ , which is an operator of dimension  $(1, 0)$ , and  $\gamma^i = \phi^i$ , of dimension  $(0, 0)$ , then the  $Q$ -cohomology of operators regular in  $U$  is represented by arbitrary local functions of  $\beta$  and  $\gamma$ , of the form  $F(\gamma, \partial_z \gamma, \partial_z^2 \gamma, \dots; \beta, \partial_z \beta, \partial_z^2 \beta, \dots)$ . The operators  $\beta$  and  $\gamma$  have the operator products of a standard  $\beta\gamma$  system. The products  $\beta \cdot \beta$  and  $\gamma \cdot \gamma$  are nonsingular, while

$$\beta(z)\gamma(w) = -\frac{1}{z-w} + \text{regular}. \quad (3.6)$$

These statements can be deduced from the flat action (3.5) by standard methods. We can write down an action for fields  $\beta$  and  $\gamma$ , regarded as elementary fields, which reproduces these OPE's. It is simply the standard action of the  $\beta\gamma$  system:

$$I_{\beta\gamma} = \frac{1}{2\pi} \int |d^2z| \sum_i \beta_i \partial_z \gamma^i. \quad (3.7)$$

The equations of motion derived from this action assert that  $\partial_z \gamma = \partial_z \beta = 0$ . So a general local operator of this system is of the form  $\bar{F}(\gamma, \partial_z \gamma, \dots; \beta, \partial_z \beta, \dots)$ . Since the theory constructed from the action  $I_{\beta\gamma}$

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<sup>11</sup>Once again, we can ignore the action of  $Q$  on derivatives of  $\bar{\phi}$  because of the equation of motion  $\partial_z \alpha = 0$ .

of the  $\beta\gamma$  system reproduces the appropriate list of operators and OPE's of the sigma model, it follows that the chiral algebra of the  $Q$ -cohomology in a small open set  $U$  is the same as the chiral algebra of the  $\beta\gamma$  system, restricted to the same open set. (Restriction to  $U$  just means that the operator  $F$  or  $\tilde{F}$ , in its dependence on the zero mode of  $\gamma = \phi$ , is required to be holomorphic in  $U$ , but not necessarily throughout  $X$  or  $\mathbb{C}^n$ ).

Does the  $\beta\gamma$  system reproduce the  $Q$ -cohomology globally, or only in a small open set  $U$ ? First of all, classically, the action (3.7) makes sense globally if we interpret the fields  $\beta$  and  $\gamma$  correctly.  $\gamma$  defines a map  $\gamma : \Sigma \rightarrow X$ , and  $\beta$  is a  $(1, 0)$ -form on  $\Sigma$  with values in the pull back  $\gamma^*(T^*X)$ . With this interpretation, (3.7) becomes the action of what we might call a nonlinear  $\beta\gamma$  system. Though nonlinear, this action can be made linear, locally, by choosing local coordinates  $\gamma^i$  on a small open set  $U \subset X$ . Although this sort of nonlinear  $\beta\gamma$  system is not widely studied in physics, there are examples where it has been studied, for example in covariant quantization of the superstring [23]. In that application,  $X$  is a  $\mathbb{C}^*$  bundle over the homogeneous space  $SO(10)/U(5)$ .

Granted that the classical action of the  $\beta\gamma$  system makes sense globally, what happens quantum mechanically? The anomalies that enter in the sigma model also appear in the nonlinear  $\beta\gamma$  system. Expand around a classical solution of the nonlinear  $\beta\gamma$  system, represented by a holomorphic map  $\gamma_0 : \Sigma \rightarrow X$ . Setting  $\gamma = \gamma_0 + \gamma'$ , the action, expanded to quadratic order about this solution, is  $(1/2\pi)(\beta, \overline{D}\gamma')$ . Here, the kinetic operator is the  $\overline{D}$  operator on sections of  $\gamma_0^*(TX)$ ; it is the complex conjugate of the operator whose anomalies we encountered in Section 2.3. Complex conjugation reverses the sign of the anomalies, but here the fields are bosonic, while in Section 2.3, they were fermionic; this gives a second sign change. So the nonlinear  $\beta\gamma$  system has exactly the same anomalies as the underlying sigma model. In effect, in going from the sigma model to the nonlinear  $\beta\gamma$  system, we have canceled antiholomorphic bosons and fermions that do not contribute to the  $Q$ -cohomology and whose net contribution to anomalies also vanishes.

On the other hand, the nonlinear  $\beta\gamma$  system lacks the  $U(1)$   $R$ -charge  $q$  of the sigma model. While locally the  $Q$ -cohomology is supported at  $q = 0$ , globally there is generically cohomology in higher degrees. How would we use the nonlinear  $\beta\gamma$  system to describe this higher cohomology? The answer should be clear from Section 3.2. In the  $\beta\gamma$  description, we do not have a close analog of  $\overline{\partial}$  cohomology at our disposal, but we can use Čech cohomology. We cover  $X$  by small open sets  $U_a$ , and, as explained in Section 3.2, we describe the  $Q$ -cohomology classes of positive degree by Čech cocycles. Though this is an unusual procedure (in the present context) from the point of view of physicists, it has been taken mathematically as the starting



point for the present subject [10]. (The subject has also been developed mathematically with less emphasis on the Čech point of view [13]).

Perhaps a more severe problem with the nonlinear  $\beta\gamma$  action (3.7) is that in this framework, it is difficult to see all the moduli of the sigma model in the classical action. As we saw in Section 2.2, those moduli are the complex structure of  $X$  and also a class in  $H^1(X, \Omega^{2,cl}(X))$ . The complex structure is built into the classical action (3.7), but it does not seem possible to build a class in  $H^1(X, \Omega^{2,cl}(X))$  into the action in this framework. In the usual mathematical approach [10], the class in  $H^1(X, \Omega^{2,cl}(X))$  is instead incorporated into the definition of Čech cocycles, as we explain in Section 3.5.

Finally, in a quantum field theory, one wants to do more than define  $Q$ -cohomology classes or a sheaf of chiral algebras. One wants to compute correlation functions of operators representing these cohomology classes as well as, possibly, other local operators. For the sigma model, there is a clear procedure to compute correlation functions, while for the nonlinear  $\beta\gamma$  system there is at first sight no natural procedure, as there is no sensible way to integrate over the zero mode of  $\gamma$ . The reason for this is clear if we consider again the charge  $q$  of the operators. In perturbation theory, on a Riemann surface  $\Sigma$  of genus  $g$ , a correlation function  $\langle \mathcal{O}_1(z_1) \dots \mathcal{O}_s(z_s) \rangle$  of operators  $\mathcal{O}_i$  of charge  $q_i$  vanishes unless  $\sum_i q_i = n(1 - g)$ . For instantons, the formula becomes  $\sum_i q_i = n(1 - g) + \int_{\Sigma} \Phi^*(c_1(X))$ . (These formulas come from the index theorem for the  $U(1)$  current associated with the  $R$ -charge. The right hand side is the dimension of instanton moduli space.) Generically, therefore, nonzero correlation functions require that the  $q_i$  do not all vanish. As operators of  $q_i \neq 0$  cannot be represented in a standard fashion in the nonlinear  $\beta\gamma$  system (but must be described by Čech cocycles), it is clear that, while in the sigma model one can compute correlation functions via a standard recipe, to do so in the nonlinear  $\beta\gamma$  system requires translating the usual recipe into the Čech language. This would be an unusual procedure, at least for physicists. (Moreover, to compute such correlation functions at the instanton level requires understanding instanton corrections to the  $Q$ -cohomology, which can radically change the picture and are beyond the scope of the present paper.)

### 3.4 Local symmetries

Having understood the local structure of the  $Q$ -cohomology, we can attempt to build a global picture by gluing together the local pieces.

We cover  $X$  by small open sets  $U_a$ . In each  $U_a$ , the  $Q$ -cohomology can be described by a free  $\beta\gamma$  system. We want to glue these local descriptions together in intersections  $U_a \cap U_b$ , so as to describe the  $Q$ -cohomology in terms of a sheaf of chiral algebras over the whole manifold  $X$ .

The gluing must be carried out by an automorphism of the free  $\beta\gamma$  system, so we must understand the symmetries of this system. The key properties can be understood by constructing the Lie algebra  $\mathfrak{g}$  of such symmetries. An element of  $\mathfrak{g}$  is the integral of a dimension one current, modulo total derivatives. The currents in the  $\beta\gamma$  system are as follows.

First, if  $V^i$  is a holomorphic vector field on  $X$ , we can make the dimension one current  $J_V = -V^i\beta_i$  and the corresponding conserved charge  $K_V = \oint J_V$ . Let  $\mathfrak{v}$  be the subspace of  $\mathfrak{g}$  generated by the  $K_V$ 's. As shown in [10], and as we will explain momentarily,  $\mathfrak{v}$  is not a Lie subalgebra of  $\mathfrak{g}$ , only a linear subspace.

By computing operator products with the elementary fields  $\gamma$ ,

$$J_V(z)\gamma^k(w) \sim \frac{V^k(w)}{z-w}, \quad (3.8)$$

we see that  $J_V$  generates the infinitesimal diffeomorphism  $\delta\gamma^k = V^k$  of  $U$ . Thus, the  $J_V$  generate the holomorphic diffeomorphisms of the target space.

The other conserved currents are as follows. Let  $B = \sum_i B_i d\gamma^i$  be a holomorphic  $(1,0)$ -form on  $X$ . Then we can make the current  $J_B = B_i\partial\gamma^i$ , and the conserved charge  $\oint J_B$ . However, if  $B$  is exact, say  $B_i = \partial_i H$  for some local holomorphic function  $H$ , then  $\oint J_B = \oint \partial_i H d\gamma^i = \oint dH = 0$ . So the conserved charge constructed from  $B$  vanishes if (and only if)  $B$  is exact. Locally,  $B$  is exact if and only if  $0 = \partial B = \partial_i B_j - \partial_j B_i$ . (As we are working in perturbation theory, it suffices to work locally.) We write  $C$  for the holomorphic  $(2,0)$ -form  $C = \partial B$ . It is annihilated by  $\partial$  and so is a local holomorphic section of  $\Omega^{2,\text{cl}}$ . For every local holomorphic section  $C$  of  $\Omega^{2,\text{cl}}$ , we find a local holomorphic  $(1,0)$ -form  $B$  with  $C = \partial B$  and write  $K_C = \oint J_B$ . Let us write  $\mathfrak{c}$  for the linear span of the  $K_C$ .

So finally, the symmetry algebra  $\mathfrak{g}$  of the  $\beta\gamma$  system in a small open set  $U$  is, as a linear space,  $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{c}$ .  $\mathfrak{c}$  is trivially a subalgebra, in fact an abelian one, because the currents  $J_B$  derived from  $(1,0)$ -forms are constructed only from  $\gamma$  (and its derivatives) and their products have no short distance singularities. So  $\mathfrak{g}$  is an extension

$$0 \rightarrow \mathfrak{c} \rightarrow \mathfrak{g} \rightarrow \mathfrak{v} \rightarrow 0. \quad (3.9)$$

In fact, (3.9) is an exact sequence of Lie algebras, since as we will see momentarily,  $[\mathfrak{v}, \mathfrak{c}] \subset \mathfrak{c}$ .

The action of  $\mathfrak{v}$  on  $\mathfrak{c}$  can be found from the OPE

$$V^i \beta_i(z) \cdot B_j \partial \gamma^j(w) = -\frac{1}{(z-w)^2} V^i B_i(w) - \frac{1}{z-w} (V^i (\partial_i B_k - \partial_k B_i) + \partial_k (V^i B_i)) \partial \gamma^k. \quad (3.10)$$

The commutator of  $K_V$  with  $K_{\partial B}$  is the residue of the simple pole on the right hand side. In the numerator, we recognize that  $V^i (\partial_i B_k - \partial_k B_i) + \partial_k (V^i B_i)$  is the same as  $(\mathcal{L}_V(B))_k$ , which represents the Lie derivative of the vector field  $V$  acting on the one-form  $B$ . This is what we might have guessed based on the result (3.8) showing that the  $J_V$  generate diffeomorphisms of  $U$ .

However, in the commutator of two elements of  $\mathfrak{v}$ , we get a surprise [10]. Let  $V$  and  $W$  be two holomorphic vector fields on  $U$ . We compute

$$J_V(z) J_W(w) \sim -\frac{\partial_j V^i \partial_i W^j(w)}{(z-w)^2} - \frac{(V^i \partial_i W^j - W^i \partial_i V^j) \beta_j}{z-w} - \frac{(\partial_k \partial_j V^i) (\partial_i W^j \partial \gamma^k)}{z-w}. \quad (3.11)$$

The first term on the right hand side, being a double pole, does not contribute to the commutator. The second and third terms take values in  $\mathfrak{v}$  and  $\mathfrak{c}$ , respectively. The second term, which comes from a single contraction of elementary fields in evaluating the OPE, is the expected result  $J_V(z) J_W(w) \sim J_{[V, W]} / (z-w)$ , where  $[V, W]^k = V^i \partial_i W^k - W^i \partial_i V^k$  is the commutator of the vector fields  $V$  and  $W$ . We would get the same result by computing the commutator of  $J_V$  and  $J_W$  via Poisson brackets in the classical  $\beta\gamma$  theory. Like all anomalies in conformal field theory, the third term comes from a multiple contraction. This last term means that  $\mathfrak{v}$  does not close upon itself as a Lie algebra — the commutator of two elements of  $\mathfrak{v}$  is not contained in  $\mathfrak{v}$ . So  $\mathfrak{g}$  is not a semidirect product of  $\mathfrak{v}$  with  $\mathfrak{c}$ , and the extension of Lie algebras in (3.9) is nontrivial.

### 3.5 Gluing the open sets together

Now take a suitable collection of small open sets  $U_a \subset \mathbb{C}^n$ . We wish to glue them together to make a good cover of a complex manifold  $X$ . On each  $U_a$ , the sheaf  $\widehat{\mathcal{A}}$  of chiral algebras is defined by a free  $\beta\gamma$  system. We want to glue together these free conformal field theories to get a globally defined sheaf of

chiral algebras. Two questions arise: is there an obstruction to this gluing? And if we can carry out the gluing, what are the moduli of the resulting sheaf?

Let us first recall how this is done geometrically. For each  $a, b$ , we pick an open set  $U_{ab} \subset U_a$ , and likewise an open set  $U_{ba} \subset U_b$ , and a holomorphic diffeomorphism  $f_{ab}$  between them

$$f_{ab} : U_{ab} \cong U_{ba}. \quad (3.12)$$

We take  $f_{ba} = f_{ab}^{-1}$ . We want to identify a point  $P \in U_{ab}$  with a point  $Q \in U_{ba}$  if  $Q = f_{ab}(P)$ . This makes sense if for any  $U_a, U_b$ , and  $U_c$ , we have

$$f_{ca} f_{bc} f_{ab} = 1 \quad (3.13)$$

wherever all the maps are defined (the space in which they are all defined is what we interpret as the triple intersection  $U_{abc}$ ). This relation says that the different pieces  $U_a$  can be glued together via the holomorphic maps  $f_{ab}$  to make a complex manifold  $X$ . Complex moduli of  $X$  appear as parameters in the  $f_{ab}$ .

Now suppose that we have a sheaf of chiral algebras on each  $U_a$ . We want to glue them together on overlaps to get a sheaf of chiral algebras on  $X$ . The gluing must be done using a symmetry not of the complex manifolds  $U_a$ , but rather using a symmetry of the conformal field theories. So for each pair  $U_a$  and  $U_b$ , we pick a conformal field theory symmetry  $\widehat{f}_{ab}$  that maps the free  $\beta\gamma$  system on  $U_a$ , restricted to  $U_{ab}$ , to the free  $\beta\gamma$  system on  $U_b$ , similarly restricted to  $U_{ba}$ . We get a global sheaf of chiral algebras if the gluing is consistent:

$$\widehat{f}_{ca} \widehat{f}_{bc} \widehat{f}_{ab} = 1. \quad (3.14)$$

If we do have a consistent gluing, it makes sense to ask what the target space is. The reason that this makes sense is that, as in (3.9), there is a Lie algebra homomorphism  $\mathfrak{g} \rightarrow \mathfrak{v}$  that “forgets” the abelian symmetries in  $\mathfrak{c}$  and only “remembers” how a symmetry acts geometrically, that is as an element of  $\mathfrak{v}$ . Similarly, at the group level, there is a map from a conformal field theory gluing operator  $\widehat{f}_{ab}$  to the corresponding geometrical symmetry  $f_{ab}$ . The relation (3.14) for the  $\widehat{f}$ 's implies a similar relation (3.13) for the  $f$ 's, so every way to glue together the conformal field theories determines a geometrical gluing of the  $U_a$  to make a complex manifold  $X$  that we call the target space of the conformal field theory.

However, if the  $f_{ab}$  are given, the  $\widehat{f}_{ab}$  are not uniquely determined. We can still pick, for each  $U_{ab}$ , an element  $C_{ab} \in H^0(U_{ab}, \Omega^{2, \text{cl}})$ , representing

an element of  $\mathfrak{c}$ . Then we transform  $\widehat{f}_{ab} \rightarrow \widehat{f}'_{ab} = \exp(C_{ab})\widehat{f}_{ab}$ . The condition that the gluing identity (3.14) is still obeyed is that in each triple intersection  $U_{abc}$  we should have

$$C_{ab} + C_{bc} + C_{ca} = 0. \quad (3.15)$$

The  $C$ 's, in other words, must define an element of the Čech cohomology group  $H^1(X, \Omega^{2,\text{cl}}(X))$ . Passing from  $\widehat{f}$  to  $\widehat{f}'$  does not change the target space  $X$  — since  $C$  is “forgotten” when we project from  $\widehat{f}_{ab}$  to the geometrical gluing data  $f_{ab}$ . So we get in this way a family of sheaves of chiral algebras, with the same target space  $X$ , and admitting an action of  $H^1(X, \Omega^{2,\text{cl}}(X))$ . (The last statement simply means that given such a sheaf and an element  $C \in H^1(X, \Omega^{2,\text{cl}}(X))$ , one can make a new sheaf by  $\widehat{f} \rightarrow \exp(C)\widehat{f}$ .)

The simple form of the cocycle condition (3.15) may require some explanation. Why can we omit the  $f$ 's in writing it? In the beginning of this section, we recalled how to build up a complex manifold  $X$  by gluing together abstract open sets  $U_a$ , using the gluing maps  $f_{ab}$ . Once this is done, by the time one gets to Čech cohomology, one usually regards the  $U_a$  as subspaces of a common space  $X$ , and then it is customary to suppress the  $f$ 's in writing the condition (3.15) of a Čech cocycle. The  $f$ 's would return in the formula if we persist in regarding the  $U_a$ 's as subsets of abstract  $\mathbb{C}^n$ 's.

### 3.5.1 The anomaly

There is also a possibility here, as in Section 2.2, for an anomaly. In the present context, this will appear as an obstruction to the gluing.

Suppose we are given a set of gluing data  $f_{ab}$  which obeys (3.13). There is no natural way to “lift” the  $f_{ab}$  to conformal field symmetries  $\widehat{f}_{ab}$ . Pick any way to do it. Though the geometrical relation  $f_{ca}f_{bc}f_{ab}$  is obeyed, the analogous lifted relation may not be. In general, we will have

$$\widehat{f}_{ca}\widehat{f}_{bc}\widehat{f}_{ab} = \exp(C_{abc}) \quad (3.16)$$

for some  $C_{abc} \in H^0(U_{abc}, \Omega^{2,\text{cl}})$ . The reason for (3.16) is that, as the left hand side maps to the identity if projected to the group of geometrical symmetries, it must be an element of the abelian group (generated by  $\mathfrak{c}$ ) that acts trivially on the coordinates  $\gamma^i$  of the  $U_a$ .

The choice of  $\widehat{f}_{ab}$  was not unique. If we transform  $\widehat{f}_{ab} \rightarrow \exp(U_{ab})\widehat{f}_{ab}$ , we get

$$C_{abc} \rightarrow C'_{abc} = C_{abc} + C_{ab} + C_{bc} + C_{ca}. \quad (3.17)$$

If it is possible to pick the  $C_{ab}$  to set all  $C'_{abc} = 0$ , then there is no anomaly and one can obtain a globally defined sheaf of chiral algebras.

In any event, in quadruple overlaps  $U_a \cap U_b \cap U_c \cap U_d$ , the  $C$ 's obey

$$C_{abc} - C_{bcd} + C_{cda} - C_{dab} = 0. \quad (3.18)$$

Along with the equivalence relation (3.17), this means that the  $C$ 's define an element of the sheaf cohomology group  $H^2(X, \Omega^{2,cl}(X))$ .

In Section 2.2, we obtained from the sigma model an anomaly measured by  $p_1(X)$ , which in de Rham cohomology (that is, in perturbation theory) represents an element of  $H^2(X, \Omega^{2,cl}(X))$ .<sup>12</sup> It has been shown [12] that the obstruction, associated with the  $C$ 's, to gluing the free  $\beta\gamma$  systems on the  $U_a$  into a global sheaf of chiral algebras is indeed given by  $p_1(X)$ . In Section 5.2, we illustrate this in an example.

### 3.5.2 The other anomaly

In Section 2.3, we really had two anomalies, one involving  $p_1(X)$  while the other was proportional to  $c_1(\Sigma)c_1(X)$ . We recall that  $\Sigma$  is the Riemann surface on which our quantum field theory is defined, and  $X$  is the target space. How do we see the second anomaly in the present discussion?

So far, we have constructed a sheaf of chiral algebras globally in  $X$ , but only locally along  $\Sigma$ . When we defined in Section 2.1 a chiral algebra on the  $Q$ -cohomology of a  $\sigma$ -model, conformal invariance was not one of the axioms (see the next-to-last paragraph of Section 2.1). The reason for this is that, generically, the  $\sigma$ -model used to construct the chiral algebra is not invariant under holomorphic reparameterizations of the Riemann surface  $\Sigma$ . We noted in Section 3.1 that the holomorphic part of the stress tensor  $T_{zz}$  does not correspond to an element of the  $Q$ -cohomology unless  $c_1(X) = 0$ . This gives an obstruction to reparameterization invariance of the  $Q$ -cohomology, and thus of the chiral algebra.

With the reformulation by a  $\beta\gamma$  system, it may appear that the problem of lack of conformal invariance has disappeared. We are now deriving the

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<sup>12</sup>This assertion was explained in a footnote in Section 2.3.

chiral algebra, locally on  $X$ , from a free  $\beta\gamma$  system. The *free*  $\beta\gamma$  system is certainly conformally invariant, and can be defined on an arbitrary global Riemann surface  $\Sigma$ .

There is no contradiction here. The anomaly we are looking for is proportional to  $c_1(\Sigma)c_1(X)$ , so it vanishes if we work locally on  $X$  (using a free  $\beta\gamma$  system) even if we work globally on  $\Sigma$ . It also vanishes if we work locally on  $\Sigma$  even if we work globally on  $X$ . The latter is what we did in our above discussion of conformal field theory gluing relations such as (3.14). We will only see the  $c_1(\Sigma)c_1(X)$  anomaly if we work globally on both  $\Sigma$  and  $X$ .

The global  $\beta\gamma$  system (3.7) with target space  $X$  is not quite conformally invariant, as we will make most explicit in Section 5.1 when we consider an example in detail. The problem arises because of normal-ordering problems in defining quantum operators corresponding to classical expressions  $F(\gamma, \partial_z \gamma, \dots; \beta, \partial_z \beta, \dots)$ . In comparing different methods of normal-ordering a given operator that classically has dimension  $d$ , with the different methods corresponding to different choices of local coordinates on  $\Sigma$  or  $X$ , we get results that differ by operators of dimension no greater than  $d$ . This corresponds to the statement that in perturbative quantum field theory, the set of all classical operators of dimension no greater than  $d$  (for any given integer  $d$ ) can be consistently renormalized, without considering operators of higher dimension. Thus, working globally on  $\Sigma$  and  $X$ , the space of quantum operators is “filtered” (but possibly not graded) by the dimension.

Suppose that we cover  $\Sigma$  with small open sets  $P_\tau$  while covering  $X$  with small open sets  $U_a$ . On each  $P_\tau$ , we define a free  $\beta\gamma$  system with target  $U_a$ . Now we want to glue together the  $U_a$ 's and  $P_\tau$ 's to get a chiral algebra, with target  $X$ , defined on all of  $\Sigma$ . By a “chiral algebra,” we mean a system of holomorphically varying local operators on  $\Sigma$ , with operator product expansions that have singularities of the usual type only on the diagonal, and obeying associativity. (Missing is the usual claim in physical discussion of chiral algebras that the chiral algebra on  $\Sigma$  comes from a universal, conformally invariant chiral algebra that is universally defined on all Riemann surfaces and has been specialized to  $\Sigma$ .)

In effect, we are covering  $X \times \Sigma$  with open sets  $W_{a\tau} = U_a \times P_\tau$ . On each such open set, we define a free  $\beta\gamma$  system and hence a chiral algebra, and then on overlaps we want to glue these together. The gluing is made using a combination of holomorphic changes of coordinate on  $\Sigma$  (since the free  $\beta\gamma$  system is conformally invariant) and holomorphic changes of coordinate in the target space  $X$  (since the free  $\beta\gamma$  system has the geometrical symmetries  $\mathfrak{v}$  generated by vector fields). There is no problem in finding a gluing map

$g_{a\tau,b\nu}$  from  $W_{a\tau}$  to  $W_{b\nu}$ , but there may be a problem in arranging on triple overlaps to get suitable relations  $g_{c\sigma,a\tau}g_{a\tau,b\nu}g_{b\nu,c\sigma} = 1$ . The obstruction will be a two-dimensional Čech cohomology class  $H^2(X \times \Sigma, \mathcal{U})$  for some sheaf  $\mathcal{U}$  that we must determine.

$\mathcal{U}$  is a sheaf of symmetries of the free  $\beta\gamma$  system, since the problem arises from the indeterminacy in the gluing maps  $g_{a\tau,b\nu}$ . As in our discussion of the anomaly involving  $p_1 \in H^2(X, \Omega^{2,\text{cl}}(X))$ ,  $\mathcal{U}$  will be a sheaf of symmetries that act trivially on the coordinates  $\gamma$  of the target space  $X$ . The reason for this is that, as the notion of a map  $\gamma : \Sigma \rightarrow X$  makes sense, there is no inconsistency in the gluing of  $\gamma$ .

We earlier identified the abelian algebra  $\mathfrak{c}$  of symmetries that act trivially on  $\gamma$ . However, in that discussion we implicitly assumed reparameterization invariance on  $\Sigma$ , which is valid locally on  $\Sigma$  but perhaps not globally. If we drop reparameterization invariance, we can write more general holomorphic operator-valued  $(1,0)$ -forms on  $\Sigma$ . We can take any holomorphic operator  $\mathcal{O}(z)$  of dimension  $(n,0)$  for any integer  $n$ , and any holomorphic section  $f(z)$  of an appropriate power of the canonical bundle of  $\Sigma$  (restricted to a suitable open set  $P_\tau$ ) and consider the symmetry generated by  $\oint f(z)\mathcal{O}(z)$ . However, because we are looking for deformations in which the local operators are filtered by dimension, there is a drastic simplification: we can limit ourselves to  $n \leq 1$ . Also, since the ambiguity in gluing comes from operators that commute with  $\gamma$ , and so are not visible geometrically, we need only consider operators constructed from  $\gamma$  and its derivatives, and not depending on  $\beta$ ; the constraint  $n \leq 1$  means that derivatives enter only via a linear dependence on  $\partial_z\gamma$ .

The most general symmetry obeying these conditions is generated by  $\oint J$ , with

$$J = B_i(\gamma, z)d\gamma^i + E(\gamma, z)dz. \quad (3.19)$$

Here  $B_i$  and  $E$  are the components of a holomorphic  $(1,0)$ -form on  $X \times \Sigma$ , namely  $Y = B_id\gamma^i + E dz$ .

The conserved charge  $K_Y = \oint (B_id\gamma^i + E dz)$  vanishes if, and only if,  $Y$  is exact,  $Y = \partial\Lambda$  for some zero-form  $\Lambda(\gamma, z)$  on  $X \times \Sigma$ . So this charge really depends only on the closed holomorphic  $(2,0)$ -form  $Z = \partial Y$  on  $X \times \Sigma$ .  $Z$  is a local holomorphic section of  $\Omega^{2,\text{cl}}(X \times \Sigma)$ . Thus, we learn finally that the appropriate sheaf of symmetries that act trivially on  $\gamma$  is isomorphic to  $\Omega^{2,\text{cl}}(X \times \Sigma)$ .

We therefore must expect that an anomaly takes values in  $H^2(X \times \Sigma, \Omega^{2,\text{cl}}(X \times \Sigma))$ . Indeed,  $c_1(X)c_1(\Sigma)$  takes values in this group, as  $c_1(X) \in$



$H^1(X, \Omega^{1,\text{cl}}(X))$ ,  $c_1(\Sigma) \in H^1(\Sigma, \Omega^{1,\text{cl}}(X))$ , and the wedge product of these takes values in  $H^2(X \times \Sigma, \Omega^{2,\text{cl}}(X \times \Sigma))$ . So it is natural for the anomaly seen using Dirac operators in the sigma model to have an alternative interpretation in terms of the Čech cohomology of the sheaf of chiral algebras. This description has been developed in the mathematical literature.

Actually, as  $\Sigma$  is of complex dimension one, we have  $\Omega^{2,\text{cl}}(X \times \Sigma) = \Omega^{2,\text{cl}}(X) \otimes \mathcal{O}_\Sigma \oplus \Omega^{1,\text{cl}}(X) \otimes \Omega^{1,\text{cl}}(\Sigma)$ . (Here  $\mathcal{O}_\Sigma$  is the sheaf of holomorphic functions on  $\Sigma$ ). Hence for compact  $\Sigma$ ,

$$\begin{aligned} H^2(X \times \Sigma, \Omega^{2,\text{cl}}(X \times \Sigma)) &= H^2(X, \Omega^{2,\text{cl}}(X)) \\ &\oplus H^1(X, \Omega^{1,\text{cl}}(X)) \otimes H^1(\Sigma, \Omega^{1,\text{cl}}(\Sigma)) \\ &\oplus \dots \end{aligned} \tag{3.20}$$

(We have used the fact that  $H^0(\Sigma, \mathcal{O}) \cong \mathbb{C}$ ). The two anomalies  $p_1(X)/2$  and  $c_1(X)c_1(\Sigma)$  take values in the two summands on the right hand side of (3.20).

Similarly, if the anomaly vanishes and there exists a global sheaf of chiral algebras on  $\Sigma$ , with target space  $X$ , then moduli of this sheaf are parameterized (apart from the obvious geometrical moduli) by  $H^1(X \times \Sigma, \Omega^{2,\text{cl}}(X \times \Sigma))$ .

## 4 (0, 2) Supersymmetry

### 4.1 Construction of models

The reason the structure explored in this paper is relevant to physics is that it arises in sigma models with (0, 2) supersymmetry. These are unitary, or physically sensible, quantum field theories, and they have applications for compactification of the heterotic string. (0, 2) supersymmetry requires considerably stronger conditions than we assumed in Sections 2 and 3, where we considered a general hermitian metric on a complex manifold  $X$  and constructed a model with a single fermionic symmetry  $Q$  obeying  $Q^2 = 0$ .<sup>13</sup> In (0, 2) supersymmetry, one has a pair of fermionic symmetries, which

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<sup>13</sup>By contrast, weaker conditions are needed for (0, 1) supersymmetry. In this case, no complex structure is required on the target space, and the curvature of the  $B$ -field is an arbitrary closed three-form. This model has one conserved supercharge, superficially like the model considered in Sections 2 and 3, but the supercharge is hermitian and its square is not zero.

are hermitian adjoints of one another and roughly are loop space analogs of the  $\bar{\partial}$  and  $\bar{\partial}^\dagger$  operators of a finite-dimensional complex manifold. Thus, mathematically, if one wants an analog of Hodge theory for CDO's,  $(0, 2)$  supersymmetry is natural.

In general, a hermitian metric  $g$  on a complex manifold  $X$  determines an associated  $(1, 1)$ -form  $\omega$ . On a Kahler manifold,  $\omega$  is closed, while for  $(0, 2)$  supersymmetry, it must obey  $\bar{\partial}\partial\omega = 0$ . The present section contains no novelty; we merely summarize some familiar results [24, 25] about  $(0, 2)$  supersymmetry in order to make clear how the subject of the present paper is related to physics.

To construct a model with  $(0, 2)$  supersymmetry, we enlarge the world-sheet  $\Sigma$  to a supermanifold  $\widehat{\Sigma}$  with bosonic coordinates  $z, \bar{z}$  and fermionic coordinates  $\theta^+, \bar{\theta}^+$ .<sup>14</sup> The supersymmetries act geometrically:

$$\begin{aligned}\bar{Q}_+ &= \frac{\partial}{\partial\theta^+} - i\theta^+ \frac{\partial}{\partial\bar{z}} \\ Q_+ &= \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+ \frac{\partial}{\partial\bar{z}}.\end{aligned}\tag{4.1}$$

Thus,  $Q_+^2 = \bar{Q}_+^2 = 0$  and  $\{Q_+, \bar{Q}_+\} = -2i\partial/\partial\bar{z}$ . To construct Lagrangians invariant under  $Q_+$  and  $\bar{Q}_+$ , we use the fact that these operators commute with the supersymmetric derivatives

$$\begin{aligned}\bar{D}_+ &= \frac{\partial}{\partial\bar{\theta}^+} + i\theta^+ \frac{\partial}{\partial\bar{z}} \\ D_+ &= \frac{\partial}{\partial\theta^+} + i\bar{\theta}^+ \frac{\partial}{\partial\bar{z}},\end{aligned}\tag{4.2}$$

as well as with  $\partial_z$  and  $\partial_{\bar{z}}$ . Moreover, the measure  $|d^2z|d\theta d\bar{\theta}$  is supersymmetric, that is, invariant under  $Q_+$  and  $\bar{Q}_+$ . So any action constructed using only the supersymmetric derivatives and measure will be supersymmetric.

We will construct a supersymmetric model of maps  $\Phi : \widehat{\Sigma} \rightarrow X$ . We consider maps that are required to be ‘‘chiral.’’ This means that if  $w$  is any local holomorphic function on  $X$ , then  $W = \Phi^*(w)$  obeys

$$\bar{D}_+ W = D_+ \bar{W} = 0.\tag{4.3}$$

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<sup>14</sup>The reason for the superscripts  $+$  is that  $\theta^+, \bar{\theta}^+$  transform as sections of one of the spin bundles of  $\Sigma$ , say the one of positive chirality. In a model with  $(2, 2)$  supersymmetry, one would have additional fermionic coordinates  $\theta^-, \bar{\theta}^-$  of the opposite type.

To write formulas, one usually picks local complex coordinates  $\phi^i$  on  $X$ , and describes the theory via “chiral superfields”  $\Phi^i = \Phi^*(\phi^i)$ , which obey  $\bar{D}_+ \Phi^i = D_+ \bar{\Phi}^{\bar{i}} = 0$ , and so admit expansions

$$\begin{aligned}\Phi^i &= \phi^i + \sqrt{2}\theta^+ \psi_+^i - i\bar{\theta}^+ \theta^+ \partial_z \phi^i \\ \bar{\Phi}^{\bar{i}} &= \bar{\phi}^{\bar{i}} - \sqrt{2}\bar{\theta}^+ \bar{\psi}_+^{\bar{i}} + i\bar{\theta}^+ \theta^+ \partial_z \bar{\phi}^{\bar{i}}.\end{aligned}\quad (4.4)$$

(The factors of  $\sqrt{2}$  are conventional.)

By acting with  $Q_+$  and  $\bar{Q}_+$ , defined as in (4.2), we can determine how the fields transform under supersymmetry. In particular,  $\bar{Q}_+$  generates the transformation

$$\begin{aligned}\delta\phi^i &= 0 \\ \delta\bar{\phi}^{\bar{i}} &= -\sqrt{2}\bar{\psi}_+^{\bar{i}} \\ \delta\psi_+^i &= -i\sqrt{2}\partial_z \phi^i \\ \delta\bar{\psi}_+^{\bar{i}} &= 0.\end{aligned}\quad (4.5)$$

If we set  $\alpha^{\bar{i}} = -\sqrt{2}\bar{\psi}_+^{\bar{i}}$ ,  $\rho^i = -i\psi_+^i/\sqrt{2}$ , then these transformations coincide with the ones we started with in equation (2.1). So  $(0, 2)$  symmetry is a specialization of the structure studied in Sections 2 and 3, with  $Q$  corresponding to  $\bar{Q}_+$ . In the specialization to  $(0, 2)$  supersymmetry, there is also a second supersymmetry  $Q_+$  that is hermitian adjoint to  $\bar{Q}_+$ . Here,  $\bar{Q}_+$  and  $Q_+$  are somewhat analogous to  $\bar{\partial}$  and  $\partial^\dagger$  on an ordinary complex manifold. The symmetry generated by  $Q_+$  is in fact, in components,

$$\begin{aligned}\delta\phi^i &= \sqrt{2}\psi_+^i \\ \delta\bar{\phi}^{\bar{i}} &= 0 \\ \delta\psi_+^i &= 0 \\ \delta\bar{\psi}_+^{\bar{i}} &= i\sqrt{2}\partial_z \bar{\phi}^{\bar{i}}.\end{aligned}\quad (4.6)$$

A Lagrangian is constructed locally by introducing a  $(1, 0)$ -form  $K = K_i d\phi^i$ , with complex conjugate  $\bar{K} = \bar{K}_{\bar{i}} d\bar{\phi}^{\bar{i}}$ , and setting

$$I = \int |d^2 z| d\bar{\theta}^+ d\theta^+ \left( -\frac{i}{2} K_i(\Phi, \bar{\Phi}) \partial_z \Phi^i + \frac{i}{2} \bar{K}_{\bar{i}}(\Phi, \bar{\Phi}) \partial_z \bar{\Phi}^{\bar{i}} \right). \quad (4.7)$$

This is regarded as a local expression for the action — so in manipulating it we are free to integrate by parts and discard exact forms. A global description will be clear shortly. The reason that (4.7) is only a local expression for the action is that there are transformations of  $K$  that only change the action density by an exact form. Hence, describing the action in terms of  $K$  is analogous to describing a Kahler manifold in terms of a “Kahler potential,” which is a locally defined zero-form  $t$  in terms of which the Kahler form can locally be written as  $\omega = -i\partial\bar{\partial}t$ .

The most obvious transformations of  $K$  that leave the action fixed are

$$K \rightarrow K + \partial\Lambda, \quad \bar{K} \rightarrow \bar{K} - \bar{\partial}\Lambda, \quad (4.8)$$

for any imaginary zero-form  $\Lambda$ . Under this change in  $K$ , the action density changes by the total derivative  $\partial_z\Lambda$ , which integrates to zero. Less obvious (but explained presently) is that the action is also invariant under  $K \rightarrow K + K'$ ,  $\bar{K} \rightarrow \bar{K} + \bar{K}'$ , where  $K'$  is a holomorphic differential.

The basic object invariant under the transformations just described and hence globally defined is the hermitian metric  $ds^2 = g_{i\bar{j}}d\phi^i d\bar{\phi}^{\bar{j}}$ , where

$$g_{i\bar{j}} = \partial_{\bar{j}}K_i + \partial_i\bar{K}_{\bar{j}}. \quad (4.9)$$

Associated to this metric is the  $(1,1)$ -form

$$\omega = \frac{i}{2}(\bar{\partial}K - \partial\bar{K}). \quad (4.10)$$

In contrast to a Kahler manifold, whose Kahler form obeys  $\partial\omega = \bar{\partial}\omega = 0$ , (4.10) implies the weaker condition

$$\bar{\partial}\partial\omega = 0, \quad (4.11)$$

which therefore characterizes  $(0,2)$  supersymmetry. (This condition might be compared with the condition defining a Guaduchon metric, which in complex dimension  $n$  is  $\partial\bar{\partial}(\omega^{n-1}) = 0$ . The two conditions coincide for  $n = 2$ .)

By virtue of (4.11), the  $(2,1)$ -form  $\mathcal{H} = 2i\partial\omega$  obeys  $\partial\mathcal{H} = \bar{\partial}\mathcal{H} = 0$ . It hence can be interpreted as a class in  $H^1(X, \Omega^{2,cl})$ . As we will see, it plays the role of the class called  $\mathcal{H}$  in Section 2.2, by which the sheaf of CDO's of a general complex manifold can be deformed. Just like the hermitian metric of  $X$ ,  $\mathcal{H}$  is severely constrained by  $(0,2)$  supersymmetry. While in the general analysis of Section 2.2,  $\mathcal{H}$  can be an arbitrary

element of  $H^1(X, \Omega^{2,\text{cl}})$ , if we wish to specialize to  $(0, 2)$  supersymmetry,  $\mathcal{H}$  must be of type  $(2, 1)$  and expressible as  $2i\partial\omega$  in terms of a positive  $(1, 1)$ -form  $\omega$ .

To justify these statements, and further explain how  $(0, 2)$  supersymmetry relates to the more general structure explored in Sections 2 and 3, we want to express (4.7) in the form  $\int |d^2z| \{Q, V\}$ , introduced in Section 2.1, and also convert it to an ordinary Lagrangian, expressed just as an ordinary integral over  $z$  and  $\bar{z}$ . The most straightforward way to do this is to simply perform the integral over  $\theta$  and  $\bar{\theta}$ . A convenient shortcut is to note that for any  $X$ , we can make the replacement

$$\begin{aligned} \int |d^2z| d\bar{\theta}^+ d\theta^+ X &= \int |d^2z| \frac{\partial^2}{\partial\bar{\theta}^+ \partial\theta^+} X \Big|_{\bar{\theta}^+ = \theta^+ = 0} \\ &= \int |d^2z| \bar{D}_+ D_+ X \Big|_{\theta^+ = \bar{\theta}^+ = 0} \\ &= - \int |d^2z| D_+ \bar{D}_+ X \Big|_{\theta^+ = \bar{\theta}^+ = 0}. \end{aligned} \quad (4.12)$$

The basis for the first step is that, for a fermionic variable  $\theta$ ,  $\int d\theta X = (\partial X / \partial \theta)|_{\theta=0}$ . The basis for the second step is that the  $D$ 's differ from the  $\partial/\partial\theta$ 's by  $\partial_z$  terms, which vanish upon integration by parts. The basis for the third step is that  $\{D_+, \bar{D}_+\} = 2i\partial_{\bar{z}}$ , which again vanishes upon integration by parts. We can, for example, now see that  $\int |d^2z| d\theta^+ d\bar{\theta}^+ X$  is invariant to  $X \rightarrow X + K'_i(\Phi) \partial_z \Phi^i$ , or to  $X \rightarrow X - \bar{K}'_{\bar{i}} \partial_z \bar{\Phi}^{\bar{i}}$ , if  $K'$  is a holomorphic one-form. For example, we have  $\bar{D}_+(K'_i(\Phi) \partial_z \Phi^i) = 0$ , as  $\bar{D}_+ \Phi = 0$  and  $K'$  is holomorphic, and hence  $\int |d^2z| D_+ \bar{D}_+ (K'_i(\Phi) \partial_z \Phi^i) = 0$ . Similarly,  $D_+(\bar{K}'_{\bar{i}} \partial_z \bar{\Phi}^{\bar{i}}) = 0$ , by virtue of which  $\int |d^2z| \bar{D}_+ D_+ (\bar{K}'_{\bar{i}} \partial_z \bar{\Phi}^{\bar{i}}) = 0$ .

To evaluate the action, a slight variant of (4.12) is more useful. We write the action as

$$\int |d^2z| \{ \bar{Q}_+, [D_+, X] \} \Big|_{\theta = \bar{\theta} = 0}. \quad (4.13)$$

This is valid because, again,  $\bar{Q}_+$  differs from  $\bar{D}_+$  and  $\partial/\partial\bar{\theta}^+$  by a total derivative. In Section 2.1, we wrote the action as  $\int |d^2z| \{Q, V\}$ . Since we are identifying  $\bar{Q}_+$  with  $Q$ , we see that  $V = D_+ X = -iD_+(K_i(\Phi, \bar{\Phi}) \partial_z \bar{\Phi}^i - \bar{K}_{\bar{i}}(\Phi, \bar{\Phi}) \partial_z \bar{\Phi}^{\bar{i}})/2$ . To evaluate this, we note that, as  $D_+ \bar{\Phi} = 0$ , we have  $D_+(K_i \partial_z \Phi - \bar{K}_{\bar{i}} \partial_z \bar{\Phi}^{\bar{i}}) = K_{i,j} D_+ \Phi^j \partial_z \Phi^i + K_i \partial_z D_+ \Phi^i - \bar{K}_{\bar{i},i} D_+ \Phi^i \partial_z \bar{\Phi}^{\bar{i}}$ . After subtracting the total derivative  $\partial_z(K_i D_+ \bar{\Phi}^i)$ , which will not contribute to the action, we get  $2iV = -(K_{i,\bar{j}} + \bar{K}_{\bar{j},i}) \partial_z \bar{\Phi}^{\bar{j}} D_+ \Phi^i + (K_{i,j} -$

$K_{j,i})D_+\Phi^j\partial_z\Phi^i$ . To set  $\bar{\theta}^+ = \theta^+ = 0$ , we just set  $\Phi^i = \phi^i$ ,  $\bar{\Phi}^{\bar{i}} = \bar{\phi}^{\bar{i}}$ , and  $D_+\Phi^j = \sqrt{2}\psi_+^j = -2i\rho^j$ , and let  $\bar{Q}_+$  act as in (4.5). So  $V = -\left((\partial_{\bar{i}}K_i + \partial_i\bar{K}_{\bar{i}})\rho^i\partial_z\bar{\phi}^{\bar{i}} - (K_{i,j} - K_{j,i})\rho^j\partial_z\phi^i\right)$ . It is now straightforward to read off the hermitian metric  $g_{\bar{i}i}$  used in Section 2.1 to construct the basic Lagrangian, as well as the field called  $T$  in Section 2.2. We have  $g_{\bar{i}i} = \partial_i\bar{K}_{\bar{i}} + \partial_{\bar{i}}K_i$ , as claimed above, and  $T_{ij} = \partial_iK_j - \partial_jK_i$ . From the last statement, it follows that the curvature of the two-form field  $T$  is  $\mathcal{H} = dT = \bar{\partial}\partial K = 2i\partial\omega$ , as asserted above.

We should note, however, that in physics one defines the two-form gauge field  $B$  and the associated field curvature  $H = dB$  a little differently. One defines  $H = \text{Re}(\mathcal{H})$ , so that locally  $H = dB$  with  $B = -(1/2)(\bar{\partial}K + \partial\bar{K})$ . The imaginary part of  $\mathcal{H}$  can, in the case of  $(0, 2)$  supersymmetry, be written as  $\text{Im}(\mathcal{H}) = d\omega$ , where  $\omega$  is globally defined, so  $\text{Im}(\mathcal{H})$  is cohomologous to zero. Thus, the interesting global information can equally well be described by  $H$  or  $\mathcal{H}$ .

Finally, the complete action can be written explicitly

$$\begin{aligned} I = \int |d^2z| & \left( K_{i,\bar{j}}\partial_{\bar{z}}\bar{\phi}^{\bar{j}}\partial_z\phi^i + K_{\bar{i},j}\partial_{\bar{z}}\phi^j\partial_z\bar{\phi}^{\bar{i}} \right. \\ & - i \left( K_{i,\bar{j}}\bar{\psi}_+^{\bar{j}}\partial_z\psi^i + K_{\bar{i},j}\psi_+^j\partial_z\bar{\psi}_+^{\bar{i}} \right) \\ & \left. + i \left( K_{i,k\bar{j}}\psi_+^k\bar{\psi}_+^{\bar{j}}\partial_z\phi^i - K_{\bar{i},j\bar{l}}\psi_+^j\bar{\psi}_+^{\bar{l}}\partial_z\bar{\phi}^{\bar{i}} \right) \right). \end{aligned} \quad (4.14)$$

## 4.2 An example

A very simple example of an essentially non-Kähler complex manifold that is the target space of a  $(0, 2)$  model is  $X = \mathbf{S}^1 \times \mathbf{S}^3$ , originally considered in this context in [26, 27]. This example is quite elementary geometrically but of considerable interest in conformal field theory.

The complex structure of  $X$  can be constructed as follows. By composing the projection onto the second factor  $X \rightarrow \mathbf{S}^3$  with the Hopf fibration  $\pi : \mathbf{S}^3 \rightarrow \mathbf{S}^2 \cong \mathbb{C}\mathbb{P}^1$ , whose fibers are copies of  $\mathbf{S}^1$ ,  $X$  can be fibered over  $\mathbb{C}\mathbb{P}^1$  with fibers  $E = \mathbf{S}^1 \times \mathbf{S}^1$ . Giving  $E$  the structure of a complex Riemann surface of genus one,  $X$  becomes a complex manifold.

Alternatively,  $X$  can be constructed as  $\mathbb{C}^2/\mathbb{Z}$ , where  $\mathbb{Z}$  acts on coordinates  $z^i$ ,  $i = 1, 2$  of  $\mathbb{C}^2$  by  $z^i \rightarrow \lambda^n z^i$ , with  $\lambda$  a nonzero complex number of modulus

less than 1. The choice of  $\lambda$  determines the complex structure of  $E$  in the other description. The two descriptions are related by simply regarding the  $z^i$  as homogeneous coordinates of  $\mathbb{CP}^1$ .

A hermitian form  $\omega$  on  $X$  that obeys  $\partial\bar{\partial}\omega = 0$  (and corresponds to real  $\lambda$ ) can be obtained as follows. We will construct  $\omega$  to be invariant under  $U(1) \times U(2)$ , where  $U(1)$  acts by rotation of  $\mathbf{S}^1$ , leaving fixed a one-form  $dt$ , and  $U(2)$  acts on  $\mathbf{S}^3$  commuting with the Hopf fibration.  $U(2)$  induces on the base  $\mathbf{S}^2$  of the Hopf fibration a rotation symmetry group  $SO(3)$ . Let  $\omega_0$  be an  $SO(3)$ -invariant form on  $\mathbf{S}^2$  that integrates to 1. When pulled back to  $\mathbf{S}^3$ ,  $\omega_0$  is topologically trivial, and in fact  $\pi^*(\omega_0) = d\zeta$  for a unique  $U(2)$ -invariant one-form  $\zeta$  (which integrates to 1 on each fiber of the Hopf fibration and is an “angular form” for this fibration). Finally, we let

$$\omega = dt \wedge \zeta + \pi^*(\omega_0). \quad (4.15)$$

To prove that  $\partial\bar{\partial}\omega = 0$ , one approach is to note that this is  $d((\bar{\partial} - \partial)\omega/2)$ , and so is cohomologically trivial and integrates to zero on the four-manifold  $\mathbf{S}^1 \times \mathbf{S}^3$ . Since it is also  $U(1) \times U(2)$ -invariant, it can only integrate to zero if it vanishes pointwise. The form  $\omega$  is of type  $(1, 1)$  for the complex structure on  $\mathbf{S}^1 \times \mathbf{S}^3$ , and the associated hermitian metric is just a “round” metric on  $\mathbf{S}^1 \times \mathbf{S}^3$ , which in fact has the full  $SO(4)$  rotation symmetry of  $\mathbf{S}^3$ , and not just the  $U(2)$  symmetry of the complex structure. One way to describe the complex structure of  $\mathbf{S}^1 \times \mathbf{S}^3$  is to say that the forms of type  $(1, 0)$  on  $\mathbf{S}^1 \times \mathbf{S}^3$  are generated by  $dt + i\zeta$  and pullbacks of  $(1, 0)$ -forms on  $\mathbb{CP}^1$ .

If we simply ask for a  $U(1) \times SO(4)$ -invariant metric on  $\mathbf{S}^1 \times \mathbf{S}^3$ , we note at once that such metrics are determined by two positive numbers, the radii of the two factors. How do these two parameters enter in the present construction? The ratio of radii of  $\mathbf{S}^1$  and  $\mathbf{S}^3$  is determined by the choice of  $dt$  (we specified it to be  $U(1)$ -invariant, but did not fix the value of  $w = \int_{\mathbf{S}^1} dt$ ). The choice of  $dt$  is also correlated with the choice of complex structure, since  $\omega$  must be of type  $(1, 1)$ . Hence, when the complex structure of  $\mathbf{S}^1 \times \mathbf{S}^3$  is chosen, the ratio of radii is fixed. On the other hand, one can rescale the  $\mathbf{S}^1$  and  $\mathbf{S}^3$  radii by a common positive constant by multiplying the action (4.14) by this constant. So the complex structure determines the ratio of radii, and leaves one overall free parameter.

Now let us compute  $H = \text{Re } \mathcal{H}$ , the curvature of the  $B$ -field. This is most conveniently done using the fact that  $H = i(\partial - \bar{\partial})\omega$ . First of all,  $\partial\pi^*(\omega_0) = \bar{\partial}\pi^*(\omega_0) = 0$ , so actually  $H = -i(\partial - \bar{\partial})(\zeta \wedge dt)$ . To compute  $(\partial - \bar{\partial})(\zeta \wedge dt)$ , we note that  $\zeta \wedge dt$  is a  $(1, 1)$ -form, so that  $d(\zeta \wedge dt) = v + \bar{v}$ , where  $v$  is a  $(2, 1)$ -form and  $\bar{v}$  is a  $(1, 2)$ -form, and finally  $(\partial - \bar{\partial})(\zeta \wedge dt) = v - \bar{v}$ . But

explicitly,  $d(\zeta \wedge dt) = dt \wedge \pi^*(\omega_0)$ . Here  $\pi^*(\omega_0)$  is of type  $(1, 1)$ , while  $dt + i\zeta$  is of type  $(1, 0)$  and  $dt - i\zeta$  is of type  $(0, 1)$ . So  $v = (1/2)(dt + i\zeta)\pi^*(\omega_0)$ ,  $\bar{v} = (1/2)(dt - i\zeta)\pi^*(\omega_0)$ , and at last  $H = -i(v - \bar{v}) = \zeta \wedge \pi^*(\omega_0)$ .

It follows that  $\int_{\mathbf{S}^3} H = 1$  and in particular  $H$  is topologically nontrivial. Therefore, to obtain a quantum theory, we cannot use an arbitrary action of the form (4.14). We must multiply this action by a constant chosen so that  $\int_{\mathbf{S}^3} H = 2\pi k$  for some integer  $k$ , which must be positive so that the hermitian metric of  $\mathbf{S}^1 \times \mathbf{S}^3$  is positive.

In fact,  $H \sim \zeta \wedge \pi^*(\omega_0)$  is invariant not just under the  $U(2)$  symmetry of the Hopf fibration but under the full  $SO(4)$  rotation symmetry of  $\mathbf{S}^3$ . The hermitian metric obtained in this construction is likewise  $SO(4)$ -invariant, as we noted above. So the full sigma model has this symmetry (and as a result [27] has  $(0, 4)$  supersymmetry, not just the  $(0, 2)$  supersymmetry that was built into our construction of it).

In fact, as explained in [26, 27], the  $U(1) \times SO(4)$ -invariant supersymmetric sigma model of  $\mathbf{S}^1 \times \mathbf{S}^3$  is simply a product of a WZW model of the group  $SU(2) \cong \mathbf{S}^3$  with a free field theory. (The latter is the product of a free model of  $\mathbf{S}^1$  times a free fermion system, which arises because the fermions  $\psi$  and  $\bar{\psi}$  in this particular example become free when expressed in a left-invariant frame on  $\mathbf{S}^3$ ). The level of the WZW model is  $k$ . In particular, the parameter  $k$ , which from the point of view of the perturbative theory of CDO's is a complex parameter associated with  $H^1(\mathbf{S}^1 \times \mathbf{S}^3, \Omega^{2,cl}) \cong \mathbb{C}$ ,<sup>15</sup> must actually be an integer in order for the model to be well-defined nonperturbatively.

We will return to this example in Section 5.4.

## 5 Examples of sheaves of CDO's

In this section, we analyze some examples of sheaves of CDO's, aiming mainly to illustrate the slightly abstract discussion of Section 3.

### 5.1 CDO's of $\mathbb{CP}^1$

For our first example, following Section 5.6 of [10], we take  $X = \mathbb{CP}^1$ . We work locally on the worldsheet  $\Sigma$ , choosing a local complex parameter  $z$  and using it, as explained below, for normal-ordering.

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<sup>15</sup>By using the fibration of  $\mathbf{S}^1 \times \mathbf{S}^3$  over  $\mathbb{CP}^1$ , one can prove that  $H^1(\mathbf{S}^1 \times \mathbf{S}^3, \Omega^{2,cl})$  is one-dimensional with  $\mathcal{H} = -2iv \wedge \pi^*(\omega_0)$  as a generator.



Of course,  $\mathbb{CP}^1$  can be regarded as the complex  $\gamma$ -plane plus a point at infinity. We can usefully cover it by two open sets,  $U_1$  and  $U_2$ , where  $U_1$  is the complex  $\gamma$ -plane, and  $U_2$  is the complex  $\tilde{\gamma}$ -plane, where  $\tilde{\gamma} = 1/\gamma$ .

In  $U_1$ , since it is isomorphic to  $\mathbb{C}$ , the sheaf of chiral operators or CDO's can be described by a single free  $\beta\gamma$  system:

$$I = \frac{1}{2\pi} \int |d^2z| \beta \bar{\partial} \gamma. \quad (5.1)$$

Here  $\beta$  and  $\gamma$  are fields of dimension  $(1, 0)$  and  $(0, 0)$  with the familiar free-field OPE's; there are no singularities in the operator products  $\beta(z) \cdot \beta(z')$  or  $\gamma(z) \cdot \gamma(z')$ , while

$$\gamma(z)\beta(z') \sim \frac{1}{z - z'}. \quad (5.2)$$

Similarly, the theory in  $U_2$  is described by a free  $\tilde{\beta}\tilde{\gamma}$  system, with the same dimensions and action

$$I = \frac{1}{2\pi} \int |d^2z| \tilde{\beta} \bar{\partial} \tilde{\gamma}, \quad (5.3)$$

and the same OPE's.

To completely describe the sheaf of chiral operator, we must explain the gluing between  $\beta, \gamma$  and  $\tilde{\beta}, \tilde{\gamma}$  in the overlap region  $U_1 \cap U_2$ . There is nothing we can do to modify the classical relation

$$\tilde{\gamma} = \frac{1}{\gamma} \quad (5.4)$$

in any essential way. The reason is that, since  $\tilde{\gamma}$  is supposed to have dimension zero, the right hand side of (5.4) must be a function of  $\gamma$  only (and cannot depend on  $\beta$  or the derivatives of  $\gamma$  or  $\beta$ ). Hence, (5.4) is a classical gluing relation that (given that it is consistent) makes an ordinary complex manifold obtained by gluing together the  $U_i$ . As  $\mathbb{CP}^1$  has no complex moduli, any modification of (5.4) would give back an equivalent result.<sup>16</sup>

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<sup>16</sup>Here, we can recall an observation from Section 3.2. Even if we consider instead of  $\mathbb{CP}^1$  a complex manifold  $X$  that *does* have complex moduli, nothing is gained by assuming quantum corrections to the classical gluing laws for the dimension zero fields. Those gluing laws build up a complex manifold  $X'$  that we can call the target of the quantum theory, and we may as well parameterize the space of quantum theories in such a way that  $X'$  is isomorphic to the underlying classical manifold  $X$ .

Since  $\beta$  has dimension 1, one might expect the appropriate gluing law for  $\beta$  to be  $\tilde{\beta} = \beta'$  where  $\beta' = (\partial\gamma/\partial\tilde{\gamma})\beta = -\gamma^2\beta$ . This formula is a little ambiguous, because of the existence of a short distance singularity in the  $\gamma - \beta$  operator product. We resolve such ambiguities, for any differential polynomial in  $\beta$  and  $\gamma$ , by normal ordering. So in this case,  $\gamma^2\beta$  is a shorthand for

$$\gamma^2\beta(z) = \lim_{z' \rightarrow z} \left( \gamma^2(z')\beta(z) - \frac{2}{z' - z}\gamma(z') \right), \quad (5.5)$$

which to be more precise is also sometimes denoted as  $:\gamma^2\beta:(z)$ . This normal ordering definition of a local operator gives results that depend on the choice of local parameter  $z$  (though the space of all operators does not depend on this choice). For this reason, as we discussed at several junctures in Sections 2 and 3, once we construct a chiral sheaf, its invariance under reparameterizations of  $\Sigma$  is not guaranteed.

However, now that we have specified exactly what it means, the gluing formula  $\tilde{\beta} = \beta'$  is not right, as we find if we compute the OPE's of  $\beta'$ :

$$\beta'(z)\beta'(z') \sim -\frac{4}{(z - z')^2}\gamma^2(z') - \frac{4}{z - z'}\gamma\partial_{z'}\gamma(z'). \quad (5.6)$$

As explained in [10], the appropriate formula is

$$\tilde{\beta}(z) = -\beta\gamma^2(z) + 2\partial_z\gamma(z). \quad (5.7)$$

$\tilde{\beta}$  has no short distance singularity with itself and has the proper short distance singularity with  $\tilde{\gamma}$ .

So this gives the appropriate description of a sheaf of CDO's that is globally defined on the target space  $\mathbb{CP}^1$  (but only locally on the worldsheet  $\Sigma$  of the conformal field theory, since we defined it using a local complex parameter  $z$ ). In this example, one might think of  $\tilde{\beta}$  as the ‘‘correctly’’ normal-ordered version of  $-\beta\gamma^2$ . In more complicated examples, anomalies (obstructing existence of the theory) or free parameters (moduli) arise in trying to find the right gluing.

### 5.1.1 Global sections of the sheaf

Having understood the sheaf  $\widehat{\mathcal{A}}$  of chiral operators, let us consider the global chiral algebra  $\mathcal{A}$  of such operators. We recall that operators in  $\mathcal{A}$  correspond to elements of  $H^i(\mathbb{CP}^1, \widehat{\mathcal{A}})$ . As  $\mathbb{CP}^1$  has complex dimension 1, we have here  $i = 0, 1$ . We write  $\mathcal{A}^i$  for  $H^i(\mathbb{CP}^1, \widehat{\mathcal{A}})$ .

First we consider  $\mathcal{A}^0$ , that is, the global sections of  $\widehat{\mathcal{A}}$ . At dimension 0, we must consider functions of  $\gamma$  only. But a holomorphic function on  $\mathbb{CP}^1$ , to have no poles anywhere, must be a constant, so the space of dimension 0 global sections is one-dimensional, generated by 1.

Typically, given a global section  $\mathcal{O}$ , we can make another one of dimension one higher by differentiating,  $\mathcal{O} \rightarrow \partial_z \mathcal{O}$ . For the case of the identity operator, this fails, as  $\partial_z 1 = 0$ .

On the other hand, there are global sections of  $\widehat{\mathcal{A}}$  of dimension 1. There are no such sections of the form  $f(z)\partial_z \gamma$ . Indeed, such an operator could be transformed purely geometrically under (5.4), by virtue of which it would correspond to a holomorphic differential  $f(\gamma)d\gamma$  on  $\mathbb{CP}^1$ . But there are no such objects.

The remaining possibility is to find an operator that is linear in  $\beta$ . In fact, we right away see an example in (5.7), as the left hand side,  $\widetilde{\beta}$ , is by definition regular in  $U_2$ , while the right hand side, being polynomial in  $\beta$ ,  $\gamma$ , and their derivatives, is manifestly regular in  $U_1$ . Their being equal means that they represent a global section of  $\widehat{\mathcal{A}}$  that we will call  $J_+$ :

$$J_+ = -\gamma^2 \beta + 2\partial\gamma = \widetilde{\beta}. \quad (5.8)$$

The construction is completely symmetric between  $U_1$  and  $U_2$ , with  $\gamma \leftrightarrow \widetilde{\gamma}$  and  $\beta \leftrightarrow \widetilde{\beta}$ , so a reciprocal formula gives another dimension one global section  $J_-$ :

$$J_-(z) = \beta(z) = -\widetilde{\gamma}^2 \widetilde{\beta} + 2\partial\widetilde{\gamma}. \quad (5.9)$$

The skeptical reader can properly define  $\gamma^2 \beta$  and similarly  $\widetilde{\gamma}^2 \widetilde{\beta}$  using (5.5) and then verify that the gluing laws defining  $\widetilde{\beta}$  and  $\widetilde{\gamma}$  in terms of  $\beta$  and  $\gamma$  can be inverted to solve for  $\beta$  and  $\gamma$  in terms of  $\widetilde{\beta}$  and  $\widetilde{\gamma}$ .

So  $J_+$  and  $J_-$  give us two-dimension one sections of the sheaf  $\widehat{\mathcal{A}}$ . Since these are global sections of a sheaf of chiral vertex operators, we can construct more from their OPE's. There are no singularities in the  $J_+ \cdot J_+$  or  $J_- \cdot J_-$  operator products, but

$$J_+(z)J_-(z') \sim \frac{2J_3}{z-z'} - \frac{2}{(z-z')^2}, \quad (5.10)$$

where  $J_3$  is another global section of dimension 1,

$$J_3(z) = -\gamma\beta(z) \quad (5.11)$$

(which we again define by normal-ordering). For operator products involving  $J_3$ , we get

$$\begin{aligned} J_3(z)J_3(z') &\sim -\frac{1}{(z-z')^2} \\ J_3(z)J_+(z') &\sim \frac{J_+(z')}{z-z'} \\ J_3(z)J_-(z') &\sim -\frac{J_-(z')}{z-z'}. \end{aligned} \tag{5.12}$$

Taken together, the  $J$ 's generate a familiar chiral algebra — the current algebra of  $\mathrm{SL}(2)$  at level  $-2$ , which here, as noted in [10], appears in the Wakimoto free field representation [28]. The space  $\mathcal{A}^0$  of global sections of  $\widehat{\mathcal{A}}$  is thus a module for this chiral algebra.

It is shown in [10] to be an irreducible module, but one that has unusual properties. In general, for any  $\mathrm{SL}(2)$  current algebra at any level  $k \neq -2$ , one can define a stress tensor

$$T(z) = \frac{:J_+J_- + J_3^2:}{k+2}. \tag{5.13}$$

For every  $k \neq 2$ ,  $T$  generates a Virasoro algebra. If we want an operator that makes sense at  $k+2=0$ , we can remove the factor of  $1/(k+2)$  and define

$$S(z) =: J_+J_- + J_3^2 :. \tag{5.14}$$

Because  $S(z) = (k+2)T(z)$ , it generates in its OPE's with any operator  $k+2$  times the transformation usually generated by the stress tensor. If  $k+2=0$ ,  $S(z)$  generates no transformation at all — it has no singularities in its OPE with any operator. Hence, in an irreducible representation of current algebra,  $S(z)$  can be represented by a  $c$ -number, and might vanish. This fact is important in conformal field theory approaches to the geometric Langlands program (for reviews, see [15]). In that context, it is important to consider a generalization in which  $S(z)$  is set to an arbitrary projective connection (or locally, a quadratic differential) on  $\Sigma$ . That is what we get here if we carry out the same construction using some other complex parameter on  $\Sigma$  instead of  $z$ . In fact, in this particular Wakimoto module, it is true that  $S(z) = 0$ . The semiclassical approximation to this statement can be verified immediately; with  $J_- = \beta$ ,  $J_+ = -\gamma^2\beta + \dots$ , and  $J_3 = -\gamma\beta$ , it is clear that, ignoring quantum contractions, the  $\beta^2$  terms in  $J_+J_- + J_3^2$  cancel.

As explained in [10], the space  $\mathcal{A}^0$  of global sections of  $\widehat{\mathcal{A}}$  is an irreducible module of  $\mathrm{SL}(2)$  current algebra at level  $-2$  that can be obtained from a free Verma module by setting to zero  $S$  and all of its derivatives.  $\mathcal{A}^0$  has the structure, roughly speaking, of a chiral algebra; it obeys all the usual physical axioms of a chiral algebra, except the existence of a stress tensor, and hence, reparameterization invariance on the  $z$ -plane.

At one level, we have already explained why there is no stress tensor: the usual definition (5.13) does not make sense at  $k = -2$ . But at another level, this may still appear perplexing. The free  $\beta\gamma$  system certainly has a stress tensor

$$T_f(z) = - : \beta \partial \gamma : (z) = - \lim_{z' \rightarrow z} \left( \beta(z') \partial \gamma(z) + \frac{1}{(z' - z)^2} \right). \quad (5.15)$$

Likewise there is a free stress tensor of the  $\widetilde{\beta}\widetilde{\gamma}$  system,

$$\widetilde{T}_f(z) = - : \widetilde{\beta} \partial \widetilde{\gamma} : (z) = - \lim_{z' \rightarrow z} \left( \widetilde{\beta}(z') \partial \widetilde{\gamma}(z) + \frac{1}{(z' - z)^2} \right). \quad (5.16)$$

Clearly  $T_f$  is regular in  $U_1$  and  $\widetilde{T}_f$  is regular in  $U_2$ . The problem is that in  $U_1 \cap U_2$ ,  $T_f \neq \widetilde{T}_f$ , and there is no way to fix this (by adding to  $T_f$  a term regular in  $U_1$  and to  $\widetilde{T}_f$  a term regular in  $U_2$ ).

If one inserts the definition of  $\widetilde{\beta}$  and  $\widetilde{\gamma}$  in the definition of  $\widetilde{T}_f$ , a small computation shows that

$$\widetilde{T}_f - T_f = \partial \left( \frac{\partial \gamma}{\gamma} \right). \quad (5.17)$$

There is no way to fix this inconsistency without spoiling the fact that  $T_f$  and  $\widetilde{T}_f$  have the OPE's of stress tensors. In fact, to preserve the OPE's  $T_f \cdot \gamma$  and  $\widetilde{T}_f \cdot \gamma$ , we must add to  $T_f$  or  $\widetilde{T}_f$  terms that depend only on  $\gamma$  or  $\widetilde{\gamma}$ . The right hand side of (5.17) is invariant under  $\gamma \rightarrow \lambda \gamma$ ,  $\lambda \in \mathbb{C}^*$ , and if it is possible to modify  $T_f$  and  $\widetilde{T}_f$  so as to agree on  $U_1 \cap U_2$ , it can be done while preserving this invariance. The only  $\mathbb{C}^*$ -invariant operators of dimension two depending only on  $\gamma$  are  $\partial^2 \gamma / \gamma$  and  $(\partial \gamma)^2 / \gamma^2$ . Any linear combination of these has a pole at both  $\gamma = 0$  and  $\widetilde{\gamma} = 0$ , so these operators are of no help in removing the anomaly.

So finally, although the free  $\beta\gamma$  and  $\widetilde{\beta}\widetilde{\gamma}$  systems have stress tensors, there is no stress tensor for the global chiral algebra  $\mathcal{A}^0$  of  $\mathbb{CP}^1$ . This is a reflection of the fact that the  $(0, 2)$  model with target space  $\mathbb{CP}^1$  is not conformally invariant — there is a nonzero one-loop beta function. However, the ability to see the failure of conformal invariance purely in terms of holomorphic

data [10] is novel from the point of view of physicists, which is why it has been expounded here in some detail. Moreover, the fact that conformal invariance is not restored even for the chiral algebra, despite its holomorphy in  $z$ , is somewhat surprising, at least for physicists. Chiral algebras without stress tensors, such as  $SL(2)$  current algebra at level  $-2$ , are important in the geometric Langlands program [15].

By contrast,  $(0, 2)$  models that are expected to flow to conformal field theories in the infrared do typically have a stress tensor in the chiral algebra, which hence is conformally invariant. This has been seen in several examples of linear sigma models [7, 29], and for nonlinear sigma models has been proved, in the present context, when  $c_1(X) = 0$  [12].

Although the chiral algebra is not invariant under arbitrary reparameterizations of  $z$ , it is invariant under affine transformations  $z \rightarrow az + b$ , as these leave fixed the normal ordering recipe. In particular, from the scaling symmetry  $z \rightarrow az$ , it follows that  $\mathcal{A}^0$  and  $\mathcal{A}^1$  are, in perturbation theory, naturally graded by dimension, even though there is no conformal invariance. Nonperturbatively, via instantons, this grading by dimension is violated, unless  $c_1(X) = 0$ .

### 5.1.2 The first cohomology

Now we move on to investigate  $\mathcal{A}^1 = H^1(\mathbb{CP}^1, \widehat{\mathcal{A}})$ .

There is no nonzero element of  $\mathcal{A}^1$  of dimension zero. Such an element would be represented as a function  $f(\gamma)$ , with possible poles at  $\gamma = 0$  and  $\gamma = \infty$ , that cannot be written as  $f_1 - f_2$ , where  $f_1$  is holomorphic in  $U_1$  and  $f_2$  in  $U_2$ . Such an  $f$  does not exist, as  $H^1(\mathbb{CP}^1, \mathcal{O}) = 0$ .

In dimension 1,  $\mathcal{A}^1$  is one-dimensional, generated by an object  $\theta = \partial\gamma/\gamma = -\partial\tilde{\gamma}/\tilde{\gamma}$  that we have essentially already encountered.  $\theta$  has a pole at  $\gamma = 0$  and one at  $\tilde{\gamma} = 0$ , and there is no way to “split” it as a difference of operators with only one pole. This statement corresponds in ordinary geometry to the fact that the differential  $d\gamma/\gamma$ , which is holomorphic in  $U_1 \cap U_2$ , generates  $H^1(\mathbb{CP}^1, K)$ , with  $K$  the sheaf of holomorphic differentials. (If  $d\gamma/\gamma$  could be “split,” the contour integral  $\oint_C d\gamma/\gamma$ , for  $C$  a circle surrounding the pole at  $\gamma = 0$ , would vanish).

What happens in dimension 2? At first it seems that we can construct the dimension two operator  $\partial\theta$  in  $\mathcal{A}^1$ . Certainly, classically, differentiating  $\theta$

with respect to  $z$  does not help us in “splitting” it between  $U_1$  and  $U_2$ . But quantum mechanically, it is a different story. In fact, this is what we get by reading (5.17) backwards:

$$\partial\theta = \tilde{T}_f - T_f. \quad (5.18)$$

Here  $\tilde{T}_f$  is holomorphic in  $U_2$ , and  $T_f$  in  $U_1$ , so  $\partial\theta$  vanishes as an element of  $\mathcal{A}^1$ .

We can construct other elements of  $\mathcal{A}^1$  by acting on  $\theta$  with the elements of  $\mathcal{A}^0$ . Some relations are obvious:  $\partial\theta = 0$ , as we have just seen, and  $S$  and all its derivatives annihilate  $\theta$ , since  $S$  actually vanishes in  $\mathcal{A}^0$ . It is shown in [10] that these are the only relations. As a result,  $\mathcal{A}^1$  is isomorphic to  $\mathcal{A}^0$ , but shifted in dimension by 1. This isomorphism maps the dimension zero generator of  $\mathcal{A}^0$ , namely 1, to the dimension one generator of  $\mathcal{A}^1$ , namely  $\theta$ . Of course, such an isomorphism, shifting the dimension of the operators, would not be compatible with conformal invariance.

The isomorphism between  $\mathcal{A}^1$  and  $\mathcal{A}^0$  is a classical starting point for an important quantum phenomenon, which will be discussed elsewhere.  $\mathcal{A}^1$  and  $\mathcal{A}^0$  have been constructed to be annihilated in perturbation theory by the differential  $Q$  that we studied in Sections 2 and 3. But nonperturbatively,  $Q$  is corrected by instantons and the corrected  $Q$  is simply the isomorphism from  $\mathcal{A}^1$  to  $\mathcal{A}^0$ . As a result, in the exact quantum theory, the cohomology of  $Q$  in the space of local operators is identically zero. In particular, there is an instanton-induced relation  $\{Q, \theta\} \sim 1$ ; the fact that the identity operator is of the form  $\{Q, \dots\}$  implies that it acts trivially on the  $\mathcal{A}$ -module  $\mathcal{V}$  given by the  $Q$ -cohomology of quantum states (this module was briefly discussed at the end of Section 2.1). So there are no supersymmetric states and supersymmetry is spontaneously broken.

Finally, relation (5.18) is an analog in Čech cohomology of a formula that in conventional physical notation is familiar to physicists and which in fact was briefly mentioned in (3.2):

$$\partial_z(R_{i\bar{j}}\partial_z\phi^i\alpha^{\bar{j}}) = \{Q, T_{zz}\}. \quad (5.19)$$

Here,  $R_{i\bar{j}}\partial_z\phi^i\alpha^{\bar{j}}$  is the counterpart of  $\theta$  in conventional physical notation, and the relation (5.19) can be read, just like (5.18), in two ways. It implies that the stress tensor  $T$  is not in the  $Q$ -cohomology, and that while  $\theta$  does represent an element of this cohomology, its derivative  $\partial\theta$  vanishes in the  $Q$ -cohomology.

## 5.2 The $p_1$ anomaly

Having illustrated in Section 5.1 how the beta function can be understood in the structure of the sheaf of chiral observables, we now proceed to do the same for chiral anomalies. We consider here the  $p_1$  anomaly, and in Section 5.3 we consider the  $c_1(X)c_1(\Sigma)$  anomaly. We will recover results obtained in the mathematical literature in [12] (which uses the Čech approach that we follow here) and also in [13] from a more abstract viewpoint.

As a simple example of a complex manifold with  $p_1 \neq 0$ , we take  $X = \mathbb{CP}^2$ , which we endow with homogeneous coordinates  $\lambda_0, \lambda_1, \lambda_2$ . We cover  $\mathbb{CP}^2$  with open sets  $U_i$ ,  $i = 0, \dots, 2$ , in which  $\lambda_i \neq 0$ . We consider the label  $i$  to take values in  $\mathbb{Z}/3\mathbb{Z}$ , so  $i = 3$  is equivalent to  $i = 0$ .

In each  $U_i$ , the sheaf of chiral operators can be described by a free field theory, now with two  $\beta\gamma$  pairs since  $\mathbb{CP}^2$  is of complex dimension 2. We denote the spin zero fields in  $U_i$  as  $v^{[i]}$  and  $w^{[i]}$ , and the spin one fields as  $V^{[i]}$  and  $W^{[i]}$ . We can take the spin zero fields to correspond to holomorphic functions on  $U_i$ , as follows:

$$\begin{aligned} v^{[i]} &\leftrightarrow \frac{\lambda_{i+1}}{\lambda_i} \\ w^{[i]} &\leftrightarrow \frac{\lambda_{i+2}}{\lambda_i}. \end{aligned} \tag{5.20}$$

The action in  $U_i$  is

$$I^{[i]} = \frac{1}{2\pi} \int |d^2z| (V^{[i]}\bar{\partial}v^{[i]} + W^{[i]}\bar{\partial}w^{[i]}). \tag{5.21}$$

The nontrivial OPE's of the fields appearing in this action are

$$\begin{aligned} V^{[i]}(z)v^{[i]}(z') &\sim -\frac{1}{z-z'} \\ W^{[i]}(z)w^{[i]}(z') &\sim -\frac{1}{z-z'}. \end{aligned} \tag{5.22}$$

Other OPE's are nonsingular. To avoid cluttering the equations too much, we also adopt a convention of writing simply  $v, w, V$ , and  $W$  as shorthand for  $v^{[0]}, w^{[0]}, V^{[0]}$ , and  $W^{[0]}$ .

To construct a global sheaf of chiral observables, we must find gluing maps  $R_i$  from operators in  $U_i$  to operators in  $U_{i+1}$ . There is no problem



in constructing any one  $R_i$ . Moreover, as we have introduced the variables in a cyclically symmetric way, the various  $R_i$  can all be represented by essentially the same formulas. An anomaly appears because the gluings are not compatible. This will show up in the fact that  $R_2 R_1 R_0 \neq 1$ .

Of course, since the complex manifold  $\mathbb{CP}^2$  does exist, there is no inconsistency in the gluing at a geometrical level. The difference  $R_2 R_1 R_0 - 1$  is a nongeometrical symmetry of the free field theory of  $v, w, V$ , and  $W$ , as described in Section 3.4.

Such a symmetry is determined, we recall, by a closed holomorphic two-form. Consider a general system of  $n$  conjugate  $\beta\gamma$  systems, with nontrivial OPE's  $\beta_i(z)\gamma^j(z') \sim -\delta_i^j/(z-z')$ . Let  $F = \frac{1}{2}f_{ij}(\gamma)d\gamma^i \wedge d\gamma^j$  be a closed holomorphic two-form. Under the symmetry associated with  $F$ , the fields transform as

$$\begin{aligned}\gamma^j &\rightarrow \gamma^j \\ \beta_i &\rightarrow \beta'_i = \beta_i + f_{ij}\partial\gamma^j.\end{aligned}\tag{5.23}$$

In the spirit of Section 3.4, one can justify this statement by constructing locally a holomorphic one-form  $A = A_i d\gamma^i$  with  $dA = F$ , and computing how the fields transform under the action of the conserved charge  $\oint A_i \partial\gamma^i$ . Alternatively, one can simply check directly that the transformation (5.23) preserves the OPE's if  $dF = 0$ . In general, the operator product of  $\beta'$  with itself gives

$$\beta'_i(z')\beta'_j(z) \sim -\frac{\partial\gamma^l}{z'-z}(\partial_i f_{kl} + \partial_k f_{li} + \partial_l f_{ik}).\tag{5.24}$$

So the free field OPE's are preserved by the transformation (5.23) if and only if  $dF = 0$ .

The anomaly will appear because  $R_2 R_1 R_0 - 1$  will be a symmetry of this nongeometrical kind, for some closed two-form  $F$  that is holomorphic in  $U_0 \cap U_1 \cap U_2$ , where the  $R$ 's are all defined. Moreover,  $F$  cannot be "split" as a sum of closed two-forms  $F_i$  that are holomorphic in  $U_i \cap U_{i+1}$ . If there were such a splitting, we would use it to correct the individual  $R_i$  so as to restore  $R_2 R_1 R_0 = 1$ .

The anomaly thus represents, as reviewed in Section 3.5, an element of  $H^2(\mathbb{CP}^2, \Omega^{2,cl})$ . This group is one-dimensional, and one can take a generator to be

$$F = \frac{dv \wedge dw}{vw},\tag{5.25}$$

which has poles when  $\lambda_0, \lambda_1$ , or  $\lambda_2$  vanishes. The anomaly therefore will appear in the fact that  $R_2 R_1 R_0$ , while leaving  $v$  and  $w$  fixed, will transform  $V$  and  $W$  by

$$\begin{aligned} V &\rightarrow V + k \frac{dw}{vw} \\ W &\rightarrow W - k \frac{dv}{vw} \end{aligned} \quad (5.26)$$

for some constant  $k$ .

### 5.2.1 The computation

Now that we know exactly what we are looking for, let us find it.

First, we find the transformation  $R_0$  from  $U_0$  to  $U_1$ . This turns out to be

$$\begin{aligned} v^{[1]} &= \frac{w}{v} \\ w^{[1]} &= \frac{1}{v} \\ V^{[1]} &= vW \\ W^{[1]} &= -v^2V - vwW + \frac{5}{2}\partial v. \end{aligned} \quad (5.27)$$

The formulas for  $v^{[1]}$  and  $w^{[1]}$  are just the classical changes of variable from  $U_0$  to  $U_1$ , and likewise the terms in  $V^{[1]}$  and  $W^{[1]}$  that are linear in  $V$  and  $W$  can be found from classical geometry. The last term in  $W^{[1]}$  was found as in Section 5.1 to ensure that the OPE's come out correctly.

As always, formulas such as those for  $V^{[1]}$  and  $W^{[1]}$  are only meaningful if a precise recipe is given for defining the operator products. There are no ambiguities for operators constructed only from  $v$  and  $w$  and their derivatives, such as  $w/v$  or  $\partial w/v^2$ . We will interpret an operator  $f(v, w)V$  to mean

$$f(v, w)V(z) = \lim_{z' \rightarrow z} \left( f(v, w)(z') \cdot V(z) - \frac{1}{z' - z} \partial_v f(v, w)(z) \right), \quad (5.28)$$

and similarly for an operator  $g(v, w)W$ . For example, in (5.27), we have

$$\begin{aligned} vW(z) &= \lim_{z' \rightarrow z} v(z')W(z) \\ v^2V(z) &= \lim_{z' \rightarrow z} \left( v^2(z')V(z) - \frac{2v(z')}{z' - z} \right). \end{aligned} \quad (5.29)$$

With this definition, the formulas in (5.27) have the correct OPE's. We use the same recipes for operator products for all operators that appear presently, for example

$$f(v^{[i]}, w^{[i]})V^{[i]}(z) = \lim_{z' \rightarrow z} \left( f(v^{[i]}, w^{[i]})(z') \cdot V^{[i]}(z) - \frac{1}{z' - z} \partial_{v^{[i]}} f(v^{[i]}, w^{[i]})(z') \right), \quad (5.30)$$

for all  $i$ .

Now, to obtain the transformation  $R_1$  from  $U_1$  to  $U_2$ , we simply repeat this process:

$$\begin{aligned} v^{[2]} &= \frac{w^{[1]}}{v^{[1]}} \\ w^{[2]} &= \frac{1}{v^{[1]}} \\ V^{[2]} &= v^{[1]}W^{[1]} \\ W^{[2]} &= -(v^{[1]})^2V^{[1]} - v^{[1]}w^{[1]}W^{[1]} + \frac{5}{2}\partial v^{[1]}. \end{aligned} \quad (5.31)$$

Next, we substitute (5.27) into (5.31), so as to express  $v^{[2]}, w^{[2]}, \dots$  in terms of the original variables  $v, w, \dots$ , and thereby get an explicit formula for the composition  $R_1R_0$ . Here we have to be quite careful in the use of (5.30). We first express the operator products on the right hand side of (5.31) in a well-defined form, using (5.30), to get a well defined formula for  $v^{[2]}, w^{[2]}, \dots$ , in terms of  $v^{[1]}, w^{[1]}, \dots$ , and then we substitute the expressions in (5.27) to re-express those formulas in terms of  $v, w, V, W$ . Upon doing this, we obtain the following formulas:

$$\begin{aligned} v^{[2]} &= \frac{1}{w} \\ w^{[2]} &= \frac{v}{w} \\ V^{[2]} &= -vwV - w^2V + \frac{3w\partial v}{2v} + \partial w \\ W^{[2]} &= wV + \frac{3\partial w}{2v}. \end{aligned} \quad (5.32)$$

One can check these formulas by verifying that the OPE's are correct.

The transformation  $R_2$  from  $U_2$  back to  $U_3 = U_0$  is, of course, defined by the same formulas:

$$\begin{aligned} v^{[3]} &= \frac{w^{[2]}}{v^{[2]}} \\ w^{[3]} &= \frac{1}{v^{[2]}} \\ V^{[3]} &= v^{[2]}W^{[2]} \\ W^{[3]} &= -(v^{[2]})^2V^{[2]} - v^{[2]}w^{[2]}W^{[2]} + \frac{5}{2}\partial v^{[2]}. \end{aligned} \quad (5.33)$$

Combining this with (5.32), and again exercising care with the definition of the operator products, we finally get an explicit formula for the action of the composition  $R_2R_1R_0$ . This transformation acts on the field variables by

$$\begin{aligned} v &\rightarrow v \\ w &\rightarrow w \\ V &\rightarrow V + \frac{3}{2}\frac{\partial w}{vw} \\ W &\rightarrow W - \frac{3}{2}\frac{\partial v}{vw}, \end{aligned} \quad (5.34)$$

exhibiting the promised anomaly.

### 5.3 The $c_1(\Sigma)c_1(X)$ anomaly

In a similar spirit, we can illustrate the  $c_1(\Sigma)c_1(X)$  anomaly. For this, we return to the example of Section 5.1,  $X = \mathbb{CP}^1$ . But now, instead of working only locally on  $\Sigma$ , as in Section 5.1, we take  $\Sigma = \mathbb{CP}^1$  and work globally on  $\Sigma$ .

In Section 5.1, we covered  $X$  by two open sets  $U_1$  and  $U_2$ , respectively, the complex  $\gamma$ -plane and  $\tilde{\gamma}$ -plane, with  $\tilde{\gamma} = 1/\gamma$ . Similarly, we can regard  $\Sigma = \mathbb{CP}^1$  as the complex  $z$ -plane glued to the complex  $y$ -plane by the gluing map  $y = 1/z$ . We denote the  $z$ -plane as  $P_1$  and the  $y$ -plane as  $P_2$ . A free  $\beta\gamma$  system defines a sheaf of chiral observables on  $P_1$  or  $P_2$  with target  $U_1$  or  $U_2$ . As long as the target space is just  $U_1$  or  $U_2$ , there is no problem in gluing together the theories defined on  $P_1$  and  $P_2$ , since the free  $\beta\gamma$  system makes sense on any Riemann surface. Similarly, as long as the Riemann surface on which we define our theory is just  $P_1$  or  $P_2$ , we learned in Section 5.1 how to glue together the theories in which the target is  $U_1$  or  $U_2$  to make a theory with target  $\mathbb{CP}^1$ . However, as discussed in general terms in Section 3.5, an anomaly arises if we try to glue in both the  $\Sigma$  and the  $X$  directions.

As usual, the anomaly involves a nongeometrical symmetry that acts only on  $\beta$ . We briefly make some remarks on such symmetries for the general case that  $X$  has complex dimension  $n$ . Nongeometrical symmetries, as we reviewed in Section 3.2, are determined by a closed holomorphic two-form  $F$  on  $X \times \Sigma$ . If  $F = \frac{1}{2}f_{ij}(\gamma, z)d\gamma^i \wedge d\gamma^j + C_i(\gamma, z)d\gamma^i \wedge dz$  is such a form, then under the symmetry associated with  $F$ , the fields transform as

$$\begin{aligned}\gamma^i &\rightarrow \gamma^i \\ \beta_j &\rightarrow \beta'_j = \beta_j + f_{ij}\partial\gamma^j + C_i.\end{aligned}\tag{5.35}$$

We get the OPE

$$\begin{aligned}\beta'_i(z')\beta'_k(z) &\sim \frac{1}{z' - z}(-\partial\gamma^l(\partial_l f_{ik} + \partial_k f_{li} + \partial_i f_{kl}) \\ &\quad - (\partial_z f_{ik} + \partial_i C_k - \partial_k C_i)),\end{aligned}\tag{5.36}$$

showing that the transformation (5.35) preserves the OPE's if and only if  $dF = 0$ .

The anomaly arises from an element of  $H^2(X \times \Sigma, \Omega^{2,cl})$  that appears as an inconsistency in gluing together various local descriptions. For  $X = \Sigma = \mathbb{CP}^1$ , this cohomology group is one-dimensional. A convenient generator is the two-form

$$F = \frac{d\gamma \wedge dz}{\gamma z},\tag{5.37}$$

which is holomorphic in all triple intersections of the open sets  $U_i \times P_j$ . Under the associated symmetry, the fields transform as

$$\begin{aligned}\gamma &\rightarrow \gamma \\ \beta &\rightarrow \beta + \frac{1}{\gamma z},\end{aligned}\tag{5.38}$$

and this is the form that the anomaly will take.

### 5.3.1 The calculation

We start with a free  $\beta\gamma$  system, where  $\gamma(z)$  describes a map from the  $z$ -plane to the  $\gamma$ -plane — that is, from the open set  $P_1 \subset \Sigma$  to the open set  $U_1 \subset X$ . In Section 5.1, we showed how to map from  $\gamma$  to  $1/\gamma$ , that is, from variables

associated with  $U_1 \times P_1$  to variables associated with  $U_2 \times P_1$ . The gluing map, which we will call  $R$ , is

$$\begin{aligned}\tilde{\gamma}(z) &= \frac{1}{\gamma(z)} \\ \tilde{\beta}(z) &= \lim_{z' \rightarrow z} \left( -\gamma^2(z')\beta(z) + \frac{2\gamma(z')}{z' - z} \right) + 2\partial_z \gamma(z).\end{aligned}\quad (5.39)$$

Likewise, it is possible, while keeping the target space as  $U_1$ , to glue together theories on which the Riemann surface is  $P_1$  or  $P_2$ . In an elementary way, we can map a free  $\beta\gamma$  system on the  $z$ -plane  $P_1$  to a similar free field theory on the  $y$ -plane  $P_2$  (where  $y = 1/z$ ). We let  $B$  and  $\Gamma$  be fields of dimension  $(1, 0)$  and  $(0, 0)$  on  $P_2$ , defined in terms of  $\beta$  and  $\gamma$  by

$$\begin{aligned}\Gamma(y) &= \gamma\left(\frac{1}{y}\right) \\ B(y) &= -\frac{1}{y^2}\beta\left(\frac{1}{y}\right).\end{aligned}\quad (5.40)$$

This transformation, which we will call  $Y$ , maps free field OPE's of  $\beta$  and  $\gamma$  to OPE's of the same form for  $B$  and  $\Gamma$ ; the nontrivial OPE is  $\Gamma(y)B(y') \sim 1/(y - y')$ .

By using  $R$  to map from  $U_1$  to  $U_2$  and  $Y$  to map from  $P_1$  to  $P_2$ , we could — if  $R$  and  $Y$  would commute — glue together four different free field descriptions associated with  $U_i \times P_j$  to make a theory that is global in both  $X$  and  $\Sigma$ .  $R$  and  $Y$  do commute in their action on  $\gamma$ . Their combined operation on  $\gamma$ , in either order, gives

$$\hat{\Gamma}(y) = \frac{1}{\gamma(1/y)}.\quad (5.41)$$

But they do not commute in their action on  $\beta$ . That is where the anomaly comes in.

Let us write  $\tilde{B}$  for the result of applying first  $R$  and then  $Y$  — in other words, first mapping from  $U_1$  to  $U_2$  via (5.39), and then from  $P_1$  to  $P_2$  via (5.40). The composition is easy to write:

$$\tilde{B}(y) = -\frac{1}{y^2} \left( \lim_{y' \rightarrow y} \left( -\gamma^2\left(\frac{1}{y'}\right)\beta\left(\frac{1}{y}\right) + \frac{2\gamma(1/y')}{1/y' - 1/y} \right) + 2\partial_z \gamma|_{z=1/y} \right).\quad (5.42)$$

And let us write  $B^*$  for the result of reversing the order of the two operations, applying first  $Y$  and then  $R$ . Here we get

$$B^*(y) = \lim_{y' \rightarrow y} \left( -\gamma^2 \left( \frac{1}{y'} \right) \left( -\beta \left( \frac{1}{y} \right) / y^2 \right) + \frac{2}{y' - y} \gamma \left( \frac{1}{y'} \right) \right) + 2\partial_y \gamma \left( \frac{1}{y} \right). \quad (5.43)$$

When we subtract these expressions, the  $\gamma^2\beta$  terms trivially cancel, and the  $\partial\gamma$  terms cancel, given that  $z = 1/y$ . The terms linear in  $\gamma$  do not cancel. We get

$$\tilde{B}(y) - B^*(y) = \frac{2\gamma(1/y)}{y} = \frac{2}{y\widehat{\Gamma}(y)}, \quad (5.44)$$

showing the form of the anomaly expected from (5.38). Of course, we get  $y\widehat{\Gamma}$  instead of  $z\gamma$  in the denominator because the equation (5.44) is written for fields on  $U_2 \times P_2$ .

## 5.4 $\mathbf{S}^1 \times \mathbf{S}^3$ Revisited

Finally, we will reexamine the  $\mathbf{S}^1 \times \mathbf{S}^3$  model, which we introduced in Section 4.2.

### 5.4.1 The WZW model

First we make a few remarks on the WZW model of  $\mathbf{S}^1 \times \mathbf{S}^3$ , which as we recall has  $(0, 2)$  supersymmetry [26, 27], leading to the possibility of constructing sheaves of CDO's on  $\mathbf{S}^1 \times \mathbf{S}^3$ .

This model is the tensor product of an  $SU(2)$  WZW model at level  $k$ , times a free field theory of  $\mathbf{S}^1$ , times four free right-moving (or in our conventions in this paper, antiholomorphic) fermions. These fermions transform in the adjoint representation of  $SU(2) \times U(1)$ .

After the twisting that is used, as in Section 2.1, in defining  $Q$ -cohomology and constructing a sheaf of CDO's, the fermions are a pair of fermionic  $\beta\gamma$  fields (called  $\rho, \alpha$  in Section 2.1) with spins 1 and 0. A single such pair has left and right central charges  $(0, -2)$ , so the fermions contribute  $(0, -4)$  to the left and right central charges of the system. The  $SU(2)$  WZW model contributes  $(3k/(k+2), 3k/(k+2))$  to the central charges, and the free theory of  $\mathbf{S}^1$  contributes  $(1, 1)$ . The total central charges are hence  $(3k/(k+2) + 1, 3k/(k+2) - 3)$ . Passing from the physical theory to the  $Q$ -cohomology does not change the difference of left and right central charges, which is

$c = 4$ . This will be the central charge of the stress tensor that appears as a global section of the sheaf of CDO's.<sup>17</sup>

Similarly, we can anticipate the central charges of the current algebra that will appear when we take global sections of the sheaf of CDO's. The underlying  $SU(2)$  WZW model has an  $SU(2)$ -valued field  $g$ , with symmetry  $SU(2)_L \times SU(2)_R$  (actually  $(SU(2)_L \times SU(2)_R)/\mathbb{Z}_2$ , where  $\mathbb{Z}_2$  is the common center of the two factors). The symmetry acts by  $g \rightarrow agb^{-1}$ ,  $a, b \in SU(2)$ . In the WZW model, the  $SU(2)_L$  symmetry is part of a holomorphic  $SU(2)$  current algebra of level  $k$ , while  $SU(2)_R$  is part of an antiholomorphic  $SU(2)$  current algebra of level  $k + 2$ . Here, "2" is the contribution of the right-moving fermions (real fermions in the adjoint representation of  $SU(2)$ ). The left and right central charges are thus  $(k, 0)$  for  $SU(2)_L$  and  $(0, k + 2)$  for  $SU(2)_R$ .

The twisting of the four real fermions of the underlying  $(0, 2)$  model of  $\mathbf{S}^1 \times \mathbf{S}^3$  to make a pair of fermionic  $\beta\gamma$  or  $\rho\alpha$  systems explicitly breaks  $SU(2)_R$  to its maximal torus  $U(1)_R$ . So the symmetry that survives for the  $Q$ -cohomology or sheaf of CDO's is  $(SU(2)_L \times U(1)_R)/\mathbb{Z}_2 = U(2)$ . The difference between left and right central charges is unchanged in passing to the sheaf of CDO's, so the level of the  $SU(2)_L$  current algebra will be  $k$  and that of the  $U(1)_R$  current algebra will be  $-k - 2$ . The only case in which they are equal is  $k = -1$ , for which the levels are both  $-1$ . This is not really a physically sensible value for the WZW model; physically sensible, unitary WZW models with convergent path integrals are restricted to integer values of  $k$  with  $k \geq 0$ . However, in the sheaf of CDO's, which corresponds to a perturbative treatment,  $k$  is an arbitrary complex parameter, as explained in Sections 3 and 4.

In the sheaf of CDO's, the symmetries are automatically complexified, so the symmetry we see will be at the Lie algebra level  $GL(2)$  rather than  $U(2)$ . Moreover,  $U(1)_R$  (which acts on the variables introduced momentarily by  $v^i \rightarrow e^{i\theta} v^i$ ) and the rotation of  $\mathbf{S}^1$  (which acts by  $v^i \rightarrow e^\chi v^i$  with real  $\chi$ ) combine together to generate the center of  $GL(2)$ . (At the Lie algebra level, the center is  $GL(1)$ , but the group structure is really that of an elliptic curve or of  $GL(1)/\mathbb{Z} = U(1) \times U(1)$ .) The rotation of  $\mathbf{S}^1$  corresponds to a  $U(1)$  current algebra with equal left and right central charges, so it does not affect the above discussion.

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<sup>17</sup>If one replaces  $\mathbf{S}^1 \times \mathbf{S}^3$  by  $\mathbb{R} \times \mathbf{S}^3 = \mathbb{C}^2 - \{0\}$ , it is possible to add a linear dilaton coupling on  $\mathbb{R}$  such that the theory becomes a superconformal theory whose left and right central charges (in the half-twisted version) are  $(4, 0)$ . In this description, the left-moving central charge is unchanged in passing to the  $Q$ -cohomology, and remains at  $c = 4$ .



### 5.4.2 Constructing a sheaf of CDO's

Now we come to our main task, which is to construct a family of sheaves of CDO's over  $\mathbf{S}^1 \times \mathbf{S}^3$ . First we construct a simple special case, and then we show how to introduce a variable parameter.

We will regard  $\mathbf{S}^1 \times \mathbf{S}^3$  as  $(\mathbb{C}^2 - \{0\})/\mathbb{Z}$ , where  $\mathbb{C}^2$  has coordinates  $v^1$ ,  $v^2$ , and  $\{0\}$  is the origin in  $\mathbb{C}^2$  (the point  $v^1 = v^2 = 0$ ) which should be removed before dividing by  $\mathbb{Z}$ . Also,  $\mathbb{Z}$  acts by  $v^i \rightarrow \gamma^n v^i$ , where  $\gamma$  is a nonzero complex number of modulus less than 1.  $\gamma$  is a modulus of  $\mathbf{S}^1 \times \mathbf{S}^3$  that we will keep fixed.

To construct the simplest sheaf of CDO's with target  $\mathbf{S}^1 \times \mathbf{S}^3$ , we simply promote the  $v^i$  to free fields of spin 0, with conjugate spin 1 fields  $V_i$ , and free action

$$I = \frac{1}{2\pi} \int |d^2 z| (V_1 \bar{\partial} v^1 + V_2 \bar{\partial} v^2). \quad (5.45)$$

The nontrivial OPE's are as usual  $V_i(z)v^j(z') \sim -\delta_j^i/(z-z')$ . We allow only operators that are invariant under  $v^i \rightarrow \gamma v^i$ ,  $V_i \rightarrow \gamma^{-1}V_i$ .

One operator that possesses this invariance is the stress tensor

$$T_{zz} = \sum_i V_i \partial v^i. \quad (5.46)$$

Hence, the chiral algebra of this theory is conformally invariant, in contrast to the chiral algebra of  $\mathbb{CP}^1$ . This reflects the conformal invariance of the underlying (0, 2) model with target  $\mathbf{S}^1 \times \mathbf{S}^3$ . A bosonic  $\beta\gamma$  system of spins 0 and 1 has  $c = 2$ , so the stress tensor  $T$  has  $c = 4$ , in agreement with what we expected from the underlying WZW model.

The chiral algebra of  $\mathbf{S}^1 \times \mathbf{S}^3$  also contains the dimension 1 currents  $J_j^i = -V_j v^i$ . These obey the OPE's

$$J_j^i(z)J_l^m(z') \sim -\frac{\delta_l^i \delta_j^m}{(z-z')^2} + \frac{\delta_j^m J_l^i - \delta_l^i J_j^m}{z-z'}. \quad (5.47)$$

This is a  $\text{GL}(2)$  current algebra at level  $-1$ .

In what follows, it will not be possible to maintain manifest  $\text{GL}(2)$  symmetry, and it will be convenient to pick a basis in the current algebra. The  $\text{SL}(2)$  subgroup is generated by  $J_3 = -\frac{1}{2}(V_1 v^1 - V_2 v^2)$ ,  $J_+ = -V_2 v^1$ ,

$J_- = -V_1 v^2$ , with nontrivial OPE's

$$\begin{aligned} J_3(z)J_3(z') &\sim -\frac{1}{2}\frac{1}{(z-z')^2} \\ J_3(z)J_\pm(z') &\sim \pm\frac{J_\pm(z')}{z-z'} \\ J_+(z)J_-(z') &\sim -\frac{1}{(z-z')^2} + \frac{2J_3(z')}{z-z'}. \end{aligned} \quad (5.48)$$

Here we recognize the  $\mathrm{SL}(2)$  current algebra at level  $-1$ . The center of  $\mathrm{GL}(2)$ , which is of course a copy of  $\mathrm{GL}(1)$ , is generated by  $K = -\frac{1}{2}(V_1 v^1 + V_2 v^2)$ , with

$$K(z)K(z') \sim -\frac{1}{2}\frac{1}{(z-z')^2}. \quad (5.49)$$

### 5.4.3 The modulus of the CDO

Now we are going to generalize the CDO of  $\mathbf{S}^1 \times \mathbf{S}^3$  that was constructed above, introducing a parameter associated with  $H^1(\mathbf{S}^1 \times \mathbf{S}^3, \Omega^{2,\mathrm{cl}}) \cong \mathbb{C}$ .

To do this, we first make a cover of  $\mathbf{S}^1 \times \mathbf{S}^3$  by two open sets  $U_1$  and  $U_2$ , where  $U_1$  is characterized by the condition  $v^1 \neq 0$ , and  $U_2$  by  $v^2 \neq 0$ . In fact, this is not a “good cover,” as  $U_1$  and  $U_2$  are topologically complicated (each is isomorphic to  $\mathbb{C} \times E$ , where  $E$  is an elliptic curve). As a result, in general, we are not guaranteed that an arbitrary cohomology class can be represented by a Čech cocycle with respect to this cover. In the case at hand, however, we have on  $U_1 \cap U_2$  a holomorphic section of  $\Omega^{2,\mathrm{cl}}$ , namely

$$F = \frac{dv^1 \wedge dv^2}{v^1 v^2}. \quad (5.50)$$

$F$  cannot be “split” as the difference of a form holomorphic in  $U_1$  and one holomorphic in  $U_2$ , so it represents an element of  $H^1(\mathbf{S}^1 \times \mathbf{S}^3, \Omega^{2,\mathrm{cl}})$ .

From (5.23), we know that the symmetry associated with  $F$  generates the following transformation:

$$\begin{aligned} v^1 &\rightarrow v^1 \\ v^2 &\rightarrow v^2 \\ V_1 &\rightarrow V_1 + t\frac{\partial v^2}{v^1 v^2} \end{aligned}$$

$$V_2 \rightarrow V_2 - t \frac{\partial v^1}{v^1 v^2}. \quad (5.51)$$

Here  $t$  is a complex parameter, which will turn out to be related to  $k$  of the WZW model. We get a family of CDO's, parameterized by  $t$ , by declaring that the fields undergo this gluing in going from  $U_1$  to  $U_2$ .

Let us determine how some important operators behave under this deformation. The stress tensor  $T = V_1 \partial v^1 + V_2 \partial v^2$  is invariant. So the deformed theory, for any  $t$ , has a stress tensor of  $c = 4$ . This is in accord with the fact that the WZW model is conformally invariant for all  $k$  and that the difference of its left and right central charges is always 4.

Next, let us consider the  $\text{GL}(1)$  current, which at  $t = 0$  was defined as  $K = -\frac{1}{2}(V_1 v^1 + V_2 v^2)$ . Under (5.51), we have

$$K \rightarrow K - \frac{t}{2} \left( \frac{\partial v_2}{v_2} - \frac{\partial v_1}{v_1} \right). \quad (5.52)$$

In contrast to what one might guess from our previous examples, the shift in  $K$  under this transformation is *not* an anomaly that spoils existence of  $K$  at  $t \neq 0$ . The reason is that this shift can be split as a difference between a term (namely  $t \partial v^1 / 2v^1$ ) that is holomorphic in  $U_1$  and a term (namely  $t \partial v^2 / 2v^2$ ) that is holomorphic in  $U_2$ . As a result, we can modify  $K$  to get a  $\text{GL}(1)$  current generator that is holomorphic in both  $U_1$  and  $U_2$ . In  $U_1$ , the current is

$$K^{[1]} = -\frac{1}{2} (V_1 v^1 + V_2 v^2) - \frac{t}{2} \frac{\partial v^1}{v^1}, \quad (5.53)$$

while in  $U_2$  it is represented by

$$K^{[2]} = -\frac{1}{2} (V_1 v^1 + V_2 v^2) - \frac{t}{2} \frac{\partial v^2}{v^2}. \quad (5.54)$$

$K^{[1]}$  is holomorphic in  $U_1$ , and transforms under (5.51) into  $K^{[2]}$ , which is holomorphic in  $U_2$ . So, for any  $t$ , the sheaf  $\widehat{\mathcal{A}}$  of chiral operators has a global section  $K$  that is represented in  $U_1$  by  $K^{[1]}$  and in  $U_2$  by  $K^{[2]}$ .

Now we can calculate the OPE singularity of  $K$  for any  $t$ :

$$K(z)K(z') \sim -\frac{1+t}{2} \frac{1}{(z-z')^2}. \quad (5.55)$$

To calculate this, we either work in  $U_1$ , setting  $K = K^{[1]}$  and computing the OPE, or we work in  $U_2$ , setting  $K = K^{[2]}$  and computing the OPE. The

answer comes out the same either way, since the transformation (5.51) is an automorphism of the CFT. Thus, the level of the  $GL(1)$  current algebra is  $-t - 1$ .

Similarly, we can work out the transformation of the  $SL(2)$  currents under (5.51). The currents as defined at  $t = 0$ , namely  $J_3 = -\frac{1}{2}(V_1v^1 - V_2v^2)$ ,  $J_+ = -V_2v^1$ ,  $J_- = -V_1v^2$ , transform as

$$\begin{aligned} J_3 &\rightarrow J_3 - \frac{t}{2} \left( \frac{\partial v^1}{v^1} + \frac{\partial v^2}{v^2} \right) \\ J_+ &\rightarrow J_+ + t \frac{\partial v^1}{v^2} \\ J_- &\rightarrow J_- - t \frac{\partial v^2}{v^1}. \end{aligned} \tag{5.56}$$

The shifts in each current can be “split” as a difference of terms holomorphic in  $U_1$  and  $U_2$ . So the currents can be defined at  $t \neq 0$ , but receive  $t$ -dependent terms. The corrected currents are

$$J_3 = \begin{cases} -\frac{1}{2}(V_1v^1 - V_2v^2) + t \partial v^1 / 2v^1 \\ -\frac{1}{2}(V_1v^1 - V_2v^2) - t \partial v^2 / 2v^2 \end{cases} \tag{5.57}$$

along with

$$J_+ = \begin{cases} -V_2v^1 \\ -V_2v^1 + t \partial v^1 / v^2 \end{cases} \tag{5.58}$$

and

$$J_- = \begin{cases} -V_1v^2 + t \partial v^2 / v^1 \\ -V_1v^2. \end{cases} \tag{5.59}$$

In each case, the upper expression holds in  $U_1$  and the lower expression holds in  $U_2$ .

We now can compute the OPE's of these operators, working in either  $U_1$  or  $U_2$ . We get an  $SL(2)$  current algebra at level  $t - 1$ :

$$\begin{aligned} J_3(z)J_3(z') &\sim \frac{t-1}{2} \frac{1}{(z-z')^2} \\ J_3(z)J_{\pm}(z') &\sim \pm \frac{J_{\pm}(z')}{z-z'} \\ J_+(z)J_-(z') &\sim \frac{t-1}{(z-z')^2} + \frac{2J_3(z')}{z-z'}. \end{aligned} \tag{5.60}$$

The  $SL(2)$  and  $GL(1)$  current algebras thus have levels  $t - 1$  and  $-t - 1$ , in agreement with expectations from the WZW model if the WZW level  $k$  is related to the CDO parameter  $t$  by  $k = t - 1$ . We will not attempt an a priori explanation of this relationship.

The  $Q$ -cohomology of  $\mathbf{S}^1 \times \mathbf{S}^3$  has no instanton corrections. For any target space  $X$ , such corrections (because they are local on the Riemann surface  $\Sigma$ , though global in  $X$ ) come only from holomorphic curves in  $X$  of genus zero. There are no such curves in  $\mathbf{S}^1 \times \mathbf{S}^3$ .

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