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Correlators in Timelike Bulk Liouville Theory

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Abstract

Liouville theory with a negative norm boson and no screening charge corresponds to an exact classical solution of closed bosonic string theory describing time-dependent bulk tachyon condensation. A simple expression for the two point function is proposed based on renormalization/analytic continuation of the known results for the ordinary (positive-norm) Liouville theory. The expression agrees exactly with the minisuperspace result for the closed string pair-production rate (which diverges at finite time). Puzzles concerning the three-point function are presented and discussed.

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1 Introduction

The endpoint - if any - of tachyon condensation in closed string theories with tachyons is a major unsolved problem. A relatively complete picture has recently been obtained of the condensation of certain *localized* closed string tachyons [1, 2, 3, 4], as have some insights into the non-localized bulk case [5].

It is interesting in this regard that there are exact, time dependent classical solutions of tachyonic string theories describing homogenous tachyon condensation. We refer to the corresponding worldsheet CFT for the case of the bosonic string as the timelike Liouville theory. It is governed by the the action (suppressing spatial dimensions and setting $\alpha' = 1$)

$$S = \frac{1}{4\pi} \int d^2\sigma \left(-(\partial X^0)^2 + 4\pi\mu e^{2\beta X^0} \right), \tag{1.1}$$

which has a negative norm boson and central charge $c = 1 - 6q^2 \quad (q \equiv \beta - 1/\beta)$. This corresponds to a real dilaton with timelike slope q, but we will mainly be interested in $q = 0$ and $\beta = 1$. The potential term in (1.1) can be interpreted a closed string tachyon field which grows exponentially in time. At early times $X^0 \to -\infty$ the tachyon field is very small and flat space is recovered.

Because this is a time dependent background, there is closed string pair production. In section 2 we compute the rate of pair production in a minisuperspace approximation [6, 7] and find that it diverges exponentially (see also [8]). This divergence is much stronger than that found in an analogous computation of open string pair production [9, 10, 11] during open string tachyon condensation[12, 13] (the related phenomenon of closed string emission was analyzed in[14, 15, 16, 17]). This result implies that the gas of pair produced closed strings will reach the string density and string perturbation theory will break down in a time of order one in string units.

This work began in as an attempt to compute the two and three point functions of the timelike Liouville theory. This sounds like a simple task but in fact the analytic continuation involved is quite subtle. Our strategy will parallel that adopted for the timelike boundary Liouville theory in

[10]. After some renormalization and analytic continuation we will produce a reasonable expression for the exact two point function in section 3 which agrees exactly with the minisuperspace particle production rate. We also discuss the three point function. We present a natural analytic continuation procedure which encounters singularities and yields a puzzling result, not obviously compatible with conformal invariance, as discussed in section 4.

The basic source of the difficulty is that the action (1.1) is not positive definite, and hence does not fully define the associated functional integral or CFT. It is natural to try to define the timelike theory[1] by analytic continuation $\phi = -iX^0$ and $b = i\beta$ from the ordinary spacelike Liouville theory with the positive definite worldsheet action

$$S = \frac{1}{4\pi}\int d^2\sigma \left((\partial\phi)^2 + 4\pi\mu e^{2b\phi}\right), \tag{1.2}$$

and background charge $Q = b + 1/b$. The central charge here is $c = 1 + 6Q^2$ and Q is the (spacelike) slope of the dilaton. The analytic continuation from real to pure imaginary b however, for the three point function, leads to an infinite accumulation of singularities. A similar analytical continuation was discussed in [18], where it was argued that the $b = i$ singularity is related to the accumulation of minimal models at $c = 1$. It was also discussed in [10] (whose treatment we closely follow) in the closely related context of timelike boundary Liouville theory[19, 20].[2]

2 The Minisuperspace Approximation

In this section we apply the minisuperspace approximation [6, 7] to the time-like bulk Liouville theory with constant dilaton ($q = 0$).[3] The analysis here is mathematically quite similar (though the final conclusion differs) to that given for the boundary tachyon interaction considered in [9]. We shall

[1] We assume here that such an exact CFT indeed exists, although it has yet to be demonstrated.

[2] For the boundary theory, the singularities accumulate already for the two point function, while for the bulk theory this difficulty first arises for the three point function.

[3] Recently the paper [8] appeared which also analyzes the minisuperspace approximation and obtains the same results.

accordingly be brief. In the minisuperspace approximation we retain only the zero mode x^0 of X^0. The on-shell condition for the closed string excitations is then the Klein-Gordon equation with time dependent mass

$$\Big[\frac{\partial^2}{(\partial x^0)^2} + 2\pi\mu e^{2x^0} + \vec{p}^2 + 2(N_L + N_R - 2)\Big]\psi_{\vec{p}}(x^0) = 0. \tag{2.1}$$

Below we define the energy by $\omega = \sqrt{\vec{p}^2 + 2(N_L + N_R - 2)}$, and $N_{L,R}$ are the left and right-moving oscillator contributions. *in* and *out* wave function $\psi^{in}_{\vec{p}}(x^0)$ and $\psi^{out}_{\vec{p}}(x^0)$ asymptotically obeying $\psi^{in}_{\vec{p}}(x^0) \sim e^{-i\omega x^0 + i\vec{p}\cdot\vec{x}}$ and $\psi^{out}_{\vec{p}}(x^0) \sim e^{-\frac{x^0}{2} - 2i\sqrt{\pi\mu}e^{x^0} + i\vec{p}\cdot\vec{x}}$ are given by Bessel and Hankel functions. This leads to the Bogolubov coefficients

$$\gamma^{in}_{\vec{p}} = \frac{\beta^{in*}_{\vec{p}}}{\alpha^{in}_{\vec{p}}} = -ie^{-\pi\omega}, \tag{2.2}$$

where the Bogolubov transformation is

$$\begin{aligned} \psi^{out}_{\vec{p}} &= \alpha_{\vec{p}}\psi^{in}_{\vec{p}} + \beta_{\vec{p}}\psi^{in*}_{-\vec{p}} \\ \psi^{in}_{\vec{p}} &= \alpha^*_{\vec{p}}\psi^{out}_{\vec{p}} - \beta_{\vec{p}}\psi^{out*}_{-\vec{p}}. \end{aligned} \tag{2.3}$$

According to the conjecture of [10] this is related to the two point function by

$$|\langle e^{-i\omega x^0} e^{-i\omega x^0}\rangle| = |\gamma^{in}_{\vec{p}}| = e^{-\pi\omega}. \tag{2.4}$$

In the following section we shall recover this result from an exact CFT analysis.

The rate of closed string pair production is determined from the Bogolubov transformations. At high frequencies ω this rate behaves as

$$\rho(\omega)|\gamma^{in}_{\vec{p}}|^2 \sim e^{\omega/T_H} e^{-2\pi\omega}. \tag{2.5}$$

Recalling the value of the Hagedorn temperature $T_H = \frac{1}{4\pi}$, we see that this expression diverges exponentially. The closed string pair production is much more rapid than in the corresponding open string case [9], where the exponentials cancelled and the divergence was at most power law.

3 The Two Point Function

In this section we define the two point function in timelike Liouville theory by analytic continuation from the spacelike Liouville theory.

The two point function in the space-like theory is given by [21][22]

$$D(\alpha) \equiv \langle e^{2\alpha\phi} e^{2\alpha\phi} \rangle = (\pi\mu\gamma(b^2))^{\frac{Q-2\alpha}{b}} \cdot \frac{\gamma(2b\alpha - b^2)}{b^2 \cdot \gamma(2 - \frac{2\alpha}{b} + \frac{1}{b^2})}, \tag{3.1}$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. This is related to the reflection coefficient

$$\begin{aligned} &\psi(\phi) \sim e^{2ip\phi} + d(2ip)e^{-2ip\phi}, \\ &d(2ip) \equiv D(ip + \frac{Q}{2}) = -(\pi\mu\gamma(b^2))^{-2ip/b} \cdot \frac{\Gamma(1+2ip/b)\Gamma(1+2ipb)}{\Gamma(1-2ip/b)\Gamma(1-2ipb)}. \end{aligned} \tag{3.2}$$

We wish to analytically continue to $\phi = -iX_0$ and $p = -i\omega/2$. We further take $b \equiv i\beta$ to be purely imaginary since we are ultimately interested in the bulk time-like Liouville theory with a real linear dilaton. The reflection coefficient for bulk time-like Liouville theory is then

$$d(\omega) = \langle e^{-i(\omega+Q)X^0} e^{-i(\omega+Q)X^0} \rangle = -(\pi\mu\gamma(-\beta^2))^{i\frac{\omega}{\beta}} \cdot \frac{\Gamma(1-i\omega/\beta)\Gamma(1+i\omega\beta)}{\Gamma(1+i\omega/\beta)\Gamma(1-i\omega\beta)}. \tag{3.3}$$

Its magnitude, which appears in the particle creation rate, is simply

$$|d(\omega)| = e^{-\frac{\pi\omega}{\beta}}. \tag{3.4}$$

This agrees exactly with the $\beta = 1$ minisuperspace result (2.4). Taking the limit $\beta \to 1$ with $\beta^2 = 1 - \epsilon$ and $\epsilon \to 0+$, we find

$$d(\omega) \to -(\pi\mu_R)^{i\omega} \cdot e^{-\pi\omega}. \tag{3.5}$$

where we have identified the renormalized coupling $\mu_R = |\mu\gamma(\beta^2)|$ from the expected invariance of the two point function (for $Q = 0$) under $X^0 \to X^0 + C, \quad \mu_R \to e^{2i\omega C}\mu_R$. Note that we must take $\mu \to 0$ to keep μ_R finite in the limit $\beta \to 1$. We will see in the next section that this renormalization procedure also removes a similar divergence in the three (and presumably higher) point function.

4 The Three Point Function

In this section we use the same analytic continuation to obtain a candidate three point functions for timelike Liouville theory. However subtleties are encountered along the way. The result has some puzzling behavior and may not be the correct three-point function.

The three point function in the spacelike theory is given by [21, 22] (we use the conventions of the latter reference)

$$C(\alpha_1,\alpha_2,\alpha_3) \equiv \langle e^{2\alpha_1\phi}e^{2\alpha_2\phi}e^{2\alpha_3\phi}\rangle = [\pi\mu\gamma(b^2)b^{2-2b^2}]^{(Q-\alpha_1-\alpha_2-\alpha_3)/b} \times \frac{\Upsilon_0\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_2+\alpha_3-\alpha_1)\Upsilon(\alpha_1+\alpha_3-\alpha_2)}, \tag{4.1}$$

where $\Upsilon_0 = \frac{d\Upsilon(x)}{dx}|_{x=0}$ and the function $\Upsilon(x)$ is defined by

$$\log\Upsilon(x) = \int_0^\infty \frac{dt}{t}\left[(Q/2-x)^2 e^{-2t} - \frac{\sinh^2(Q/2-x)t}{\sinh(bt)\sinh(t/b)}\right]. \tag{4.2}$$

The analytic continuation of the prefactor in (4.1) to $b = i\beta$ is given by

$$\left[e^{\pi i(\beta^2-1)}\cdot(\pi\mu_R\beta^{2(1+\beta^2)})^{\frac{i(\alpha_1+\alpha_2+\alpha_3)}{\beta}}\right]\cdot(\pi\mu_R\beta^{2(1+\beta^2)})^{1-1/\beta^2}\cdot e^{-\pi\beta(\alpha_1+\alpha_2+\alpha_3)}, \tag{4.3}$$

where all of the phase factors are in the first parenthesis. Taking the limit $\beta \to 1$ yields the following expression for the prefactor

$$(\pi\mu_R)^{i(\alpha_1+\alpha_2+\alpha_3)}\cdot e^{-\pi(\alpha_1+\alpha_2+\alpha_3)}. \tag{4.4}$$

The non-trivial part is the analytic continuation of the function $\Upsilon(x)$ appearing in (4.1), which acquires an infinite accumulation of poles for purely imaginary b. This difficulty was already encountered at the level of the two-point function for timelike boundary Liouville theory, and our mathematical treatment will follow the one given in [10]. We would like to perform an analytical continuation from the real value of b to the imaginary value of $b = i\beta$ by a $\pi/2$ rotation. In deforming the contour of integration of (4.2) along the real axis into one along the imaginary axis, we encounter poles at

$t = \frac{n\pi i}{b}$ $(n = 1, 2, \cdots)$. Hence one may write the integral over the real axis as an integral I along the imaginary axis plus a sum of poles P

$$\log \Upsilon(x) = I(x) + P(x). \tag{4.5}$$

The convergent integral $I(x)$ along the imaginary axis $t = i\tau$ is given by

$$\begin{aligned} I(x) &= \int_0^\infty \frac{d\tau}{\tau}\Big[(Q/2 - x)^2 e^{-2i\tau} - \frac{\sin^2((Q/2-x)\tau)}{\sin(b\tau)\sin(\tau/b)}\Big] \\ &= \int_0^\infty \frac{d\tau}{\tau}\Big[(iq/2 - x)^2 e^{-2i\tau} - \frac{\sin^2((iq/2-x)\tau)}{\sinh(\beta\tau)\sinh(\tau/\beta)}\Big], \end{aligned} \tag{4.6}$$

where we set $b = i\beta$ and $Q = i(\beta - 1/\beta) = iq$. It follows from (4.1) that we are interested in the linear combination

$$\begin{aligned} I(\alpha_1, \alpha_2, \alpha_3) &= -I(\alpha_1 + \alpha_2 + \alpha_3) - I(\alpha_1 + \alpha_2 - \alpha_3) - I(\alpha_1 - \alpha_2 + \alpha_3) \\ &\quad -I(-\alpha_1 + \alpha_2 + \alpha_3) \quad + I(2\alpha_1) + I(2\alpha_2) + I(2\alpha_3). \end{aligned} \tag{4.7}$$

in which the linear and quadratic terms with respect to the argument x cancel. Hence we can omit[4] the first term in (4.6). Setting $\beta = 1$ one finds

$$\begin{aligned} I(x) &= -\int_0^\infty \frac{d\tau}{\tau}\frac{\sin^2(x\tau)}{\sinh^2\tau} \\ &= \frac{1}{2}\int_0^\infty \frac{d\tau}{\tau}\frac{\cos(2x\tau)}{\sinh^2\tau} + const. \end{aligned} \tag{4.8}$$

To derive this we neglect the imaginary part of I since it is proportional to x^2. Note that $e^{I(x)}$ has no zero or poles for real x. However, for e.g. $x = \pm i\beta$, it has a zero.

The sum of poles is given by

$$\begin{aligned} P(x) &= -2i\sum_{n=1}^{\infty}\frac{(-1)^n \sin^2(\frac{\pi n}{b}(Q/2 - x))}{n\sin(\frac{\pi n}{b^2})} \\ &= -i\sum_{n=1}^{\infty}\frac{(-1)^n}{n\sin(\frac{\pi n}{b^2})} + i\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\cos(\frac{\pi n}{b^2})\cos(\frac{2\pi n x}{b})}{\sin(\frac{\pi n}{b^2})} + \sin(\frac{2\pi n x}{b})\right). \end{aligned} \tag{4.9}$$

[4]This leads to the divergence due to the pole $\tau = 0$ in (4.6). However, it is cancelled in the ratio (4.1).

The first term is constant and can be neglected. We drop this term and analytically continue defining $\frac{1}{b^2} = -\frac{1}{\beta^2} - i\epsilon$ and $x = -iby$ and taking y to be real. The real part ReP of $P(x)$ is

$$\mathrm{Re}P = \sum_{n=1}^{\infty} \frac{1}{n} \left(\sinh(2\pi n y) + \mathrm{Im}\Big(\cot(\frac{n\pi}{\beta^2} + in\pi\epsilon)\Big) \cosh(2\pi n y) \right). \quad (4.10)$$

The first term can be resummed by using the analytic continuation[5]

$$\sum_{n=1}^{\infty} \frac{1}{2n} e^{2\pi n y} - \sum_{n=1}^{\infty} \frac{1}{2n} e^{-2\pi n y} = -\frac{1}{2}\log(1 - e^{2\pi y}) + \frac{1}{2}\log(1 - e^{-2\pi y}) = -\pi y + \frac{\pi i}{2}. \quad (4.12)$$

These terms cancel one another in the ratio (4.1) and can be neglected except that it leads to a constant factor $-\pi$ in Υ_0. The second term in (4.10) can be rewritten as

$$-\frac{1}{2}\sum_{n=1}^{\infty} \frac{\cosh(2\pi n y)}{n} \cdot \frac{\sinh(2\pi n \epsilon)}{\sinh^2(\pi n \epsilon) + \sin^2(\frac{n\pi}{\beta^2})}. \quad (4.13)$$

Assuming β^2 is irrational as in [10] we find that this contribution vanishes in the limit $\epsilon \to 0$.

The imaginary part of the pole sum is

$$\mathrm{Im}P = -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Re}\Big(\cot(\frac{n\pi}{\beta^2} + in\pi\epsilon)\Big) \cosh(2\pi n y). \quad (4.14)$$

If the β^2 is rational such that $\beta^2 = q/p$, then

$$\mathrm{Re}\Big(\cot(\frac{n\pi}{\beta^2} + in\pi\epsilon)\Big) = \frac{1}{2} \frac{\sin(2np\pi/q)}{\sinh^2(\pi n \epsilon) + \sin^2(\frac{np\pi}{q})}. \quad (4.15)$$

The second term can also be rewritten after we take the limit $\epsilon \to 0$

$$-\sum_{n_0=1}^{q-1} \sum_{m=0}^{\infty} \frac{\cosh(2\pi n y)}{mq + n_0} \cot(n_0 \pi \frac{p}{q}). \quad (4.16)$$

[5]This follows from

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{n} = (\pi - z)/2,$$
$$\sum_{n=1}^{\infty} \frac{\cos(nz)}{n} = -\log(2\sin(z/2)), \quad (4.11)$$

by substituting $z = 2\pi i y$.

For the specific value $\beta = 1$ this contribution also vanishes.

Putting this all together leads to the very simple formula for the three-point function at $\beta = 1$ ($Q = 0$)

$$\begin{aligned} C(\omega_1, \omega_2, \omega_3) &\equiv \langle e^{-i\omega_1 X^0} e^{-i\omega_1 X^0} e^{-i\omega_1 X^0} \rangle \\ &= -\pi(\pi\mu_R)^{i(\omega_1+\omega_2+\omega_3)} \cdot e^{-\pi(\omega_1+\omega_2+\omega_3)/2} \cdot e^{I(\omega_1/2, \omega_2/2, \omega_3/2)}. \end{aligned} \tag{4.17}$$

As a check we note that (4.17) reduces to the two-point function (3.5) up to a constant when we set $\omega_3 = 0$ and $\omega_1 = \omega_2 = \omega$.

However the candidate three point function has a disturbing feature which leads us to suspect its validity. Conformal invariance would lead us to expect that it should vanish for $\omega_3 = 0$ and $\omega_1 \neq \omega_2$, but it does not do so. It can be seen that the original three point function (4.1) indeed vanishes for such momenta when Re$b > 0$.

How is it possible that such correlators vanish for all $\epsilon \neq 0$, but not at $\epsilon = 0$? The answer lies in the order of limits used in defining the continued expressions. Suppose that we do not take the limit $\epsilon \to 0$ before $n \to \infty$ in (4.10). In this case we can write

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Im}\left(\cot(\frac{n\pi}{\beta^2} + in\pi\epsilon) \right) \cosh(2\pi n y) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Re}\left(\frac{e^{in\pi/\beta^2 - n\pi\epsilon} + e^{-in\pi/\beta^2 + n\pi\epsilon}}{e^{in\pi/\beta^2 - n\pi\epsilon} - e^{-in\pi/\beta^2 + n\pi\epsilon}} \right) \cosh(2\pi n y). \end{aligned} \tag{4.18}$$

Since we can replace the fraction with -1 for large n and $\epsilon > 0$, we expect that the sum at $y = 0$ is divergent even though the value at non-zero y is zero in the $\epsilon \to 0$ limit. Thus we could have Re$P \to -\infty$ at $y = 0$, producing the expected zero in the three-point correlator.

This raises the possibility that the mathematical continuation procedure used to compute the three point correlator (4.17) is not the physically correct one. However we have not been able to find an alternate procedure which yields correlators that are analytic functions of the momenta and vanish at the expected places. We hope that future work will either provide an explanation of the unusual behavior of (4.17) or an alternate expression.

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