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Covariant Hamiltonian formalism for the calculus of variations with several variables: Lepage–Dedecker versus De Donder–Weyl

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Abstract

The main purpose in the present paper is to build a Hamiltonian theory for fields which is consistent with the principles of relativity. For this we consider detailed geometric pictures of Lepage theories in the spirit of Dedecker and try to stress out the interplay between the Lepage-Dedecker (LP) description and the (more usual) De Donder-Weyl (DDW) one. One of the main points is the fact that the Legendre transform in the DDW approach is replaced by a Legendre correspondence in the LP theory (this correspondence behaves differently: ignoring the singularities whenever the Lagrangian is degenerate).

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1 Introduction

1.1 Presentation

Multisymplectic formalisms are finite dimensional descriptions of variational problems with several variables (or field theories for physicists) analogue to the well-known Hamiltonian theory of point mechanics. For example consider on the set of maps $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ a Lagrangian action of the type

$$\mathcal{L}[u] = \int_{\mathbb{R}^n} L(x, u(x), \nabla u(x)) dx^1 \cdots dx^n.$$

Then it is well-known that the maps which are critical points of $\mathcal{L}$ are characterized by the Euler–Lagrange equation $\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial(\partial_\mu u)}\right) = \frac{\partial L}{\partial u}$. By analogy with the Hamiltonian theory we can do the change of variables $p^\mu := \frac{\partial L}{\partial(\partial_\mu u)}$ and define the Hamiltonian function

$$H(x, u, p) := p^\mu \frac{\partial u}{\partial x^\mu} - L(x, u, \nabla u),$$

where here $\nabla u = \left(\frac{\partial u}{\partial x^\mu}\right)$ is a function of (x, u, p) defined implicitly by $p^\mu := \frac{\partial L}{\partial(\partial_\mu u)}(x, u, \nabla u)$. Then the Euler-Lagrange equation is equivalent to the generalized Hamilton system of equations

$$\left\{ \begin{array}{rcl} \dfrac{\partial u}{\partial x^\mu} & = & \dfrac{\partial H}{\partial p^\mu}(x, u, p) \\ \displaystyle\sum_\mu \frac{\partial p^\mu}{\partial x^\mu} & = & -\dfrac{\partial H}{\partial u}(x, u, p). \end{array} \right. \tag{1}$$

This simple observation is the basis of a theory discovered by T. De Donder [3] and H. Weyl [20] independently in 1935. This theory can be formulated in a geometric setting, an analogue of the symplectic geometry, which is governed by the Poincaré–Cartan n-form $\theta := e\omega + p^\mu du \wedge \omega_\mu$ (where $\omega := dx^1 \wedge \cdots \wedge dx^n$ and $\omega_\mu := \partial_\mu \lrcorner\, \omega$) and its differential $\Omega := d\theta$, often called multisymplectic (or polysymplectic form).

Although similar to mechanics this theory shows up deep differences. In particular there exist other theories which are analogues of Hamilton's one as for instance the first historical one, constructed by C. Carathéodory in 1929 [2]. In fact, as realized by T. Lepage in 1936 [16], there are infinitely many theories, due to the fact that one could fix arbitrary the value of some tensor in the Legendre transform (see also [18], [6]). Much later on, in 1953, P. Dedecker [4] built a geometrical framework in which all Lepage theories are embedded. The present paper, which is a continuation of [9], is devoted to the study of the Lepage–Dedecker theory. We also want to compare this formalism with the more popular De Donder–Weyl theory.

First recall that the range of application of the De Donder–Weyl theory is restricted in principle to variational problems on sections of a bundle $\mathcal{F}$. The right framework for it, as expounded e.g. in [8], consists in using the first jet bundle $J^1\mathcal{F}$ and its affine dual $\left(J^1\right)^*\mathcal{F}$ as analogues of the tangent and the cotangent bundles for mechanics respectively. For non degenerate variational problems the Legendre transform induces an immersion of $J^1\mathcal{F}$ in $\left(J^1\right)^*\mathcal{F}$. In contrast the Lepage theories can be applied to more general situations but involve, in general, many more variables and so are more complicated to deal with, as noticed in [15]. This is probably the reason why most papers on the subject focus on the De Donder–Weyl theory, e.g. [14], [8]. The general idea of Dedecker in [4] for describing Lepage's theories is the following: if we view variational problems as being defined on n-dimensional submanifolds embedded in a $(n+k)$-dimensional manifold $\mathcal{N}$, then what plays the role of the (projective) tangent bundle to space-time in mechanics is the Grassmann bundle $Gr^n\mathcal{N}$ of oriented n-dimensional subspaces of tangent spaces to $\mathcal{N}$. The analogue of the cotangent bundle in mechanics is $\Lambda^nT^*\mathcal{N}$. Note that $\mathrm{dim}Gr^n\mathcal{N} = n+k+nk$ so that $\mathrm{dim}\Lambda^nT^*\mathcal{N} = n+k+\frac{(n+k)!}{n!k!}$ is strictly larger than $\mathrm{dim}Gr^n\mathcal{N}+1$ unless $n=1$ (classical mechanics) or $k=1$ (submanifolds are hypersurfaces). This difference between the dimensions explains the multiplicity of Lepage theories: as shown in [4], we substitute to the Legendre transform a Legendre correspondence which associates to each n-subspace $T\in Gr^n_q\mathcal{N}$ (a "generalized velocity") an affine subspace of $\Lambda^nT^*_q\mathcal{N}$ called *pseudofibre* by Dedecker. Then two points in the same pseudofiber do actually represent the same physical (infinitesimal) state, so that the coordinates on $\Lambda^nT^*\mathcal{N}$, called *momentoïdes* by Dedecker do not represent physically observable quantities. In this picture any choice of a Lepage theory corresponds to a selection of a submanifold of $\Lambda^nT^*\mathcal{N}$, which — when the induced Legendre transform is a well-defined map — intersects transversally each pseudofiber at one point (see Figure 1.1): so the Legendre correspondence specializes to a Legendre transform. For instance the De Donder–Weyl theory can be recovered in this setting by the restriction to some submanifold of $\Lambda^nT^*\mathcal{N}$ (see Section 2.2).

In [9] and in the present paper we consider a geometric pictures of Lepage theories in the spirit of Dedecker and we try to stress out the interplay between the Lepage–Dedecker description and the De Donder–Weyl one. Roughly speaking a comparison between these two points of view shows up some analogy with some aspects of the projective geometry, for which there

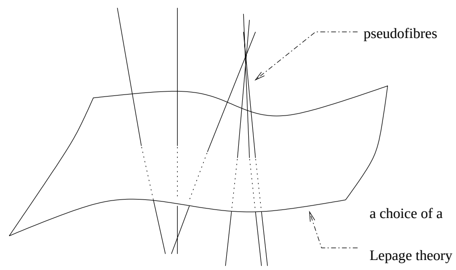

Figure 1: Pseudofibers which intersect a submanifold corresponding to the choice of a Lepage theory

is no perfect system of coordinates, but basically two: the homogeneous ones, more symmetric but redundant (analogue to the Dedecker description) and the local ones (analogue to the choice of a particular Lepage theory like e.g. the De Donder–Weyl one). Note that both points of view are based on the same geometrical framework, a multisymplectic manifold:

Definition 1.1. *Let $\mathcal{M}$ be a differential manifold. Let $n \in \mathbb{N}$ be some positive integer. A smooth $(n+1)$-form Ω on $\mathcal{M}$ is a* **multisymplectic** *form if and only if*

(i) *Ω is non degenerate, i.e. $\forall m \in \mathcal{M}$, $\forall \xi \in T_m\mathcal{M}$, if $\xi \lrcorner \Omega_m = 0$, then $\xi = 0$*

(ii) *Ω is closed, i.e. $d\Omega = 0$.*

Any manifold $\mathcal{M}$ equipped with a multisymplectic form Ω will be called a **multisymplectic** *manifold.*

For the De Donder–Weyl theory we choose $\mathcal{M}$ to be $\left(J^1\right)^* \mathcal{F}$ and for the Lepage–Dedecker theory $\mathcal{M}$ is $\Lambda^n T^*\mathcal{N}$. In both descriptions solutions of the variational problem correspond to n-dimensional submanifolds Γ (analogues of Hamiltonian trajectories: we call them **Hamiltonian n-curves**) and are characterized by the Hamilton equation $X \lrcorner \Omega = (-1)^n d\mathcal{H}$, where X is a n-multivector tangent to Γ, $\mathcal{H}$ is a (Hamiltonian) function defined on $\mathcal{M}$ and by " $\lrcorner$ " we mean the interior product.

We may insist on the point that many contributions on the De Donder–Weyl theory are devoted to the construction of multisymplectic manifolds having the same dimension as the Lagrangian formulation configuration space, i.e. $J^1\mathcal{F}$, either by pulling back the multisymplectic form by the Legendre map as in [8], or by working on a quotient or a submanifold of $(J^1)^*\mathcal{F}$ as for instance in [7] (see [5] for a comparaison between the different points of view). However when dealing with Lepage–Dedecker theories, one is forced to abandon these points of view and to work with multisymplectic manifolds whose dimension is larger than the number of physical variables. The advantage is however is that we do not need for any extra structure, like connections, and in particular in our setting the Hamiltonian function is thought as a global function on $\mathcal{M}$.

Consequently, in Section 2 we present a complete derivation of the (Dedecker) Legendre correspondence and of the generalized Hamilton equations, using a method that does not rely on any trivialization or connection on the Grassmannian bundle. A remarkable property, which is illustrated in this paper through the examples given in Paragraph 2.2.2, is that when n and k are greater than 2, the Legendre correspondence is generically never degenerate. The more spectacular example is when the Lagrangian density is a constant function — the most degenerate situation one can think about — then the Legendre correspondence is well-defined almost everywhere except precisely along the De Donder–Weyl submanifold. We believe that such a phenomenon was not noticed before; it however may be useful when one deals for example with the bosonic string theory with a skewsymmetric 2-form on the target manifold (a "B-field", as discussed in [9] and in subsection 2.2, example 5) or with the Yang–Mills action in 4 dimensions with a topological term in the Lagrangian: then the De Donder–Weyl formalism may fail but one can cure this degenerateness by using another Lepage theory or by working in the full Dedecker setting.

In this paper we also stress out another aspect of the (Dedecker) Legendre correspondence: one expects that the resulting Hamiltonian function on $\Lambda^nT^*\mathcal{N}$ should satisfy some condition expressing the "projective" invariance along each pseudofiber. This is indeed the case. On the one hand we observe in Section 2.1 that any smoothly continuous deformation of a Hamiltonian n-curve along directions tangent to the pseudofibers remains a Hamiltonian n-curve[1] (Corollary 2.1). On the other hand we give in Section 4.3 an intrinsic characterization of the subspaces tangent to pseudofibers. This motivates the definition given in Section 3.3 of the *generalized pseudofiber directions* on any multisymplectic manifold.

[1]A property quite similar to a gauge theory behavior although of different meaning. Here we are interested by desingularizing the theory and avoid the problems related to the presence of a constraints.

Beside these properties in this paper and in its companion paper [11] we wish to address other kind of questions related to the physical gain of these theories: the main advantage of multisymplectic formalisms is to offer us a Hamiltonian theory which is consistent with the principles of Relativity, i.e. being *covariant*. Recall for instance that for all the multisymplectic formalisms which have been proposed one does not need to use a privilege time coordinate. One of our ambitions in this paper was to try to extend this democracy between space and time coordinates to the coordinates on fiber manifolds (i.e. along the fields themselves). This is quite in the spirit of the Kaluza–Klein theory and its modern avatars: 11-dimensional supergravity, string theory and M-theory. This concern leads us naturally to replace De Donder–Weyl by the Dedecker theory. In particular we do not need in our formalism to split the variables into the horizontal (i.e. corresponding to space-time coordinates) and vertical (i.e. non horizontal) categories.

Moreover we may think that we start from a (hypothetical) geometrical model where space-time and fields variables would not be distinguished *a priori* and then ask how to make sense of a space-time coordinate function (that we call a "r-regular" in Section 3.2). A variant of this question would be how to define a constant time hypersurface (that we call a "slice" in Section 3.2) without referring to a given space-time background. We propose in Section 3.2 a definition of r-regular functions and of slices which, roughly speaking, requires a slice to be transversal to all Hamiltonian n-curves. Here the idea is that the dynamics only (i.e. the Hamiltonian equation) should determine what are the slices. We give in Section 4.2 a characterization of these slices in the case where the multisymplectic manifold is $\Lambda^n T^*\mathcal{N}$.

These questions are connected to the concept of observable functionals over the set of solutions of the Hamilton equation. First because by using a codimension r slice Σ and an $(n-r)$-form F on the multisymplectic manifold one can define such a functional by integrating F over the the intersection of Σ with a Hamiltonian curve. And second because one is then led to impose conditions on F in such a way that the resulting functional carries only dynamical information. The analysis of these conditions is the subject of our companion paper [11]. And we believe that the conditions required on these forms are connected with the definitions of r-regular functions given in this paper, although we have not completely elucidated this point.

Lastly in a future paper [12] we investigate gauge theories, addressing the question of how to formulate a fully covariant multisymplectic for them.

Note that the Lepage–Dedecker theory expounded here does not answer this question completely, because a connection cannot be seen as a submanifold. We will show there that it is possible to adapt this theory and that a convenient covariant framework consists in looking at gauge fields as *equivariant* submanifolds over the principal bundle of the theory, i.e. satisfying some suitable zeroth and first order differential constraints.

1.2 Notations

The Kronecker symbol δ^μ_ν is equal to 1 if $\mu = \nu$ and equal to 0 otherwise. We shall also set

$$\delta^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p} := \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \dots & \delta^{\mu_1}_{\nu_p} \\ \vdots & & \vdots \\ \delta^{\mu_p}_{\nu_1} & \dots & \delta^{\mu_p}_{\nu_p} \end{vmatrix}.$$

In most examples, $\eta_{\mu\nu}$ is a constant metric tensor on $\mathbb{R}^n$ (which may be Euclidean or Minkowskian). The metric on his dual space his $\eta^{\mu\nu}$. Also, ω will often denote a volume form on some space-time: in local coordinates $\omega = dx^1 \wedge \cdots \wedge dx^n$ and we will use several times the notation $\omega_\mu := \frac{\partial}{\partial x^\mu} \lrcorner\, \omega$, $\omega_{\mu\nu} := \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu} \lrcorner\, \omega$, etc. Partial derivatives $\frac{\partial}{\partial x^\mu}$ and $\frac{\partial}{\partial p_{\alpha_1\cdots\alpha_n}}$ will be sometime abbreviated by ∂_μ and $\partial^{\alpha_1\cdots\alpha_n}$ respectively.

When an index or a symbol is omitted in the middle of a sequence of indices or symbols, we denote this omission by $\widehat{\ }$. For example $a_{i_1\cdots\widehat{i_p}\cdots i_n} := a_{i_1\cdots i_{p-1}i_{p+1}\cdots i_n}$, $dx^{\alpha_1} \wedge \cdots \wedge \widehat{dx^{\alpha_\mu}} \wedge \cdots \wedge dx^{\alpha_n} := dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{\mu-1}} \wedge dx^{\alpha_{\mu+1}} \wedge \cdots \wedge dx^{\alpha_n}$.

If $\mathcal{N}$ is a manifold and $\mathcal{FN}$ a fiber bundle over $\mathcal{N}$, we denote by $\Gamma(\mathcal{N}, \mathcal{FN})$ the set of smooth sections of $\mathcal{FN}$. Lastly we use the following notations concerning the exterior algebra of multivectors and differential forms. If $\mathcal{N}$ is a differential N-dimensional manifold and $0 \leq k \leq N$, $\Lambda^k T\mathcal{N}$ is the bundle over $\mathcal{N}$ of k-multivectors (k-vectors in short) and $\Lambda^k T^*\mathcal{N}$ is the bundle of differential forms of degree k (k-forms in short). Setting $\Lambda T\mathcal{N} := \oplus_{k=0}^N \Lambda^k T\mathcal{N}$ and $\Lambda T^*\mathcal{N} := \oplus_{k=0}^N \Lambda^k T^*\mathcal{N}$, there exists a unique duality evaluation map between $\Lambda T\mathcal{N}$ and $\Lambda T^*\mathcal{N}$ such that for every decomposable k-vector field X, i.e. of the form $X = X_1 \wedge \cdots \wedge X_k$, and for every l-form μ, then $\langle X, \mu \rangle = \mu(X_1, \cdots, X_k)$ if $k = l$ and $= 0$ otherwise. Then interior products $\lrcorner$ and $\llcorner$ are operations defined as follows. If $k \leq l$, the product $\lrcorner : \Gamma(\mathcal{N}, \Lambda^k T\mathcal{N}) \times \Gamma(\mathcal{N}, \Lambda^l T^*\mathcal{N}) \longrightarrow \Gamma(\mathcal{N}, \Lambda^{l-k} T^*\mathcal{N})$ is given by

$$\langle Y, X \lrcorner \mu \rangle = \langle X \wedge Y, \mu \rangle, \quad \forall (l-k)\text{-vector } Y.$$

And if $k \geq l$, the product $\llcorner : \Gamma(\mathcal{N}, \Lambda^k T\mathcal{N}) \times \Gamma(\mathcal{N}, \Lambda^l T^*\mathcal{N}) \longrightarrow \Gamma(\mathcal{N}, \Lambda^{k-l} T\mathcal{N})$ is given by

$$\langle X \llcorner \mu, \nu \rangle = \langle X, \mu \wedge \nu \rangle, \quad \forall (k-l)\text{-form } \nu.$$

2 The Lepage–Dedecker theory

We expound here a Hamiltonian formulation of a large class of first order variational problems in an intrinsic way. Details and computations in coordinates can be found in [14], [9].

2.1 Hamiltonian formulation of variational problems with several variables

2.1.1 Lagrangian formulation

The category of Lagrangian variational problems we start with is described as follows. We consider $n, k \in \mathbb{N}^*$ and a smooth manifold $\mathcal{N}$ of dimension $n+k$; $\mathcal{N}$ will be equipped with a closed nowhere vanishing "space-time volume" n-form ω. We define

- the Grassmannian bundle $Gr^n\mathcal{N}$, it is the fiber bundle over $\mathcal{N}$ whose fiber over $q \in \mathcal{N}$ is $Gr^n_q\mathcal{N}$, the set of all oriented n-dimensional vector subspaces of $T_q\mathcal{N}$.
- the subbundle $Gr^\omega\mathcal{N} := \{(q,T) \in Gr^n\mathcal{N} / \omega_{q|T} > 0\}$.
- the set $\mathcal{G}^\omega$, it is the set of all oriented n-dimensional submanifolds $G \subset \mathcal{N}$, such that $\forall q \in G$, $T_qG \in Gr^\omega_q\mathcal{N}$ (i.e. the restriction of ω on G is positive everywhere).

Lastly we consider any Lagrangian function L, i.e. a smooth function $L : Gr^\omega\mathcal{N} \longmapsto \mathbb{R}$. Then the Lagrangian of any $G \in \mathcal{G}^\omega$ is the integral

$$\mathcal{L}[G] := \int_G L\left(q, T_qG\right)\omega \qquad (2)$$

We say that a submanifold $G \in \mathcal{G}^\omega$ is a **critical point of** $\mathcal{L}$ if and only if, for any compact $K \subset \mathcal{N}$, $G \cap K$ is a critical point of $\mathcal{L}_K[G] := \int_{G\cap K} L\left(q, T_qG\right)\omega$

with respect to variations with support in K.

It will be useful to represent $Gr^n\mathcal{N}$ differently, by means of n-vectors. For any $q \in \mathcal{N}$, we define $D^n_q\mathcal{N}$ to be the set of decomposable n-vectors[2], i.e. elements $z \in \Lambda^n T_q\mathcal{N}$ such that there exist n vectors $z_1,...,z_n \in T_q\mathcal{N}$ satisfying $z = z_1 \wedge \cdots \wedge z_n$. Then $D^n\mathcal{N}$ is the fiber bundle whose fiber at each $q \in \mathcal{N}$ is $D^n_q\mathcal{N}$. Moreover the map

$$\begin{array}{ccc} D^n_q\mathcal{N} & \longrightarrow & Gr^n_q\mathcal{N} \\ z_1 \wedge \cdots \wedge z_n & \longmapsto & T(z_1, \cdots, z_n), \end{array}$$

where $T(z_1, \cdots, z_n)$ is the vector space spanned and oriented by $(z_1, \cdots, z_n)$, induces a diffeomorphism between $\left(D^n_q\mathcal{N} \setminus \{0\}\right)/\mathbb{R}^*_+$ and $Gr^n_q\mathcal{N}$. If we set also $D^\omega_q\mathcal{N} := \{(q,z) \in D^n_q\mathcal{N}/\omega_q(z) = 1\}$, the same map allow us also to identify $Gr^\omega_q\mathcal{N}$ with $D^\omega_q\mathcal{N}$.

This framework includes a large variety of situations as illustrated below.

Example 1 — *Classical point mechanics* — The motion of a point moving in a manifold $\mathcal{Y}$ can be represented by its graph $G \subset \mathcal{N} := \mathbb{R} \times \mathcal{Y}$. If $\pi : \mathcal{N} \longrightarrow \mathbb{R}$ is the canonical projection and t is the time coordinate on $\mathbb{R}$, then $\omega := \pi^* dt$.

Example 2 — *Maps between manifolds* — We consider maps $u : \mathcal{X} \longrightarrow \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are manifolds of dimension n and k respectively and $\mathcal{X}$ is equipped with some non vanishing volume form ω. A first order Lagrangian density can represented as a function $l : T\mathcal{Y} \otimes_{\mathcal{X}\times\mathcal{Y}} T^*\mathcal{X} \longmapsto \mathbb{R}$, where $T\mathcal{Y} \otimes_{\mathcal{X}\times\mathcal{Y}} T^*\mathcal{X} := \{(x,y,v)/(x,y) \in \mathcal{X} \times \mathcal{Y}, v \in T_y\mathcal{Y} \otimes T^*_x\mathcal{X}\}$. (We use here a notation which exploits the canonical identification of $T_y\mathcal{Y} \otimes T^*_x\mathcal{X}$ with the set of linear mappings from $T_x\mathcal{X}$ to $T_y\mathcal{Y}$; note that the bundle $T\mathcal{Y} \otimes_{\mathcal{X}\times\mathcal{Y}} T^*\mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{Y}$ is diffeomorphic to the first jet bundle $J^1\mathcal{F} \longrightarrow \mathcal{F}$, where $\mathcal{F} = \mathcal{X} \times \mathcal{Y}$ is a trivial bundle over $\mathcal{X}$). The action of a map u is

$$\ell[u] := \int_{\mathcal{X}} l(x, u(x), du(x))\omega.$$

In local coordinates x^μ such that $\omega = dx^1 \wedge \cdots \wedge dx^n$, critical points of ℓ satisfy the Euler-Lagrange equation $\sum_{\mu=1}^n \frac{\partial}{\partial x^\mu}\left(\frac{\partial l}{\partial v^i_\mu}(x, u(x), du(x))\right) = \frac{\partial l}{\partial y^i}(x, u(x), du(x))$, $\forall i = 1, \cdots, k$..

Then we set $\mathcal{N} := \mathcal{X} \times \mathcal{Y}$ and denoting by $\pi : \mathcal{N} \longrightarrow \mathcal{X}$ the canonical projection, we use the volume form $\omega \simeq \pi^*\omega$. Any map u can be represented by

[2] another notation for this set would be $D\Lambda^n T_q\mathcal{N}$, for it reminds that it is a subset of $\Lambda^n T_q\mathcal{N}$, but we have chosen to lighten the notation.

its graph $G_u := \{(x, u(x))/\, x \in \mathcal{X}\} \in \mathcal{G}^\omega$, (and conversely if $G \in \mathcal{G}^\omega$ then the condition $\omega_{|G} > 0$ forces G to be the graph of some map). For all $(x, y) \in \mathcal{N}$ we also have a diffeomorphism

$$\begin{array}{ccc} T_y\mathcal{Y} \otimes T_x^*\mathcal{X} & \longrightarrow & Gr^\omega_{(x,y)}\mathcal{N} \simeq D^\omega_{(x,y)}\mathcal{N} \\ v & \longmapsto & T(v), \end{array}$$

where $T(v)$ is the graph of the linear map $v : T_x\mathcal{X} \longrightarrow T_y\mathcal{Y}$. Then if we set $L(x, y, T(v)) := l(x, y, v)$, the action defined by (2) coincides with ℓ.

Example 3 — *Sections of a fiber bundle* — This is a particular case of our setting, where $\mathcal{N}$ is the total space of a fiber bundle with base manifold $\mathcal{X}$. The set $\mathcal{G}^\omega$ is then just the set of smooth sections.

2.1.2 The Legendre correspondence

Now we consider the manifold $\Lambda^nT^*\mathcal{N}$ and the projection mapping $\Pi : \Lambda^nT^*\mathcal{N} \longrightarrow \mathcal{N}$. We shall denote by p an n-form in the fiber $\Lambda^nT_q^*\mathcal{N}$. There is a canonical n-form θ called the *Poincaré–Cartan* form defined on $\Lambda^nT^*\mathcal{N}$ as follows: $\forall (q, p) \in \Lambda^nT^*\mathcal{N}$, $\forall X_1, \cdots, X_n \in T_{(q,p)}\left(\Lambda^nT^*\mathcal{N}\right)$,

$$\theta_{(q,p)}(X_1, \cdots, X_n) := p\left(\Pi_*X_1, \cdots, \Pi_*X_n\right) = \langle \Pi_*X_1 \wedge \cdots \wedge \Pi_*n, p\rangle,$$

where $\Pi_*X_\mu := d\Pi_{(q,p)}(X_\mu)$. If we use local coordinates $(q^\alpha)_{1\leq\alpha\leq n+k}$ on $\mathcal{N}$, then a basis of $\Lambda^nT_q^*\mathcal{N}$ is the family $(dq^{\alpha_1} \wedge \cdots \wedge dq^{\alpha_n})_{1\leq\alpha_1<\cdots<\alpha_n\leq n+k}$ and we denote by $p_{\alpha_1\cdots\alpha_n}$ the coordinates on $\Lambda^nT_q^*\mathcal{N}$ in this basis. Then θ writes

$$\theta := \sum_{1\leq\alpha_1<\cdots<\alpha_n\leq n+k} p_{\alpha_1\cdots\alpha_n} dq^{\alpha_1} \wedge \cdots \wedge dq^{\alpha_n}. \tag{3}$$

Its differential is the **multisymplectic** form $\Omega := d\theta$ and will play the role of generalized symplectic form.

In order to build the analogue of the Legendre transform we consider the fiber bundle $Gr^\omega\mathcal{N} \times_\mathcal{N} \Lambda^nT^*\mathcal{N} := \{(q, z, p)/q \in \mathcal{N}, z \in Gr^\omega_q\mathcal{N} \simeq D^\omega_q\mathcal{N}, p \in \Lambda^nT_q^*\mathcal{N}\}$ and we denote by $\widehat{\Pi} : Gr^\omega\mathcal{N} \times_\mathcal{N} \Lambda^nT^*\mathcal{N} \longrightarrow \mathcal{N}$ the canonical projection. To summarize:

$$\begin{tikzcd}
 & Gr^\omega\mathcal{N} \times_\mathcal{N} \Lambda^nT^*\mathcal{N} \arrow[r, "\Pi^H"] \arrow[d, "\Pi^L"] \arrow[dr, "\widehat{\Pi}"] & \Lambda^nT^*\mathcal{N} \arrow[d, "\Pi"] & \mathcal{M} \arrow[l, "\imath"'] \arrow[dl, "\Pi_{|\mathcal{M}}"] \\
Gr^n\mathcal{N} & Gr^\omega\mathcal{N} \arrow[l, "\imath"] \arrow[r] & \mathcal{N} &
\end{tikzcd}$$

We define on $Gr^\omega\mathcal{N} \times_{\mathcal{N}} \Lambda^n T^*\mathcal{N}$ the function

$$W(q,z,p) := \langle z, p\rangle - L(q,z).$$

Note that for each (q,z,p) there a vertical subspace $V_{(q,z,p)} \subset T_{(q,z,p)}(Gr^\omega\mathcal{N} \times_{\mathcal{N}} \Lambda^n T^*\mathcal{N})$, which is canonically defined as the kernel of

$$d\widehat{\Pi}_{(q,z,p)} : T_{(q,z,p)}\left(Gr^\omega\mathcal{N} \times_{\mathcal{N}} \Lambda^n T^*\mathcal{N}\right) \longrightarrow T_q\mathcal{N}.$$

We can further split $V_{(q,z,p)} \simeq T_zD_q^\omega\mathcal{N} \oplus T_p\Lambda^n T_q^*\mathcal{N}$, where $T_zD_q^\omega\mathcal{N} \simeq \mathrm{Ker}d\Pi^H_{(q,z,p)}$ and $T_p\Lambda^n T_q^*\mathcal{N} \simeq \mathrm{Ker}d\Pi^L_{(q,z,p)}$. Then, for any function F defined on $Gr^\omega\mathcal{N} \times_{\mathcal{N}} \Lambda^n T^*\mathcal{N}$, we denote respectively by $\partial F/\partial z(q,z,p)$ and $\partial F/\partial p(q,z,p)$ the restrictions of the differential[3] $dF_{(q,z,p)}$ on respectively $T_zD_q^\omega\mathcal{N}$ and $T_p\Lambda^n T_q^*\mathcal{N}$.

Instead of a Legendre transform we shall rather use a Legendre correspondence: we write

$$(q,z) \longleftrightarrow (q,p) \quad \text{if and only if} \quad \frac{\partial W}{\partial z}(q,z,p) = 0. \tag{4}$$

Let us try to picture geometrically the situation (see figure 2.1.2): $D_q^\omega\mathcal{N}$ is

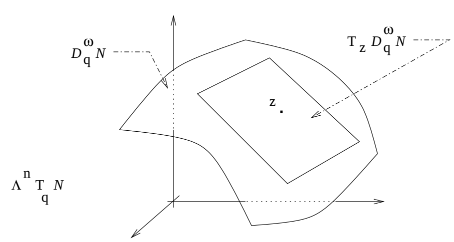

Figure 2: $T_zD_q^\omega\mathcal{N}$ is a vector subspace of $\Lambda^n T_q\mathcal{N}$

a smooth submanifold of dimension nk of the vector space $\Lambda^n T_q\mathcal{N}$, which is of dimension $\frac{(n+k)!}{n!k!}$; $T_zD_q^\omega\mathcal{N}$ is thus a vector subspace of $\Lambda^n T_q\mathcal{N}$. And $\frac{\partial L}{\partial z}(q,z)$ or $\frac{\partial W}{\partial z}(q,z,p)$ can be understood as linear forms on $T_zD_q^\omega\mathcal{N}$ whereas $p \in \Lambda^n T_q^*\mathcal{N}$ as a linear form on $\Lambda^n T_q\mathcal{N}$. So the meaning of the right hand side of (4) is that the restriction of p at $T_zD_q^\omega\mathcal{N}$ coincides with $\frac{\partial L}{\partial z}(q,z,p)$:

$$p_{|T_zD_q^\omega\mathcal{N}} = \frac{\partial L}{\partial z}(q,z). \tag{5}$$

[3]However in order to make sense of "$\partial F/\partial q(q,z,p)$" we would need to define a "horizontal" subspace of $T_{(q,z,p)}\left(Gr^\omega\mathcal{N} \times_{\mathcal{N}} \Lambda^n T^*\mathcal{N}\right)$, which requires for instance the use of a connection on the bundle $Gr^\omega\mathcal{N} \times_{\mathcal{N}} \Lambda^n T^*\mathcal{N} \longrightarrow \mathcal{N}$. Indeed such a horizontal subspace prescribes a inertial law on $\mathcal{N}$, such a law would have a sense on a Galilee or Minkowski space-time but not in general relativity.

Given $(q,z) \in Gr^\omega \mathcal{N}$ we define the **enlarged pseudofiber** in q to be:

$$P_q(z) := \{p \in \Lambda^n T_q^*\mathcal{N} / \frac{\partial W}{\partial z}(q,z,p) = 0\}.$$

In other words, $p \in P_q(z)$ if it is a solution of (5). Obviously $P_q(z)$ is not empty; moreover given some $p_0 \in P_q(z)$,

$$p_1 \in P_q(z), \iff p_1 - p_0 \in \left(T_z D_q^\omega \mathcal{N}\right)^\perp := \{p \in \Lambda^n T_q^*\mathcal{N} / \forall \zeta \in T_z D_q^\omega \mathcal{N}, p(\zeta) = 0\}. \tag{6}$$

So $P_q(z)$ is an affine subspace of $\Lambda^n T_q^*\mathcal{N}$ of dimension $\frac{(n+k)!}{n!k!} - nk$. Note that in case where $n = 1$ (the classical mechanics of point) then $\dim P_q(z) = 1$: this is due to the fact that we are still free to fix arbitrarily the momentum component dual to the time (i.e. the energy)[4].

We now define

$$\mathcal{P}_q := \bigcup_{z \in D_q^\omega \mathcal{N}} P_q(z) \subset \Lambda^n T_q^*\mathcal{N}, \quad \forall q \in \mathcal{N}$$

and we denote by $\mathcal{P} := \cup_{q \in \mathcal{N}} \mathcal{P}_q$ the associated bundle over $\mathcal{N}$. We also let, for all $(q,p) \in \Lambda^n T^*\mathcal{N}$,

$$Z_q(p) := \{z \in Gr_q^\omega \mathcal{N} / p \in P_q(z)\}.$$

It is clear that $Z_q(p) \neq \emptyset \iff p \in \mathcal{P}_q$. Now in order to go further we need to choose some submanifold $\mathcal{M}_q \subset \mathcal{P}_q$, its dimension is not fixed *a priori*.

Legendre Correspondence Hypothesis — *We assume that there exists a subbundle manifold $\mathcal{M} \subset \mathcal{P} \subset \Lambda^n T^*\mathcal{N}$ over $\mathcal{N}$ where* $\dim \mathcal{M} =: M$ *such that,*

- *for all $q \in \mathcal{N}$ the fiber $\mathcal{M}_q$ is a smooth submanifold, possibly with boundary, of dimension $1 \leq M - n - k \leq \frac{(n+k)!}{n!k!}$*
- *for any $(q,p) \in \mathcal{M}$, $Z_q(p)$ is a non empty smooth connected submanifold of $Gr_q^\omega \mathcal{N}$*

[4] a simple but more interesting example is provided by variational problems on maps $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Then one is led to the multisymplectic manifold $\Lambda^2 T^*\mathbb{R}^4$. And given any $(q,z) \in Gr^\omega \mathbb{R}^4$ the enlarged pseudofiber $P_q(z) \subset \Lambda^2 T^*\mathbb{R}^4$ is an affine plane parallel to $\mathbb{R}\left[\left(v_1^1 v_2^2 - v_1^2 v_2^1\right) dx^1 \wedge dx^2 - \epsilon_{ij} v_\nu^j dy^i \wedge dx^\nu + dy^1 \wedge dy^2\right] \oplus \mathbb{R} dx^1 \wedge dx^2$, where (using the notations of Example 2) $T(v) = z$. For details see Paragraph 2.2.2.

- *if* $z_0 \in Z_q(p)$*, then we have* $Z_q(p) = \{z \in D^\omega_q\mathcal{N} / \forall \dot{p} \in T_p\mathcal{M}_q, \langle z - z_0, \dot{p}\rangle = 0\}$.

Remark — In the case where $M = \frac{(n+k)!}{n!k!} + n + k$, then $\mathcal{M}_q$ is an open subset of $\Lambda^n T^*_q\mathcal{N}$ and so $T_p\mathcal{M}_q \simeq \Lambda^n T^*_q\mathcal{N}$. Hence the last assumption of the Legendre Correspondence Hypothesis means that $Z_q(p)$ is reduced to a point. In general this condition will imply that the inverse correspondence can be rebuild by using the Hamiltonian function (see Lemma 2.2 below).

Lemma 2.1. *Assume that the Legendre correspondence hypothesis is true. Then for all* $(q,p) \in \mathcal{M}$*, the restriction of* W *to* $\{q\}\times Z_q(p)\times\{p\}$ *is constant.*

Proof — Since $Z_q(p)$ is smooth and connected, it suffices to prove that W is constant along any smooth path inside $\{(q,z,p)/q, p \text{ fixed }, z \in Z_q(p)\}$. Let $s \longmapsto z(s)$ be a smooth path with values into $Z_q(p)$, then

$$\frac{d}{ds}\left(W(q,z(s),p)\right) = \frac{\partial W}{\partial z}(q,z(s),p)\left(\frac{dz}{ds}\right) = 0,$$

because of (4). ■

A straightforward consequence of Lemma 2.1 is that we can define the **Hamiltonian function** $\mathcal{H} : \mathcal{M} \longrightarrow \mathbb{R}$ by

$$\mathcal{H}(q,p) := W(q,z,p), \quad \text{where } z \in Z_q(p), \text{ i.e. } \frac{\partial W}{\partial z}(q,z,p) = 0.$$

Any function f constructed this way will be called **Legendre Image Hamiltonian function**. In the following, for all $(q,p) \in \mathcal{M}$ and for all $z \in D^n_q\mathcal{N}$ we denote by

$$\begin{array}{rccc} z_{|T_p\mathcal{M}_q} : & T_p\mathcal{M}_q & \longrightarrow & \mathbb{R} \\ & \dot{p} & \longmapsto & \langle z, \dot{p}\rangle \end{array}$$

the linear map induced by z on $T_p\mathcal{M}_q$. Then:

Lemma 2.2. *Assume that the Legendre Correspondence Hypothesis is true. Then*[5]

[5] The advised Reader may expect to have also the relation "$\frac{\partial \mathcal{H}}{\partial q}(q,p) = -\frac{\partial L}{\partial q}(q,z)$". But as remarked above the meaning of $\frac{\partial \mathcal{H}}{\partial q}$ and $\frac{\partial L}{\partial q}$ is not clearly defined, because we did not introduce a connection on the bundle $Gr^\omega\mathcal{N} \times_{\mathcal{N}} \Lambda^n T^*\mathcal{N}$. This does not matter and we shall make the economy of this relation later ! (cf footnote 2)

(i) $\forall(q,p) \in \mathcal{M}$ *and* $\forall z \in Z_q(p)$,

$$\frac{\partial \mathcal{H}}{\partial p}(q,p) = z_{|T_p\mathcal{M}_q}. \tag{7}$$

As a corollary of the above formula, $z_{|T_p\mathcal{M}_q}$ *does not depend on the choice of* $z \in Z_q(p)$.

(ii) Conversely if $(q,p) \in \mathcal{M}$ *and* $z \in D^\omega_q\mathcal{N}$ *satisfy condition (7), then* $z \in Z_q(p)$ *or equivalently* $p \in P_q(z)$.

Proof — Let $(q,p) \in \mathcal{M}$ and $(0,\dot{p}) \in T_{(q,p)}\mathcal{M}$, where $\dot{p} \in T_p\mathcal{M}_q$. In order to compute $d\mathcal{H}_{(q,p)}(0,\dot{p})$, we consider a smooth path $s \longmapsto (q,p(s))$ with values into $\mathcal{M}_q$ whose derivative at $s=0$ coincides with $(0,\dot{p})$. We can further lift this path into another one $s \longmapsto (q,z(s),p(s))$ with values into $Gr^\omega_q\mathcal{N} \times \mathcal{M}_q$, in such a way that $z(s) \in Z_q(p(s))$, $\forall s$. Then using (5) we obtain

$$\begin{aligned} \frac{d}{ds}\left(\mathcal{H}(q,p(s))\right)_{|s=0} &= \frac{d}{ds}\left(\langle z(s),p(s)\rangle - L(q,p(s))\ \right)_{|s=0} \\ &= \langle \dot{z},p\rangle + \langle z,\dot{p}\rangle - \frac{\partial L}{\partial z}(q,z)(\dot{z}) = \langle z,\dot{p}\rangle, \end{aligned}$$

from which (7) follows. This proves (i).
The proof of (ii) uses the Legendre Correspondence Hypothesis: consider $z, z_0 \in D^n_q\mathcal{N}$ and assume that $z_0 \in Z_q(p)$ and that z satisfies (7). Then by applying the conclusion (i) of the Lemma to z_0 we deduce that $\partial\mathcal{H}/\partial p(q,p) = z_{0|T_p\mathcal{M}_q}$ and thus $(z-z_0)_{|T_p\mathcal{M}_q} = 0$. Hence by the Legendre Correspondence Hypothesis we deduce that $z \in Z_q(p)$. ■

A further property is that, given $(q,z) \in D^\omega\mathcal{N}$, it is possible to find a $p \in P_q(z)$ and to choose the value of $\mathcal{H}(q,p)$ simultaneously. This property will be useful in the following in order to simplify the Hamilton equations. For that purpose we define, for all $h \in \mathbb{R}$, the **pseudofiber**:

$$P^h_q(z) := \{p \in P_q(z) / \mathcal{H}(q,p) = h\}.$$

We then have:

Lemma 2.3. *For all* $(q,z) \in Gr^\omega\mathcal{N}$ *the pseudofiber* $P^h_q(z)$ *is a affine subspace*[6] *of* $\Lambda^nT^*_q\mathcal{N}$ *parallel to* $\left(T_zD^n_q\mathcal{N}\right)^\perp$. *Hence dim* $P^h_q(z) =$ *dim* $P_q(z) - 1 = \frac{(n+k)!}{n!k!} - nk - 1$.

[6]again in the instance of variational problems on maps $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and the multisymplectic manifold $\Lambda^2T^*\mathbb{R}^4$, for any $(q,z) \in Gr^\omega\mathbb{R}^4$ the pseudofiber $P^h_q(z) \subset \Lambda^2T^*\mathbb{R}^4$ is an affine line parallel to $\mathbb{R}\left[\left(v^1_1v^2_2 - v^2_1v^1_2\right)dx^1 \wedge dx^2 - \epsilon_{ij}v^j_\nu dy^i \wedge dx^\nu + dy^1 \wedge dy^2\right]$, where $T(v) = z$. (See also Paragraph 2.2.2.)

Proof — We first remark that, $\forall q \in \mathcal{N}$ and $\forall z \in D^\omega_q\mathcal{N}$, ω_q belongs to $\left(T_zD^\omega_q\mathcal{N}\right)^\perp$, because of the definition of $D^\omega_q\mathcal{N}$. So $\forall \lambda \in \mathbb{R}$, $\forall p \in P_q(z)$, we deduce from (6) that $p + \lambda\omega_q \in P_q(z)$ and thus

$$\begin{array}{rcl} \mathcal{H}(q, p + \lambda\omega_q) & = & \langle z, p + \lambda\omega_q\rangle - L(q,z) \\ & = & \mathcal{H}(q,p) + \lambda\langle z, \omega_q\rangle = \mathcal{H}(q,p) + \lambda. \end{array}$$

Hence we deduce that $\forall h \in \mathbb{R}$, $\forall p \in P_q(z)$, $\exists!\lambda \in \mathbb{R}$ such that

$$\mathcal{H}(q, p + \lambda\omega_q) = h,$$

so that $P^h_q(z)$ is non empty. Moreover if $p_0 \in P^h_q(z)$ then $p_1 \in P^h_q(z)$ if and only if $p_1 - p_0 \in \left(T_zD^\omega_q\mathcal{N}\right)^\perp \cap z^\perp$, where $z^\perp := \{p \in \Lambda^nT^*_q\mathcal{N} / \langle z, p\rangle = 0\}$. In order to conclude observe that $\left(T_zD^\omega_q\mathcal{N}\right)^\perp \cap z^\perp = \left(T_zD^n_q\mathcal{N}\right)^\perp$. ∎

2.1.3 Critical points

We now look at critical points of the Lagrangian functional using the above framework. Instead of the usual approach using jet bundles and contact structure, we shall derive Hamilton equations directly, without writing the Euler–Lagrange equation.

First we extend the form ω on $\mathcal{M}$ by setting $\omega \simeq \Pi^*\omega$, where $\Pi : \mathcal{M} \longrightarrow \mathcal{N}$ is the bundle projection, and we define $\widehat{\mathcal{G}}^\omega$ to be the set of oriented n-dimensional submanifolds Γ of $\mathcal{M}$, such that $\omega_{|\Gamma} > 0$ everywhere. A consequence of this inequality is that the restriction of the projection Π to any $\Gamma \in \widehat{\mathcal{G}}^\omega$ is an embedding into $\mathcal{N}$: we denote by $\Pi(\Gamma)$ its image. It is clear that $\Pi(\Gamma) \in \mathcal{G}^\omega$. Then we can view Γ as (the graph of) a section $q \longmapsto p(q)$ of the pull-back of the bundle $\mathcal{M} \longrightarrow \mathcal{N}$ by the inclusion $\Pi(\Gamma) \subset \mathcal{N}$.

Second, we define the subclass $\mathfrak{p}\widehat{\mathcal{G}}^\omega \subset \widehat{\mathcal{G}}^\omega$ as the set of $\Gamma \in \widehat{\mathcal{G}}^\omega$ such that, $\forall (q,p) \in \Gamma$, $p \in P_q(T_q\Pi(\Gamma))$ (a contact condition). [As we will see later it can be viewed as the subset of $\Gamma \in \widehat{\mathcal{G}}^\omega$ which satisfy half of the Hamilton equations.] And given some $G \in \mathcal{G}^\omega$, we denote by $\mathfrak{p}\widehat{G} \subset \mathfrak{p}\widehat{\mathcal{G}}^\omega$ the family of submanifolds $\Gamma \in \mathfrak{p}\widehat{\mathcal{G}}^\omega$ such that $\Pi(\Gamma) = G$ and we say that $\mathfrak{p}\widehat{G}$ is the set of **Legendre lifts** of G. We hence have $\mathfrak{p}\widehat{\mathcal{G}}^\omega = \cup_{G \in \mathcal{G}^\omega}\mathfrak{p}\widehat{G}$.

Lastly, we define the functional on $\widehat{\mathcal{G}}^\omega$

$$\mathcal{I}[\Gamma] := \int_\Gamma \theta - \mathcal{H}\omega.$$

Properties of the restriction of $\mathcal{I}$ to $\mathfrak{p}\widehat{\mathcal{G}}^\omega$ — First we claim that

$$\mathcal{I}[\Gamma] = \mathcal{L}[G], \quad \forall G \in \mathcal{G}^\omega, \forall \Gamma \in \mathfrak{p}\widehat{G}. \tag{8}$$

This follows from

$$\begin{aligned}\int_\Gamma \theta - \mathcal{H}\omega &= \int_G \langle z_G, p(q)\rangle\omega - \mathcal{H}(q,p(q))\omega \\ &= \int_G \left(\langle z_G, p(q)\rangle - \langle z_G, p(q)\rangle + L(q,z_G)\right)\omega = \int_G L(q,z_G)\omega,\end{aligned}$$

where $G \longrightarrow \mathcal{M} : q \longmapsto (q, p(q))$ is the parametrization of Γ and where z_G is the unique n-vector in $D^\omega_q\mathcal{N}$ (for $q \in G$) which spans T_qG.
Second let us exploit relation (8) to compute the first variation of $\mathcal{I}$ at any submanifold $\Gamma \in \mathfrak{p}\widehat{G}$, i.e. a Legendre lift of $G \in \mathcal{G}^\omega$. We let $\xi \in \Gamma(\mathcal{N}, T\mathcal{N})$ be a smooth vector field with compact support and G_s, for $s \in \mathbb{R}$, be the image of G by the flow diffeomorphism $e^{s\xi}$. For small values of s, G_s is still in $\mathcal{G}^\omega$ and for all $q_s := e^{s\xi}(q) \in G_s$ we shall denote by z_s the unique n-vector in $D^\omega_{q_s}\mathcal{N}$ which spans $T_{q_s}G_s$. Then we choose a smooth section $(s, q_s) \longmapsto p(q)_s$ in such a way that $p(q)_s \in P_{q_s}(z_s)$. This builds a family of Legendre lifts $\Gamma_s = \{(q_s, p(q)_s)\}$. We can now use relation (8): $\mathcal{I}[\Gamma_s] = \mathcal{L}[G_s]$ and derivate it with respect to s. Denoting by $\widehat{\xi} \in T_{(q,p(q))}\mathcal{M}$ the vector $d(q_s, p(q)_s)/ds_{|s=0}$, we obtain

$$\delta\mathcal{I}[\Gamma](\widehat{\xi}) = \frac{d}{ds}\mathcal{I}[\Gamma_s]_{|s=0} = \frac{d}{ds}\mathcal{L}[G_s]_{|s=0} = \delta\mathcal{L}[G](\xi). \tag{9}$$

Variations of $\mathcal{I}$ along $T_p\mathcal{M}_q$ — On the other hand for all $\Gamma \in \widehat{\mathcal{G}}^\omega$ and for all **vertical** tangent vector field along Γ ζ, i.e. such that $d\Pi_{(q,p)}(\zeta) = 0$ or such that $\zeta \in T_p\mathcal{M}_q \subset T_{(q,p)}\mathcal{M}$, we have

$$\delta\mathcal{I}[\Gamma](\zeta) = \int_\Gamma \left(\langle z_{\Pi(\Gamma)}, \zeta\rangle - \frac{\partial\mathcal{H}}{\partial p}(q,p)(\zeta)\right)\omega, \tag{10}$$

where $z_{\Pi(\Gamma)}$ is the unique n-vector in $D^\omega_q\mathcal{N}$ (for $q \in G(\Gamma)$) which spans $T_q\Pi(\Gamma)$. Note that in the special case where $\Gamma \in \mathfrak{p}\widehat{\mathcal{G}}^\omega$, we have $z_{\Pi(\Gamma)} \in Z_q(p)$, so we deduce from (7) and (10) that $\delta\mathcal{I}[\Gamma](\zeta) = 0$. And the converse is true. So $\mathfrak{p}\widehat{\mathcal{G}}^\omega$ can be characterized by requiring that condition (10) is true for all vertical vector fields ζ.

Conclusion — The key point is now that any vector field along Γ can be written $\widehat{\xi} + \zeta$, where $\widehat{\xi}$ and ζ are as above. And for any $G \in \mathcal{G}^\omega$ and for all $\Gamma \in \mathfrak{p}\widehat{G}$, the first variation of $\mathcal{I}$ at Γ with respect to a vector field $\widehat{\xi} + \zeta$, where locally $\widehat{\xi}$ lifts $\xi \in T_q\mathcal{N}$ and $\zeta \in T_p\mathcal{M}_q$, satisfies

$$\delta\mathcal{I}[\Gamma](\widehat{\xi} + \zeta) = \delta\mathcal{L}[G](\xi). \tag{11}$$

We deduce the following.

Theorem 2.1. *(i) For any $G \in \mathcal{G}^\omega$ and for all Legendre lift $\Gamma \in \mathfrak{p}\widehat{G}$, G is a critical point of $\mathcal{L}$ if and only if Γ is a critical point of $\mathcal{I}$.*
(ii) Moreover for all $\Gamma \in \widehat{\mathcal{G}}^\omega$, if Γ is a critical point of $\mathcal{I}$ then Γ is a Legendre lift, i.e. $\Gamma \in \mathfrak{p}\widehat{\Pi(\Gamma)}$ and $\Pi(\Gamma)$ is a critical point of $\mathcal{L}$.

Proof — (i) is a straightforward consequence of (11). Let us prove (ii): if $\Gamma \in \widehat{\mathcal{G}}^\omega$ is a critical point of $\mathcal{I}$, then in particular for all vertical tangent vector field $\zeta \in T_p\mathcal{M}_q$, $\delta\mathcal{I}[\Gamma](\zeta) = 0$ and by (10) this implies $(z_{\Pi(\Gamma)})_{|T_p^*\mathcal{M}_q} = (\partial\mathcal{H}/\partial p)(q,p)$. Then by applying Lemma 2.2–(ii) we deduce that $z_{\Pi(\Gamma)} \in Z_q(p)$. Hence Γ is a Legendre lift. Lastly we use the conclusion of the part (i) of the Theorem to conclude that $G(\Gamma)$ is a critical point of $\mathcal{L}$. ■

Corollary 2.1. *Let $\Gamma \in \widehat{\mathcal{G}}^\omega$ be a critical point of $\mathcal{I}$ and let $\psi : \Gamma \longrightarrow \Lambda^n T^*\mathcal{N}$ be a smooth map satisfy:*

(i) $\Pi \circ \psi = Id_\Gamma$ *(so ψ is a section of the pull-back of $\Lambda^n T^*\mathcal{N}$ by the inclusion map $\iota : \Gamma \longrightarrow \Lambda^n T^*\mathcal{N}$);*

(ii) $\forall (q,p) \in \Gamma$, $\psi(q,p) \simeq \psi(q) \in \left(T_z D_q^\omega \mathcal{N}\right)^\perp$ *(where $z \in Z_q(p)$).*

Then $\tilde{\Gamma} := \{(q, p + \psi(q))/(q,p) \in \Gamma\}$ is another critical point of $\mathcal{I}$.

Proof — By using Theorem 2.1–(ii) we deduce that Γ has the form $\Gamma = \{(q,p)/q \in \Pi(\Gamma), p \in P_q(z_{\Pi(\Gamma)})\}$ and thus $\tilde{\Gamma} = \{(q, p + \psi(q))/q \in \Pi(\Gamma), p \in P_q(z_{\Pi(\Gamma)})\}$. This implies, by using (6), that $\tilde{\Gamma} \in \mathfrak{p}\widehat{\Pi(\Gamma)}$; then $\tilde{\Gamma}$ is also a critical point of $\mathcal{I}$ because of Theorem 2.1–(i). ■

Note that, for any constant $h \in \mathbb{R}$, by choosing $\psi(q) = (h - \mathcal{H}(q,p))\,\omega_q$ (see the proof of Lemma 2.3) in the above Corollary we deform any critical point Γ of $\mathcal{I}$ $\Gamma \in \widehat{\mathcal{G}}^\omega$ into a critical point $\tilde{\Gamma}$ of $\mathcal{I}$ contained in $\mathcal{M}^h := \{m \in \mathcal{M}/\mathcal{H}(m) = h\}$.

Definition 2.1. *An* **Hamiltonian** n**-curve** *is a critical point Γ of $\mathcal{I}$ such that there exists a constant $h \in \mathbb{R}$ such that $\Gamma \subset \mathcal{M}^h$.*

2.1.4 Hamilton equations

We now end this section by looking at the equation satisfied by critical points of $\mathcal{I}$. Let $\Gamma \in \widehat{\mathcal{G}}^\omega$ and $\xi \in \Gamma(\mathcal{M}, T\mathcal{M})$ be a smooth vector field with compact support. We let $e^{s\xi}$ be the flow mapping of ξ and Γ_s be the image of Γ

by $e^{s\xi}$. We let $\mathcal{X}$ be an n-dimensional manifold diffeomorphic to Γ and we denote by

$$\begin{array}{rccc}\sigma: & (0,1)\times\mathcal{X} & \longrightarrow & \mathcal{M}\\ & (s,x) & \longmapsto & \sigma(s,x)\end{array}$$

a map such that if $\gamma_s : x \longmapsto \sigma(s,x)$, then $\gamma = \gamma_0$ is a parametrization of Γ, γ_s is a parametrization of Γ_s and $\frac{\partial}{\partial s}(\sigma(s,x)) = \xi(\sigma(s,x))$. Then

$$\begin{aligned}\mathcal{I}[\Gamma_s] - \mathcal{I}[\Gamma] &= \int_{\mathcal{X}} \gamma_s^*(\theta - \mathcal{H}\omega) - \gamma^*(\theta - \mathcal{H}\omega)\\ &= \int_{\partial((0,s)\times\mathcal{X})} \sigma^*(\theta - \mathcal{H}\omega) = \int_{(0,s)\times\mathcal{X}} d\left(\sigma^*(\theta - \mathcal{H}\omega)\right)\\ &= \int_{(0,s)\times\mathcal{X}} \sigma^*(\Omega - d\mathcal{H}\wedge\omega)).\end{aligned}$$

Thus

$$\begin{aligned}\lim_{s\to 0}\frac{\mathcal{I}[\Gamma_s] - \mathcal{I}[\Gamma]}{s} &= \lim_{s\to 0}\frac{1}{s}\int_{(0,s)\times\mathcal{X}} \sigma^*(\Omega - d\mathcal{H}\wedge\omega)\\ &= \int_{\mathcal{X}} \frac{\partial}{\partial s} \lrcorner\, \sigma^*(\Omega - d\mathcal{H}\wedge\omega) = \int_{\mathcal{X}} \gamma^*(\xi \lrcorner (\Omega - d\mathcal{H}\wedge\omega))\\ &= \int_{\Gamma} \xi \lrcorner (\Omega - d\mathcal{H}\wedge\omega).\end{aligned}$$

We hence conclude that Γ is a critical point of $\mathcal{I}$ if and only if $\forall m \in \Gamma$, $\forall \xi \in T_m\mathcal{M}$, $\forall X \in \Lambda^n T_m\Gamma$,

$$\xi \lrcorner (\Omega - d\mathcal{H}\wedge\omega)(X) = 0 \quad\Longleftrightarrow\quad X \lrcorner (\Omega - d\mathcal{H}\wedge\omega)(\xi) = 0.$$

We thus deduce the following.

Theorem 2.2. *A submanifold $\Gamma \in \widehat{\mathcal{G}}^\omega$ is a critical point of $\mathcal{I}$ if and only if*

$$\forall m \in \Gamma, \forall X \in \Lambda^n T_m\Gamma, \quad X \lrcorner (\Omega - d\mathcal{H}\wedge\omega) = 0. \tag{12}$$

Moreover, if there exists some $h \in \mathbb{R}$ such that $\Gamma \subset \mathcal{M}^h$ (i.e. Γ is a Hamiltonian n-curve) then

$$\forall m \in \Gamma, \exists! X \in \Lambda^n T_m\Gamma, \quad X \lrcorner \Omega = (-1)^n d\mathcal{H}. \tag{13}$$

Recall that, because of Lemma 2.3 and Corollary 2.1, it is always possible to deform a Hamiltonian n-curve $\Gamma \longmapsto \tilde{\Gamma}$ in such a way that $\mathcal{H}$ be constant on $\tilde{\Gamma}$ and $\Pi(\Gamma) = \Pi(\tilde{\Gamma})$.

Proof — We just need to check (13). Let $\Gamma \subset \mathcal{M}^h$. Since $d\mathcal{H}_{|\Gamma} = 0$, $\forall X \in \Lambda^n T_m\Gamma$, $X \lrcorner d\mathcal{H}\wedge\omega = (-1)^n\langle X,\omega\rangle d\mathcal{H}$.. So by choosing the unique X such that $\langle X,\omega\rangle = 1$, we obtain $X \lrcorner d\mathcal{H}\wedge\omega = (-1)^n d\mathcal{H}$. Then (12) is equivalent to (13). ■

2.2 Some examples

We pause to study on some simple examples how the Legendre correspondence and the Hamilton work. In particular in the construction of $\mathcal{M}$ we let a large freedom in the dimension of the fibers $\mathcal{M}_q$, having just the constraint that $\dim \mathcal{M}_q \leq \dim \mathcal{P}_q = \frac{(n+k)!}{n!k!}$. This leads to a large choice of approaches between two opposite ones: the first one consists in using as less variables as possible, i.e. to choose $\mathcal{M}$ to be of minimal dimension (for example the De Donder–Weyl theory), the other one consists in using the largest number of variables, i.e. to choose $\mathcal{M}$ to be equal to the interior of $\mathcal{P}$ (the advantage will be that in some circumstances we avoid degenerate situations).

We focus here on special cases of Example 2 of the previous Section: we consider maps $u : \mathcal{X} \longrightarrow \mathcal{Y}$. We denote by $q^\mu = x^\mu$, if $1 \leq \mu \leq n$, coordinates on $\mathcal{X}$ and by $q^{n+i} = y^i$, if $1 \leq i \leq k$, coordinates on $\mathcal{Y}$. Recall that $\forall x \in \mathcal{X}$, $\forall y \in \mathcal{Y}$, the set of linear maps v from $T_x^*\mathcal{X}$ to $T_y\mathcal{Y}$ can be identified with $T_y\mathcal{Y} \otimes T_x^*\mathcal{X}$. And coordinates representing some $v \in T_y\mathcal{Y} \otimes T_x^*\mathcal{X}$ are denoted by v_μ^i, in such a way that $v = \sum_\alpha \sum_i v_\mu^i \frac{\partial}{\partial y^i} \otimes dx^\mu$. Then through the diffeomorphism $T_y\mathcal{Y} \otimes T_x^*\mathcal{X} \ni v \longmapsto T(v) \in Gr^\omega_{(x,y)}\mathcal{N}$ (where $\mathcal{N} = \mathcal{X} \times \mathcal{Y}$) we obtain coordinates on $Gr^\omega_q\mathcal{N} \simeq D^\omega_q\mathcal{N}$. We also denote by $e := p_{1\cdots n}$, $p_i^\mu := p_{1\cdots(\mu-1)i(\mu+1)\cdots n}$, $p_{i_1 i_2}^{\mu_1\mu_2} := p_{1\cdots(\mu_1-1)i_1(\mu_1+1)\cdots(\mu_2-1)i_2(\mu_2+1)\cdots n}$, etc., so that

$$\Omega = de \wedge \omega + \sum_{j=1}^n \sum_{\mu_1 < \cdots < \mu_j} \sum_{i_1 < \cdots < i_j} dp_{i_1\cdots i_j}^{\mu_1\cdots\mu_j} \wedge \omega_{\mu_1\cdots\mu_j}^{i_1\cdots i_j},$$

where, for $1 \leq p \leq n$,

$$\begin{array}{rcl} \omega & := & dx^1 \wedge \cdots \wedge dx^n \\ \omega_{\mu_1\cdots\mu_p}^{i_1\cdots i_p} & := & dy^{i_1} \wedge \cdots \wedge dy^{i_p} \wedge \left(\frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_p}} \lrcorner\, \omega\right). \end{array}$$

Remark — It can be checked (see for instance [9]) that, by denoting by p^* all coordinates $p_{i_1\cdots i_j}^{\mu_1\cdots\mu_j}$ for $j \geq 1$, the Hamiltonian function has always the form $\mathcal{H}(q, e, p^*) = e + H(q, p^*)$.

2.2.1 The De Donder–Weyl formalism

In the special case of the De Donder–Weyl theory, $\mathcal{M}_q^{DDW}$ is the submanifold of $\Lambda^n T_q^*\mathcal{N}$ defined by the constraints $p_{i_1\cdots i_j}^{\mu_1\cdots\mu_j} = 0$, for all $j \geq 2$ (Observe that these constraints are invariant by a change of coordinates, so that they have an intrinsic meaning.) We thus have

$$\Omega^{DDW} = de \wedge \omega + \sum_\mu \sum_i dp_i^\mu \wedge \omega^i_\mu..$$

Then the equation $\partial W/\partial z(q,z,p) = 0$ is equivalent to $p_i^\mu = \partial l/\partial v_\mu^i(q,v)$, so that the Legendre Correspondence Hypothesis holds if and only if $(q,v) \longmapsto (q, \partial l/\partial v(q,p))$ is an invertible map. Note that then the enlarged pseudofibers $P_q(z)$ intersect $\mathcal{M}_q^{DDW}$ along lines $\{e\omega + \frac{\partial l}{\partial v^i_\mu}(q,v)\omega^i_\mu / e \in \mathbb{R}\}$. So since $\dim \Lambda^n T_q^*\mathcal{N} = \frac{(n+k)!}{n!k!}$, $\dim \mathcal{M}_q^{DDW} = nk+1$ and $\dim P_q(z) = \frac{(n+k)!}{n!k!} - nk$, the Legendre Correspondence Hypothesis can be rephrased by saying that each $P_q(z)$ meets $\mathcal{M}_q^{DDW}$ transversally along a line. Moreover $Z_q(e\omega + p_i^\mu \omega^i_\mu)$ is then reduced to one point, namely $T(v)$, where v is the solution to $p_i^\mu = \frac{\partial l}{\partial v^i_\mu}(q,v)$.

For more details and a description using local coordinates, see [9].

2.2.2 Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ via the Lepage–Dedecker point of view

Let us consider a simple situation where $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$ and $\mathcal{M} \subset \Lambda^2 T^*\mathbb{R}^4$. It corresponds to variational problems on maps $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. For any point $(x,y) \in \mathbb{R}^4$, we denote by (e, p^i_μ, r) the coordinates on $\Lambda^2 T_{(x,y)}\mathbb{R}^4$, such that $\theta = e\, dx^1 \wedge dx^2 + p^1_i dy^i \wedge dx^2 + p^2_i dx^1 \wedge dy^i + r\, dy^1 \wedge dy^2$. An explicit parametrization of $\{z \in D^2_{(x,y)}\mathbb{R}^4 / \omega(z) > 0\}$ is given by the coordinates (t, v^i_μ) through

$$z = t^2 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + t\, \epsilon^{\mu\nu} v^i_\mu \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial x^\nu} + (v^1_1 v^2_2 - v^1_2 v^2_1) \frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial y^2},$$

where $\epsilon^{12} = -\epsilon^{21} = 1$ and $\epsilon^{11} = \epsilon^{22} = 0$. One then finds that $\left(T_z D^2_q \mathbb{R}^4\right)^\perp$ is $\mathbb{R}\left[\left(v^1_1 v^2_2 - v^2_1 v^1_2\right) dx^1 \wedge dx^2 - \epsilon_{ij} v^j_\nu dy^i \wedge dx^\nu + dy^1 \wedge dy^2\right]$, whereas $\left(T_z D^\omega_q \mathbb{R}^4\right)^\perp$ is $\left(T_z D^2_q \mathbb{R}^4\right)^\perp \oplus \mathbb{R} dx^1 \wedge dx^2$.

We deduce that the sets $P_q(z)$ and $P^h_q(z)$ form a family of non parallel affine subspaces so we expect that on the one hand these subspaces will intersect, causing obstructions there for the invertibility of the Legendre mapping, and on the other hand they will fill “almost” all of $\Lambda^2 T^*_{(x,y)}\mathbb{R}^4$, giving rise to the phenomenon that the Legendre correspondence is “generically everywhere” well defined.

Example 4 — *The trivial variational problem* — We just take $l = 0$, so that any map map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a critical point of ℓ ! This example is motivated by gauge theories where the gauge invariance gives rise to constraints. In this case the sets $P_q(z)$ are exactly $\left(T_z D_q^\omega \mathbb{R}^4\right)^\perp$ and $\cup_z P_q(z)$ is equal to $\mathcal{P}_q := \{(e, p_i^\mu, r) \in \Lambda^2 T_q^*\mathbb{R}^4 / r \neq 0\} \cup \{(e, 0, 0)/e \in \mathbb{R}\}$. If we assume that $r \neq 0$ and choose $\mathcal{M}_q = \{(e, p_i^\mu, r) \in \Lambda^2 T_q^*\mathbb{R}^4 / r \neq 0\}$, then

$$\mathcal{H}(q,p) = e - \frac{p_1^1 p_2^2 - p_2^1 p_1^2}{r}.$$

One can then check that all Hamiltonian 2-curves are of the form

$$\Gamma = \left\{ \left(x, u(x), e(x)dx^1 \wedge dx^2 + \epsilon_{\mu\nu} p_i^\mu(x) dy^i \wedge dx^\nu + r(x) dy^1 \wedge dy^2\right) / x \in \mathbb{R}^2 \right\},$$

where $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an *arbitrary* smooth function, $r : \mathbb{R}^2 \longrightarrow \mathbb{R}^*$ is also an arbitrary smooth function, $e(x) = r(x)\left(\frac{\partial u^1}{\partial x^1}(x)\frac{\partial u^2}{\partial x^2}(x) - \frac{\partial u^1}{\partial x^2}(x)\frac{\partial u^1}{\partial x^2}(x)\right) + h$, (for some constant $h \in \mathbb{R}$) and $p_i^\mu(x) = -r(x)\epsilon_{ij}\epsilon^{\mu\nu}\frac{\partial u^j}{\partial x^\nu}(x)$.

Example 5 — *The elliptic Dirichlet integral (see also [9])* — The Lagrangian is $l(x, y, v) = \frac{1}{2}|v|^2 + B(v_1^1 v_2^2 - v_1^2 v_2^1)$ where[7] $|v|^2 := (v_1^1)^2 + (v_2^1)^2 + (v_1^2)^2 + (v_2^2)^2$. We then find that

$$\mathcal{H}(q,p) = e + \frac{1}{1 - (r - B)^2}\left(\frac{|p|^2}{2} + (r - B)(p_1^1 p_2^2 - p_2^1 p_1^2)\right).$$

Example 6 — *Maxwell equations in two dimensions* — We choose $l(x, y, v) = -\frac{1}{2}\left(v_2^1 - v_1^2\right)^2$, so that by identifying (u^1, u^2) with the components (A_1, A_2) of a Maxwell gauge potential, we recover the usual Lagrangian $l(dA) = -\frac{1}{4}\sum_{\mu,\nu}\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right)^2$ for Maxwell fields without charges. We then obtain

$$\mathcal{H}(q,p) = e + \frac{(p_2^1 + p_1^2)^2 - 4p_1^1 p_2^2}{4r} - \frac{1}{4}\frac{(p_2^1 - p_1^2)^2}{2 + r}.$$

Conclusion — It is worth looking at the differences between the Lepage–Dedecker and the De Donder–Weyl theories through these examples. Indeed the De Donder–Weyl theory can be simply recovered by letting $r = 0$. One sees immediately that for the trivial variational problem this forces $p_1^1 p_2^2 - p_2^1 p_1^2$ to be 0: actually a more careful inspection shows that all pseudofibers intersect along $p_i^\mu = 0$ so that all these components must be set to 0 in the De Donder–Weyl theory. In the example of the elliptic Dirichlet functional no constraint appears unless $B = \pm 1$. And for the Maxwell equations all pseudofibers intersect along the subspace $p_2^1 + p_1^2 = p_1^1 = p_2^2 = 0$ and so we recover the constraints already observed in [14] and [9] in the De Donder–Weyl formulation.

[7] There B could be interpreted as a B-field of a bosonic string theory.

2.3 Invariance properties along pseudofibers

We have seen that for all $q \in \mathcal{N}$, for $h \in \mathbb{R}$ and $z \in D_q^\omega \mathcal{N}$, the pseudofiber $P_q^h(z)$ is an affine subspace of $\Lambda^n T_q^* \mathcal{N}$ parallel to $\left(T_z D_q^n \mathcal{N}\right)^\perp$. Let us assume that $\mathcal{M}_q$ is an open subset of $\Lambda^n T_q^* \mathcal{N}$: then the Legendre Correspondence Hypothesis implies that $\forall (p,q) \in \mathcal{M}$, $Z_q(p)$ is reduced to one point that we shall denote by $Z(q,p)$. Hence we can define the distribution of subspaces on $\mathcal{M}$ by:

$$\forall (q,p) \in \mathcal{M}, \quad L^{\mathcal{H}}_{(q,p)} := \left(T_{Z(q,p)} D_q^n \mathcal{N}\right)^\perp .$$

It is actually the subspace tangent to the pseudo-fiber passing through (q,p). In Section 3.3 we will propose a generalization of the definition of $L^{\mathcal{H}}_{(q,p)}$ which makes sense on an arbitrary multisymplectic manifold. We will prove in Section 4.3 that this generalized definition coincides with the first one in the case where the multisymplectic manifold is $\Lambda^n T^* \mathcal{N}$. Lastly Lemma 2.3 and Corollary 2.1 can be rephrased as

Theorem 2.3. *Let $\mathcal{M}$ be an open subset of $\Lambda^n T^* \mathcal{N}$ and let $\mathcal{H}$ be a Legendre image Hamiltonian function on $\mathcal{M}$ (by means of the Legendre correspondence). Then*

$$\forall (q,p) \in \mathcal{M}, \forall \xi \in L^{\mathcal{H}}_{(q,p)}, \quad d\mathcal{H}_{(q,p)}(\xi) = 0. \tag{14}$$

And if $\Gamma \in \widehat{\mathcal{G}}^\omega$ is a Hamiltonian n-curve and if ξ a vector field which is a smooth section of $L^{\mathcal{H}}$, then denoting by $e^{s\xi}$ the flow mapping of ξ

$$\forall s \in \mathbb{R}, \text{ small enough }, e^{s\xi}(\Gamma) \text{ is a Hamiltonian } n\text{−curve.} \tag{15}$$

2.4 Gauge theories

The above theory can be adapted for variational theories on gauge fields (connections) by using a local trivialization. More precisely, given a $\mathfrak{g}$-connection ∇^0 acting on a trivial bundle with structure group $\mathfrak{G}$ (and Lie algebra $\mathfrak{g}$) any other connection ∇ can be identified with the $\mathfrak{g}$-valued 1-form A on the base manifold $\mathcal{X}$ such that $\nabla = \nabla^0 + A$. We may couple A to a Higgs field $\varphi : \mathcal{X} \longrightarrow \Phi$, where Φ is a vector space on which $\mathfrak{G}$ is acting. Then any choice of a field (A, φ) is equivalent to the data of an n-dimensional submanifold Γ in $\mathcal{M} := (\mathfrak{g} \otimes T^* \mathcal{X}) \times \Phi$ which is a section of this fiber bundle over $\mathcal{X}$. An example of this approach is the one that we use for the Maxwell field at the end of this paper.

But if we wish to study more general gauge theories and in particular connections on a non trivial bundle we need a more general and more covariant framework. Such a setting can consist in viewing a connection as a $\mathfrak{g}$-valued 1-form a on a principal bundle $\mathcal{F}$ over the space-time satisfying some equivariance conditions (under some action of the group $\mathfrak{G}$). Similarly the Higgs field, a section of an associated bundle, can be viewed as an equivariant map ϕ on $\mathcal{F}$ with values in a fixed space. Thus the pair (a,ϕ) can be pictured geometrically as a section Γ, i.e. a submanifold of some fiber bundle $\mathcal{N}$ over $\mathcal{F}$, satisfying two kinds of constraints:

- Γ is contained in a submanifold $\mathcal{N}_{\mathfrak{g}}$ (a geometrical translation of the constraints "the restriction of a_f to the subspace tangent to the fiber $\mathcal{F}_f$ is $-dg \cdot g^{-1}$") and
- Γ is invariant by an action of $\mathfrak{G}$ on $\mathcal{N}$ which preserves $\mathcal{N}_{\mathfrak{g}}$.

Within this more abstract framework we are reduced to a situation similar to the one studied in the beginning of this Section, but we need to understand what are the consequence of the two equivariance conditions. (In particular this will imply that there is a canonical distribution of subspaces which is tangent to all pseudofibers). This will be done in details in [12]. In particular we compare this abstract point of view with the more naive one expounded above.

3 Multisymplectic manifolds

We now set up a general framework extending the situation encountered in the previous Section.

3.1 Definitions

Recall that, given a differential manifold $\mathcal{M}$ and $n \in \mathbb{N}$ a smooth $(n+1)$-form Ω on $\mathcal{M}$ is a **multisymplectic** form if and only if (i) Ω is non degenerate, i.e. $\forall m \in \mathcal{M}$, $\forall \xi \in T_m\mathcal{M}$, if $\xi \lrcorner \Omega_m = 0$, then $\xi = 0$ (ii) Ω is closed, i.e. $d\Omega = 0$. And we call any manifold $\mathcal{M}$ equipped with a multisymplectic form Ω a **multisymplectic** manifold. (See Definition 1.1.) In the following, N denotes the dimension of $\mathcal{M}$. For any $m \in \mathcal{M}$ we define the set

$$D^n_m\mathcal{M} := \{X_1 \wedge \cdots \wedge X_n \in \Lambda^n T_m\mathcal{M} / X_1, \cdots, X_n \in T_m\mathcal{M}\},$$

of **decomposable** n-vectors and denote by $D^n\mathcal{M}$ the associated bundle.

Definition 3.1. *Let $\mathcal{H}$ be a smooth real valued function defined over a multisymplectic manifold $(\mathcal{M},\Omega)$. A Hamiltonian n-curve Γ is a n-dimensional submanifold of $\mathcal{M}$ such that for any $m\in\Gamma$, there exists a n-vector X in $\Lambda^nT_m\Gamma$ which satisfies*

$$X \,\lrcorner\, \Omega = (-1)^n d\mathcal{H}.$$

We denote by $\mathcal{E}^{\mathcal{H}}$ the set of all such Hamiltonian n-curves. We shall also write for all $m\in\mathcal{M}$, $[X]^{\mathcal{H}}_m := \{X\in D^n_m\mathcal{M}/X \,\lrcorner\, \Omega = (-1)^n d\mathcal{H}_m\}$.

A Hamiltonian n-curve is automatically oriented by the n-vector X involved in the Hamilton equation. Remark also that it may happen that no Hamiltonian n-curve exist. An example is $\mathcal{M} := \Lambda^2T^*\mathbb{R}^4$ with $\Omega = \sum_{1\leq\mu<\nu\leq 4} dp_{\mu\nu}\wedge dq^\mu\wedge dq^\nu$ for the case $\mathcal{H}(q,p) = p_{12}+p_{34}$. Assume that a Hamiltonian 2-curve Γ would exist and let $X:(t^1,t^2)\longmapsto X(t^1,t^2)$ be a parametrization of Γ such that $\frac{\partial X}{\partial t^1}\wedge\frac{\partial X}{\partial t^2} \,\lrcorner\, \Omega = (-1)^2 d\mathcal{H}$. Then, denoting by $X_\mu := \frac{\partial X}{\partial t^\mu}$, we would have $dx^\mu\wedge dx^\nu(X_1,X_2) = \frac{\partial\mathcal{H}}{\partial p_{\mu\nu}}$, which is equal to ± 1 if $\{\mu,\nu\} = \{1,2\}$ or $\{3,4\}$ and to 0 otherwise. But this would contradict the fact that $X_1\wedge X_2$ is decomposable. Hence there is no Hamiltonian 2-curve in this case.

Note that beside the the Lepage–Dedecker multisymplectic manifold $(\Lambda^nT^*\mathcal{N},\Omega)$ studied in the previous Section, other examples of multisymplectic manifolds arises naturally as for example a multisymplectic structure associated to the Palatini formulation of pure gravity in 4-dimensional space-time (see [10], [11], [17]).

In the following we address questions related to the following general problematic, set in the spirit of the general relativity: assume that a field theory (and in particular including a space-time description) is modelled by a multisymplectic manifold $(\mathcal{M},\Omega)$ and possibly a Hamiltonian $\mathcal{H}$. How could we recover its physical properties, i.e. understand how space-time coordinates merge out, how momenta and energy appear, without using *ad hoc* hypotheses ? We probably do not know enough to be able to answer such questions and in the following we will content ourself with partial answers.

3.2 The notion of r-regular functions

This question is motivated by the search for understanding space-time coordinates. One could characterize components of a space-time chart as functions which: (i) are defined for all possible dynamics, (ii) allow us to separate any pair of different points on space-time. The easiest way to fulfill the first requirement is to assume that any coordinate function is obtained as the restriction of a function $f : \mathcal{M} \longrightarrow \mathbb{R}$ on the Hamiltonian n-curve describing the dynamics. The infinitesimal version of the second requirement is then to assume that the restriction of the n functions chosen $f^1, \cdots, f^n$ on any Hamiltonian n-curve is locally a diffeomorphism. This motivates the following

Definition 3.2. *Let $(\mathcal{M}, \Omega)$ be a multisymplectic manifold and $\mathcal{H} \in \mathcal{C}^\infty(\mathcal{M})$ a Hamiltonian function. Let $1 \leq r \leq n$ be an integer. A function $f \in \mathcal{C}^1(\mathcal{M}, \mathbb{R}^r)$ is called* r**-regular** *if and only if for any Hamiltonian n-curve $\Gamma \subset \mathcal{M}$ the restriction $f_{|\Gamma}$ is a submersion.*

The dual notion is:

Definition 3.3. *Let $\mathcal{H}$ be a smooth real valued function defined over a multisymplectic manifold $(\mathcal{M}, \Omega)$. A* **slice of codimension** *r is a cooriented submanifold Σ of $\mathcal{M}$ of codimension r such that for any $\Gamma \in \mathcal{E}^{\mathcal{H}}$, Σ is transverse to Γ. By* **cooriented** *we mean that for each $m \in \Sigma$, the quotient space $T_m\mathcal{M}/T_m\Sigma$ is oriented continuously in function of m.*

Indeed it is clear that the level sets of a r-regular function $f : \mathcal{M} \longrightarrow \mathbb{R}^r$ are slices of codimension r.

Example 7 — *The case when $\mathcal{M} = \Lambda^n T^*(\mathcal{X} \times \mathcal{Y})$ and that $\mathcal{H}(x, y, p) = e + H(x, y, p^*)$ as in Section 2.2* —Let $\Pi_{\mathcal{X}} : \mathcal{M} \longrightarrow \mathcal{X}$ be the natural projection. Then for any function $\varphi \in \mathcal{C}^1(\mathcal{X}, \mathbb{R}^r)$ without critical point (i.e. $d\varphi$ is of rank r everywhere) the function $\varphi \circ \Pi_{\mathcal{X}} : \mathcal{M} \longrightarrow \mathbb{R}^r$ a r-regular function. Indeed because of the particular dependance of $\mathcal{H}$ on e a Hamiltonian n-curve is always a graph over $\mathcal{X}$. A particular case is when $r = 1$, then any level set Σ of φ is a codimension 1 slice and a (class of) vector $\tau \in T_m\mathcal{M}/T_m\Sigma$ is positively oriented if and only if $d\varphi(\tau) > 0$.

Note that in this framework an event in space-time can be represented by a slice of codimension n. The notion of slice is also important because it helps to construct observable functionals on the set of solutions $\mathcal{E}^{\mathcal{H}}$. Indeed if F is a $(n-1)$-form on $\mathcal{M}$ and if Σ is a slice of codimension 1 we define the *functional* denoted symbolically by $\int_\Sigma F : \mathcal{E}^{\mathcal{H}} \longmapsto \mathbb{R}$ by:

$$\Gamma \quad \longmapsto \int_{\Sigma \cap \Gamma} F.$$

Here the intersection $\Sigma \cap \Gamma$ is oriented as follows: assume that $\alpha \in T_m^*\mathcal{M}$ is such that $\alpha_{|T_m\Sigma} = 0$ and $\alpha > 0$ on $T_m\mathcal{M}/T_m\Sigma$ and let $X \in \Lambda^n T_m\Gamma$ be positively oriented. Then we require that $X \llcorner \alpha \in \Lambda^{n-1}T_m(\Sigma \cap \Gamma)$ is positively oriented. We can further assume restrictions on the choice of F in order to guarantee the fact that the resulting functional is physically observable. Such a situation is achieved if for example F is so that $dF_{|T_m\Gamma}$ depends only on $d\mathcal{H}_m$ (see [11] for details).

In the next Section we will study a characterization of r-regular functions in the special case where $\mathcal{M} = \Lambda^n T^*\mathcal{N}$.

3.3 Pataplectic invariant Hamiltonian functions

In Section 2.3 we gave a definition of the subspaces tangent to the pseudofibers $L_m^{\mathcal{H}}$ which was directly deduced from our analysis of pseudofibers. In Section 4.3 we will prove that an alternative characterization of $L_m^{\mathcal{H}}$ in $\Lambda^n T^*\mathcal{N}$ exists and is more intrinsic. It motivates the following definition: given an arbitrary multisymplectic manifold $(\mathcal{M}, \Omega)$ and a Hamiltonian function $\mathcal{H} : \mathcal{M} \longrightarrow \mathbb{R}$ and for all $m \in \mathcal{M}$ we define the **generalized pseudofiber direction** to be

$$\begin{aligned} L_m^{\mathcal{H}} &:= \left(T_{[X]_m^{\mathcal{H}}} D_m^n \mathcal{M} \,\lrcorner\, \Omega\right)^{\perp} \\ &:= \{\xi \in T_m\mathcal{M} / \forall X \in [X]_m^{\mathcal{H}}, \forall \delta X \in T_X D_m^n \mathcal{M}, \xi \,\lrcorner\, \Omega(\delta X) = 0\}. \end{aligned} \tag{16}$$

And we write $L^{\mathcal{H}} := \cup_{m\in\mathcal{M}} L_m^{\mathcal{H}} \subset T\mathcal{M}$ for the associated distribution of subspaces.

Note that if we choose an arbitrary Hamiltonian function $\mathcal{H}$, there is no reason for the conclusions of Theorem 2.3 to be true, unless we know that $\mathcal{H}$ was created out of a Legendre correspondence. This motivates the following definition[8]:

Definition 3.4. *We say that $\mathcal{H}$ is* **pataplectic invariant** *if*

(i) $\forall \xi \in L_m^{\mathcal{H}}$, $d\mathcal{H}_m(\xi) = 0$

(ii) for all Hamiltonian n-curve $\Gamma \in \mathcal{E}^{\mathcal{H}}$, for all vector field ξ which is a smooth section of $L^{\mathcal{H}}$, then, for $s \in \mathbb{R}$ sufficiently small, $\Gamma_s := e^{s\xi}(\Gamma)$ is also a Hamiltonian n-curve.

[8] In the following if ξ is a smooth vector field, we denote by $e^{s\xi}$ (for $s \in I$, where I is an interval of $\mathbb{R}$) its flow mapping. And if E is any subset of $\mathcal{M}$, we denote by $E_s := e^{s\xi}(E)$ its image by $e^{s\xi}$.

In [11] we prove that, if $\mathcal{H}$ is pataplectic invariant and if some further hypotheses are fulfilled, functionals of the type $\int_\Sigma F$ are invariant by deformations along $L^{\mathcal{H}}$.

4 The study of $\Lambda^n T^*\mathcal{N}$

In this Section we analyze in details the special case where $\mathcal{M}$ is an open subset of $\Lambda^n T^*\mathcal{N}$. Since we are interested here in local properties of $\mathcal{M}$, we will use local coordinates $m = (q, p) = (q^\alpha, p_{\alpha_1\cdots\alpha_n})$ on $\mathcal{M}$, and the multisymplectic form reads $\Omega = \sum_{\alpha_1<\cdots<\alpha_n} dp_{\alpha_1\cdots\alpha_n} \wedge dq^{\alpha_1} \wedge \cdots \wedge dq^{\alpha_n}$. For $m = (q, p)$, we write

$$d_q\mathcal{H} := \sum_{1\leq\alpha\leq n+k} \frac{\partial\mathcal{H}}{\partial q^\alpha} dq^\alpha, \quad d_p\mathcal{H} := \sum_{1\leq\alpha_1<\cdots<\alpha_n\leq n+k} \frac{\partial\mathcal{H}}{\partial p_{\alpha_1\cdots\alpha_n}} dp_{\alpha_1\cdots\alpha_n},$$

so that $d\mathcal{H} = d_q\mathcal{H} + d_p\mathcal{H}$.

4.1 The structure of $[X]^{\mathcal{H}}_m$

Here we are given some Hamiltonian function $\mathcal{H} : \mathcal{M} \longrightarrow \mathbb{R}$ and a point $m \in \mathcal{M}$ such that $[X]^{\mathcal{H}}_m \neq \emptyset$ and[9] $d_p\mathcal{H}_m \neq 0$. Given any $X = X_1 \wedge \cdots \wedge X_n \in D^n_m\mathcal{M}$ and any form $a \in T^*_m\mathcal{M}$ we will write that $a_{|X} \neq 0$ (resp. $a_{|X} = 0$) if and only if $(a(X_1), \cdots, a(X_n)) \neq 0$ (resp. $(a(X_1), \cdots, a(X_n)) = 0$). We will say that a form $a \in T^*_m\mathcal{M}$ is **proper on** $[X]^{\mathcal{H}}_m$ if and only if it's either a *point-slice*

$$\forall X \in [X]^{\mathcal{H}}_m, \quad a_{|X} \neq 0, \tag{17}$$

or a *co-isotropic*

$$\forall X \in [X]^{\mathcal{H}}_m, \quad a_{|X} = 0. \tag{18}$$

We are interested in characterizing all proper 1-forms on $[X]^{\mathcal{H}}_m$. We show in this section the following.

Lemma 4.1. *Let $\mathcal{M}$ be an open subset of $\Lambda^n T^*\mathcal{N}$ endowed with its standard multisymplectic form Ω, let $\mathcal{H} : \mathcal{M} \longrightarrow \mathbb{R}$ be a smooth Hamiltonian function. Let $m \in \mathcal{M}$ such that $d_p\mathcal{H}_m \neq 0$ and $[X]^{\mathcal{H}}_m \neq \emptyset$. Then*
(i) the $n+k$ forms $dq^1, \cdots, dq^{n+k}$ are proper on $[X]^{\mathcal{H}}_m$ and satisfy the following property: $\forall X \in [X]^{\mathcal{H}}_m$ and for all $Y, Z \in T_m\mathcal{M}$ which are in the vector

[9] observe that, although the splitting $d\mathcal{H} = d_q\mathcal{H} + d_p\mathcal{H}$ depends on a trivialization of $\Lambda^n T^*\mathcal{N}$, the condition $d_p\mathcal{H}_m \neq 0$ is intrinsic: indeed it is equivalent to $d\mathcal{H}_{m|\mathrm{Ker}\, d\Pi_m} \neq 0$, where $\Pi : \Lambda^n T^*\mathcal{N} \longrightarrow \mathcal{N}$.

space spanned by X, if $dq^\alpha(Y) = dq^\alpha(Z)$, $\forall\alpha = 1, \cdots, n+k$, then $Y = Z$.
(ii) Moreover for all $a \in T^*_m\mathcal{M}$ which is proper on $[X]^{\mathcal{H}}_m$ we have

$$\exists! \lambda \in \mathbb{R}, \exists!(a_1, \cdots, a_{n+k}) \in \mathbb{R}^{n+k}, \quad a = \lambda d\mathcal{H}_m + \sum_{\alpha=1}^{n+k} a_\alpha dq^\alpha. \tag{19}$$

(iii) Up to a change of coordinates on $\mathcal{N}$ we can assume that $dq^1, \cdots, dq^n$ are point-slices and that $dq^{n+1}, \cdots, dq^{n+k}$ satisfy (18). Then $a \in T^*\mathcal{M}$ is a point-slice if and only if (19) occurs with $(a_1, \cdots, a_n) \neq 0$.

Proof — First step — analysis of $[X]^{\mathcal{H}}_m$. We start by introducing some extra notations: each vector $Y \in T_m\mathcal{M}$ can be decomposed into a "vertical" part Y^V and a "horizontal" part Y^H as follows: for any $Y = \sum_{1\leq\alpha\leq n+k} Y^\alpha \frac{\partial}{\partial q^\alpha} + \sum_{1\leq\alpha_1<\cdots<\alpha_n\leq n+k} Y_{\alpha_1\cdots\alpha_n} \frac{\partial}{\partial p_{\alpha_1\cdots\alpha_n}}$, set $Y^H := \sum_{1\leq\alpha\leq n+k} Y^\alpha \frac{\partial}{\partial q^\alpha}$ and $Y^V := \sum_{1\leq\alpha_1<\cdots<\alpha_n\leq n+k} Y_{\alpha_1\cdots\alpha_n} \frac{\partial}{\partial p_{\alpha_1\cdots\alpha_n}}$. Let $X = X_1 \wedge \cdots \wedge X_n \in D^n_m(\Lambda^n T^*\mathcal{N})$ and let us use this decomposition to each X_μ: then X can be split as $X = \sum_{j=0}^n X_{(j)}$, where each $X_{(j)}$ is homogeneous of degree j in the variables X^V_μ and homogeneous of degree $n-j$ in the variables X^H_μ.

Recall that a decomposable n-vector X is in $[X]^{\mathcal{H}}_m$ if and only if $X \lrcorner \Omega = (-1)^n d\mathcal{H}$. This equation actually splits as

$$X_{(0)} \lrcorner \Omega = (-1)^n d_p\mathcal{H} \tag{20}$$

and

$$X_{(1)} \lrcorner \Omega = (-1)^n d_q\mathcal{H}. \tag{21}$$

Equation (20) determines in an unique way $X_{(0)} \in D^n_q\mathcal{N}$. The condition $d_p\mathcal{H} \neq 0$ implies that necessarily[10] $X_{(0)} \neq 0$. At this stage we can choose a family of n linearly independent vectors $X^0_1, \cdots, X^0_n$ in $T_q\mathcal{N}$ such that $X^0_1 \wedge \cdots \wedge X^0_n = X_{(0)}$. Thus the forms dq^α are proper on $[X]^{\mathcal{H}}_m$, since their restriction on X are fully determined by their restriction on the vector subspace spanned by $X^0_1, \cdots, X^0_n$. Furthermore the subspace of $T_m\mathcal{M}$ spanned by X is a graph over the subspace of $T_q\mathcal{N}$ spanned by $X_{(0)}$. This proves the part (i) of the Lemma..

Proving (ii) and (iii) requires more work. First we deduce that there exists a unique family $(X_1, \cdots, X_n)$ of vectors in $T_m\mathcal{M}$ such that $\forall\mu$, $X^H_\mu = X^0_\mu$ and

[10] Note also that (20) implies that $d_p\mathcal{H}$ must satisfy some compatibility conditions since $X_{(0)}$ is decomposable.

$X_1 \wedge \cdots \wedge X_n = X$. And Equation (21) consists in further underdetermined conditions on the vertical components $X_{\mu,\alpha_1\cdots\alpha_n}$ of the X_μ's, namely

$$\sum_\mu \sum_{\alpha_1<\cdots<\alpha_n} C_\beta^{\mu,\alpha_1\cdots\alpha_n} X_{\mu,\alpha_1\cdots\alpha_n} = -\frac{\partial \mathcal{H}}{\partial q^\beta},$$

where

$$C_\beta^{\mu,\alpha_1\cdots\alpha_n} := \sum_\nu \delta_\beta^{\alpha_\nu}(-1)^{\mu+\nu}\Delta^{\alpha_1\cdots\widehat{\alpha_\nu}\cdots\alpha_n}_{1\cdots\widehat{\mu}\cdots n}$$

and

$$\Delta^{\alpha_1\cdots\alpha_{n-1}}_{\mu_1\cdots\mu_{n-1}} := \begin{vmatrix} X^{\alpha_1}_{\mu_1} & \dots & X^{\alpha_1}_{\mu_{n-1}} \\ \vdots & & \vdots \\ X^{\alpha_n}_{\mu_1} & \dots & X^{\alpha_{n-1}}_{\mu_{n-1}} \end{vmatrix}.$$

Step2 — Local coordinates. To further understand these relations we choose suitable coordinates q^α in such a way that $d_p\mathcal{H}_m = dp_{1\cdots n}$ and

$$X_\mu^H = \frac{\partial}{\partial q^\mu} \quad \text{for} \quad \mu = 1, ..., n, \tag{22}$$

so that (20) is automatically satisfied. In this setting we also have

$$(-1)^n X_{(1)} \lrcorner\, \Omega = -\sum_\mu X_{\mu,1\cdots n} dq^\mu - (-1)^n \sum_\mu \sum_{n<\beta} (-1)^\mu X_{\mu,1\cdots\widehat{\mu}\cdots n\beta} dq^\beta,$$

and so (21) is equivalent to

$$\left\{\begin{array}{rcl} X_{\mu,1\cdots n} & = & -\dfrac{\partial\mathcal{H}}{\partial q^\mu},\ \text{for } 1 \leq \mu \leq n \\ & & \\ (-1)^n \displaystyle\sum_\mu (-1)^\mu X_{\mu,1\cdots\widehat{\mu}\cdots n\beta} & = & -\dfrac{\partial\mathcal{H}}{\partial q^\beta},\ \text{for } n+1 \leq \beta \leq n+k. \end{array}\right. \tag{23}$$

Let us introduce some notations: $I := \{(\alpha_1, \cdots, \alpha_n)/1 \leq \alpha_1 < \cdots \leq \alpha_n \leq n+k\}$, $I^0 := \{(1, \cdots, n)\}$, $I^* := \{(\alpha_1, \cdots, \alpha_{n-1}, \beta)/1 \leq \alpha_1 < \cdots < \alpha_{n-1} \leq n, n+1 \leq \beta \leq n+k\}$, $I^{**} := I \setminus \left(I^0 \cup I^*\right)$. We note also $M_\mu := \sum_{(\alpha_1,\cdots,\alpha_n)\in I^*} X_{\mu,\alpha_1\cdots\alpha_n}\partial^{\alpha_1\cdots\alpha_n}$, $R_\mu := \sum_{(\alpha_1,\cdots,\alpha_n)\in I^{**}} X_{\mu,\alpha_1\cdots\alpha_n}\partial^{\alpha_1\cdots\alpha_n}$ and $M^\nu_{\mu,\beta} := (-1)^{n+\nu} X_{\mu,1\cdots\widehat{\nu}\cdots n\beta}$. Then the set of solutions of (20) and (21) satisfying (22) is

$$X_\mu = \frac{\partial}{\partial q^\mu} - \frac{\partial\mathcal{H}}{\partial q^\mu}\frac{\partial}{\partial p_{1\cdots n}} + M_\mu + R_\mu, \tag{24}$$

where the components of R_μ are arbitrary, and the coefficients of M_μ are only subject to the constraint

$$\sum_\mu M^\mu_{\mu,\beta} = -\frac{\partial\mathcal{H}}{\partial q^\beta}, \quad \text{for } n+1 \leq \beta \leq n+k. \tag{25}$$

Step 3 — The search of all proper 1-forms on $[X]^{\mathcal{H}}_m$. Now let $a \in T^*_m\mathcal{M}$ and let us look at necessary and sufficient conditions for a to be a proper 1-form on $[X]^{\mathcal{H}}_m$. We write

$$a = \sum_\alpha a_\alpha dq^\alpha + \sum_{\alpha_1 < \cdots < \alpha_n} a^{\alpha_1 \cdots \alpha_n} dp_{\alpha_1 \cdots \alpha_n}.$$

Let us write $a^* := (a^{\alpha_1 \cdots \alpha_n})_{(\alpha_1, \cdots, \alpha_n) \in I^*}$, $a^{**} := (a^{\alpha_1 \cdots \alpha_n})_{(\alpha_1, \cdots, \alpha_n) \in I^{**}}$ and

$$\langle M_\mu, a^* \rangle := \sum_\nu \sum_{n < \beta} (-1)^{n+\nu} M^\nu_{\mu,\beta} a^{1 \cdots \widehat{\nu} \cdots n \beta},$$

and

$$\langle R_\mu, a^{**} \rangle := \sum_{(\alpha_1, \cdots, \alpha_n) \in I^{**}} X_{\mu, \alpha_1 \cdots \alpha_n} a^{\alpha_1 \cdots \alpha_n}.$$

Using (24) we obtain that

$$a(X_\mu) = a_\mu - \frac{\partial \mathcal{H}}{\partial q^\mu} a^{1 \cdots n} + \langle M_\mu, a^* \rangle + \langle R_\mu, a^{**} \rangle .$$

Lemma 4.2. *Condition (17) (resp. (18)) is equivalent to the two following conditions:*

$$a^* = a^{**} = 0 \tag{26}$$

and

$$\left(a_1 - \frac{\partial \mathcal{H}}{\partial q^1} a^{1 \cdots n}, \ \cdots \ , a_n - \frac{\partial \mathcal{H}}{\partial q^n} a^{1 \cdots n} \right) \neq 0 \quad (\textit{resp.} \ = 0). \tag{27}$$

Proof — We first look at necessary and sufficient conditions on for a to be a point-slice, i.e. to satisfy (17). Let us denote by $\vec{A} := \left(a_\mu - \frac{\partial \mathcal{H}}{\partial q^\mu} a^{1 \cdots n} \right)_\mu$ and $\vec{M} := (M_\mu)_\mu$, $\vec{R} := (R_\mu)_\mu$. We want conditions on $a^{\alpha_1 \cdots \alpha_n}$ in order that the image of the affine map $(\vec{M}, \vec{R}) \longmapsto \vec{\mathcal{A}}(\vec{M}, \vec{R}) := \vec{A} + \langle \vec{M}, a^* \rangle + \langle \vec{R}, a^{**} \rangle$ does not contain 0 (assuming that $\vec{M}$ satisfies the constraint (25)). We see immediately that if a^{**} would be different from 0, then by choosing $\vec{M} = 0$ and $\vec{R}$ suitably, we could have $\vec{\mathcal{A}}(\vec{M}, \vec{R}) = 0$. Thus $a^{**} = 0$. Similarly, assume by contradiction that a^* is different from 0. Up to a change of coordinates, we can assume that $\left(a^{1 \cdots \widehat{\nu} \cdots n(n+1)}\right)_{1 \leq \nu \leq n} \neq 0$. And by another change of coordinates, we can further assume that $a^{2 \cdots n(n+1)} = \lambda \neq 0$ and $a^{1 \cdots \widehat{\nu} \cdots n(n+1)} = 0$, if $\nu \geq 1$. Then choose $M^\nu_{\mu,\beta} = 0$ if $\beta \geq n+2$, and

$$\begin{pmatrix} M^1_{1,n+1} & M^1_{2,n+1} & M^1_{3,n+1} & \cdots & M^1_{n,n+1} \\ M^2_{1,n+1} & M^2_{2,n+1} & M^2_{3,n+1} & \cdots & M^2_{n,n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ M^n_{1,n+1} & M^n_{2,n+1} & M^n_{3,n+1} & \cdots & M^n_{n,n+1} \end{pmatrix} = \begin{pmatrix} t_1 & t_2 & t_3 & \cdots & t_n \\ 0 & s & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $s = -t_1 - \partial\mathcal{H}/\partial q^{n+1}$. Then we find that $\mathcal{A}_\mu(\vec{M},\vec{R}) = A_\mu + (-1)^{n+1}\lambda t_\mu$, so that this expression vanishes for a suitable choice of the t_μ's. Hence we get a contradiction. Thus we conclude that $a^* = 0$ and $\vec{A} \neq 0$. The analysis of 1-forms which satisfies (18) is similar: this condition is equivalent to $a^* = 0$ and $\vec{A} = 0$. ■

Conclusion. We translate the conclusion of Lemma 4.2 without using local coordinates: it gives relation (19). ■

4.2 Slices and r-regular functions

As an application of the above analysis we can give a characterization of r-regular functions. We first consider the case $r = 1$.

Indeed any smooth function $f : \mathcal{M} \longrightarrow \mathbb{R}$ is 1-regular if and only if $\forall m \in \mathcal{M}$, df_m is a point-slice. Using Lemma 4.2 we obtain two conditions on df_m: the condition (26) can be restated as follows: for all $m \in \mathcal{M}$ there exists a real number $\lambda(m)$ such that $d_pf_m = \lambda(m)d_p\mathcal{H}_m$. Condition (27) is equivalent to: $\exists(\alpha_1,\cdots,\alpha_n) \in I$, $\exists 1 \leq \mu \leq n$,

$$\{\mathcal{H},f\}^{\alpha_1\cdots\alpha_n}_{\alpha_\mu}(m) := \frac{\partial\mathcal{H}}{\partial p_{\alpha_1\cdots\alpha_n}}(m)\frac{\partial f}{\partial q^{\alpha_\mu}}(m) - \frac{\partial f}{\partial p_{\alpha_1\cdots\alpha_n}}(m)\frac{\partial\mathcal{H}}{\partial q^{\alpha_\mu}}(m) \neq 0. \tag{28}$$

[Alternatively using Lemma 4.1, df_m is a point-slice if and only if $\exists\lambda(m) \in \mathbb{R}$, $\exists(a_1,\cdots,a_{n+k}) \in \mathbb{R}^{n+k}$ such that $df_m = \lambda(m)d\mathcal{H}_m + \sum_{\alpha=1}^{n+k} a_\alpha dq^\alpha$ and $(a_1,\cdots,a_n) \neq 0$.] Now we remark that $d_pf_m = \lambda(m)d_p\mathcal{H}_m$ everywhere if and only if there exists a function $\widehat{f}$ of the variables $(q,h) \in \mathcal{N}\times\mathbb{R}$ such that $f(q,p) = \widehat{f}(q,\mathcal{H}(q,p))$. So we deduce the following.

Theorem 4.1. *Let $\mathcal{M}$ be an open subset of $\Lambda^nT^*\mathcal{N}$ endowed with its standard multisymplectic form Ω, let $\mathcal{H} : \mathcal{M} \longrightarrow \mathbb{R}$ be a smooth Hamiltonian function and let $f : \mathcal{M} \longrightarrow \mathbb{R}$ be a smooth function. Assume that $d_p\mathcal{H} \neq 0$ and $[X]^{\mathcal{H}} \neq \emptyset$ everywhere. Then f is 1-regular if and only if there exists a smooth function $\widehat{f} : \mathcal{N}\times\mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$f(q,p) = \widehat{f}(q,\mathcal{H}(q,p)), \quad \forall(q,p) \in \mathcal{M}$$

and $\forall m \in \mathcal{M}$,

$$\exists(\alpha_1,\cdots,\alpha_n) \in I, \exists 1 \leq \mu \leq n, \qquad \{\mathcal{H},f\}^{\alpha_1\cdots\alpha_n}_{\alpha_\mu}(m) \neq 0.$$

By the same token this result gives sufficient conditions for a hypersurface defined as the level set $f^{-1}(s) := \{m \in \mathcal{M}/f(m) = s\}$ of a given function to be a slice: it suffices that the above condition be true along $f^{-1}(s)$.

Example 8 — We come back here to critical points $u : \mathcal{X} \longrightarrow \mathcal{Y}$ of a Lagrangian functional l. We use the notations of Section 2.2 and denote by p^* the set of coordinates $p^{\mu_1\cdots\mu_j}_{i_1\cdots i_j}$ for $j \geq 1$, so that $\mathcal{H}(q, e, p^*) = e + H(q, p^*)$. Let us assume that, $\forall q \in \mathcal{N} = \mathcal{X} \times \mathcal{Y}$, there exists some value p^*_0 of p^* such that $\partial H/\partial p^*(q, p^*_0) = 0$. Note that this situation arises in almost all standard situation (if in particular the Lagrangian $l(x, u, v)$ has a quadratic dependence in v). Assume further the hypotheses of Theorem 4.1 and consider a 1-regular function $f \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$. We note that $f(q, p) = \widehat{f}(q, \mathcal{H}(q, p))$ implies that $\{\mathcal{H}, f\}^{\alpha_1\cdots\alpha_n}_{\alpha_\mu}(q, p) = \frac{\partial \mathcal{H}}{\partial p_{\alpha_1\cdots\alpha_n}}(q, p)\frac{\partial \widehat{f}}{\partial q^{\alpha_\mu}}(q, \mathcal{H}(q, p))$. Now for all $(q, h) \in \mathcal{N} \times \mathbb{R}$, let p^*_0 be such that $\partial H/\partial p^*(q, p^*_0) = 0$ and let $e_0 := h - H(q, p^*_0)$. Since $\frac{\partial \mathcal{H}}{\partial p^*}(q, e_0, p^*_0) = 0$ and $\frac{\partial \mathcal{H}}{\partial e} = 1$, condition (28) at $m = (q, e_0, p^*_0)$ means that $\exists \mu$ with $1 \leq \mu \leq n$ such that $\frac{\partial \widehat{f}}{\partial x^\mu}(q, h) = \frac{\partial \widehat{f}}{\partial x^\mu}(q, \mathcal{H}(q, e_0, p^*_0)) \neq 0$. This singles out **space-time coordinates**: they are the functions on $\mathcal{M}$ needed to build slices.

We now turn to the case where $1 \leq r \leq n$. We consider a map $f = (f^1, \cdots, f^r)$ from $\mathcal{M}$ to $\mathbb{R}^r$ and look for necessary and sufficient conditions on f for being r-regular. We still assume that $d_p\mathcal{H} \neq 0$ and $[X]^{\mathcal{H}} \neq \emptyset$. We first analyze the situation locally. Given a point $m \in \mathcal{M}$, the property "$X \in [X]^{\mathcal{H}} \Longrightarrow df_{m|X}$ is of rank r" is equivalent to:

$$\forall (t_1, \cdots, t_r) \in \mathbb{R}^r \setminus \{0\}, \quad X \in [X]^{\mathcal{H}} \Longrightarrow \sum_{i=1}^r t_i df^i_{m|X} \neq 0.$$

Hence by using Lemma 4.1 we deduce that the property $X \in [X]^{\mathcal{H}} \Longrightarrow$ rank $df_{m|X} = r$ is equivalent to

- $\forall (t_1, \cdots, t_r) \in \mathbb{R}^r \setminus \{0\}$, $\exists \lambda(m) \in \mathbb{R}$, $\sum_{i=1}^r t_i d_p f^i_m = \lambda(m) d_p \mathcal{H}_m$. And then one easily deduce that $\exists \lambda^1(m), \cdots, \lambda^r(m) \in \mathbb{R}$, such that $\lambda(m) = \sum_{i=1}^r t_i \lambda^i(m)$.
- $\forall (t_1, \cdots, t_r) \in \mathbb{R}^r \setminus \{0\}$, $\exists (\alpha_1, \cdots, \alpha_n) \in I, \exists 1 \leq \mu \leq n$, $\{\mathcal{H}, \sum_{i=1}^r t_i f^i\}^{\alpha_1\cdots\alpha_n}_{\alpha_\mu}(m) \neq 0$.

Now the second condition translates as $\forall (t_1, \cdots, t_r) \in \mathbb{R}^r \setminus \{0\}$, $\exists (\alpha_1, \cdots, \alpha_n) \in I, \exists 1 \leq \mu \leq n$,

$$\sum_{i=1}^r t_i \frac{\partial \mathcal{H}}{\partial p_{\alpha_1\cdots\alpha_n}} \left(\frac{\partial f^i}{\partial q^{\alpha_\mu}} - \lambda^i \frac{\partial \mathcal{H}}{\partial q^{\alpha_\mu}} \right) \neq 0.$$

This condition can be expressed in terms of minors of size r from the matrix $\left(\frac{\partial f^i}{\partial q^{\alpha_\mu}} - \lambda^i \frac{\partial \mathcal{H}}{\partial q^{\alpha_\mu}}\right)_{i,\alpha_\mu}$. For that purpose let us denote by

$$\{\{\mathcal{H}, f^1, \cdots, f^r\}\} := \sum_{1\leq\alpha_1<\cdots<\alpha_n\leq n+k} \sum_{1\leq\mu_1<\cdots<\mu_r\leq n}$$

$$\left\langle \frac{\partial}{\partial p_{\alpha_1\cdots\alpha_n}} \wedge \frac{\partial}{\partial q^{\alpha_{\mu_1}}} \wedge \cdots \wedge \frac{\partial}{\partial q^{\alpha_{\mu_r}}}, d\mathcal{H}\wedge df^1 \wedge \cdots \wedge df^r \right\rangle dp_{\alpha_1\cdots\alpha_n} \wedge dq^{\alpha_{\mu_1}} \wedge \cdots \wedge dq^{\alpha_{\mu_r}}.$$

We deduce the following.

Proposition 4.1. *Let $\mathcal{M}$ be an open subset of $\Lambda^n T^*\mathcal{N}$ endowed with its standard multisymplectic form Ω, let $\mathcal{H}:\mathcal{M}\longrightarrow\mathbb{R}$ be a smooth Hamiltonian function and let $f:\mathcal{M}\longrightarrow\mathbb{R}^r$ be a smooth function. Let $m\in\mathcal{M}$ and assume that $d_p\mathcal{H}\neq 0$ and $[X]^{\mathcal{H}}\neq\emptyset$ everywhere. Then $X\in[X]^{\mathcal{H}} \Longrightarrow df_{m|X}$ is of rank r if and only if*

- $\exists\lambda^1(m),\cdots,\lambda^r(m)\in\mathbb{R}$, $\forall 1\leq i\leq r$, $d_p f^i_m = \lambda^i(m) d_p\mathcal{H}_m$.
- $\{\{\mathcal{H}, f^1, \cdots, f^r\}\}(m)\neq 0$.

And we deduce the global result:

Theorem 4.2. *Let $\mathcal{M}$ be an open subset of $\Lambda^n T^*\mathcal{N}$ endowed with its standard multisymplectic form Ω, let $\mathcal{H}:\mathcal{M}\longrightarrow\mathbb{R}$ be a smooth Hamiltonian function and let $f:\mathcal{M}\longrightarrow\mathbb{R}^r$ be a smooth function. Assume that $d_p\mathcal{H}\neq 0$ and $[X]^{\mathcal{H}}\neq\emptyset$ everywhere. Then f is r-regular if and only if there exists a smooth function $\widehat{f}:\mathcal{N}\times\mathbb{R}\longrightarrow\mathbb{R}^r$ such that $f(q,p)=\widehat{f}(q,\mathcal{H}(q,p))$ and $\forall m\in\mathcal{M}$, $\{\{\mathcal{H}, f^1, \cdots, f^r\}\}(m)\neq 0$.*

4.3 Generalized pseudofibers directions

We are now able to prove the equivalence in (an open subset of) $\mathcal{M}=\Lambda^n T^*\mathcal{N}$ between the two possible definitions of $L^{\mathcal{H}}_m$: either $\left(T_z D^n_q\mathcal{N}\right)^\perp$ or

$$\begin{aligned}&\left(T_{[X]^{\mathcal{H}}_m} D^n_m\mathcal{M} \lrcorner\, \Omega\right)^\perp\\ &:= \{\xi\in T_{(q,p)}\mathcal{M}/\ \forall X\in\ [X]^{\mathcal{H}}_{(q,p)}, \forall\delta X\in T_X D^n_{(q,p)}\mathcal{M},\ \xi\lrcorner\,\Omega(\delta X)=0\}\end{aligned}$$

as presented in Sections 2.3 and 3.3. First recall that the Legendre correspondence hypothesis implies here that $Z_q(p)$ is reduced to a point that we shall denote by $Z(p,q)$. As a preliminary we prove the following:

Lemma 4.3. *Let $\mathcal{M}$ be an open subset of $\Lambda^n T^*\mathcal{N}$ and let $\mathcal{H}$ be an* arbitrary *smooth function from $\mathcal{M}$ to $\mathbb{R}$, such that $d_p\mathcal{H}$ never vanishes. Let $\xi \in L^{\mathcal{H}}_{(q,p)}$, then $dq^\alpha(\xi) = 0$, $\forall\alpha$, i.e.*

$$\xi = \sum_{\alpha_1<\cdots<\alpha_n} \xi_{\alpha_1\cdots\alpha_n} \frac{\partial}{\partial p_{\alpha_1\cdots\alpha_n}}.$$

Proof — We use the results proved in Section 4.1: we know that we can assume w.l.g. that $d_p\mathcal{H} = dp_{1\cdots n}$. Then any n-vector $X \in D^n_{(q,p)}\mathcal{M}$ such that $(-1)^n X \lrcorner \Omega = d\mathcal{H}$ can be written $X = X_1 \wedge \cdots \wedge X_n$, where each vector X_μ is given by (24) with the conditions on $M^\nu_{\mu,\beta}$ and R_μ described in Section 4.1. We construct a solution X of $(-1)^n X \lrcorner \Omega = d\mathcal{H} = \sum_\alpha \frac{\partial\mathcal{H}}{\partial q^\alpha} dq^\alpha + dp_{1\cdots n}$ by choosing

- $R_\mu = 0$, $\forall 1 \leq \mu \leq n$
- $M^\nu_{\mu,\beta} = 0$ if $(\mu,\nu) \neq (1,1)$
- $M^1_{1,\beta} = -\frac{\partial\mathcal{H}}{\partial q^\beta}$, $\forall n+1 \leq \beta \leq n+k$

in relations (24). It corresponds to

$$\left\{ \begin{array}{lcl} X_1 & = & \dfrac{\partial}{\partial q^1} - \dfrac{\partial\mathcal{H}}{\partial q^1}\dfrac{\partial}{\partial p_{1\cdots n}} + (-1)^n \displaystyle\sum_{\beta=n+1}^{n+k} \dfrac{\partial\mathcal{H}}{\partial q^\beta}\dfrac{\partial}{\partial p_{2\cdots n\beta}} \\ X_\mu & = & \dfrac{\partial}{\partial q^\mu} - \dfrac{\partial\mathcal{H}}{\partial q^\mu}\dfrac{\partial}{\partial p_{1\cdots n}}, \quad \text{if } 2 \leq \mu \leq n. \end{array} \right.$$

We first choose $\delta X^{(1)} \in T_X D^n_{(q,p)}\mathcal{M}$ to be $\delta X^{(1)} := \delta X^{(1)}_1 \wedge X_2 \wedge \cdots \wedge X_n$, where $\delta X^{(1)}_1 := \frac{\partial}{\partial p_{1\cdots n}}$. It gives

$$\delta X^{(1)} = \frac{\partial}{\partial p_{1\cdots n}} \wedge \frac{\partial}{\partial q^2} \wedge \cdots \wedge \frac{\partial}{\partial q^n}.$$

Now let $\xi \in L^{\mathcal{H}}_{(q,p)}$, we must have $\xi \lrcorner \Omega(\delta X^{(1)}) = 0$. But a computation gives

$$\xi \lrcorner \Omega(\delta X^{(1)}) = (-1)^n \delta X^{(1)} \lrcorner \Omega(\xi) = -dq^1(\xi),$$

so that $dq^1(\xi) = 0$.

For $n+1 \leq \beta \leq n+k$, consider $\delta X^{(\beta)} := \delta X^{(\beta)}_1 \wedge X_2 \wedge \cdots \wedge X_n \in T_X D^n_{(q,p)}\mathcal{M}$, where $\delta X^{(\beta)}_1 := \frac{\partial}{\partial p_{2\cdots n\beta}}$. Then we compute that $\delta X^{(\beta)} \lrcorner \Omega = dq^\beta$. Hence, by a similar reasoning, the relation $\xi \lrcorner \Omega(\delta X^{(\beta)}) = 0$ is equivalent to $dq^\beta(\xi) = 0$.

Lastly by considering another solution $X \in D^n_{(q,p)}\mathcal{M}$ to the Hamilton equation, where the role of X_1 has been exchanged with the role of X_μ, for some $2 \leq \mu \leq n$, we can prove that $dq^\mu(\xi) = 0$, as well. ■

Recall that the tangent space $T_{(q,p)}\left(\Lambda^n T^*\mathcal{N}\right)$ possesses a canonical "vertical" subspace $\mathrm{Ker}d\Pi_{(p,q)} \simeq \Lambda^n T^*_q\mathcal{N}$: Lemma 4.3 can be rephrased by saying that, if $d_p\mathcal{H} \neq 0$ everywhere, then $L^{\mathcal{H}}_{(q,p)}$ can be identified with a vector subspace of this vertical subspace.

Proposition 4.2. *Let $\mathcal{M}$ be an open subset of $\Lambda^n T^*\mathcal{N}$ and let $\mathcal{H}$ be a Hamiltonian function on $\mathcal{M}$ built from a Lagrangian function L by means of the Legendre correspondence. Then, through the identification $Kerd\Pi_{(p,q)} \simeq \Lambda^n T^*_q\mathcal{N}$, $\left(T_{[X]^{\mathcal{H}}_m} D^n_m\mathcal{M} \lrcorner \Omega\right)^\perp$ coincides with $\left(T_{Z(q,p)}D^n_q\mathcal{N}\right)^\perp$.*

Proof — First we remark that the hypotheses imply that $d_p\mathcal{H}$ never vanishes (because $d\mathcal{H}(0,\omega) = 1$). Let $\xi \in \left(T_{[X]^{\mathcal{H}}_m} D^n_m\mathcal{M} \lrcorner \Omega\right)^\perp$, using the preceding remark we can associate a n-form $\pi \in \Lambda^n T^*_q\mathcal{N}$ to ξ with coordinates $\pi_{\alpha_1\cdots\alpha_n} = \xi_{\alpha_1\cdots\alpha_n}$, simply by the relation $\pi = \xi \lrcorner \Omega$. Now let us look at the condition:

$$\forall X \in [X]^{\mathcal{H}}_{(q,p)}, \quad \forall \delta X \in T_X D^n_{(q,p)}\mathcal{M}, \quad \xi \lrcorner \Omega(\delta X) = 0. \tag{29}$$

By the analysis of section 4.1 we know that the "horizontal" part $X_{(0)}$ of X is fully determined by $\mathcal{H}$: it is actually $X_{(0)} = Z(q,p)$. Now take any $\delta X \in T_X D^n_{(q,p)}\mathcal{M}$ and split it into its horizontal part $\delta z \in T_{Z(q,p)}D^n_q\mathcal{N}$ and a vertical part δX^V. We remark that

- $\delta z \in T_{Z(q,p)}D^n_q\mathcal{N}$
- $\xi \lrcorner \Omega(\delta X) = \pi(\delta X) = \pi(\delta z)$.

Hence (29) means that $\pi \in \left(T_{Z(q,p)}D^n_q\mathcal{N}\right)^\perp$. So the result follows. ■

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