Effective Stringy Description of Schwarzschild Black Holes

Kirill Krasnov

Albert Einstein Institute

14476 Golm/Potsdam, Germany

krasnov@aei.mpg.de

Sergey N. Soloduhkin

Theoretische Physik

Ludwig-Maximilians Universitat

Theresienstrasse 37, 80333 Munich, Germany

and

International University Bremen

P.O.Box 750561, 28759 Bremen, Germany

s.soloduhkin@iu-bremen.de

Abstract

We start by pointing out that certain Riemann surfaces appear rather naturally in the context of wave equations in the black hole background. For a given black hole there are two closely related surfaces. One is the Riemann surface of complexified "tortoise" coordinate. The other Riemann surface appears when the radial wave equation is interpreted as the Fuchsian differential equation. We study these surfaces in detail for the BTZ and Schwarzschild black holes in four and higher dimensions. Topologically, in all cases both surfaces are a sphere with a set of marked points; for BTZ and 4D Schwarzschild black holes there is 3 marked points. In certain limits the surfaces can be characterized very explicitly. We then show how properties of the wave equation (quasi-normal modes) in such limits are encoded in the geometry of the corresponding surfaces. In particular, for the Schwarzschild black hole in the high damping limit we describe the Riemann surface in question and use this to derive the quasi-normal mode frequencies with the $\log 3$ as the real part. We then argue that the surfaces one finds this way signal an appearance of an effective string. We propose that a description of this effective string propagating in the black hole background can be given in terms of the Liouville theory living on the corresponding Riemann surface. We give such a stringy description for the Schwarzschild black hole in the limit of high damping and show that the quasi-normal modes emerge naturally as the poles in 3-point correlation function in the effective conformal theory.

1 Introduction

It has been suggested some time ago by 't Hooft [1] that there is an analogy between string theory and black hole (BH) physics: emission and absorption of particles by a black hole is described by an effective string living on the 2d BH horizon. More recently AdS/CFT correspondence provided other instances of stringy description of black holes. Another glimpse of relation between strings and BH's comes from the effective near-horizon description proposed recently in [2]. In the present paper we describe yet another relation between strings and black holes. We show that in certain regimes an effective stringy (or, more precisely, conformal field theoretic) description of black hole arises rather naturally and propose certain (yet to be completed) rules of this description. The starting point in our approach is the BH wave equation and its geometric interpretation in terms of two-dimensional Riemann surfaces.

The black hole wave equations is a fascinating object to study. It has been frequently suggested that the wave equation can serve as a window into the quantum mechanical description of the black hole. Indeed, according to Hawking, a black hole emits particles with rate:

$$\Gamma(w) \sim \frac{\sigma_{abs}(w)}{e^{w/T_H} - 1}. \tag{1.1}$$

Here w is the frequency of a quantum emitted, T_H is the Hawking temperature and we have omitted the density of final states factor $d^n k/(2\pi)^n$ that depends on the number n of spacetime dimensions that we are in. The quantity $\sigma_{abs}(w)$ is the black hole absorption cross-section which also can be interpreted as the absolute value squared of the transmission coefficient for quanta emitted from the horizon. It is a complicated function of the frequency which accounts for the transmission of the emitted quanta through the potential outside of the BH. In a quantum mechanical description one attempts to replace the whole BH spacetime by a quantum system that would interact with the asymptotic radiation in exactly the same way as the BH does. In other words, one attempts to describe the BH together with the potential outside. The scattering of quanta by the potential, emission and absorption processes would then all be described as processes of interaction of the radiation with a quantum system that replaced the hole. Such a quantum mechanical description must in particular reproduce (1.1) as the rate of spontaneous emission from the quantum system in question. There is by now a famous example when this requirement basically determines what the quantum system must be. The absorption cross-section for a large class of near-extremal black holes was found to be:

$$\sigma_{abs}(w) \sim \frac{e^{w/T_H} - 1}{(e^{w/2T_L} - 1)(e^{w/2T_R} - 1)}. \tag{1.2}$$

This formula is valid for the range of frequencies in which the near-horizon approximation can be used (small frequencies, low energy). Here T_L, T_R are the so-called left and right temperatures, see, e.g. [3] for more detail. The thermal factors in (1.2) are assembled exactly in such a way that the resulting rate is:

$$\Gamma(w) \sim \frac{1}{(e^{w/2T_L} - 1)(e^{w/2T_R} - 1)}. \qquad (1.3)$$

This is an emission rate from a quantum system containing two types of bosons, each of them at the corresponding temperature. Such behavior is characteristic of 2D conformal field theories, where T_L, T_R are the temperatures of left and right movers. This suggests that the quantum mechanical description of the BH is given by a 2D CFT. This description is valid for small frequencies, that is, in the near-horizon regime. Thus, in this regime a near-extremal BH behaves like a 2D CFT! The behavior of the BH absorption cross-section suggested what quantum system must be used to describe the hole.

Another instance of a possible glimpse into the black hole quantum physics comes from the study of the BH quasi-normal modes. It has been speculated in [4] that the behavior of the QNM spectrum in the limit of high damping can provide some hints as to the BH horizon area spectrum quantization. The proposal of Hod is intriguing, and has recently attracted some attention. In particular Neitzke [5] has calculated the absorption cross-section for the Schwarzschild black hole (in any dimension) in the same high damping limit and found a suggestive result:

$$\sigma_{abs}(w) \sim \frac{e^{w/T_H} - 1}{e^{w/T_H} + 3}. \qquad (1.4)$$

Thus, in the regime of high damping $w \to i\infty$ the black hole emits particles with the rate:

$$\Gamma(w) \sim \frac{1}{e^{w/T_H} + 3}. \qquad (1.5)$$

This answer suggests to search for a system with such statistics. So far, however, no genuine quantum system with such properties have been found.

Thus, properties of the black hole wave equation can be quite instrumental in determining what quantum system describes the hole. In this paper we study wave equations for various black holes and find that certain Riemann surfaces arise very naturally in this context. We then use these surfaces to propose an effective conformal field theoretic description of black holes in various limits. Our proposal is that the BH transmission coefficient

is related to a certain Liouville theory n-point function on the sphere; the number of insertions depends on the topological type of the Riemann surface that corresponds to the BH. Thus, the description we propose is essentially a stringy one in that it deals with Riemann surfaces. Our main application is the limit of high damping, or equivalently high energy $|w| \to \infty$. In this case the transmission coefficient is related to a certain Liouville 3-point function.

Few remarks are in order. First of all, the effective description that we propose in this paper is not of the black hole itself in that the question of what are the fundamental degrees of freedom of a BH is not addressed. The story we present is relevant for describing not the BH degrees of freedom but propagation of waves through the black hole background in various limits. In this sense it is similar to e.g. an effective low energy description of rotating black holes found in [6]. We hope that availability of an effective description along the lines of this paper will one day aid in finding the fundamental theory describing black holes quantum mechanically.

Another remark is that the term "stringy" description that appears in the title must be understood in a loose sense. What is implied is really a conformal field theoretic description in that properties of the wave equation are encoded in certain Liouville theory correlation functions. Even though in the case of higher dimensional black holes the Riemann surfaces that appear come with moduli, we are not making any claim as to the necessity of integrating over these moduli.

The paper is organized as follows. We start by reviewing the black hole wave equations in section 2. In section 3 we analyze the mapping given by the "tortoise" coordinate and describe the Riemann surfaces that appear. Section 4 deals with various limits of the wave equation. We show how in such limits the wave equation becomes solvable and find corresponding Riemann surfaces. These surfaces are then used in section 5 to propose an effective Liouville theory description. We conclude with a discussion of the results obtained.

2   Black hole wave equation and the "tortoise" coordinate

In this paper we shall study non-rotating black holes in various dimensions. The metric for such a BH is of the form:

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d^2\Omega. \tag{2.1}$$

Here $d^2\Omega$ is the metric on the unit sphere of the appropriate dimension. The function $g(r)$ vanishes on the BH horizon. The wave equation for scalar perturbations of mass m in this background is:

$$\frac{1}{\sqrt{-\det}}\partial_i\sqrt{-\det}\,g^{ij}\partial_j\varphi = m^2\varphi. \tag{2.2}$$

It takes the form:

$$\left(-\frac{1}{g(r)}\partial_t^2 + \frac{1}{r^{n-2}}\partial_r r^{n-2}g(r)\partial_r + \frac{1}{r^2}\Delta_\Omega\right)\varphi = m^2\varphi. \tag{2.3}$$

Here n is the number of spacetime dimensions and Δ_Ω is the Laplacian on the unit sphere. Taking the ansatz:

$$\varphi = e^{-iwt}\frac{\psi(r)}{r^{(n-2)/2}}Y_{lM}(\Omega), \tag{2.4}$$

where l is the angular momentum of the spherical harmonic mode, and M is a collective index of an appropriate length, we get the radial wave equation:

$$\left(-\partial_z^2 + V(r(z)) - w^2\right)\psi(z) = 0. \tag{2.5}$$

We have introduced the "tortoise" coordinate:

$$z(r) = \int \frac{dr}{g(r)}. \tag{2.6}$$

This coordinate is defined up to a constant. The potential in (2.5) is given by:

$$V(r) = g(r)\left(\frac{l(l+n-3)}{r^2} + \frac{n-2}{2}\frac{1}{r^{(n-2)/2}}\partial_r(g(r)r^{(n-4)/2}) + m^2\right). \tag{2.7}$$

Thus, the radial wave equation is the Schroedinger equation for a particle moving in the potential $V(r(z))$. For non-extremal BH's the function $g(r)$ has a first order zero at the horizon. Thus, near horizon

$$z(r) \sim \log(r - 1), \tag{2.8}$$

and therefore $z \to -\infty$ as $r \to 1$. This is why the coordinate is called the "tortoise": it takes an infinite amount of the z-coordinate to reach a finite in the r-coordinate distance to the horizon. For asymptotically flat BH's the metric at infinity should approach the flat metric. Thus, $g(r) \to 1$ as $r \to \infty$. Therefore, for large values of r the tortoise coordinate coincides with the radial one: $z \sim r, r \to \infty$. Thus, in the asymptotically flat case the effective quantum mechanical problem is one on the real line: $z \in (-\infty, \infty)$.

The potential $V(r(z))$ vanishes at the horizon: $V(z) \to 0, z \to -\infty$ and approaches constant $V(z) \sim m^2$ at infinity. For functions $g(r)$ coming from the BH metrics the potential is everywhere positive. For massless fields the potential is zero at both ends and therefore looks like the snake that ate an elephant from Antoine de Saint Exúpery's book "The Little Prince". Its hight in the maximum depends on l. The absorption cross-section σ_{abs} must be obtained by considering the transmission in this potential. The quasi-normal modes are poles of σ_{abs}.

Apart from the Schroedinger equation the wave equation has yet another useful interpretation. It is that of the Fuchsian differential equation from the theory of conformal mappings. We turn to this interpretation in the next section.

3   Riemann surfaces

The following differential equation appears prominently in the theory of conformal mappings:

$$\partial_z^2 \psi + \frac{1}{2} T \psi = 0. \tag{3.1}$$

Here $T = T(z)$ is a holomorphic function of z. The equation (3.1) is known as the Fuchsian differential equation. Its characteristic property is that the ratio of two linearly independent solutions:

$$w(z) := \psi_1(z)/\psi_2(z) \tag{3.2}$$

is a map from the complex z-plane to the complex w-plane such that its Schwarzian derivative equals to T:

$$\mathcal{S}(w; z) := \frac{w'''}{w'} - \frac{3}{2} \left(\frac{w''}{w'} \right)^2 = T(z). \tag{3.3}$$

The problem of solving the differential equation (3.1) can be reduced to the problem of finding the image of the complex z-plane under the map $w(z)$: on the Riemann surface, that is the image under this map, the equation (3.1) becomes the trivial equation:

$$\partial_w^2 \psi = 0. \tag{3.4}$$

Thus, one can immediately write down the general solution of (3.1) if the map $w(z)$ is known. The problem of solving the wave equation thus reduces to the problem of finding the corresponding Riemann surface.

Thus, there are two Riemann surfaces to consider. One is the complex z-plane, where the equation is given. Singular points of the potential are special points on the z-plane. Having this points marked gives a Riemann surface of a particular topological type. We shall refer to this surface as the Z-surface. The other surface is the surface that solves the Fuchsian differential equation. It is the surface of the w-coordinate. We shall refer to it as the W-surface.

It is important to note that the topological type of the Z and W-surfaces is the same. Indeed, consider the fundamental group $\pi(Z)$ of the Z-surface. When one goes around a cycle on this surface the two linearly independent solutions transform into linear combinations of the solutions. Then the w-coordinate (3.2) undergoes a fractional linear transformation. Such a fractional linear transformation corresponds to an element of the group $\mathrm{SL}(2, \mathbb{C})$. We thus get a map from $\pi(Z)$ to $\mathrm{SL}(2, \mathbb{C})$. The W-surface is the quotient of the complex plane by the image of this map. Topologically the fundamental group of the W-surface is therefore the same as that of the Z-surface.

In the present paper we would like to apply the Fuchsian equation point of view to the black hole wave equations, and describe the corresponding Riemann surfaces. Since the wave equation becomes trivial on the corresponding surface, the surface itself characterizes the scattering problem completely and is thus of fundamental importance. Let us start from the simplest example in which the wave equation can be solved exactly: the case of the 2+1 dimensional BTZ BH.

3.1 BTZ BH wave equation

The BTZ BH is not asymptotically flat; the corresponding function $g(r)$ is given by:

$$g(r) = r^2 - 1. \tag{3.5}$$

We have again placed the horizon at $r_+ = 1$. The potential for this BH is given by:

$$V(r) = (r^2 - 1) \left(\frac{l^2}{r^2} - \frac{1}{4} \frac{r^2 - 1}{r^2} + 1 + m^2 \right). \tag{3.6}$$

Here l is the angular momentum quantum number on the circle. Unlike the asymptotically flat case, the potential grows at infinity as $V(r) \sim r^2(3/4 + m^2)$. We note that neither of these functions is sensitive to the sign of r.
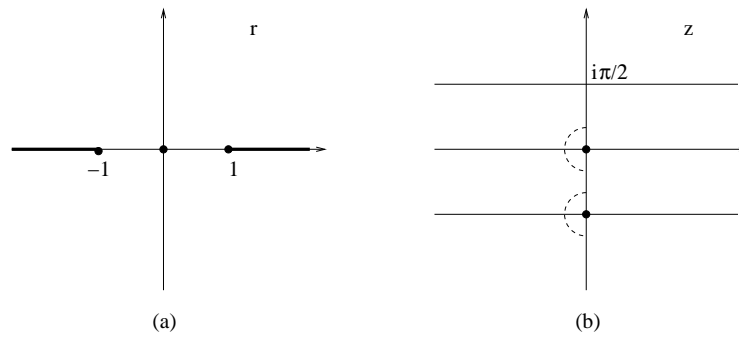
Figure 1: The tortoise map for the BTZ BH. (a) The complex r-plane is shown together with the branch cut (bold line); (b) The image of the r-plane on the z-plane is the strip of width π. Both horizons are mapped to infinity, zero is mapped to zero, and infinity is mapped to the point $z = -i\pi/2$. Closed contours around the singularity and infinity are shown.

The tortoise coordinate (2.6) is given by:

$$z(r) = \frac{1}{2} \log \frac{r-1}{r+1} - \frac{i\pi}{2}. \qquad (3.7)$$

We have chosen the additive constant so that $z(r) = 0$ when $r = 0$. This choice will prove convenient for what follows. Let us note that both $g(r)$ and $V(r)$ are functions of the r^2. Note also that $z(-r) = -z(r) \bmod(i\pi)$. This suggests that r must be identified with $-r$ and z with $-z$. As we shall see, this identification is necessary to get surfaces of the right topological type (namely, those related to hypergeometric equation).

Let us now see how the complex r-plane is mapped to the z-plane. Since $\pm r$ are to be identified, it is enough to consider only the right half-plane. However, as usual, it is easier to work with the covering space. Let us first give the picture of the mapping of the covering space. The logarithm function requires a branch cut. We shall place the cut along the real axis, from $-\infty$ to -1, and from 1 to ∞. Thus, on the compactified r-plane this is simply a cut from -1 to 1 passing through infinity. The r-plane is mapped into the strip $\operatorname{Im} z \in [-\pi/2, \pi/2]$ in the complex z-plane. The horizons $r = \pm 1$ are mapped to infinity, and the singularity $r = 0$ is mapped to zero. The infinity in the r-plane is mapped to the point $z = -i\pi/2$. See Fig.1.

Thus, if not for the identification $r \sim -r$, the image of the r-plane under the tortoise coordinate map would be a strip with its two edges identified, that is a cylinder. Let us now describe the result of the identification. The action of the identification on the z-plane is: $z \sim -z$. The quotient is therefore half the cylinder, with one of the end holes sealed. The resulting
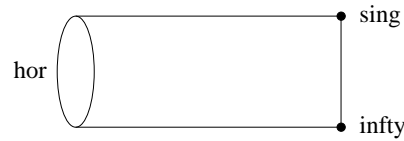
Figure 2: The surface obtained after all the identifications are performed.

Z-surface is shown in Fig.2. Topologically this is a sphere with a hole and two conical singularities of angle deficit π each.

Thus, we get a wave equation in the complex z-plane. In the case at hand the map $z(r)$ is not very complicated; it can be explicitly inverted: $r = \tanh(-z)$. It is then easy to write the wave equation directly in terms of the z-coordinate. One gets an equation solvable in terms of the hypergeometric function. To bring it into the explicitly hypergeometric form one makes a further change of variables that maps the infinity on the z-plane to zero, and the point $z = i\pi/2$ to one. Instead of describing this change of coordinates let us follow another route. Let us directly get the Riemann surface that "solves" the wave equation. We shall use the following reasoning. The wave equation is one on the surface shown in Fig.2. The potential has 3 singular points: horizon, singularity and the infinity. As one can easily check, the leading singularity at these points is that of a double pole. This is suggestive of the Fuchsian equation that corresponds to the hypergeometric function; we review this equation in the appendix. Knowing that the equation is of the hypergeometric type one can find the parameters of the hypergeometric function by calculating the monodromy around the singular points. This way one determines the parameters directly, without having to do a coordinate transformation.

Thus, we would like to calculate the monodromy of the BTZ BH wave equation around the horizon, singularity and infinity.

Horizon. At the horizon the potential vanishes, the wave equation becomes:

$$(\partial_z^2 + w^2)\psi = 0. \tag{3.8}$$

Its two linearly independent solutions are: $\psi_{1,2} = e^{\pm iwz}$. A rotation by 2π around the horizon is a shift $z \to z + i\pi$. The two solutions transform: $\psi_1 \to e^{-\pi w}\psi_1, \psi_2 \to e^{\pi w}\psi_2$. The corresponding monodromy matrix is $\mathbf{M}_h = \text{diag}(e^{-\pi w}, e^{\pi w})$. The trace is equal to:

$$\frac{1}{2}\text{Tr}\mathbf{M}_h = \cosh \pi w. \tag{3.9}$$

Singularity. The tortoise coordinate map $z(r)$ behaves near the singularity as: $z = -r + O(r^2)$. Thus, keeping only the leading term of the potential, the equation becomes:

$$\left(\partial_z^2 + \frac{1 + 4l^2}{4z^2} + w^2\right) = 0 \tag{3.10}$$

One gets the equation solvable in terms of the Bessel functions. The two linearly independent solutions are:

$$\psi_{1,2} = \sqrt{z} J_{\pm il}(wz). \tag{3.11}$$

A tricky point is that the closed contour around the singularity is a rotation by π, not by 2π. This is due to the identification $z \sim -z$ that must be performed on the z-plane. Thus, a rotation around $z = 0$ by π results in: $\psi_1 \to e^{i\pi/2 - \pi l}\psi_1, \psi_2 \to e^{i\pi/2 + \pi l}\psi_2$. Thus, the corresponding monodromy matrix is: $\mathrm{diag}(ie^{-\pi l}, ie^{\pi l})$. As it stands, its determinant is not equal to one. We have to divide this matrix by the square root of its determinant. The correctly normalized monodromy matrix is given by: $\mathbf{M}_s = \mathrm{diag}(e^{-\pi l}, e^{\pi l})$; its trace is given by:

$$\frac{1}{2}\mathrm{Tr}\mathbf{M}_s = \cosh(\pi l). \tag{3.12}$$

Infinity. Infinity in the complex r-plane is mapped to $z = -i\pi/2$. Introducing a new coordinate $z = -i\pi/2 + y$, and again keeping only the leading term of the potential $V(y) = y^2(3/4 + m^2)$, we get an equation:

$$\left(\partial_y^2 - \frac{3 + 4m^2}{4y^2} + w^2\right) = 0. \tag{3.13}$$

It is solution is given in terms of the Bessel function:

$$\psi_1 = \sqrt{y} J_\nu(wy), \psi_2 = \sqrt{y} J_{-\nu}(wy). \tag{3.14}$$

Here

$$\nu = \sqrt{1 + m^2}. \tag{3.15}$$

We have assumed that ν is not an integer and one can use $J_{\pm\nu}$ as two linearly independent solutions. If this is not so, one has to use the Bessel function of the second kind as the second linearly independent solution. In any case, a closed contour around infinity is a rotation by π around $y = 0$. The monodromy matrix is calculated using (B.3). One gets:

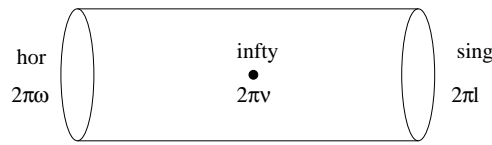$$\frac{1}{2}\mathrm{Tr}\mathbf{M}_\infty = \cos \pi\nu. \tag{3.16}$$

Figure 3: The W-surface for the BTZ BH

This result holds independently of whether ν is an integer or not.

Having calculated the monodromies of the wave equation, it is easy to write the hypergeometric equation with the same monodromies. One uses the fact that the monodromies around the singular points of the hypergeometric equation are given by $\cos(\pi a_i)$. One immediately gets:

$$a_1 = iw, \qquad a_2 = \nu, \qquad a_3 = il. \tag{3.17}$$

Notice that $a_1 = 2\pi T_H i\omega$ in terms of the Hawking temperature $T_H = 1/2\pi$. Written in this form the expression for the monodromy around the horizon is also valid for the Schwarzschild black hole in any dimension, as we show later in the paper. The quantities (3.17) are correct parameters that one can obtain, for example, by doing a coordinate transformation that brings the wave equation into the hypergeometric form, see [7, 8]. The resulting W-surface, or, in other words, the surface that solves the wave equation is shown in Fig.3.

Thus, a solution of the resulting wave equation is the hypergeometric function $F(\alpha, \beta; \gamma; y)$ with singular points at $y = 0$ (singularity), $y = 1$ (infinity) and $y = \infty$ (horizon) and parameters:

$$\alpha = \frac{1}{2}(1 - \nu + i(l - w)), \quad \beta = \frac{1}{2}(1 - \nu - i(l + w)), \quad \gamma = 1 - iw. \tag{3.18}$$

as determined in terms of monodromies (3.17) by (A.3).

Having determined the parameters (3.17), (3.18) it is quite easy to determine the quasi-normal modes. One has to impose the following boundary conditions: the solution should vanish at infinity and be in-going at the horizon. It is possible to satisfy these boundary conditions only at a discrete set of frequencies: the quasi-normal modes. As the solution that is purely in-going at the horizon one takes, see [8], the hypergeometric function $F(\alpha, \beta; \gamma; y)$ multiplied by y to certain power. Here y is a coordinate that is different from the tortoise coordinate. It is the one in which the wave equation has an explicitly hypergeometric form. The horizon is located at $y = 0$ in this coordinates. The relation between y and z is $y = -1/\sinh^2 z$. Note that our method works with monodromies only and we never need to

use this relation. This solution can be expanded into two linearly indepen-
dent solutions near $y = 1$ (infinity). One has to use the formula (A.7) given
in the Appendix. The second term in proportional to $(1 - y)$ and vanishes
at $y = 1$. The condition that the first term vanishes is satisfied at the poles
of the Γ functions in the denominator. We get two sets of poles:

$$\frac{1}{2} + \frac{\nu}{2} - \frac{iw}{2} - \frac{il}{2} = -n, \qquad \frac{1}{2} + \frac{\nu}{2} - \frac{iw}{2} + \frac{il}{2} = -n. \qquad (3.19)$$

In terms of the monodromies (3.17) these two sets are represented as follows

$$1 + a_2 - a_1 - a_3 = -2n \ , \quad 1 + a_2 - a_1 + a_3 = -2n \ . \qquad (3.20)$$

This gives the correct, see [7, 8], quasi-normal modes. Recalling that the
Hawking temperature for the BTZ (in the units when $r_+ = 1$) is equal to
$T_H = 1/2\pi$ the frequencies can be rewritten in a more familiar form:

$$w = \pm l - 4\pi i T_H(n + h), \qquad (3.21)$$

where we have introduced:

$$h = \frac{1}{2}(1 + \nu) = \frac{1}{2}(1 + \sqrt{1 + m^2}). \qquad (3.22)$$

This is usual expression for the conformal dimension of a field of mass m in
AdS/CFT correspondence.

We would like to point out that there is a quicker root to the quasi-
normal modes. Let us consider the case $m = 0$. Then, quite remarkably,
the QNM condition can be expressed geometrically as the condition that the
sizes of the two holes of the W-surface are equal:

$$\cos \pi a_1 = \cos \pi a_3. \qquad (3.23)$$

This gives the correct expression $w = \pm l - 2ni$. We shall see how a similar
geometric condition leads to the quasi-normal modes for the Schwarzschild
BH. Equivalently, the equation (3.23) can be written in the form

$$e^{i\pi a_1} \ e^{i\pi a_3} = 1 \ \ .$$

This is the form which can be further generalized to include the case of non-
vanishing mass m when the monodromy a_2 around the infinity is non-trivial.
The generalization can be read from the condition (3.20) and takes the form

$$e^{i\pi a_1} \ e^{-i\pi a_2} \ e^{i\pi a_3} = e^{i\pi} \ \ . \qquad (3.24)$$

We show later in the paper that the condition (3.20) and respectively (3.24)
emerge naturally as a condition for location of poles in the Liouville 3-point

function. The condition similar to (3.24) was suggested in [9] for monodromies around inner and outer horizons of the rotating BTZ black hole.

Thus, we have learned that the wave equation has two closely related Riemann surfaces associated with it. One surface, which we shall refer to as the Z-surface, is simply the image of the r-plane under the tortoise coordinate map $z(r)$. In the case of the BTZ BH this surface is shown in Fig.2. The BH wave equation is the Fuchsian equation on this surface. In the case of the BTZ BH this is the hypergeometric equation with the parameters given by (3.17). The ratio $w(z) = \psi_1/\psi_2$ of two linearly independent solutions of the hypergeometric equation maps the complex z-plane into a sphere with either holes or conical singularities. In our case this is a sphere with two holes of sizes $2\pi w$ and $2\pi l$, and a single "conical defect" whose total angle at the tip is $2\pi\nu$. The Z-surface codes only the properties of the tortoise coordinate map. It does not know anything about the potential. The wave equation is an equation on the Z-surface. The W-surface "solves" this wave equation in the sense that if one knows the map $w(z)$ and its inverse one can explicitly write down the solution on the Z-surface. The W-surface knows about the potential. Its moduli depend on the parameters of the potential, which are the parameters of the BH. It is the W-surface that is the main object of interest.

Let us now show how considerations of this subsection carry over to more interesting black holes.

3.2 Schwarzschild BH wave equation

For the Schwarzschild BH in $n = 4$ spacetime dimensions the function $g(r)$ is given by (we put the horizon at $r_+ = 1$):

$$g(r) = 1 - \frac{1}{r}, \tag{3.25}$$

and one gets the familiar potential:

$$V(r) = \left(1 - \frac{1}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{1}{r^3}\right). \tag{3.26}$$

The tortoise coordinate (2.6) is given by:

$$z(r) = r + \log(r - 1) - i\pi. \tag{3.27}$$

Again, we have chosen the additive constant so that the singularity $r = 0$ is mapped to zero. The map (3.27) is now much more involved than in the
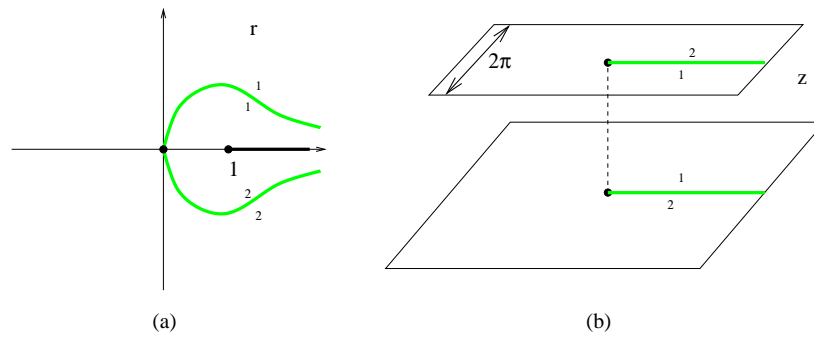
Figure 4: The tortoise map for the 4d Schwarzschild BH. (a) The complex r-plane with a cut placed from the horizon to infinity. The region inside the green curve is mapped into the physical sheet (the strip). The green curve itself becomes the cut on the z-plane (b) The Z-surface has two sheets. The physical sheet is the strip of width 2π; the unphysical sheet is a copy of the complex plane. The sheets are glued as indicated.

case of the BTZ BH. To understand the image of the complex r-plane it is instructive to consider what the circles $r = 1 + Re^{i\phi}$ are mapped to. The presence of the logarithm requires a branch cut. We put it along the positive real axis from the horizon $r = 1$ to infinity. Then small circles $R << 1$ are mapped into almost vertical segments of size 2π. The image of the circle gets more complicated as R increases. When $R = 1$ the image develops a kink. This kink is located exactly at the singularity $z = 0$. For larger radii the image of the circle has a self-intersection. The self-intersection is located on the positive z-axis. A moment of reflection reveals the structure shown in Fig.4. One has a strip of width 2π; with a cut along the positive real axis. The sides of the strip are to be identified. Below the strip one attaches a copy of the complex plane. It is also cut along the positive real axis. One identifies the sides of the cuts as is shown in Fig.4. Note that the rotation by 2π around the singularity is now a rotation by 4π in the z-plane. This can be seen by expanding the logarithm near $r = 0$. One finds: $z = -r^2/2 + O(r^3)$. Half of this rotation occurs on the upper sheet, the other half is on the lower sheet. We shall refer to the two sheets in Fig.4 as the physical (upper) and unphysical (lower). The two sheets glued together form the Z-surface for the Schwarzschild BH.

Let us now discuss the topological type of the Z-surface. The physical sheet is simply a cylinder. One attaches to it a copy of the complex plane. However, this does not change the topological type. Thus, topologically, the Z-surface is simply a cylinder with an additional marked point- the singularity. Let us note however that the two ends of the cylinder will
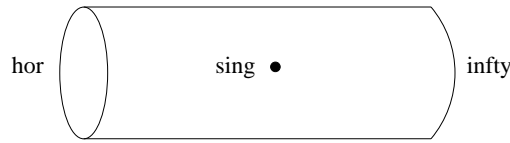
Figure 5: The topological type of the Schwarzschild BH Z-surface is that of a cylinder with one marked point. The marked point corresponds to the singularity.

have different monodromy properties when we consider the wave equation on the Z-surface. Thus, the ends of the cylinder are different. One of them corresponds to the horizon, the other is the infinity. The corresponding surface is shown in Fig.5.

Let us now discuss the wave equation on the Z-surface. Unfortunately, now the potential is too complicated to be solved exactly. Thus, even though the Z-surface is still topologically a sphere with 3 marked points as in the case of BTZ black hole, no exact solution in terms of the hypergeometric function is possible. However, the hypergeometric equation can be used as a good approximation in certain regimes. Such approximations are studied in the next section. Let us now turn to the higher dimensional Schwarzschild black holes, and see what kind of surfaces one gets.

3.3 Higher-dimensional Schwarzschild BH wave equations

The function $g(r)$ characterizing a BH in higher dimensions is given by:

$$g(r) = 1 - \frac{1}{r^{n-3}}. \tag{3.28}$$

As before, the horizon is at $r_+ = 1$. The potential that one gets is:

$$V(r) = \left(1 - \frac{1}{r^{n-3}}\right)\left(\frac{l(l+n-3)}{r^2} + \frac{n-2}{4}\left(\frac{n-4}{r^2} + \frac{n-2}{r^{n-1}}\right)\right). \tag{3.29}$$

The tortoise coordinate is given by:

$$z(r) = r + \sum_{j=0}^{n-4} \frac{e^{2\pi i j/(n-3)}}{n-3} \log\left(1 - re^{-2\pi i j/(n-3)}\right). \tag{3.30}$$

As before, we would first like to understand the Z-surface that appears as the result of the tortoise coordinate mapping. Let us consider the case
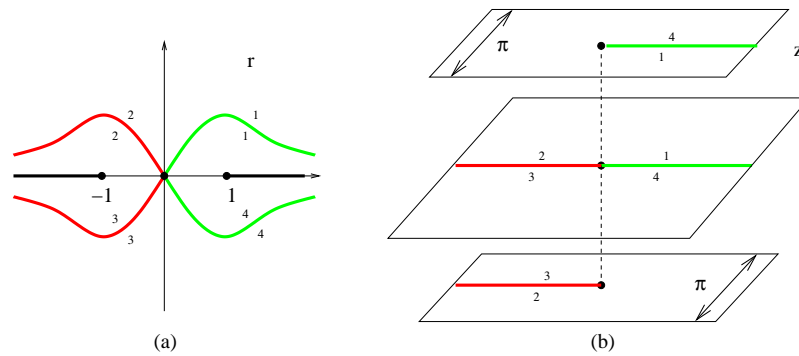
Figure 6: The tortoise map for the 5d Schwarzschild BH. (a) The complex r-plane has now a cut placed between the horizons going through infinity. The green and red curves are mapped to the cuts in the z-plane; (b) The Z-surface of the 5-dimensional Schwarzschild BH consists of 3 sheets. Two of the sheets are strips of width π, the other sheets is a copy of the complex plane. The gluing identifications are indicated.

$n = 5$; the general pattern will then become clear. In this case the tortoise map becomes:

$$z(r) = r + \frac{1}{2} \log \frac{1-r}{1+r}. \tag{3.31}$$

This is "almost" the BTZ BH tortoise coordinate (3.7), if not for the first term. It is the presence of this term that makes the Riemann surface very involved. As in the BTZ case, the presence of the logarithm requires a branch cut. We place it from the horizon at $r = 1$ to the other, unphysical horizon at $r = -1$ along the real axis and avoiding the singularity. Thus, the cut goes through the infinity. Near the singularity the tortoise coordinate behaves as:

$$z = -\frac{r^3}{3} + O(r^4). \tag{3.32}$$

Thus, the singularity $r = 0$ is mapped to $z = 0$, and near the singularity the tortoise map is a 1 to 3 covering. One should expect that the relevant Riemann surface will contain 3 sheets. The surface is shown in Fig.6.

The topological type of the Z-surface can be determined by thinking about the geometry in Fig.6(b). It is a cylinder, with two ends corresponding to the two horizons, and a handle with one of the cycles completely degenerated. The degenerated cycle is at infinity. We believe that it is essentially right to replace the degenerated handle by a marked point, denoted as infinity in Fig.7. The other marked point is the singularity.
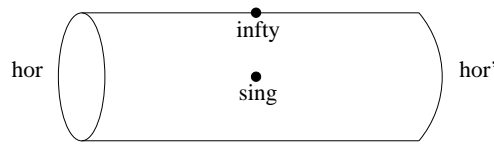
Figure 7: The topological type of the 5d Schwarzschild BH Z-surface is that of a cylinder with 2 marked points.

Being topologically a sphere with 4 marked points the surface is too complicated to approximate the corresponding W-surface by the hypergeometric equation. Thus, no analysis similar to the one performed in the BTZ case is possible. However, simplifications are possible in certain limits, as we shall see in the next section.

Let us now turn to the discussion of Z-surfaces for Schwarzschild black holes in higher dimensions. Expanding the tortoise map near the singularity one finds, in general:

$$z(r) = -\frac{r^{n-2}}{n-2} + O(r^{n-1}). \tag{3.33}$$

Thus, the map near the singularity is 1 to $n-2$. Therefore, the Z-surface consists of $n-2$ sheets. One of these sheets is the physical sheets containing the horizon. It is always a strip of width $2\pi/(n-3)$. Topologically, the Z-surface is a sphere with $n-1$ marked points. Out of these points $n-3$ are holes corresponding to the horizons. The other marked points are the singularity and the infinity. For n even infinity is a hole, for n odd it is just a marked point on the surface.

The wave equations are too complicated to be treated exactly. However, since the singularities of the potential coincide with the singularities (marked points) of the Z-surface, one does not expect that the map $w(z)$ will change the type of the Riemann surface. Thus, it is to be expected that the W-surface that "solves" the wave equation is of the same topological type as the Z-surface. Since the potential is coded in the W-surface, so are the BH parameters.

4 Limits in which the wave equation becomes solvable

In this section we will apply the techniques described to solve the wave equation in certain limits. In order to obtain a solvable wave equation one

has to approximate the potential by a solvable one. In practice this is done by keeping only the leading term of the potential in some limit, and ignoring the other terms. One can classify such approximations according to where on the r-plane the term kept is the leading singularity. Thus, there is an approximation that is good near the horizon or near singularity. Each of this approximations leads to a different wave equation.

Another, more geometrical point of view on this is as follows. As we have explained, the wave equation is "solved" by a particular W-surface. Thus, an approximate solution is obtained by approximating the real W-surface with another one, which can be described explicitly. Approximating the potential by its behavior near one of the singular points on the Z-surface, one essentially zooms onto the region close to this singular point. Thus, the W-surface that arises as the result of the approximation is a part of the "correct" W-surface near one of the singularities. Such an approximation is not good for all frequencies. However, in certain limits, the approximated W-surface captures all physics.

The main limit that we are interested in is that of high damping $w \to i\infty$. However, it is instructive to describe the other possible limits as well. We shall consider two interesting limits:

- The limit of low energy $|w| \to 0$. This corresponds to keeping only the terms most relevant near the horizon of the BH.

- The limit of high energy $|w| \to \infty$. In this limit the relevant region is that where the potential becomes large. One keeps only the leading terms near the singularity.

We restrict our analysis to the case of the 4d Schwarzschild BH; some comments on the general case will be given after this analysis. Let us first consider the low energy limit. It has been analyzed in detail in [2] so we shall be very sketchy.

4.1   Low energy limit.

In this approximation it is convenient to define the tortoise coordinate a bit differently:

$$\tilde{z} = r - 1 + \log{(r - 1)}. \tag{4.1}$$

This amounts to a different choice of the integration constant in (2.6). The potential for the 4d Schwarzschild BH is given by (3.26). Near the horizon it can be approximated by:

$$V(z) = \delta^2 e^{\tilde{z}} - \lambda^2 e^{2\tilde{z}}, \tag{4.2}$$

where $\delta^2 = l(l+1)+m^2+1, \lambda = 4l(l+1)+2m^2+5$. The expansion whose first two terms are (4.2) is in powers of $(w/l)^2$. Thus, it is good when $|w| << l$. Near the horizon $e^{\tilde{z}} = r-1+O((r-1)^2)$. In the near-horizon approximation one keeps only the first term. Then $\tilde{z} = \log(r-1)$. The corresponding Z-surface is a cylinder of circumference 2π. Restricting potential (4.2) to the first term we get a classical Liouville equation while the second term gives a perturbation.

The wave equation with (4.2) can be solved in terms of confluent hypergeometric functions. Let us introduce: $\phi = e^{-\tilde{z}/2}W$ and a new variable $y = 2i\lambda e^{\tilde{z}}$. The wave equation becomes:

$$\frac{d^2W}{dy^2} + \left(-\frac{1}{4} + \frac{i\delta^2}{2\lambda y} + \frac{\frac{1}{4} + w^2}{y^2}\right) W = 0. \tag{4.3}$$

This is the confluent hypergeometric equation in the form (C.1). Thus, two linearly independent solutions of the original wave equation are:

$$\phi_1 = e^{-\tilde{z}/2}W_{\frac{i\delta^2}{2\lambda},iw}(2i\lambda e^{\tilde{z}}), \qquad \phi_2 = e^{-\tilde{z}/2}W_{-\frac{i\delta^2}{2\lambda},iw}(-2i\lambda e^{\tilde{z}}). \tag{4.4}$$

It is now easy to evaluate the monodromies.

Horizon. Near the horizon the potential vanishes. The wave equation becomes (3.8). The monodromy calculation is then unchanged from the BTZ case. The only difference is that now the size of the strip is 2π instead of π. This results in the monodromy:

$$\frac{1}{2}\text{Tr}\mathbf{M}_h = \cosh 2\pi w. \tag{4.5}$$

Infinity. The asymptotic behavior of the Whittaker's functions is given by (C.2). This results in the following monodromy:

$$\frac{1}{2}\text{Tr}\mathbf{M}_\infty = -\cosh \pi\delta^2/\lambda. \tag{4.6}$$

Notice an additional minus sign in front of the cosh that comes from the factor $e^{-\tilde{z}/2}$ multiplying Whittaker's functions in (4.4).
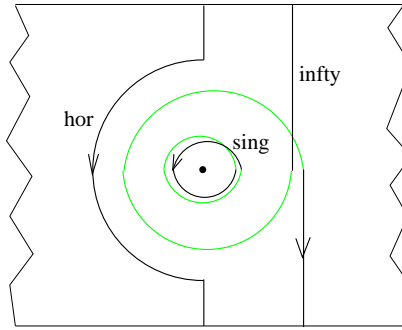
Figure 8:   Contours for evaluation of monodromies in the case of Schwarzschild black hole high energy limit. Parts of contours shown in green are on the unphysical sheet.

The resulting W-surface is now a cylinder with sizes of holes $4\pi w$ and $2\pi\delta^2/\lambda$. As in the BTZ case, the quasi-normal modes can be determined from a geometric condition saying that the sizes of holes are equal. Thus equating monodromies (4.5) and (4.6) this gives the correct, see [2], expression for the QNM frequency in the potential (4.2):

$$w = 2\pi T_H \delta^2/\lambda - 4\pi i T_H(n + \frac{1}{2}) \quad . \tag{4.7}$$

4.2   High energy limit.

As it was demonstrated in [10], in the limit $|w| \to \infty$ it is correct to keep only the leading singularity of the potential near $r = 0$. For the 4d Schwarzschild, this singularity is $V(r) \sim -1/r^4 \sim -1/4z^2$. It can be shown, see [11], that the sub-leading term is $-[l(l+1)+1/3]/(-2z)^{3/2}$. One can rewrite the wave equation in terms of the rescaled coordinate $y = wz$. Then the sub-leading term will be of order $[l(l+1)+1/3]/w^{1/2}$, and can be dropped in the large $|w|$ limit. Thus, the high energy limit is that of $|w| >> l^4$. In this approximation one has to use:

$$V(z) = -\frac{1}{4z^2} \tag{4.8}$$

for all values of z, not necessarily small. Actually, one has to be a bit careful here because the potential must be periodic on the physical sheet (the strip) and (4.8) is not. Thus, to be more precise, we approximate the potential by (4.8) only on the unphysical sheet.

Having spelled out the approximation that we are going to use, it is easy to determine a solvable wave equation with the same properties. Indeed, the Z-surface in question is a sphere with 3 marked points. A solvable wave equation in this case must be of the hypergeometric type. It is therefore completely determined by its monodromies. Thus, we only have to compute the monodromies around the singularities.

Thus, the potential in this approximation is given by (4.8). The wave equation is of the Bessel type. Two linearly independent solutions are given by $\psi_1 = \sqrt{z}J_0(wz), \psi_2 = \sqrt{z}N_0(wz)$. The contours for evaluating monodromies are shown in Fig.8. The calculation is as follows.

Horizon. In the high energy approximation one zooms onto the singularity region. Thus, the contour around the singular point that corresponds to the horizon looks as shown in Fig.8. In particular, this closed contour involves not only the shift $z \to z + 2\pi i$, but also a rotation around singularity by π in the anti clock-wise direction. Thus, one must first perform the shift $z \to z + i\pi$, then do a rotation by π around the singularity, and then perform another shift $z \to z + i\pi$. The asymptotic of the Bessel functions for large $|z|$ is given by (B.5). The computation of the monodromy is then straightforward:

$$
\begin{aligned}
J_\nu(zw + i\pi w) &= \cosh \pi w J_\nu(zw) - i \sinh \pi w N_\nu(zw), \\
N_\nu(zw + i\pi w) &= i \sinh \pi w J_\nu(zw) + \cosh \pi w N_\nu(zw).
\end{aligned}
\tag{4.9}
$$

Thus, the monodromy matrix for the shift is:

$$
\mathbf{M}_{shift} = \begin{pmatrix} \cosh \pi w & -i \sinh \pi w \\ i \sinh \pi w & \cosh \pi w \end{pmatrix}.
\tag{4.10}
$$

Let us now calculate the monodromy for the rotation. This is an anti clock-wise rotation by π. The solutions transform as:

$$
J_\nu(ze^{\pi i}) = e^{\pi i \nu} J_\nu(z),
\tag{4.11}
$$

$$
N_\nu(ze^{\pi i}) = e^{-\pi i \nu} N_\nu(z) + 2i \frac{\sin \pi \nu}{\tan \pi \nu} J_\nu(z).
$$

The Bessel functions we need are for $\nu = 0$. Then the last term in the transformation formula for N_ν has $0/0$ type singularity. One has to understand the expression in the limiting sense. One then gets the following monodromy matrix:

$$
\mathbf{M}_{rot} = \begin{pmatrix} 1 & 0 \\ 2i & 1 \end{pmatrix}.
\tag{4.12}
$$

The total monodromy matrix is $\mathbf{M}_h = \mathbf{M}_{shift}\mathbf{M}_{rot}\mathbf{M}_{shift}$, and its trace is given by:

$$\frac{1}{2}\mathrm{Tr}\mathbf{M}_h = \cosh 2\pi w + \sinh 2\pi w = e^{2\pi w}. \tag{4.13}$$

Singularity. A closed cycle around the singularity is a rotation by 4π in the z-plane. The monodromy one gets is:

$$\frac{1}{2}\mathrm{Tr}\mathbf{M}_s = 1. \tag{4.14}$$

Infinity. As before, the cycle around infinity is non-trivial, see Fig.8. One must first perform the shift $z \to z+i\pi$, then do a rotation, and then perform another shift $z \to z+i\pi$. The monodromy matrix for the shift was calculated above. The computation of the monodromy for rotation is also similar, but now the rotation is by angle 2π in the clock-wise direction. We get:

$$\mathbf{M}_{rot} = \begin{pmatrix} 1 & 0 \\ -4i & 1 \end{pmatrix}. \tag{4.15}$$

The total monodromy matrix is $\mathbf{M}_\infty = \mathbf{M}_{shift}\mathbf{M}_{rot}\mathbf{M}_{shift}$, and its trace is given by:

$$\frac{1}{2}\mathrm{Tr}\mathbf{M}_\infty = \cosh 2\pi w - 2\sinh 2\pi w. \tag{4.16}$$

We now have all three monodromies, so we can immediately obtain the W-surface that arises in this approximation. The corresponding wave equation is the hypergeometric one, with the parameters given by:

$$\cos \pi a_1 = \cosh 2\pi w + \sinh 2\pi w, \qquad \cos \pi a_2 = \cosh 2\pi w - 2\sinh 2\pi w. \tag{4.17}$$

and $a_3 = 0$. Let us also explicitly write down the parameters of the hypergeometric function:

$$\alpha = \beta = \frac{1}{2} - \frac{a_1}{2} - \frac{a_2}{2}, \qquad \gamma = 1 - a_1. \tag{4.18}$$

Thus, the W-surface in this approximation is the sphere with two holes: horizon plus infinity. The object monodromy around which is unity is the so-called puncture: it is a point of angle deficit 2π. Thus, one gets the W-surface shown in Fig.9.
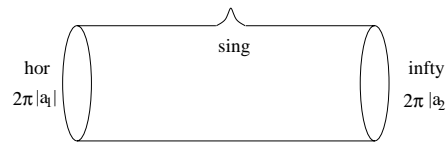
Figure 9: The W-surface in the high energy limit has two holes and one puncture.

Let us now discuss the quasi-normal modes. Once we have determined the wave equation to be the hypergeometric one it is straightforward to get the quasi-normal modes. They are zeros of the coefficient in front of the term in (A.7) that is interpreted as the wave incoming at infinity. It is easy to see that such zeros can only come from the poles of $\Gamma(\alpha)$ and $\Gamma(\beta)$. The poles of the Γ-function are located at negative integers. Thus, we have an equation for the quasi-normal modes:

$$1 - a_1 - a_2 = -2\,n, \tag{4.19}$$

This is quite similar to the condition (3.20) we had in the case of BTZ black hole (notice that in the present case the third monodromy a_3 vanishes). Let us multiply this by π and take the exponent. We find:

$$\cos \pi a_2 = -\cos \pi a_1. \tag{4.20}$$

One can now use the formula (4.17) to get:

$$e^{4\pi w} = -3. \tag{4.21}$$

So that the QNM frequencies fall into the set

$$w_n = T_H \ln 3 - 2\pi T_H i(n + \frac{1}{2})\quad. \tag{4.22}$$

This is the log 3 behavior at high damping derived by a different but related method in [10]. Let us also note that, as in the examples considered before, the condition for QNM's has a geometrical interpretation. The equation (4.20) simply says that the sizes of holes are equal. The sign here is important. Compare (4.20) to the previous low-energy limit case; there the surface is simply a two-punctured sphere and one equates the monodromies without an extra sign. The sign in (4.20) comes from the additional puncture that is inserted into the W-surface at the point corresponding to the singularity, see Fig.9. This is especially clear from the analysis of the poles of certain Liouville theory 3-point function that is to be performed in the next section.

4.3   Reflection/transmission coefficients.

In the limits discussed when the wave equation can be solved exactly one can find not only the quasi-normal modes but also calculate explicitly the reflection (or equivalently, the transmission) coefficient. These coefficients determine what is called the grey-body factor which shows how the Hawking radiation deviates from the purely thermal spectrum. The quasi-normal modes have standard interpretation as poles in the reflection amplitude. The latter however contains more information than just the location of its poles. In the low energy limit the reflection/transmission coefficients have been calculated in [2]. Here we focus on the high energy limit. The calculation of the grey-body factors for BTZ black hole was done in [12].

The general solution to the hypergeometric equation is a combination of two functions:

$$\phi(y) = A\ F(\alpha, \beta; \gamma; y) + B\ y^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; y), \quad (4.23)$$

where in terms of the y coordinate the asymptotic infinity is at $y = 1$ while the horizon is at $y = 0$. Expanding the solution near $y = 1$ and using equation (A.7) and (A.6) we find that $\phi(y)$ is a combination of a constant function and $(1 - y)^{\gamma - \alpha - \beta}$. For parameters (4.18) we find that $\gamma - \alpha - \beta = a_2(\omega)$. To identify in- and out-going modes let us consider large real frequencies. In this limit $a_1(w) \sim a_2(w) \sim 2iw$. Thus, the mode $y^{1-\gamma}$ is out-going at the horizon, and the mode $(1 - y)^{\gamma - \alpha - \beta}$ is in-going at infinity. Assuming that a particle comes from the horizon, scatters and then goes to infinity we expect only out-going modes to appear at infinity. Thus, the in-going at infinity mode should vanish. This gives a condition on constants A and B in (4.23):

$$A \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} + B \frac{\Gamma(2 - \gamma)}{\Gamma(\alpha - \gamma + 1)\Gamma(\beta - \gamma + 1)} = 0 \quad . \qquad (4.24)$$

Now we should look at the solution at the horizon, i.e. when $y \to 0$. The solution (4.23) behaves there as follows

$$\phi(y) = A + B\ y^{1-\gamma}, \qquad (4.25)$$

where A and B are same coefficients as in (4.23) and (4.24); it is a combination of the constant function and a mode $y^{1-\gamma}$. Since $\gamma = 1 - 2iw$ the latter mode is out-going at the horizon while the constant function represents the in-going mode. The reflection amplitude for a particle emitted form horizon is given by the ratio, $S(w) \sim A/B$. Using the relation (4.24) and the definition (4.18) for the parameters of the hypergeometric function we find the

expression for the reflection amplitude

$$S(w) \sim -\frac{\Gamma(1 + a_1(w))}{\Gamma(1 - a_1(w))} \left(\frac{\Gamma(\frac{1}{2} - \frac{a_1(w)}{2} - \frac{a_2(w)}{2})}{\Gamma(\frac{1}{2} + \frac{a_1(w)}{2} - \frac{a_2(w)}{2})} \right)^2 . \qquad (4.26)$$

This can be compared to the expression found in [2] in the low energy case. Two expressions have a similar structure. Note, however, that the poles in (4.26) are double poles, while the reflection amplitude in the low energy case has only simple poles. In the case of non-rotating BTZ black hole the poles are doubled as well [12] for zero angular momentum l. This degeneracy is removed when either rotation or non-vanishing angular momentum is included. The location of the poles in (4.26) gives the QNM as expected. It would be of interest to compare (4.26) with the expression found in [5]. We shall not pursue this in the present paper.

5 Effective theory

We are now in a position to interpret all the results obtained. We have seen how two types of surfaces naturally appear when one considers the BH wave equation. One surface is the Riemann surface of the tortoise map. The wave equation is defined on this surface. The wave equation can be interpreted as the Fuchsian equation. This interpretation gives rise to the W-surface that solves the equation. It is this surface that is fundamental for the interpretation that we are going to propose.

We have seen that the role of the arising W-surface is two-fold. First of all, the W-surface encodes properties of the potential. It contains information about the background black hole geometry. Importantly, this information is not just about the part of space-time which is out-side the horizon but also about the BH interior. This is manifested by the fact that marked points on the W-surface correspond to space-time infinity, black hole singularity and (possibly more than one) horizon. On the other hand, this surface encodes information about the wave propagation in the black hole background. In certain sense (explained in detail in section 3) the surface itself is a solution to the wave equation. Information about the modes emitted from horizon and propagating to infinity is encoded in the monodromies around the horizon and space-time infinity marked points respectively. Another observation we have made is that the quasi-normal modes arise from the geometric condition that these monodromies are the same. This condition is non-trivial since another marked point representing the black hole singularity with some non-vanishing monodromy is always present on the

surface: the QNM's are thus a sort of resonant states between these three marked points on the surface.

Now, as a step in the direction of an effective theory interpretation let us make the following observation. In all (solvable) examples that we have considered the W-surface is topologically a sphere with 3 marked points, and we completely characterized the surface by its monodromies (sizes of holes). One can obtain the same W-surfaces in the CFT context. Indeed, the semi-classical limit of Liouville theory is intimately related with the uniformization and Riemann surfaces. This is explained, for example, in the classical paper [13] that proposed an answer for the Liouville theory 3-point function. The basic idea of this relation is that in the semi-classical limit insertion of Liouville vertex operators modifies the Liouville equation by introducing sources into it. The solution of the resulting Liouville equation can be interpreted as giving a metric with a hole or conical singularity at the point of insertion of the vertex operator. The precise identification is as follows. Consider "heavy" vertex operators $V_\alpha = e^{\alpha\phi(z)}$ with $\alpha = \eta/b$. Here η is a parameter that survives in the semi-classical approximation, $\phi(z)$ is the Liouville field, and b is the Liouville coupling constant. The semi-classical limit corresponds to $b \to 0$. For real $\eta < 1/2$ such an operator will create a conical singularity of the deficit angle $4\pi\eta$. For a general η the monodromy around the "object" created is: $\cos \pi a$ where $a = 1 - 2\eta$.

Consider now the 3-point function of Liouville theory where the vertex operators that one inserts are such that in the semi-classical limit the monodromies created would exactly match the monodromies of the W-surface. Our observation is that the poles of such 3-point function are exactly the QNM of the corresponding BH. Let us demonstrate this for the BTZ black hole.

5.1   Liouville 3-point function: BTZ black hole.

We take the expression (D.1) for the 3-point function and substitute into it the parameters:

$$\alpha_1 = \frac{1}{2b} - \frac{iw}{2b}, \qquad \alpha_2 = \frac{1-\nu}{2b}, \qquad \alpha_3 = \frac{1}{2b} - \frac{il}{2b}. \tag{5.1}$$

Then the 3-point function takes the form:

$$C(\alpha_1, \alpha_2, \alpha_3) \sim \tag{5.2}$$

$$\frac{\Upsilon(\frac{1-iw}{b})\Upsilon(\frac{1-\nu}{b})\Upsilon(\frac{1-il}{b})}{\Upsilon(\frac{1-\nu}{2b} - \frac{i(w+l)}{2b})\Upsilon(\frac{1-\nu}{2b} - \frac{i(w-l)}{2b})\Upsilon(\frac{1-\nu}{2b} + \frac{i(w-l)}{2b})\Upsilon(\frac{1+\nu}{2b} - \frac{i(w+l)}{2b})}.$$

We work in the semi-classical approximation in which $Q \sim 1/b$. Taking into account the position (D.2) of zeros of the function $\Upsilon(x)$ we see that the poles of (5.2) occur in the following 4 series:

$$w = \pm l - 2i(n + h), \qquad w = \pm l + 2i(n + h), \qquad (5.3)$$
$$w = \pm l - 2i(n + \tilde{h}), \qquad w = \pm l + 2i(n + \tilde{h}).$$

Here everywhere $n \in \mathbb{Z}^{\geq 0}$ and we have introduced:

$$\tilde{h} = \frac{1}{2}(1 - \nu) = \frac{1}{2}(1 - \sqrt{1 + m^2}). \qquad (5.4)$$

The first line in (5.3) is the correct set of QNM frequencies (3.21) together with their complex conjugates. The second line are similar frequencies in which the conformal dimension h is replaced by the conjugate conformal dimension $\tilde{h}$. These two sets do not overlap. Because $\tilde{h} = 1 - h$ the two sets of frequencies can be combined as follows:

$$w = \pm l + 2i(n + h), \qquad w = \pm l - 2i(n + h), \qquad n \in \mathbb{Z}. \qquad (5.5)$$

If $l \neq 0, m \neq 0$ then there is (for a generic m) no zeros in the numerator and (5.5) are indeed the poles of the whole 3-point function. For a special case that m is such that h is a positive integer or half-integer, there is a zero in the numerator, and the two sets (5.5) coincide. The 3-point function has to be understood in the generalized function sense. We shall not treat this case.

Thus, the poles of the 3-point function are the correct quasi-normal modes of the BTZ black hole (plus the complex conjugate frequencies), together with a similar set of modes with the conformal dimension h replaced by the conjugate dimension $\tilde{h}$. This has the following explanation. We have two linearly independent solutions of the wave equation. In asymptotically AdS space, such as the BTZ black hole, one of them grows and the other dies off at infinity. Quasi-normal modes are frequencies when there is no growing mode (provided there is only the in-going mode at the horizon). However, there is another set of frequencies for which there is no decaying mode. The 3-point function treats both linearly independent solutions on equal footing. Thus, its poles contain both sets of frequencies .The fact that (5.2) contains also poles with positive real part might be related to the fact we consider the Euclidean correlation function. In Minkowski version we would have to use the retarded correlation function as was suggested in [14]. Then the poles with positive imaginary part as well as some unphysical poles may actually disappear.

5.2   Liouville 3-point function: Schwarzschild black hole. High energy limit

In the case of the Schwarzschild black hole the parameters of the Liouville vertex operators read:

$$\alpha_1 = \frac{1}{2b} - \frac{a_1}{2b}, \ \ \alpha_2 = \frac{1}{2b} - \frac{a_2}{2b}, \ \ \alpha_3 = \frac{1}{2b}. \tag{5.6}$$

We see that one of the operators to be inserted is that corresponding to a puncture. The correct puncture vertex operator is: $\phi(z)e^{Q\phi(z)/2} = (\partial V_\alpha/\partial\alpha)_{\alpha=Q/2}$. Thus, the 3-point function must be differentiated with respect to α_3 before setting it to the value (5.6). This removes $\Upsilon(1/b) = 0$ that would otherwise be present in the the numerator of (D.1). The 3-point function then takes the form

$$C(\alpha_1, \alpha_2, \alpha_3) \sim \tag{5.7}$$

$$\frac{\Upsilon(\frac{1-a_1}{b})\Upsilon(\frac{1-a_2}{b})}{\Upsilon(\frac{1}{2b} - \frac{a_1}{2b} - \frac{a_2}{2b})\Upsilon(\frac{1}{2b} - \frac{a_1}{2b} - \frac{a_2}{2b})\Upsilon(\frac{1}{2b} + \frac{a_1}{2b} - \frac{a_2}{2b})\Upsilon(\frac{1}{2b} - \frac{a_1}{2b} + \frac{a_2}{2b})}$$

The poles of the 3-point function (5.7) fall into the following two sets:

$$-\frac{a_1}{2} - \frac{a_2}{2} = \pm(n + \frac{1}{2}), \qquad \frac{a_1}{2} - \frac{a_2}{2} = \pm(n + \frac{1}{2}), \tag{5.8}$$

where n is non-negative integer. Notice that each pole appears twice in the denominator of (5.7). It is easy to see that both sets satisfy (4.21). The two sets of modes have the following interpretation: half of the frequencies are the correct QNM's and their complex conjugates. Each monodromy a_1 and a_2 is a multi-sheet function of w, and for the other half of the modes the monodromies are taken on a different sheet. Presumably, this other set corresponds to "unphysical" boundary conditions, such as purely outgoing boundary conditions at the horizon and purely in-going conditions at infinity. We shall not attempt to demonstrate this.

We conclude that the Liouville 3-point function (5.7) reproduces the structure of the poles of the reflection amplitude for Schwarzschild black hole we have calculated in section 4.3. Both in the reflection amplitude and in the 3-point function (5.7) the QNM's appear as double poles. However, a precise relation between the reflection amplitude $S(w)$ given in section 4.3 and the Liouville 3-point function just discussed is not clear at the moment. Presumably it should be a generalization of the reflection property (noticed in [13]) of Liouville 3-point function, $C(Q - \alpha_1, \alpha_2, \alpha_3) = C(\alpha_1, \alpha_2, \alpha_3)S_L(i\alpha_1 - iQ/2)$ where $S_L(P)$ is the Liouville reflection amplitude.

Analogously, one can analyze the case of the low energy limit. This case is described by a Riemann surface with two of the singularities coinciding into an essential singularity. However, one can still use a certain 3-point function. The same observation holds: the poles of this 3-point function are the black hole quasi-normal modes. We shall not consider this case in detail.

5.3   The interpretation

Thus we have used the well-known fact that a Riemann surface with prescribed monodromies at marked points can be obtained in the semi-classical Liouville theory. To this end one inserts certain Liouville vertex operators into the sphere. The parameter of the vertex operator is determined by the monodromy one wants to create. We then found that quasi-normal modes are the poles of the resulting Liouville theory correlators.

A description of these vertex operators is as follows. At the horizon and infinity there is a family of vertex operators parameterized by the frequency w that is naturally associated with continuous family of propagating degrees of freedom there. There are no propagating degrees of freedom at the black hole singularity and so it is represented by a single vertex operator–puncture operator playing special role in the Liouville theory. Remarkably, these Liouville operators not just contain local information about the states emitted and detected but also about the way the propagation goes between the emission and absorption. This we can see from the fact that in the semi-classical limit the location of the poles in 3-point function of Liouville operators inserted give exactly the quasi-normal modes. The latter fact has been verified for the BTZ black hole and for the Schwarzschild black hole in the high-energy (also, called high damping) limit. We have also claimed that the same holds for the low energy limit. In all these cases the relevant Riemann surface is a 3-punctured sphere. In this case monodromies around the marked points determine the geometry of the whole Riemann surface.

These are the different pieces which emerge from our analysis and we now suggest that those pieces are actually indications of a bigger, more fundamental, picture which should have a string theoretic interpretation. The interpretation we give is inevitably speculative at the present stage and more evidence should be collected to support it. We however believe that our interpretation is rather natural.

We propose that the Riemann surface which arises in the process of solving the radial wave equation can be given an interpretation of a worldsheet of an effective string. Indeed, the form of the vertex operators inserted sug-

gests that in all cases the W-surface describes propagation. Thus, we have seen that the horizon vertex operators are always functions of the frequency w:

$$V_\omega = e^{\alpha(w)\phi}. \tag{5.9}$$

For the case of BTZ and the low energy limit of the Schwarzschild BH the dependence on the frequency is:

$$\alpha(w) = Q/2 + (i/2b)(w/2\pi T_H). \tag{5.10}$$

This vertex operator can be interpreted as the vertex operator describing the emission of a quantum with frequency w from the horizon. For the BTZ black hole, other vertex operators inserted only depend on the angular momentum of the wave, and not on the frequency. This is because the potential of the corresponding wave equation is growing at infinity and therefore does not admit propagating asymptotic modes. Therefore, one does not insert the vertex operator describing a propagating mode at infinity. Thus, in the case of BTZ black hole a mode of frequency w that appears at the horizon travels through the potential and gets converted to a mode of conformal dimension h at infinity. Exactly this is described by the BTZ W-surface. In the case of the low energy limit of Schwarzschild one again has a mode of frequency w emitted from the horizon, and the W-surface describes its propagation through the potential. In the high energy limit the approximation used is that of zooming on the near-singularity region. In doing this one cuts away both the near-horizon and near-infinity regions. The resulting wave equation has propagating solutions in what replaced both the horizon and infinity. Thus, the vertex operators inserted at the horizon and infinity do depend on frequency. However, neither one of them is of the simple form (5.9).

Thus, the form of the Liouville vertex operators inserted, together with the observation that the Liouville correlator encodes the quasi-normal modes suggests that the W-surfaces we described are not just a convenient mathematical tool for solving the differential equation, but are of a more fundamental significance. Similarly, the same facts suggest that the Liouville theory appears not just because in the semi-classical approximation it gives rise to the same Riemann surfaces, but has a more deep meaning as describing an effective string. This is, in short, the interpretation we are about to propose.

To phrase our proposal a bit more precisely, we shall interpret the Z-surface as the worldsheet of an effective string. By "worldsheet" we mean here a complex plane where the conformal field theory describing the string

lives. We shall interpret the W-plane, together with its moduli that depend on the potential and the wave frequency, as corresponding to a particular profile of the Liouville field, or, in other words, as a particular classical solution of the effective string equations of motion. We shall expand on this interpretations below.

Using the term "string" we simply mean that there is a 2D conformal field theory on this worldsheet. Thus, the term is used in a rather loose sense; at present we do not claim that one has to integrate over the moduli of the arising surfaces. The term "effective" is used to emphasize the fact that it is not a string that moves in the full black hole background. Such a string would be described by a sigma model with the number of fields equal to the dimension of the spacetime. The theory that we used, namely Liouville CFT, has a single field in it. This is why we use the term "effective" description: it is the description of the motion of the worldsheet in the radial coordinate only. It is nevertheless a "stringy" description because in order to calculate the propagation amplitudes we have to follow the standard string theory recipe: insert vertex operators describing the in- and out-going modes and compute the correlation functions. We have shown that the 3-point correlation function of the effective string does describe the propagation in certain limits, in that it encodes the quasi-normal modes.

It might be considered surprising that the Liouville theory is relevant not always but only in certain limits. Indeed, far away from the solvable limits of low and high energy limits the wave equation is complicated, and no description in terms of Liouville or any other CFT does seem possible. Our interpretation for this is as follows. As we have said, we believe that the CFT arising, namely the Liouville theory, has the interpretation of the radial part of the sigma model describing the motion of the string worldsheet in the black hole background. The background is very complicated, and the sigma model is not conformal. Even truncating this sigma model to its radial part (to get an "effective" description) the theory one gets is too complicated to be conformal. However, in certain limits, like the ones described in this paper, the effective theory does become conformal, and the Liouville theory becomes relevant. It would be desirable to find more evidence for this interpretation, but we believe that the observations we presented in this paper strongly hint in this direction.

5.4 Comments

Few more comments on the relation between the fundamental and the "effective" string are in order. Consider the (truncated to the excitations along

the radial direction only) fundamental string on the background of a black hole. In the semi-classical limit $\alpha' \to 0$ the quantum dynamics of the string reduces to that of the zero mode (representing the location of the center of mass of the string in the radial direction). The quantum state of the string in this limit is given by a function of the zero mode that satisfies a differential equation which is identical to the radial equation for a scalar field in the black hole background. The solution to this equation, as we described in this paper, is given by a certain Riemann surface. Thus, semi-classically the analytic continuation of the wave function of the string is a function on the Riemann surface of complex r. Thus, in this sense, at least semi-classically, the fundamental string does "see" the complex values of r. Note that, in spite of the fact that the relevant Riemann surface is topologically always a sphere with 3 marked points, the solution of the zero mode wave equation is not characterized just by the monodromies around singularities.

On the other hand, we have the semi-classical Liouville theory. The profile of the Liouville field on a sphere with 3 marked points is completely determined by the parameters of the operators inserted, or by the monodromies. This configuration of the Liouville field is related (in the standard way used in the uniformization theory) to a solution of another (Fuchsian) wave equation. By construction, we have chosen the Liouville vertex operators in such a way that the monodromies of the two wave equations –the one for the string zero mode and the Fuchsian one for the Liouville theory– are the same. However, the Fuchsian differential equation of Liouville theory is not identical to the equation describing the radial motion of the zero mode of the string.

Now, the limits which we analyzed in detail in the present paper, namely, the low-energy (near-horizon) limit and the high-energy limit are remarkable in that in these cases the relevant Riemann surface is completely determined by the monodromies and thus is the same both for the effective string and the (semi-classical limit of) fundamental string. This is the origin of our proposal that in these special limits the Liouville theory is directly related to the (truncated to the excitations along the radial direction only) fundamental string. Of course, a further study on the relation in general case is needed. We plan to return to this problem in subsequent publications. In the case of BTZ black hole the presence of the large symmetry guarantees that the relevant Riemann surface is always determined by the monodromies about the special points so that in this case one expects that the Liouville theory is related to the fundamental string even beyond these limits. Indeed, the analysis in section 5.1 shows no restriction on the range of the frequency ω.

From the analysis that we have presented it is impossible to determine the

Liouville coupling constant. In the semi-classical approximation b completely drops out of all the formulas. However, the Liouville theory that appears in the low energy limit can be further characterized, see [2]. Indeed, the near horizon wave equation with the potential (4.2) is the Liouville equation. This allows to identify the Liouville coupling constant with the horizon radius measured in the units of string length:

$$1/b = 2r_+/\sqrt{\alpha'}. \tag{5.11}$$

One introduces the string length $\sqrt{\alpha'}$ on dimensional grounds; it is an arbitrary parameter that characterizes when the semi-classical regime $b << 1$ becomes valid. The exact poles (at finite b) of 3-point function then can be considered as an α'-modification of the semi-classical QNM. The effective theory in the high energy limit can probably be thought of as the Liouville theory which is perturbed. Indeed, removing the strictly near-horizon restriction one introduces the term e^{2z} to the potential. That perturbs the Liouville theory. The theory may start to flow, and become a new theory in the ultra-violet. This would be the theory describing the string in the high energy limit. Our analysis shows that this theory is also related to the Liouville theory: the string propagation is described by a particular 3-point function. However, the Liouville coupling constant may flow, so the identification (5.11) may become invalid.

Yet another interpretational point seems worth mentioning. From the fact that one of the marked points on our Riemann surfaces represent the singularity it might seem that these surfaces describe the effective string moving inside the black hole horizon. Moreover, since points on these Riemann surfaces correspond to complex values of the radial coordinate it might seem that the effective string moves in the complex radial coordinate. We would like to emphasize that such an interpretation would be misleading. What we argue for in this paper that the string moves in the real radial coordinate, and this corresponds to having one real Liouville field on the worldsheet. Moreover, as the example of the low energy limit shows, see [2] for more detail, the Liouville field is identified directly with the tortoise coordinate. Thus, all possible values of it correspond to string being *outside* of the black hole horizon. The Riemann surface, marked points on it and the vertex operators inserted only code the analytic properties of the potential outside of the hole, and this is where the string described is moving.

Thus, we have argued that the W-surface describes string propagation. Note, however, that the 3-point function is not equal to the transmission or scattering amplitude, even though all these objects have partially overlapping sets of poles. This is not surprising; one needs to come up with further rules on how the string amplitudes are to be extracted from the Liouville

correlators. In the case of the BTZ black hole there is a dual theory, and the full Hawking rate can be obtained by performing the Fourier transform on the thermal 2-point function of the dual theory. The QNM then can be identified with the poles in 2-point function in the dual conformal theory, as was shown in [14]. No such rules have been proposed in our case. Moreover, apart from the asymptotically AdS case, one does not expect a dual CFT to exist. To get the physically interesting quantities one would have to do the worldsheet CFT calculation. It would be of considerable interest to supplement the interpretation proposed in this paper with concrete rules on how to extract physically interesting amplitudes from Liouville CFT. We leave development of such rules for future work.

6   Discussion

In this paper we have introduced a notion of Z and W-surfaces for the black hole wave equations. The Z-surface arises as the image of the complex r-plane under the tortoise map. We have shown how this is in general a sphere with a number of holes corresponding to horizons, plus two marked points corresponding to the singularity and infinity. The black hole wave equation is the Fuchsian differential equation on the Z-surface. The ratio of two linearly independent solutions of this equation gives a map $w(z)$ from the Z-surface to the W-surface, which is of the same topological type. The W-surface "solves" the wave equation, in that one can write down an explicit solution if the map $w(z)$ (and its inverse) are known. The Z-surface does not know anything about the potential. This information is encoded in the W-surface.

For 4d Schwarzschild BH the Z-surface is a sphere with 3 points. Thus, the hypergeometric equation can be used as an approximation to the wave equation is some limits. We have analyzed two such limits: that of low and high energy. It is easy to write down the quasi-normal modes of the resulting exactly solvable potential. In the limit of high energy this results in QNM's with $\log 3$ as the real part of the frequency. We have explained that taking a limit in which the wave equation becomes solvable amounts to zooming onto a part of the W-surface. Thus, the low energy limit is the near horizon limit, and the high energy limit is the near singularity limit. In such limits the W-surface is completely characterized by its monodromies, and is easy to determine.

For higher-dimensional BH's the Z-surface has more than 3 marked points. Thus, the hypergeometric equation is no longer applicable. In the

high energy limit $|w| \to \infty$ one zooms onto the region near the singularity. This effectively cuts away all the holes except the one corresponding to the horizon. The problem reduces to the already analyzed problem of 4d Schwarzschild. Thus, this limit is insensitive to the increased complexity of the surface. It would be of interest to find some other limits that lead to exactly solvable systems for which the presence of other holes is important. Thus, as we have seen, the surface corresponding to the 5d Schwarzschild has 4 points. It is natural to expect that there are limits in which the effective theory describing the string is well approximated by the Liouville theory on this 4-punctured sphere. The insertions depend on the monodromies of the corresponding Riemann surface.

The main novelty of the present paper is our proposal that Z, W Riemann surfaces signal the appearance of an effective string. The motion of the string in the BH background (in the radial direction) is described by a theory on the corresponding Riemann surface. This theory is complicated, there is no reason to expect that it is conformal. However, in certain limits (such as the low and high-energy ones analyzed) the theory simplifies, and can be described using the Liouville CFT. Thus, our proposal is that the propagation of the string through the potential is described by the Liouville 3-point function, where the operators to be inserted have to chosen as explained in the main text. We have shown how the poles of such 3-point function contain the BH quasi-normal modes.

The description we gave is very general. It can be applied to any spherically symmetric BH. It would be of interest to extend this description to more complicated BH's, such as rotating ones. Experience with the rotating BTZ BH suggests that inclusion of rotation will simply make all the monodromies complex. The imaginary part will encode the angular momentum. It would be very interesting to see whether this is indeed the case. It would also be important to clarify the meaning of the moduli that start appearing for dimensions $n \geq 5$. In this case the surfaces have more than 3 marked points. The number of arising moduli is $2n - 8$. It would be interesting to give an interpretation of this moduli in terms of the BH wave equation.

The stringy description we proposed is an effective one in two different ways. First of all, it only describes the motion of the worldsheet in the radial direction. Second, it does not answer the question what are the fundamental degrees of freedom of the BH. We are only saying that whatever the fundamental theory describing the BH is, there must be the Liouville mode in it, so that the semi-classical properties such as the Hawking emission are reproduced as sketched here. Thus, analysis of the BH wave equation is not sufficient to determine how to describe the BH quantum mechanically.

To find such a description an additional input from some quantum gravity theory is necessary. However, the effective structure we have found puts constraints on the fundamental theory. In particular, it is clear that, whatever the fundamental theory is, it should have interpretation in CFT terms. We believe that the structure unraveled in this paper will be important in future searches for a fundamental theory describing black holes quantum mechanically.

Acknowledgments

A   Some facts on the hypergeometric equation

The Fuchsian form of the hypergeometric equation is given by:

$$T(z) = \frac{1 - a_1^2}{z^2} + \frac{1 - a_2^2}{(z-1)^2} + \frac{a_1^2 + a_2^2 - a_3^2 - 1}{z(z-1)}. \tag{A.1}$$

It can be reduced to the usual hypergeometric form:

$$z(1-z)\frac{d^2 u}{dz^2} + [\gamma - (\alpha + \beta + 1)z]\frac{du}{dz} - \alpha\beta u = 0 \tag{A.2}$$

where the parameters are related to those of the equation of the Fuchsian form by:

$$a_1^2 = (1 - \gamma)^2, \qquad a_2^2 = (\gamma - \alpha - \beta)^2, \qquad a_3^2 = (\alpha - \beta)^2. \tag{A.3}$$

As two linearly independent solutions around zero one usually takes:

$$F(\alpha, \beta; \gamma; z), \qquad z^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z). \tag{A.4}$$

It is easy to show that the monodromies around singular points, which are $0, 1, \infty$, are given by:

$$\frac{1}{2}\mathrm{Tr}\mathbf{M}_i = \cos \pi a_i. \tag{A.5}$$

The ratio of two linearly independent solutions maps the complex plane minus $0, 1, \infty$ to a sphere with 3 holes or conical singularities, depending on what a_i are. Thus, if the trace of the monodromy matrix is real and less than two one gets a conical singularity. If it is real and larger than two one gets a hole. For complex trace the "object" on the W-surface is neither a hole or a conical singularity. In a sense, it is a mixture of the two.

Near $z = 0$ the characteristic expansion is:

$$F(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha\beta}{\gamma}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2}z^2 + .. \tag{A.6}$$

We also need the formula for the analytic continuation of the hypergeometric function from $z = 0$ to $z = 1$. One has:

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha, \beta; \alpha+\beta-\gamma+1; 1-z) + \tag{A.7}$$
$$(1-z)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}F(\gamma-\alpha, \gamma-\beta; \gamma-\alpha-\beta+1; 1-z).$$

B Some facts on the Bessel functions

The differential equation:

$$u'' + \left(\beta^2 - \frac{4\nu^2 - 1}{4z^2}\right)u = 0 \tag{B.1}$$

is solved in terms of Bessel functions via:

$$u = \sqrt{z}J_\nu(\beta z). \tag{B.2}$$

The function $J_\nu(z)$ have a branch cut along the negative real axis. The values on opposite sides of the cut are related by:

$$J_\nu(e^{i\pi k}z) = e^{i\pi k\nu}J_\nu(z). \tag{B.3}$$

Similarly, for the Bessel function of the second kind one has:

$$N_\nu(e^{i\pi k}z) = e^{-i\pi k\nu}N_\nu(z) + 2i\frac{\sin k\nu\pi}{\tan \nu\pi}J_\nu(z). \tag{B.4}$$

The asymptotic of the Bessel functions is given by:

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(1/|z|), \qquad (B.5)$$

$$N_\nu(z) = \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(1/|z|).$$

C   Some facts on the confluent hypergeometric function

The confluent hypergeometric equation is obtained from the usual hypergeometric equation when the singular point at $z = 1$ is taken to infinity. The equation one gets as the result of such limit is, when written in the Fuchsian form:

$$\frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{\lambda}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right) W = 0. \qquad (C.1)$$

As two linearly independent solutions of this equation one can take Whittaker's functions: $W_{\lambda,\mu}(z), W_{-\lambda,\mu}(-z)$.

For Whittaker's functions $z = 0$ is a branch point and $z = \infty$ is an essential singular point.

For large values of $|z|$ we have the following asymptotic behavior:

$$W_{\lambda,\mu}(z) \sim e^{-z/2} z^\lambda. \qquad (C.2)$$

D   Liouville 3-point function

The expression for the Liouville 3-point function, as proposed in [13], is given by:

$$C(\alpha_1, \alpha_2, \alpha_3) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{(Q-\sum \alpha_i)/b} \times \qquad (D.1)$$

$$\frac{\Upsilon_0\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1 + \alpha_2 + \alpha_3 - Q)\Upsilon(\alpha_1 + \alpha_2 - \alpha_3)\Upsilon(\alpha_2 + \alpha_3 - \alpha_1)\Upsilon(\alpha_3 + \alpha_1 - \alpha_2)}.$$

Here μ, b are the Liouville "cosmological" and the coupling constants correspondingly, $Q = b + 1/b$, $\gamma(x) = \Gamma(x)/\Gamma(1-x)$, and $\Upsilon(x)$ is an entire function of x with zeros located at:

$$x = -m/b - nb, \qquad x = Q + m/b + nb, \qquad m, n \in \mathbb{Z}^{\geq 0}. \qquad (D.2)$$

An explicit expression for it can be found in [13]. The quantity Υ_0 is defined as:

$$\Upsilon_0 = \left(\frac{d\Upsilon(x)}{dx} \right)_{x=0}.$$ (D.3)

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