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State distribution and reliability of some multi-state systems with complex configurations

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Abstract. In this paper[1], our goal is to investigate, first, the multi-state consecutive k_n-out-of-m_n : G series systems and second, the multi-state consecutive k_n-out-of-m_n : G parallel systems (see definitions 1 and 2). We begin by giving a non recursive formula which calculates the state distribution and the reliability of multi-state consecutive k-out-of-n : G system in the case where the number k of functioning components depends on the system state level (see agreement 1), then we extend the used method to the multi-state consecutive k_n-out-of-m_n : G series and multi-state consecutive k_n-out-of-m_n : G parallel systems. In the end, we illustrate the obtained theoretical results by a numerical application.

Résumé. Dans ce papier, notre but est d'étudier, premièrement, les systèmes k_n-consécutifs-sur-m_n : G série à multi-états, et deuxièment, les systèmes k_n-consécutifs-sur-m_n : G parallèle à multi-états (voir définitions 1 et 2). Nous commençons par donner une formule non récursive permettant le calcul de la distribution d'état des systèmes k-consécutifs-sur-n : G à multi-états et donc de déduire leur fiabilité dans le cas où le nombre k des composants qui fonctionnent dépend du niveau d'état du système (voir agreement 1). Ensuite, nous faisons une extension de la méthode utilisée aux systèmes k_n-consécutifs-sur-m_n : G série à multi-états et k_n-consécutifs-sur-m_n : G parallèle à multi-états. Enfin, nous illustrons les résultats théoriques obtenus par une application numérique.

Key words: Reliability; Multi-state system; Consecutive k-out-of-n : G system; Consecutive k_n-out-of-m_n : G series system; Consecutive k_n-out-of-m_n: G parallel system.
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1. Introduction

A consecutive k-out-of-$n : G$ system is a system with n linearly ordered components, this system works if and only if at least k ($k \leq n$) consecutive components work Bon (1980), Kontoléon (1980), Kuo *et al.* (1990). Many engineers and researchers were interested by such system and its related systems because of their wide range of applications (telecommunications, transport network, oil and gas pipelines, distribution water systems,...). Most researches in the reliability theory deal with binary systems of binary components, where there exist only two states: functioning or failed. El-Neweihi *et al.* (1978) is an example of some recent work that has been done on the extensions to multi-state components.

Until 2003, all researchers considered that the number k of functioning components in a binary or a multi-state consecutive k-out-of-$n : G$ system is independent of the state of the system. However, Huang *et al.* (2003) proposed more general definitions of the multi-state consecutive k-out-of-$n : F$ and G systems where the number k of functioning components can take different values k_j for the different states j ($j = 1, 2, ..., M$) of the systems. According to them, a multi-state consecutive k-out-of-$n : G$ system is at the state j or above ($j = 1, 2, ..., M$) if and only if at least k_l consecutive components are at the state l or above for all l ($1 \leq l \leq j$), and a multi-state consecutive k-out-of-$n : F$ system is below the state j ($j = 1, 2, ..., M$) if and only if at least k_l consecutive components are below the state l for all l ($1 \leq l \leq j$). They provided an algorithm for evaluating system state distribution of decreasing multi-state consecutive k-out-of-$n : F$ system (i.e. $k_1 \geq k_2 \geq ... \geq k_M$), and another algorithm to bound system state distribution of increasing multi-state consecutive k-out-of-$n : F$ system (i.e. $k_1 \leq k_2 \leq ... \leq k_M$). Zuo *et al.* (2003) proposed a recursive formula for evaluating the system state distribution of a consecutive k-out-of-$n : G$ system when the system has only three states ($M = 3$), and for the other case ($M \geq 4$), they provided an algorithm to bound the system state distribution. Belaloui and Ksir (2007) established a non recursive formula for computing the exact reliability of multi-state consecutive k-out-of-$n : G$ systems without constraints on M or on k_j for all j ($1 \leq j \leq M$). Usually most of the studies assume independence among components, because computation of reliability characteristics of a system that consists of dependent components is difficult especially when a specific type of dependence is not known Destercke *et al.* (2014), Eryilmaz (2009).

In the present work, since each one of the both (consecutive k_n-out-of-$m_n : G$ series and consecutive k_n-out-of-$m_n : G$ parallel) systems is compound of blocks, and in order to reach the previous declared goal, we proceed as follows: we evaluate the reliability of each block which has the multi-state consecutive k_n-out-of-$m_n : G$ configuration ($n = 1, 2, ..., N$), then we extend the method to get the reliability of the global system.

Before that, let us begin with some notations and assumptions that we need afterwards.

1.1. Notations

N : number of blocks in the system.
m_n : number of components in block n, $n \in \{1, 2, ..., N\}$.
$M + 1$: number of states of components, blocks and the global system.
M : perfect functioning, 0 : complete failure.
$S = \{0, 1, ..., M\}$.

X_{ni} : state of component i in block n, $X_{ni} \in S$, $i \in \{1, 2, ..., m_n\}$.
$\boldsymbol{X}_n := (X_{n1}, X_{n2}, ..., X_{nm_n})$ vector of states of components in block n.
k_{nj} : minimum number of consecutive components in block n with $X_{ni} \geq j$, $i \in \{1, 2, ..., m_n\}$.
$P_{i,j} = P\{X_{ni} \geq j\}$.
$Q_{i,j} = 1 - P_{i,j} = P\{X_{ni} < j\}$.
$p_{i,j} = P\{X_{ni} = j\}$.
ϕ_n (res. ϕ_s) : structure function of block n (resp. the system), ϕ_n, $\phi_s \in S$.
$R_{n,j} = P(\phi_n(\boldsymbol{X}_n) = j)$, $j = 1, 2, ..., M$, $n = 1, 2, ..., N$.
$P_s(j) = P(\phi_s = j)$.

$$R_j(b, a) = P\left\{\begin{array}{c}\text{exactly } b \text{ components in a block are at state } j\text{, which}\\ \text{include among them at least } k_j \text{ consecutive components}\\ \text{and the other } a - b \text{ components are below } j\end{array}\right\}.$$

1.2. ***Assumptions***

1. The system is multi-state monotone decreasing (blocks and global system).
2. The X_{ni} are mutually independent.
3. Each component, each block and the system can assume $M + 1$ states.
4. The possible states of each component, each block and the global system are ordered:

$$state\ 0 \leq state\ 1 \leq state\ 2 \leq ... \leq state\ M$$

5. The components states distributions in each block are known.

Definition 1. A multi-state consecutive k_n-out-of-$m_n : G$ series system is a system with N blocks, linearly disposed. Each block has the multi-state consecutive k_n-out-of-$m_n : G$ system configuration ($n = 1, 2, ..., N$). The system works if all the blocks work, and each block works if and only if at least k_n consecutive components work ($k_n \leq m_n$ for $n = 1, 2, ..., N$).

Definition 2. A multi-state consecutive k_n-out-of-$m_n : G$ parallel system is a system with N blocks, parallely disposed. Each block has the multi-state consecutive k_n-out-of-$m_n : G$ system configuration ($n = 1, 2, ..., N$). The system works if at least one block works, and each block works if and only if at least k_n consecutive components work ($k_n \leq m_n$ for $n = 1, 2, ..., N$).

According to the definitions and assumptions above :

A block n is a multi-state consecutive k_n-out-of-$m_n : G$ system which has m_n linearly ordered components ($n = 1, 2, ..., N$), this block functions if and only if there are at least k_n consecutive components which are working ($k_n \leq m_n$ for $n = 1, 2, ..., N$).

Agreement 1 : In our further considerations, we suppose that the number k_n is not constant, but it depends on the system state level j, in other words : where in mainting at least a certain system state level might require a different number of consecutive components to be at a certain state or above.

2. Evaluation of multi-state consecutive k_n-out-of-m_n:G system's reliability

By extending the binary theory results, we present an approach which evaluates the reliability of multi-state consecutive k_n-out-of-m_n : G system because each block has a such configuration.

By considering:

$$\begin{cases} \phi_n(\boldsymbol{X}_n) \geq j \Longrightarrow \text{ block } n \text{ works} \\ \phi_n(\boldsymbol{X}_n) < j \Longrightarrow \text{ block } n \text{ fails,} \end{cases}$$

and similarly :

$$\begin{cases} X_{ni} \geq j \Longrightarrow \text{ component } i \text{ of block } n \text{ works} \\ X_{ni} < j \Longrightarrow \text{ component } i \text{ of block } n \text{ fails.} \end{cases}$$

Our aim is to calculate the state distribution of block n, i.e. :

$$R_{n,j} = P(\phi_n(\boldsymbol{X}_n) = j), \qquad j = 0, 1, 2, ..., M \quad \text{and} \quad n = 1, 2, ..., N$$

for this, we use a non recursive formula based on a combinatorial approach Belaloui and Ksir (2007), and then we deduce reliability of block n as following :

$$\begin{aligned} P(\phi_n(\boldsymbol{X}_n) \geq j) &= P(\phi_n(\boldsymbol{X}_n) = j) + P(\phi_n(\boldsymbol{X}_n) = j+1) + ... + P(\phi_n(\boldsymbol{X}_n) = M) \\ &= \sum_{\alpha=j}^{M} R_{n,\alpha} \quad , \quad j = 0, 1, 2, ..., M. \end{aligned}$$

Definition 3. Huang *et al.* (2003) $\phi_n(\boldsymbol{X}_n) = j$ if and only if at least k_{nj} consecutive components are in state j or above and at most $k_{nh} - 1$ components are in state h for $h = j+1, j+2, ..., M, \quad j = 1, 2, ..., M$.

The formula which computes $R_{n,j}$ i.e. the probability that the block n is at state j is given in Belaloui and Ksir (2007) by:

$$R_{n,j} = P(\phi_n(\boldsymbol{X}_n) = j) = \sum_{k_n = k_{nj}}^{m_n} \left[R_j(k_n, m_n) + \sum_{h=j+1, k_{nh}>1}^{M} H_{k_n}^{j}(h) \right] \tag{1}$$

where $R_j(k_n, m_n)$ and $H_{k_n}^{j}(h)$ are probabilities defined as following :

1. $R_j(k_n, m_n)$ is the probability that exactly k_n components are at state j, among them there are at least k_{nj} consecutive components, and the other $(m_n - k_n)$ components are $< j$.
2. $H_{k_n}^{j}(h)$ is the probability that :
   - At least 1 and at most $k_{nh} - 1$ components are at state h $(h > j)$.
   - At most $k_{nu} - 1$ components are at state u for $j < u < h$.
   - The total number of components $\geq j$ is k_n, which include among them at least k_{nj} consecutive components.
   - $m_n - k_n$ components are at states $< j$.

– The block n is at state j. $H_{k_n}^{j}(h)$ can be expressed by :

$$H_{k_n}^{j}(h) = \sum_{i_1=1}^{k_{nh}-1} \sum_{i_2=0}^{k_{n(h-1)}-1-I_1} \sum_{i_3=0}^{k_{n(h-2)}-1-I_2} \cdots \sum_{i_{n(h-j)}=0}^{k_{n(j+1)}-1-I_{n(h-j-1)}} R\left[\left(h^{i_1}, (h-1)^{i_2}, ..., j^{k_n - i_{n(h-j)}}\right), m_n\right],$$

where

$$I_a = \sum_{m=1}^{a} i_m \qquad for\ a = 1, 2, ..., h-j$$

and

$$R\left[\left(h^{i_1}, (h-1)^{i_2}, ..., j^{k_n - i_{n(h-j)}}\right), m_n\right]$$

is the probability that there is :

– i_1 components at state h.
– i_2 components at state $h-1$.
– $i_{n(h-j)}$ components at state $j+1$.
– $k_n - i_{n(h-j)}$ components at state j.
– The remaining $(m_n - k_n)$ components are $< j$.

3. Reliability of multi-state consecutive k_n-out-of-m_n: G series system

Since we have N blocks joined in series, so the system's state depends on the worst block, and according to (1), the system's reliability is given by :

$$\begin{aligned} P(\phi_s \geq j) &= P\left(\min_{1 \leq n \leq N} \phi_n(\boldsymbol{X}_n) \geq j\right) \\ &= \prod_{n=1}^{N} P(\phi_n(\boldsymbol{X}_n) \geq j) \\ &= \prod_{n=1}^{N} \left(\sum_{\alpha=j}^{M} R_{n,\alpha}\right). \end{aligned} \tag{2}$$

Knowing the state distribution of any block at the different possible states, we can deduce the state distribution of the global system which is given by:

$$\begin{aligned} P_s(j) &= P(\phi_s \geq j) - P(\phi_s \geq j+1) \\ &= \prod_{n=1}^{N} \left(\sum_{\alpha=j}^{M} R_{n,\alpha}\right) - \prod_{n=1}^{N} \left(\sum_{\alpha=j+1}^{M} R_{n,\alpha}\right). \end{aligned} \tag{3}$$

4. Reliability of multi-state consecutive k_n-out-of-m_n : G parallel system

Counter to the series case, and because we have N blocks joined in parallel, the system's state depends on the best block. Using (1), the system's reliability is given by :

$$\begin{aligned} P(\phi_s \geq j) &= P\left(\max_{1\leq n\leq N} \phi_n(\boldsymbol{X}_n) \geq j\right) \\ &= 1 - P\left(\max_{1\leq n\leq N} \phi_n(\boldsymbol{X}_n) < j\right) \\ &= 1 - \prod_{n=1}^{N}\left(1 - \sum_{\alpha=j}^{M} R_{n,\alpha}\right). \end{aligned} \tag{4}$$

In the same way of the previous case, we obtain the state distribution of the global system as follows:

$$\begin{aligned} P_s(j) &= P(\phi_s \geq j) - P(\phi_s \geq j+1) \\ &= \prod_{n=1}^{N}\left(1 - \sum_{\alpha=j+1}^{M} R_{n,\alpha}\right) - \prod_{n=1}^{N}\left(1 - \sum_{\alpha=j}^{M} R_{n,\alpha}\right) \end{aligned} \tag{5}$$

5. Numerical application

The obtained results can be illustrated over the two following examples.

Example 1 :

Let us consider a system that has a consecutive k_n-out-of-m_n : G series configuration compound of 3 blocks ($n = 1, 2, 3$) with $m_1 = 3$, $m_2 = 4$, $m_3 = 4$ (the second and the third blocks are identical).

For the first block, we consider the following given data :

$$\begin{aligned} m_1 &= 3 \;\;,\;\; M = 2 \;\;,\;\; k_{11} = 2 \;\;,\;\; k_{12} = 3 \\ p_{1,0} &= 0,2 \;\;,\;\; p_{1,1} = 0,1 \;\;,\;\; p_{1,2} = 0,7 \\ p_{2,0} &= 0,1 \;\;,\;\; p_{2,1} = 0,3 \;\;,\;\; p_{2,2} = 0,6 \\ p_{3,0} &= 0,2 \;\;,\;\; p_{3,1} = 0,3 \;\;,\;\; p_{3,2} = 0,5 \end{aligned}$$

One can notice that this block has the structure :

- 3 consecutive components out-of-3 : G (series structure) at level 2.
- 2 consecutive components out-of-3 : G at level 1.

For the second block (even the third with the necessary changes), we consider the following given data :

$$\begin{aligned}
m_2 &= 3 \ , \quad M=3 \ , \quad k_{21}=2 \ , \quad k_{22}=3 \ , \quad k_{23}=4 \\
p_{1,0} &= 0,2 \ , \quad p_{1,1}=0,1 \ , \quad p_{1,2}=0,4 \ , \quad p_{1,3}=0,3 \\
p_{2,0} &= 0,1 \ , \quad p_{2,1}=0,1 \ , \quad p_{2,2}=0,3 \ , \quad p_{2,3}=0,5 \\
p_{3,0} &= 0,2 \ , \quad p_{3,1}=0,1 \ , \quad p_{3,2}=0,2 \ , \quad p_{3,3}=0,5 \\
p_{4,0} &= 0,1 \ , \quad p_{4,1}=0,1 \ , \quad p_{4,2}=0,4 \ , \quad p_{4,3}=0,4
\end{aligned}$$

We can say that the second (even the third) block has the structure :

- 4 consecutive components out-of-4 : G (series structure) at level 3.
- 3 consecutive components out-of-4 : G at level 2.
- 2 consecutive components out-of-4 : G at level 1.

Using formula (1), we can compute the state distribution of the first block as the following :

level 2 : $j=2 \ , \quad k_{12}=3$

$$R_{1,2}=\sum_{k_1=k_{12}}^{3}\left[R_2\left(k_1,m_1\right)+\sum_{h=3,k_{1h}>1}^{2}H_{k_1}^{2}\left(h\right)\right]=\sum_{k_1=3}^{3}R_2\left(k_1,m_1\right)=R_2\left(3,3\right)$$

with

$$R_2\left(3,3\right)=p_{1,2}p_{2,2}p_{3,2}=0,21$$

level 1 : $j=1 \ , \quad k_{11}=2 \ , \quad k_{1h}=k_{12}=3>1$

$$\begin{aligned}
R_{1,1} &= \sum_{k_1=k_{11}}^{3}\left[R_1\left(k_1,m_1\right)+\sum_{h=2,k_{1h}>1}^{2}H_{k_1}^{1}\left(h\right)\right] \\
&= R_1\left(2,3\right)+H_2^1\left(2\right)+R_1\left(3,3\right)+H_3^1\left(2\right)
\end{aligned}$$

with

$$\begin{aligned}
R_1\left(2,3\right) &= p_{1,0}p_{2,1}p_{3,1}+p_{1,1}p_{2,1}p_{3,0} \\
R_1\left(3,3\right) &= p_{1,1}p_{2,1}p_{3,1}
\end{aligned}$$

$$H_2^1\left(2\right)=\sum_{i_1=1}^{2}R\left[\left(2^{i_1},1^{2-i_1}\right),3\right]=R\left[\left(2^1,1^1\right),3\right]+R\left[\left(2^2,1^0\right),3\right]$$

$$H_3^1\left(2\right)=\sum_{i_1=1}^{2}R\left[\left(2^{i_1},1^{3-i_1}\right),3\right]=R\left[\left(2^1,1^2\right),3\right]+R\left[\left(2^2,1^1\right),3\right]$$

as a result

$$R_{1,1}=0,715$$

level 0 : using the properties, we have :

$$R_{1,0} = 1 - (R_{1,1} + R_{1,2}) = 0,075$$

Using again formula (1), we can compute the state distribution of the second (even the third) block as follows :

level 3 : $j = 3$, $k_{23} = 4$

$$R_{2,3} = R_3(4,4) = p_{1,3}p_{2,3}p_{3,3}p_{4,3} = 0,03$$

level 2 : $j = 2$, $k_{22} = 3$, $k_{2h} = k_{23} = 4 > 1$

$$\begin{aligned} R_{2,2} &= \sum_{k_2=k_{22}}^{4} \left[R_2(k_2,m_2) + \sum_{h=3,k_{2h}>1}^{3} H_{k_2}^2(h) \right] = \sum_{k_2=3}^{4} \left[R_2(k_2,m_2) + H_{k_2}^2(3) \right] \\ &= R_2(3,4) + H_3^2(3) + R_2(4,4) + H_4^2(3) \end{aligned}$$

with

$$\begin{aligned} R_2(3,4) &= p_{1,2}p_{2,2}p_{3,2}Q_{4,2} + Q_{1,2}p_{2,2}p_{3,2}p_{4,2} \\ R_2(4,4) &= p_{1,2}p_{2,2}p_{3,2}p_{4,2} \end{aligned}$$

$$H_3^2(3) = \sum_{i_1=1}^{3} R\left[\left(3^{i_1}, 2^{3-i_1}\right), 4\right] = R\left[\left(3^1, 2^2\right), 4\right] + R\left[\left(3^2, 2^1\right), 4\right] + R\left[\left(3^3, 2^0\right), 4\right]$$

$$H_4^2(3) = \sum_{i_1=1}^{3} R\left[\left(3^{i_1}, 2^{4-i_1}\right), 4\right] = R\left[\left(3^1, 2^3\right), 4\right] + R\left[\left(3^2, 2^2\right), 4\right] + R\left[\left(3^3, 2^1\right), 4\right]$$

then

$$R_{2,2} = 0,4964.$$

level 1 : $j = 1$, $k_{21} = 2$, $k_{2h} = k_{22} = 3$, $k_{23} = 4 > 1$

$$\begin{aligned} R_{2,1} &= \sum_{k_2=k_{21}}^{4} \left[R_1(k_2,m_2) + \sum_{h=2,k_{2h}>1}^{3} H_{k_2}^1(h) \right] \\ &= R_1(2,4) + H_2^1(2) + H_2^1(3) + R_1(3,4) + H_3^1(2) + H_3^1(3) + R_1(4,4) \\ &\quad + H_4^1(2) + H_4^1(3) \end{aligned}$$

with

$$\begin{aligned} R_1(2,4) &= p_{1,1}p_{2,1}p_{3,0}p_{4,0} + p_{1,0}(p_{2,0}p_{3,1}p_{4,1} + p_{2,1}p_{3,1}p_{4,0}) \\ R_1(3,4) &= p_{1,1}p_{2,1}(p_{3,1}p_{4,0} + p_{3,0}p_{4,1}) + p_{3,1}p_{4,1}(p_{1,1}p_{2,0} + p_{1,0}p_{2,1}) \\ R_1(4,4) &= p_{1,1}p_{2,1}p_{3,1}p_{4,1} \end{aligned}$$

$$H_2^1(2) = \sum_{i_1=1}^{2} R\left[\left(2^{i_1}, 1^{2-i_1}\right), 4\right] = R\left[\left(2^1, 1^1\right), 4\right] + R\left[\left(2^2, 1^0\right), 4\right],$$

$$H_2^1(3) = \sum_{i_1=1}^{3} \sum_{i_2=0}^{2-i_1} R\left[\left(3^{i_1}, 2^{i_2}, 1^{2-i_1}\right), 4\right] = R\left[\left(2^1, 1^1\right), 4\right] + R\left[\left(2^2, 1^0\right), 4\right],$$

$$H_3^1(2) = \sum_{i_1=1}^{2} R\left[\left(2^{i_1}, 1^{3-i_1}\right), 4\right] = R\left[\left(2^1, 1^2\right), 4\right] + R\left[\left(2^2, 1^1\right), 4\right],$$

$$H_4^1(2) = \sum_{i_1=1}^{2} R\left[\left(2^{i_1}, 1^{4-i_1}\right), 4\right] = R\left[\left(2^1, 1^3\right), 4\right] + R\left[\left(2^2, 1^2\right), 4\right],$$

$$\begin{aligned} H_3^1(3) &= \sum_{i_1=1}^{3} \sum_{i_2=0}^{2-i_1} R\left[\left(3^{i_1}, 2^{i_2}, 1^{3-i_1}\right), 4\right] \\ &= R\left[\left(3^1, 2^0, 1^2\right), 4\right] + R\left[\left(3^1, 2^1, 1^1\right), 4\right] + R\left[\left(3^2, 2^0, 1^1\right), 4\right], \end{aligned}$$

$$\begin{aligned} H_4^1(3) &= \sum_{i_1=1}^{3} \sum_{i_2=0}^{2-i_1} R\left[\left(3^{i_1}, 2^{i_2}, 1^{4-i_1}\right), 4\right] \\ &= R\left[\left(3^1, 2^0, 1^3\right), 4\right] + R\left[\left(3^1, 2^1, 1^2\right), 4\right] + R\left[\left(3^2, 2^0, 1^2\right), 4\right], \end{aligned}$$

then

$$R_{2,1} = 0,2286,$$

level 0 : we have

$$\begin{aligned} R_{2,0} &= 1 - (R_{2,1} + R_{2,2} + R_{2,3}) \\ &= 0,245. \end{aligned}$$

Now, using formula (2), we can calculate the global system's reliability as the following :

$j = 0$:

$$P(\phi_s \geq 0) = \prod_{n=1}^{3} \left(\sum_{\alpha=0}^{M} R_{n,\alpha} \right) = 1.$$

$j = 1$:

$$P(\phi_s \geq 1) = \prod_{n=1}^{3} \left(\sum_{\alpha=1}^{M} R_{n,\alpha} \right) = 0,527273125.$$

$j = 2$:

$$P(\phi_s \geq 2) = \prod_{n=1}^{3} \left(\sum_{\alpha=2}^{M} R_{n,\alpha} \right) = 0,0581903616,$$

therefore, with formula (3), we can get the state distribution of the global system at different levels:

$j = 0$:

$$\begin{aligned} P_s(0) &= P(\phi_s \geq 0) - P(\phi_s \geq 1) \\ &= \prod_{n=1}^{3}\left(\sum_{\alpha=0}^{M} R_{n,\alpha}\right) - \prod_{n=1}^{3}\left(\sum_{\alpha=1}^{M} R_{n,\alpha}\right) \\ &= 0,472726875. \end{aligned}$$

$j = 1$:

$$\begin{aligned} P_s(1) &= P(\phi_s \geq 1) - P(\phi_s \geq 2) \\ &= \prod_{n=1}^{3}\left(\sum_{\alpha=1}^{M} R_{n,\alpha}\right) - \prod_{n=1}^{3}\left(\sum_{\alpha=2}^{M} R_{n,\alpha}\right) \\ &= 0,4690827634. \end{aligned}$$

$j = 2$:

$$\begin{aligned} P_s(2) &= P(\phi_s \geq 2) - P(\phi_s \geq 3) \\ &= \prod_{n=1}^{3}\left(\sum_{\alpha=2}^{M} R_{n,\alpha}\right) - \prod_{n=1}^{3}\left(\sum_{\alpha=3}^{M} R_{n,\alpha}\right) \\ &= 0,0581903616. \end{aligned}$$

$j = 3$:

$$\begin{aligned} P_s(3) &= 1 - P(\phi_s < 3) \\ &= 0. \end{aligned}$$

Example 2 :

In this example, we consider a consecutive k_n-out-of-$m_n : G$ parallel system compound of 3 blocks $(n = 1, 2, 3)$ similar to the blocks in the first example.

By using formula (1), we can compute the state distributions of the three blocks, and we get the same results as in the first example. However, for the global system's reliability, by using formula (4) we get the following results :

$j = 0$:

$$P(\phi_s \geq 0) = 1 - \prod_{n=1}^{3}\left(1 - \sum_{\alpha=0}^{M} R_{n,\alpha}\right) = 1.$$

$j = 1$:

$$P(\phi_s \geq 1) = 1 - \prod_{n=1}^{3}\left(1 - \sum_{\alpha=1}^{M} R_{n,\alpha}\right) = 0,995498125.$$

$j = 2$:

$$P(\phi_s \geq 2) = 1 - \prod_{n=1}^{3}\left(1 - \sum_{\alpha=2}^{M} R_{n,\alpha}\right) = 0,8228054016.$$

$j = 3:$

$$P(\phi_s \geq 3) = 1 - \prod_{n=1}^{3}\left(1 - \sum_{\alpha=3}^{M} R_{n,\alpha}\right) = 0,0591.$$

Hence, according to formula (5), we deduce the state distribution of the global system at different levels:

$j = 0:$

$$\begin{aligned} P_s(0) &= P(\phi_s \geq 0) - P(\phi_s \geq 1) \\ &= \prod_{n=1}^{3}\left(1 - \sum_{\alpha=1}^{M} R_{n,\alpha}\right) - \prod_{n=1}^{3}\left(1 - \sum_{\alpha=0}^{M} R_{n,\alpha}\right) \\ &= 0,004501875. \end{aligned}$$

$j = 1:$

$$\begin{aligned} P_s(1) &= P(\phi_s \geq 1) - P(\phi_s \geq 2) \\ &= \prod_{n=1}^{3}\left(1 - \sum_{\alpha=2}^{M} R_{n,\alpha}\right) - \prod_{n=1}^{3}\left(1 - \sum_{\alpha=1}^{M} R_{n,\alpha}\right) \\ &= 0,1726927234. \end{aligned}$$

$j = 2:$

$$\begin{aligned} P_s(2) &= P(\phi_s \geq 2) - P(\phi_s \geq 3) \\ &= \prod_{n=1}^{3}\left(1 - \sum_{\alpha=3}^{M} R_{n,\alpha}\right) - \prod_{n=1}^{3}\left(1 - \sum_{\alpha=2}^{M} R_{n,\alpha}\right) \\ &= 0,7637054016. \end{aligned}$$

$j = 3:$

$$\begin{aligned} P_s(3) &= 1 - P(\phi_s < 3) \\ &= 0,0591. \end{aligned}$$

Remarks

1. In the application, we treat a more general (and realistic) case, where M is different from a block to another.
2. In the first example, since the system has a series structure, so it can't be at state 3 because the first block has only three states 0, 1, 2.
3. We can see that changing the structure of the first system from series to parallel configuration improved the reliability of the system.

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