ASIAN J. MATH.
Vol. 17, No. 3, pp. 569–582, September 2013
© 2013 International Press
008

LOGARITHMIC SOBOLEV TRACE INEQUALITIES*

FILOMENA FEO† AND MARIA ROSARIA POSTERARO‡

Abstract. We prove a logarithmic Sobolev trace inequality and we study the trace operator in the weighted Sobolev space $W^{1,p}(\Omega,\gamma)$ for sufficiently regular domain, where γ is the Gauss measure. Applications to PDE are also considered.

Key words. Trace inequality, Gauss measure, embedding theorems.

AMS subject classifications. 46E35, 46E30.

1. Introduction. Sobolev Logarithmic inequality states that

$$\int_{\mathbb{R}^N} |u|^p \log |u| \, d\gamma \leq \frac{p}{2} \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{p-2} \text{sign } u \, d\gamma + \|u\|^p_{L^p(\mathbb{R}^N,\gamma)} \log \|u\|_{L^p(\mathbb{R}^N,\gamma)} , \quad (1.1)$$

where $1 < p < +\infty$, γ is the Gauss measure and $L^p\left(\mathbb{R}^N,\gamma\right)$ is the weighted Lebesgue space (see §2 for the definitions). This inequality was first proved in [18] (see also [2] for more general probability measure). It has many applications in quantum field theory and unlike the classical Sobolev inequality it is independent on dimension and easily extends to the infinite dimensional case.

In terms of functional spaces inequality (1.1) implies the imbedding of weighted Sobolev space $W^{1,p}(\mathbb{R}^N,\gamma)$ into the weighted Zygmund space $L^p(\log L)^{\frac{1}{2}}\left(\mathbb{R}^N,\gamma\right)$. The imbedding holds also for $p = 1$ and it is connected with gaussian isoperimetric inequality (see [20], [15] and [23]).
This kind of imbeddings are also studied in [9] in the more general case of rearrangement-invariant spaces. In [11] a set $\Omega \subseteq \mathbb{R}^N$ with $\gamma(\Omega) < 1$ and the space $W_0^{1,p}(\Omega,\gamma)$ are considered using properties of rearrangements of functions; the authors prove that if $u \in W_0^{1,p}(\Omega,\gamma)$ with $1 \leq p < +\infty$, then $u \in L^p(\log L)^{\frac{1}{2}}(\Omega,\gamma)$ and

$$\|u\|_{L^p(\log L)^{\frac{1}{2}}(\Omega,\gamma)} \leq C \|\nabla u\|_{L^p(\Omega,\gamma)} , \quad (1.2)$$

for some constant C depending only on p and $\gamma(\Omega)$. Analogous inequalities have been obtained in infinite dimensional case and in the Lorentz-Zygmund spaces (see the appendix of [17]). Estimates in the spirit of (1.2) are also obtained for equations related to Gauss measure (see [12] and references therein).
If $\gamma(\Omega) = 1$ inequality (1.2) has to be replaced by

$$\|u\|_{L^p(\log L)^{\frac{1}{2}}(\Omega,\gamma)} \leq C \|\nabla u\|_{L^p(\Omega,\gamma)} + \|u\|_{L^p(\Omega,\gamma)} .$$

A first result of our paper is to obtain (1.2) when $u \in W^{1,p}(\gamma,\Omega)$ (see §3); in this case, as one can expect, smoothness assumption on $\partial\Omega$ has to be made. Besides the continuity also the compactness of the imbedding of $W^{1,p}(\Omega,\gamma)$ in a Zygmund

*Received January 11, 2012; accepted for publication March 6, 2013.
†Dipartimento di Ingegneria, Università degli Studi di Napoli "Pathenope", Centro Direzionale Isola C4 80143 Napoli, Italia (filomena.feo@uniparthenope.it)
‡Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli Federico II, Complesso Universitario Monte S. Angelo - via Cintia 80126 Napoli, Italia (posterar@unina.it).

space is studied. As a consequence we obtain a Poincaré-Wirtinger type inequality. Applications of these results to PDE are also considered.

The results explained above are used to investigate Sobolev trace inequalities. This kind of inequalities play a fundamental role in problems with nonlinear boundary conditions. In the euclidean case the Sobolev trace inequality (cf. e.g. [19]) tells us that if Ω is smooth enough and $1 \leq p < N$, then there exists a constant C (depending only on Ω and on p) such that

$$\|Tu\|_{L^{\frac{p(N-1)}{N-p}}(\partial\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)} \qquad \text{for every } u \in W^{1,p}(\Omega), \tag{1.3}$$

where T is the trace operator. This kind of inequalities have been developed via different methods and in different settings by various authors including Besov [8], Gagliardo [13], Lions and Magenes [22]. Trace inequalities that involve rearrangement-invariant norms are considered in [10]. Moreover trace imbedding (1.3) admits an improvement in terms of Lorentz spaces (see [3]).

To investigate about trace operator in the weighted Sobolev space $W^{1,p}(\Omega,\gamma)$ we need a Sobolev trace inequality in §4. We prove that if Ω is a smooth domain and $u \in C^\infty(\overline{\Omega})$ then

$$\int_{\partial\Omega} |u|^p \log^{\frac{p}{2p'}}(2+|u|)\,\varphi\, dS \leq C \|u\|^p_{W^{1,p}(\Omega,\gamma)}, \tag{1.4}$$

where $\varphi(x) = (2\pi)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{2}\right)$ is the density of the Gauss measure and the integral on the left-hand side of (1.4) is a surface integral. This inequality captures the spirit of the Gross inequalities: the logarithmic function replaces the powers in this case too.

Using (1.4), we can define a trace operator and we prove that it is continuous and compact from $W^{1,p}(\Omega,\gamma)$ into $L^p(\partial\Omega,\gamma)$ for sufficiently regular domain $\Omega \subseteq \mathbb{R}^N$. Moreover a Poincaré trace inequality is obtained in a suitable subspace of $W^{1,p}(\Omega,\gamma)$. We give also some applications of these results to PDE.

Another Sobolev trace inequality is obtained in [24] as limit case of the classical trace Sobolev inequality.

2. Preliminaries. In this section we recall some definitions and results which will be useful in the following.

2.1. Gauss measure and rearrangements. Let γ be the N-dimensional Gauss measure on $\mathbb{R}^N$ defined by

$$d\gamma = \varphi(x)\,dx = (2\pi)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{2}\right)\, dx, \qquad x \in \mathbb{R}^N$$

normalized by $\gamma(\mathbb{R}^N) = 1$.
We will denote by $\Phi(\tau)$ the Gauss measure of the half-space $\left\{x \in \mathbb{R}^N : x_N < \tau\right\}$:

$$\Phi(\tau) = \gamma\left(\left\{x \in \mathbb{R}^N : x_N < \tau\right\}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\tau} \exp\left(-\frac{t^2}{2}\right) dt \quad \forall \tau \in \mathbb{R} \cup \{-\infty, +\infty\}.$$

We define the decreasing rearrangement with respect to Gauss measure (see e.g. [14]) of a measurable function u in Ω as the function

$$u^{\circledast}(s) = \inf\{t \geq 0 : \gamma_u(t) \leq s\} \qquad s \in]0,1],$$

where $\gamma_u(t) = \gamma(\{x \in \Omega : |u| > t\})$ is the distribution function of u.

2.2. Sobolev and Zygmund spaces. The weighted Lebesgue space $L^p(\Omega,\gamma)$ is the space of the measurable functions u on Ω such that $\int_\Omega |u|^p\,d\gamma < +\infty$. We recall also that the weighted Sobolev space $W^{1,p}(\Omega,\gamma)$ for $1 \le p < +\infty$ is defined as the space of the measurable functions $u \in L^p(\Omega,\gamma)$ such that there exist $g_1,\ldots,g_N \in L^p(\Omega,\gamma)$ that verify

$$\int_\Omega u\frac{\partial}{\partial x_i}\psi\varphi\,dx - \int_\Omega u\psi x_i\varphi\,dx = \int_\Omega g_i\psi\varphi\,dx \qquad i=1,\ldots,N \qquad \forall\psi\in D(\Omega).$$

We stress that $u \in W^{1,p}(\Omega,\gamma)$ is a Banach space with respect to the norm $\|u\|_{W^{1,p}(\Omega,\gamma)} = \|u\|_{L^p(\Omega,\gamma)} + \|\nabla u\|_{L^p(\Omega,\gamma)}$.

The Zygmund space $L^p(\log L)^\alpha(\Omega,\gamma)$ for $1 \le p \le +\infty$ and $\alpha \in \mathbb{R}$ is the space of the measurable functions on Ω such that the quantity

$$||u||_{L^p(\log L)^\alpha(\Omega,\gamma)} = \begin{cases} \left(\displaystyle\int_0^{\gamma(\Omega)} [(1-\log t)^\alpha u^\circledast(t)]^p\,dt\right)^{\frac{1}{p}} & \text{if } 1 \le p < +\infty \\ \displaystyle\sup_{t\in(0,\gamma(\Omega))} [(1-\log t)^\alpha u^\circledast(t)] & \text{if } p = +\infty \end{cases} \tag{2.1}$$

is finite. The space $L^p(\log L)^\alpha(\Omega,\gamma)$ is not trivial if and only if $p < +\infty$ or $p = +\infty$ and $\alpha \le 0$.
The Zygmund spaces are the natural spaces in the context of Gauss measure, because of the following property of isoperimetric function is (see [21]):

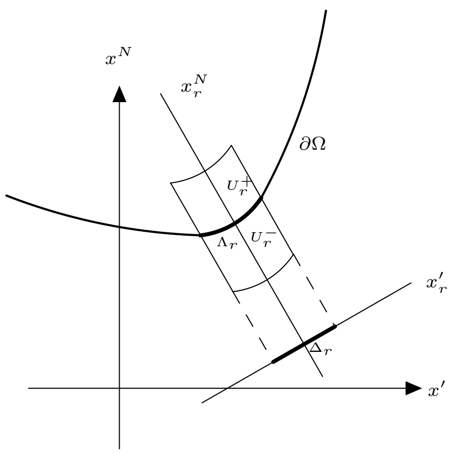
$$\varphi_1 \circ \Phi^{-1}(t) \sim t(2\log\frac{1}{t})^{\frac{1}{2}} \quad \text{for } t \to 0^+ \text{ and } t \to 1^-. \tag{2.2}$$

We recall same inclusion relations among Zygmund spaces. If $1 \le r < p \le +\infty$ and $-\infty < \alpha,\beta < +\infty$, then we get

$$L^p(\log L)^\alpha(\Omega,\gamma) \subseteq L^r(\log L)^\beta(\Omega,\gamma)\ .$$

It is clear from definition (2.1) that the space $L^p(\log L)^\alpha(\Omega,\gamma)$ decreases as α increases. For more properties we refer to [7].

2.3. Smoothness assumptions on the domain. In this paper we deal with integrals involving the values of a $W^{1,p}-$function on $\partial\Omega$. To this aim we need to have a suitable local description of the set Ω and $\partial\Omega$ is a finite union of graphs. More precisely we will consider smooth domain Ω which verifies the following condition (cfr. Chapter 6 of [19] for bounded domain).



Condition 2.1. Let $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ be a domain such that there exist
i) $m \in \mathbb{N}$ coordinate systems $X_r = (x'_r, x_r^N)$ where $x'_r = (x_r^1, ..., x_r^{N-1})$
for $r = 1, 2, ..., m$;
ii) $a_i, b_i \in \mathbb{R} \cup \{\pm\infty\}$ for $i = 1, ..., N-1$ and m Lipschitz functions a_r in
$\overline{\Delta_r} = \left\{x'_r : x_r^i \in (a_i, b_i) \ \textit{for } i = 1, ..., N-1\right\}$
for $r = 1, ..., N$;
iii) a number $\beta > 0$ such that the sets

$$\Lambda_r = \left\{(x'_r, x_r^N) \in \mathbb{R}^N : x'_r \in \Delta_r \ \textit{and } x_r^N = a_r(x'_r)\right\}$$

are subsets of $\partial\Omega$, $\partial\Omega = \bigcup\limits_{r=1}^{m} \Lambda_r$ and the sets

$$U_r^+ = \left\{(x'_r, x_r^N) \in \mathbb{R}^N : x'_r \in \Delta_r \ \textit{and } a_r(x'_r) < x_r^N < a_r(x'_r) + \beta\right\}$$

$$U_r^- = \left\{(x'_r, x_r^N) \in \mathbb{R}^N : x'_r \in \Delta_r \ \textit{and } a_r(x'_r) - \beta < x_r^N < a_r(x'_r)\right\}$$

are subset of Ω (after a suitable orthonormal transformation of coordinates).

We observe that the set $U_r = U_r^+ \cup U_r^-$ is an open subset of $\mathbb{R}^N$ and there exists an open set $U_0 \subseteq \overline{U_0} \subset \Omega$ such that the collection $\{U_r\}_{r=0}^m$ is a open cover of Ω. Moreover the collection $\{U_r\}_{r=1}^m$ is a open cover of $\partial\Omega$.

3. Sobolev logarithmic inequalities in $W^{1,p}(\Omega, \gamma)$. In this section we prove continuity and compactness of imbedding of $W^{1,p}(\Omega, \gamma)$ into $L^p(\log L)^{\frac{1}{2}}(\Omega, \gamma)$. The first step is to obtain the analogue of (1.2) when $u \in W^{1,p}(\Omega, \gamma)$ for $1 \leq p < +\infty$.

Theorem 3.1. *(Continuity) If $u \in W^{1,p}(\Omega, \gamma)$ for $1 \leq p < +\infty$ and Ω satisfies condition 2.1, then there exists a positive constant C depending only on p and Ω such that*

$$\|u\|_{L^p(\log L)^{\frac{1}{2}}(\Omega,\gamma)} \leq C \|u\|_{W^{1,p}(\Omega,\gamma)}, \tag{3.1}$$

i.e. the embedding of weighted Sobolev space $W^{1,p}(\Omega, \gamma)$ into the weighted Zygmund space $L^p(\log L)^{\frac{1}{2}}(\Omega, \gamma)$ is continuous for $1 \leq p < +\infty$.

To prove Theorem 3.1 we need an extension operator P from $W^{1,p}(\Omega, \gamma)$ into $W^{1,p}(\mathbb{R}^N, \gamma)$. When $u \in W_0^{1,p}(\Omega, \gamma)$ the natural extension by zero outside Ω is continuous without any assumptions on the regularity of the boundary. Working with the space $W^{1,p}(\Omega, \gamma)$ the situation is more delicate and the regularity of the boundary of Ω plays a crucial role.

Lemma 3.1. *Let Ω satisfy condition 2.1, then there exists an extension operator $P : W^{1,p}(\Omega, \gamma) \to W^{1,p}(\mathbb{R}^N, \gamma)$ which is linear and continuous. More precisely for every $u \in W^{1,p}(\Omega, \gamma)$*
j) $P(u)|_\Omega = u$,
jj) $\|P(u)\|_{L^p(\mathbb{R}^N,\gamma)} \leq c \|u\|_{L^p(\Omega,\gamma)}$,
jjj) $\|P(u)\|_{W^{1,p}(\mathbb{R}^N,\gamma)} \leq c \|u\|_{W^{1,p}(\Omega,\gamma)}$,
for some constant $c = c(p, \Omega)$.

The proof of this lemma can be done by adapting the classical tools of the Lebesgue case (see e.g. [6]). We give a sketch of the construction of the operator P.

Proof. If $\Omega = \mathbb{R}^N_+$ the extension is obtained by reflection. If we consider a general open set using classical tools we can reduce ourselves to the case $\mathbb{R}^N_+$. Indeed there exists a finite number of open sets $G_0, G_1, ..., G_h$ such that $\overline{\Omega} \subset \bigcup_{i=0}^{K} G_i$ with $\overline{G_0} \subset \Omega$, for each $i = 1, .., K$ a system of local coordinates $g_i : B(0,1) \to G_i$ and for each $i = 0, 1, .., K$ a function $\alpha_i \in C^\infty(G_i)$ such that $\sum_{i=0}^{K} \alpha_i = 1$ on $\overline{\Omega}$. Given $u \in W^{1,p}(\Omega, \gamma)$ we have $u = \sum_{i=0}^{k} u_i$, where $u_i = \alpha_i u$. It is enough to extend u_i to the whole of $\mathbb{R}^N$. The function u_0 is natural extended by zero outside Ω. For $i = 1, .., K$ we define

$$\widetilde{u}_i = \begin{cases} u_i \circ g_i & \text{on } B(0,1) \cap \mathbb{R}^N_+ \\ 0 & \text{on } \mathbb{R}^N_+ - B(0,1) \end{cases}.$$

This function belongs to $W^{1,p}(\mathbb{R}^N_+, \gamma)$, then we can use the extension operator P from $W^{1,p}(\mathbb{R}^N_+, \gamma)$ to $W^{1,p}(\mathbb{R}^N, \gamma)$ and return to G_i using g_i^{-1}. □

Remark 3.1. We note that the extension operator allows us to prove the density (for the classical case see e.g. [6]) of $C^\infty(\overline{\Omega})$ in $W^{1,p}(\Omega, \gamma)$.

Proof of Theorem 3.1. We consider the extension operator P and using (1.2) we obtain for some constant c

$$\begin{aligned} \|u\|_{L^p(\log L)^{\frac{1}{2}}(\Omega,\gamma)} &\le c\, \|Pu\|_{L^p(\log L)^{\frac{1}{2}}(\mathbb{R}^N,\gamma)} \\ &\le c\, \|Pu\|_{W^{1,p}(\mathbb{R}^N,\gamma)} \le c\, \|u\|_{W^{1,p}(\Omega,\gamma)} \end{aligned}$$

for $u \in W^{1,p}(\Omega, \gamma)$. □

Remark 3.2. Theorem 3.1 implies the continuity of the embedding of Sobolev space $W^{m,p}(\Omega,\gamma)$ $m \ge 1$ into the Zygmund space $L^p(\log L)^{m\alpha}(\Omega,\gamma)$ for $\alpha \le \frac{1}{2}$. A similar result for $\Omega = \mathbb{R}^N$ is proved in [16].

Proposition 3.1. *(Compactness) Let $1 \le p < +\infty$ and let Ω satisfy condition 2.1. Then the embedding of $W^{1,p}(\Omega, \gamma)$ into $L^p \log L^\alpha(\Omega, \gamma)$ is compact if $\alpha < \frac{1}{2}$.*

Proof. It is enough to prove the compactness of the embedding of $W^{1,p}(\Omega,\gamma)$ into $L^1(\Omega,\gamma)$. Indeed we have that any bounded set of $L^p(\log L)^{\frac{1}{2}}(\Omega,\gamma)$ which is precompact in $L^1(\Omega, \gamma)$ is also precompact in $L^p \log L^\alpha(\Omega, \gamma)$ with $\alpha < \frac{1}{2}$ (see e.g. Theorem 8.23 of [1]).
Let be S bounded set in $W^{1,p}(\Omega, \gamma)$, then S is bounded in $L^1(\Omega, \gamma)$ too. Using a characterization of precompact sets of Lebesgue spaces (see e.g. Theorem 2.21 of [1]) we have to prove that for any number $\varepsilon > 0$ there exists a number $\delta > 0$ and a subset $G \subset\subset \Omega$ such that for any $u \in S$ and every $h \in \mathbb{R}^N$ with $|h| < \delta$ the following conditions hold:

$$a) \int_\Omega |\widetilde{u}(x+h)\varphi(x+h) - \widetilde{u}(x)\varphi(x)|\, dx < \varepsilon \tag{3.2}$$

$$b) \int_{\Omega \setminus \overline{G}} |u|\, d\gamma < \varepsilon, \tag{3.3}$$

where $\widetilde{u}$ is the zero extension of u outside Ω.
Let $\varepsilon > 0$ and $\Omega_j = \left\{ x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \frac{1}{j} \right\}$ for $j \in \mathbb{N}$. By (3.1) we have for some constant c

$$\int_{\Omega \setminus \Omega_j} |u| \, d\gamma \le \left(\int_0^{\gamma(\Omega \setminus \Omega_j)} \left[(1 - \log t)^{\frac{1}{2}} u^{\circledast}(t) \right]^p dt \right)^{\frac{1}{p}} \left(\int_{\Omega \setminus \Omega_j} (1 - \log t)^{-\frac{p'}{2}} dt \right)^{\frac{1}{p'}}$$
$$\le c \, \|u\|_{W^{1,p}(\Omega,\gamma)} \left(\int_{\Omega \setminus \Omega_j} (1 - \log t)^{-\frac{p'}{2}} dt \right)^{\frac{1}{p'}}.$$

Since the Gauss measure of Ω is finite, we can choose j big enough to have

$$\int_{\Omega \setminus \Omega_j} |u| \, d\gamma < \varepsilon, \tag{3.4}$$

(i.e. (3.3) holds) and for $h \in \mathbb{R}^N$

$$\int_{\Omega \setminus \Omega_j} |\widetilde{u}(x+h)\varphi(x+h) - \widetilde{u}(x)\varphi(x)| \, dx < \frac{\varepsilon}{2}. \tag{3.5}$$

Let $|h| \le \frac{1}{j}$, then $x + th \in \Omega_{2j}$ if $x \in \Omega$ and $t \in [0,1]$. Let $u \in C^\infty(\overline{\Omega})$, we have for some constant c

$$\begin{aligned}
&\int_{\Omega_j} |\widetilde{u}(x+h)\varphi(x+h) - \widetilde{u}(x)\varphi(x)| \, dx \\
&\le \int_{\Omega_j} \int_0^1 \left| \frac{d}{dt} \widetilde{u}(x+th)\varphi(x+th) \right| dt dx \\
&\le \int_{\Omega_j} \int_0^1 |\nabla \widetilde{u}(x+th) h \varphi(x+th) - \widetilde{u}(x+th)\varphi(x+th)\,(x+th)\, h| \, dt dx \\
&\le |h| \left(\int_{\Omega_{2j}} |\nabla \widetilde{u}(y)\varphi(y)| \, dy + \int_{\Omega_{2j}} |\widetilde{u}(y)\varphi(y) y| \, dy \right) \\
&\le c\,|h| \left(\|\nabla u\|^p_{L^p(\Omega,\gamma)} + \|u\|^p_{L^p(\log L)^{\frac{1}{2}}(\Omega,\gamma)} \right) \le c\,|h|\, \|u\|^p_{W^{1,p}(\Omega,\gamma)}.
\end{aligned} \tag{3.6}$$

In the last inequalities we have used (3.1) and the fact that $f(x) = |x| \in L^{p'}(\log L)^{-\frac{1}{2}}(\Omega, \gamma)$. Indeed since $\gamma_f(t) = 1 - \gamma(B(0,t))$, one can easily check that

$$\int_0^{\gamma(\Omega)} (1 - \log s)^{-\frac{p'}{2}} \left[(|x|)^{\circledast}(s) \right]^{p'} ds = \int_0^{+\infty} t^{p'} (1 - \log \gamma_f(t))^{-\frac{p'}{2}} \gamma_f'(t) \, dt < +\infty.$$

Because of the density of $C^\infty(\overline{\Omega})$ in $W^{1,p}(\Omega,\gamma)$, (3.6) holds for every u in $W^{1,p}(\Omega,\gamma)$ and then for $|h|$ small enough by (3.5) and (3.6) we obtain (3.2) □

Remark 3.3. Obviously the compactness result holds for $W_0^{1,p}(\Omega,\gamma)$ for any domain Ω.

Remark 3.4. The compactness proved in Proposition 3.1 implies the compact embedding of Sobolev space $W^{m,p}(\Omega,\gamma)$ $m \ge 1$ into the Zygmund space $L^p(\log L)^{m\alpha}(\Omega,\gamma)$ for $\alpha < \frac{1}{2}$.

The compactness can be used to obtain a Poincaré-Wirtinger type inequality (see also [5] for $p = 2$).

PROPOSITION 3.2. *Let Ω be a connected domain satisfying condition 2.1. Assume $1 \leq p < +\infty$. Then there exists a positive constant C, depending only on p and Ω, such that*

$$\|u - u_\Omega\|_{L^p(\Omega,\gamma)} \leq C \|\nabla u\|_{L^p(\Omega,\gamma)} \tag{3.7}$$

for any $u \in W^{1,p}(\Omega, \gamma)$, where $u_\Omega = \frac{1}{\gamma(\Omega)} \int_\Omega u d\gamma$.

REMARK 3.5. Using Theorem 3.1 and Proposition 3.2 it follows that there exists a positive constant C, depending only on p and Ω, such that

$$\|u - u_\Omega\|_{L^p(\log L)^{\frac{1}{2}}(\Omega,\gamma)} \leq C \|\nabla u\|_{L^p(\Omega,\gamma)}$$

for any $u \in W^{1,p}(\Omega, \gamma)$. See [9] for the case $\Omega = \mathbb{R}^N$.

Proof of Proposition 3.2. We proceed as in the classical case. We argue by contradiction, then there would exist for any $k \in \mathbb{N}$ a function $u_k \in W^{1,p}(\Omega, \gamma)$ such that

$$\|u_k - (u_k)_\Omega\|_{L^p(\Omega,\gamma)} > k \|\nabla u_k\|_{L^p(\Omega,\gamma)} .$$

We renormalize by defining

$$v_k = \frac{u_k - (u_k)_\Omega}{\|u_k - (u_k)_\Omega\|_{L^p(\Omega,\gamma)}}. \tag{3.8}$$

Then

$$(v_k)_\Omega = 0, \|v_k\|_{L^p(\Omega,\gamma)} = 1$$

and

$$\text{and } \|\nabla v_k\|_{L^p(\Omega,\gamma)} < \frac{1}{k}. \tag{3.9}$$

In particular the functions $\{v_k\}_{k\in\mathbb{N}}$ are bounded in $W^{1,p}(\Omega, \gamma)$. Then by the previous theorem there exists a subsequence still denoted by $\{v_k\}_{k\in\mathbb{N}}$ and a function v such that

$$v_k \to v \quad \text{ in } L^p(\Omega, \gamma).$$

Moreover by (3.8) it follows that

$$v_\Omega = 0 \text{ and } \|v\|_{L^p(\Omega,\gamma)} = 1. \tag{3.10}$$

On the other hand, (3.9) implies for any $\psi \in C_0^\infty(\Omega)$ and $i = 1, ..., N$

$$\begin{aligned}\int_\Omega v \frac{\partial \psi}{\partial x_i} \varphi dx - \int_\Omega v \psi x_i \varphi dx &= \lim_{k\to+\infty} \left(\int_\Omega v_k \frac{\partial \psi}{\partial x_i} \varphi dx - \int_\Omega v_k \psi x_i \varphi dx \right) \\ &= \lim_{k\to+\infty} - \int_\Omega \frac{\partial v_k}{\partial x_i} \psi \varphi dx = 0.\end{aligned}$$

Consequently $v \in W^{1,p}(\Omega,\gamma)$ and $\nabla v = 0$ a.e. Then v is constant since Ω is connected. In particular by the first estimate in (3.10) we must have $v \equiv 0$; in which case $\|v\|_{L^p(\Omega,\gamma)} = 0$. This contradiction establishes the estimate (3.7). $\square$

REMARK 3.6. The previous proof works in a more general case. Let Ω be a connected domain satisfying condition 2.1 and let $V \subset W^{1,p}(\Omega,\gamma)$ be a linear subspace of $W^{1,p}(\Omega,\gamma)$ with $1 \le p < +\infty$ which is closed and such that the only constant function belonging to V is the function which is identically zero. Then there exists a positive constant C, depending only on p and Ω, such that

$$\|v\|_{L^p(\Omega,\gamma)} \le C \left(\int_\Omega \sum_{i=1}^N \left| \frac{\partial v}{\partial x_i} \right|^p d\gamma \right)^{\frac{1}{p}} \qquad \forall v \in V.$$

REMARK 3.7. (*Application to PDE*) Let Ω be a connected domain satisfying condition 2.1. Let us consider the semicoercive homogeneous Neumann problem

$$\begin{cases} -\left(u_{x_i}\varphi\right)_{x_i} = f\varphi & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.11}$$

where $f \in L^2(\log L)^{-\frac{1}{2}}(\Omega,\gamma)$ and ν is the external normal. Using classical tools (see e.g. [6] Theorem 6.2.3) and inequalities (3.1) and (3.7) it follows that problem (3.11) has a weak solution in $W^{1,2}(\Omega,\gamma)$ if and only if $\int_\Omega f d\gamma = 0$. In particular there exists a unique weak solution in $X = \left\{u \in W^{1,2}(\Omega,\gamma) : \int_\Omega u d\gamma = 0\right\}$ by Lax-Milgram theorem.

We consider also the following eigenvalue problem related to the equation of quantum harmonic oscillator

$$\begin{cases} -\left(u_{x_i}\varphi\right)_{x_i} = \lambda u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

Arguing in a classical way (see e.g. [6] Theorem 8.6.1), using inequality (3.7) and the compactness of the embedding from $W^{1,2}(\Omega,\gamma)$ into $L^2(\Omega,\gamma)$, it follows that there exists an increasing sequence of eigenvalues of the problem (3.12) which tends to infinity and a Hilbertian basis of eigenfunctions in $L^2(\Omega,\gamma)$. Moreover for $\lambda_1 = 0$, the corresponding eigenfunction $u_1 = const \neq 0$ and the first nontrivial eigenvalue λ_2 has the following characterization

$$\lambda_2 = \min\left\{ \frac{\|\nabla u\|_{L^2(\Omega,\gamma)}}{\|u\|_{L^2(\Omega,\gamma)}}, u \in W^{1,2}(\Omega,\gamma) : \int_\Omega u d\gamma = 0 \right\}.$$

A sharp lower bound for λ_2 is proved in [4].

4. Sobolev logarithmic trace inequalities. In this section we deal with integrals involving the values of a $C^\infty-$function on $\partial\Omega$. We prove that a certain integral of the function on $\partial\Omega$ is bounded by the $W^{1,p}-$norm on Ω. This inequality will be crucial to define trace operator (see §5).

THEOREM 4.1. *Let Ω be a domain satisfying condition* 2.1 *and* $1 \le p < +\infty$. *For every $u \in C^\infty(\overline{\Omega})$ there exists a positive constant C depending only on p and Ω such that*

$$\int_{\partial\Omega} |u|^p \log^{\frac{p-1}{2}}(2+|u|)\,\varphi\, dS \le C\,\|u\|^p_{W^{1,p}(\Omega,\gamma)}\,. \tag{4.1}$$

REMARK 4.1. We obtain the same result if we replace the first member of (4.1) with the quantity $\int_{\partial\Omega} u^p \left(\log^+(|u|)\right)^{\frac{p-1}{2}} \varphi\, dS$.

Proof of Theorem 4.1. Following classical tools (see Chapter 6 of [19]) it is enough to prove the existence of a constant $C_T > 0$ such that for any function $u \in C^\infty(\overline{\Omega})$ whose support is in $\Lambda_r \cup U_r^+$ we have (4.1). After suitable transformation that maps $\Delta_r \times]0,\beta[$ onto U_r^+ and $\Delta_r \times \{0\}$ onto Λ_r, we can reduce to consider u such that the support is in $\Delta_r \times [0,\beta[$. Then it is sufficient to prove the existence of a constant $C > 0$ such that for any function $u \in C^\infty(\overline{\Delta_r} \times [0,\beta[)$ whose support is in $\Delta_r \times [0,\beta[$

$$\int_{\Delta_r} |u(x_r',0)|^p \log^{\frac{p-1}{2}}(2+|u(x_r',0)|)\varphi(x_r',0)\ dx_r' \le C\,\|u\|^p_{W^{1,p}(\Delta_r\times]0,\beta[,\gamma)}\,. \tag{4.2}$$

holds. In (4.2) we have denoted by u the composition of u with the change of coordinates.

Now we prove (4.2) for $1 < p < +\infty$. The case $p = 1$ can be obtained in the same (but more direct) way. For some constant c that can vary from line to line we have

$$\begin{aligned}&\int_{\Delta_r} |u(x_r',0)|^p \log^{\frac{p-1}{2}}(2+|u(x_r',0)|)\ \varphi(x_r',0)\ dx_r' \\ &\le c(A_1+A_2+A_3)\end{aligned} \tag{4.3}$$

where

$$A_1 = \int_{\Delta_r}\int_\beta^0 p\left|u(x_r',x_r^N)\right|^{p-1} \log^{\frac{p-1}{2}}(2+\left|u(x_r',x_r^N)\right|)\left|\frac{\partial u}{\partial x_r^N}(x_r',x_r^N)\right|\varphi(x_r',x_r^N)\ dx_r^N\ dx_r'$$

$$A_2 = \int_{\Delta_r}\int_\beta^0 \frac{p-1}{2}\left|u(x_r',x_r^N)\right|^p \frac{\log^{\frac{p-3}{2}}(2+|u(x_r',x_r^N)|)}{2+|u(x_r',x_r^N)|}\left|\frac{\partial u}{\partial x_r^N}(x_r',x_r^N)\right|\varphi(x_r',x_r^N)\ dx_r^N\ dx_r'$$

$$A_3 = \int_{\Delta_r}\int_\beta^0 \left|u(x_r',x_r^N)\right|^p \log^{\frac{p-1}{2}}(2+\left|u(x_r',x_r^N)\right|)\varphi(x_r',x_r^N)\ \left|x_r^N\right|\ dx_r^N\ dx_r'.$$

We observe that the function $f(x) = x_r^N \in L^\infty(\log L)^{-\frac{1}{2}}(\Delta_r \times]0,\beta[\,,\gamma)$. Indeed $\gamma_f(t) = 2\Phi(-t)$ and using (2.2) we have

$$\begin{aligned}\sup_{t\in(0,\gamma(\Delta_r\times]0,\beta[))}(1-\log t)^{-\frac{1}{2}} f^{\circledast}(t) &= \sup_{t\in(0,\gamma(\Delta_r\times]0,\beta[))}(1-\log t)^{-\frac{1}{2}}\left(-\Phi^{-1}\left(\frac{t}{2}\right)\right) \\ &\le c \sup_{t\in(0,\gamma(\Delta_r\times]0,\beta[))}(1-\log t)^{-\frac{1}{2}}(2\log\frac{2}{t})^{\frac{1}{2}} < +\infty.\end{aligned}$$

Then we obtain

$$A_3 \le c\,\|u\|^p_{L^p(\log L)^{\frac{1}{2p'}}(\Delta_r\times]0,\beta[,\gamma)}\left\|x_r^N\right\|_{L^\infty(\log L)^{-\frac{1}{2}}(\Delta_r\times]0,\beta[,\gamma)}\,. \tag{4.4}$$

Moreover using Hölder inequality, we obtain

$$A_1 \le c\left(\int_{\Delta_r\times]0,\beta[}\left|u(x_r',x_r^N)\right|^p \log^{\frac{p}{2}}(2+\left|u(x_r',x_r^N)\right|)\varphi(x_r',x_r^N)\ dx_r^N\ dx_r'\right)^{\frac{1}{p'}} \tag{4.5}$$
$$\times\left(\int_\Omega\left|\frac{\partial u}{\partial x_r^N}(x_r',x_r^N)\right|^p\varphi(x_r',x_r^N)\ dx_r^N\ dx_r'\right)^{\frac{1}{p}}$$

and

$$A_2 \le c\left(\int_{\Delta_r\times]0,\beta[}\left|u(x_r',x_r^N)\right|^p \log^{\frac{p-3}{2}p'}(2+\left|u(x_r',x_r^N)\right|)\varphi(x_r',x_r^N)\ dx_r^N\ dx_r'\right)^{\frac{1}{p'}}$$
$$\times\left(\int_\Omega\left|\frac{\partial u}{\partial x_r^N}(x_r',x_r^N)\right|^p\varphi(x_r',x_r^N)\ dx_r^N\ dx_r'\right)^{\frac{1}{p}}. \tag{4.6}$$

We observe that

$$\int_{\Delta_r\times]0,\beta[}\left|u(x_r',x_r^N)\right|^p \log^{\frac{p-3}{2}p'}(2+\left|u(x_r',x_r^N)\right|)\varphi(x_r',x_r^N)\ dx_r^N\ dx_r'$$
$$\le c\int_{\Delta_r\times]0,\beta[}\left|u(x_r',x_r^N)\right|^p \log^{\frac{p}{2}}(2+\left|u(x_r',x_r^N)\right|)\varphi(x_r',x_r^N)\ dx_r^N\ dx_r'$$

and

$$\left(\int_{\Delta_r\times]0,\beta[}\left|u(x_r',x_r^N)\right|^p \log^{\frac{p}{2}}(2+\left|u(x_r',x_r^N)\right|)\varphi(x_r',x_r^N)\ dx_r^N\ dx_r'\right) \le \tag{4.7}$$
$$= \int_0^{\gamma(\Delta_r\times]0,\beta[)}\left[u^{\circledast}(t)\log^{\frac{1}{2}}(2+u^{\circledast}(t))\right]^p dt$$
$$\le c\left(\int_0^{\gamma(\Delta_r\times]0,\beta[)}\left[(1-\log t)^{\frac{1}{2}}u^{\circledast}(t)\right]^p dt\right),$$

because $\log(2+u^{\circledast}(t))$ is dominated by a multiple of $(1-\log t)$. Indeed $L^p(\log L)^{\frac{1}{2}}\subset L^p\subset L^{p,\infty}$, then $u^{\circledast}(t)\le ct^{-\frac{1}{p}}$ for some positive constant.

Putting (4.4)-(4.7) in (4.3) and using Proposition 3.1 we have

$$\int_{\Delta_r}|u(x_r',0)|^p\log^{\frac{p-1}{2}}(2+|u(x_r',0)|)\varphi(x_r',0)dx_r'$$
$$\le c\|u\|^{p-1}_{L^p(\log L)^{\frac{1}{2}}(\Delta_r\times]0,\beta[,\gamma)}\|\nabla u\|_{L^p(\Delta_r\times]0,\beta[,\gamma)}$$
$$+c\,\|u\|^p_{L^p(\log L)^{\frac{1}{2p'}}(\Delta_r\times]0,\beta[,\gamma)}\left\|x_r^N\right\|_{L^\infty(\log L)^{-\frac{1}{2}}(\Delta_r\times]0,\beta[,\gamma)}$$
$$\le c\,\|u\|^{p-1}_{W^{1,p}(\Delta_r\times]0,\beta[,\gamma)}\|\nabla u\|_{L^p(\Delta_r\times]0,\beta[,\gamma)}+c\,\|u\|^p_{W^{1,p}(\Delta_r\times]0,\beta[,\gamma)}$$
$$\le c\,\|u\|^p_{W^{1,p}(\Delta_r\times]0,\beta[,\gamma)}.$$

□

5. Trace operator. In this section the “boundary values” or trace of functions in Sobolev spaces are studied.

If Ω is a domain satisfying condition *2.1*, given a smooth function $u \in C^\infty(\overline{\Omega}) \subset W^{1,p}(\Omega,\gamma)$ the restriction to the boundary $u|_{\partial\Omega}$ is well defined. This restriction operator can be extended from smooth functions to $W^{1,p}(\Omega,\gamma)$ giving a linear continuous operator from $W^{1,p}(\Omega,\gamma)$ to $L^p(\partial\Omega,\gamma)$, the space of the measurable functions defined almost everywhere on $\partial\Omega$ such that

$$\int_{\partial\Omega} |u|^p \,\varphi \, d\mathcal{H}^{N-1} < +\infty.$$

We stress that $L^p(\partial\Omega,\gamma)$ is a Banach space with respect to the norm $\|u\|_{L^p(\partial\Omega,\gamma)} = \left(\int_{\partial\Omega} |u|^p \,\varphi \, d\mathcal{H}^{N-1}\right)^{\frac{1}{p}}$.

Using the logarithmic Sobolev inequalities (4.1), there exists a constant $C > 0$ such that for every $u \in C^\infty(\overline{\Omega})$

$$\|u\|_{L^p(\partial\Omega,\gamma)} \le C \, \|u\|^p_{W^{1,p}(\Omega,\gamma)} \,. \tag{5.1}$$

It follows that the operator

$$\begin{aligned} &T : C^\infty(\overline{\Omega}) \to L^p(\partial\Omega,\gamma) \\ &u \to Tu = u/\partial\Omega \end{aligned}$$

is linear and continuous from $\left(C^\infty(\overline{\Omega}), \|\|_{W^{1,p}(\Omega,\gamma)}\right)$ into $\left(L^p(\partial\Omega,\gamma), \|\|_{L^p(\partial\Omega,\gamma)}\right)$.

By the Hahn-Banach theorem and the density of $C^\infty(\overline{\Omega})$ in $W^{1,p}(\Omega,\gamma)$ the operator can be extended to $W^{1,p}(\Omega,\gamma)$. This linear continuous operator from $W^{1,p}(\Omega,\gamma)$ to $L^p(\partial\Omega,\gamma)$ is called trace operator of u on $\partial\Omega$. Then there exists a constant $C > 0$ such that

$$\|Tu\|_{L^p(\partial\Omega,\gamma)} \le C \, \|u\|_{W^{1,p}(\Omega,\gamma)} \quad \text{for every } u \in W^{1,p}(\Omega,\gamma), \tag{5.2}$$

that implies that $W^{1,p}(\Omega,\gamma)$ is continuous imbedded in $L^p(\partial\Omega,\gamma)$.

We remark that the trace operator can be defined also from $W^{1,p}(\Omega,\gamma)$ to $L^p(\log L)^{\frac{1}{2p'}}(\partial\Omega,\gamma)$, that is the space of the measurable function u such that $\displaystyle\int_{\partial\Omega} |u|^p \log^{\frac{p-1}{2}}(2+|u|)\varphi \, d\mathcal{H}^{N-1} < +\infty$.

Moreover the trace operator is compact for $1 \le p < +\infty$. Indeed let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $W^{1,p}(\Omega,\gamma)$, we will prove the existence of a Cauchy subsequence in $L^p(\partial\Omega,\gamma)$. By Proposition 3.1, there exists a Cauchy subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, in $L^p\,(\log L)^{\frac{1}{2p'}}\,(\Omega,\gamma)$.. Moreover arguing as in the proof of the inequality (4.1) we have

$$\begin{aligned} \|Tu_n - Tu_m\|^p_{L^p(\partial\Omega,\gamma)} &\le \int_{\partial\Omega} |Tu_n - Tu_m|^p \log^{\frac{p-1}{2}}(2+|Tu_n - Tu_m|)\varphi \, d\mathcal{H}^{N-1} \\ &\le c \, \|u_n - u_m\|^{p-1}_{L^p(\log L)^{\frac{1}{2}}(\Omega,\gamma)} \|\nabla\,(u_n - u_m)\|_{L^p(\Omega,\gamma)} \\ &\quad + c\,\|u_n - u_m\|^p_{L^p(\log L)^{\frac{1}{2p'}}(\Omega,\gamma)} \|x_N\|_{L^\infty(\log L)^{\frac{1}{2}}(\Omega,\gamma)}\,, \end{aligned}$$

then $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\partial\Omega,\gamma)$ too.

The norm of the trace operator is given by

$$\inf_{u\in W^{1,p}(\Omega,\gamma)-W_0^{1,p}(\Omega,\gamma)} \frac{\|u\|^p_{W^{1,p}(\Omega,\gamma)}}{\|Tu\|^p_{L^p(\partial\Omega,\gamma)}} \tag{5.3}$$

and this value is the best constant in the trace inequality (5.2). The trace operator is compact, therefore an easy compactness arguments prove that there exist extremals in (5.3). These extremals turn out to be the weak solution of

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u\varphi) = |u|^{p-2}u\varphi & \textit{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2} u & \textit{on } \partial\Omega, \end{cases} \tag{5.4}$$

where λ is the first nontrivial eigenvalue.
When $p = 2$ and Ω is a connected domain satisfying condition 2.1, using classical tools, compactness of the trace operator from $W^{1,2}(\Omega,\gamma)$ to $L^2(\partial\Omega,\gamma)$ and (3.7) it follows that there exists an increasing sequence of eigenvalues of the problem (5.4) which tends to infinity and a Hilbertian basis of eigenfunctions in $L^2(\Omega,\gamma)$.

Moreover the continuity of the trace operator from $W^{1,2}(\Omega,\gamma)$ to $L^2(\partial\Omega,\gamma)$ and (3.7) allow us to investigate about the existence of a weak solution of the following semicoercive nonhomogeneous Neumann problem

$$\begin{cases} -(u_{x_i}\varphi)_{x_i} = f\varphi & \textit{in } \Omega \\ \frac{\partial u}{\partial \nu} = g & \textit{on } \partial\Omega, \end{cases}$$

where Ω is a connected domain satisfying condition 2.1, $f \in L^2(\log L)^{\frac{1}{2}}(\Omega,\gamma)$ and $g \in L^2(\partial\Omega,\gamma)$. Indeed using classical tools (see e.g. [6] Theorem 6.2.5) we obtain that there exists a weak solution in $W^{1,2}(\Omega,\gamma)$ if and only if $\int_\Omega f d\gamma + \int_{\partial\Omega} g\varphi\, dH^{N-1} = 0$. In particular there exists a unique weak solution in $X = \left\{u \in W^{1,2}(\Omega,\gamma) : \int_\Omega v d\gamma = 0\right\}$ by Lax-Milgram theorem.

6. Poincaré trace inequality. Arguing as in Proposition 3.2 (see Remark 3.6 too), we prove the following Poincaré type inequality.

PROPOSITION 6.1. *Let Ω be a connected domain satisfying condition 2.1 and $1 \le p < +\infty$. Then there exists a positive constant C, depending only on p and Ω, such that*

$$\|u\|_{L^p(\Omega,\gamma)} \le C\, \|\nabla u\|_{L^p(\Omega,\gamma)} \tag{6.1}$$

for any $u \in X = \left\{ u \in W^{1,p}(\Omega,\gamma) : \int_{\partial\Omega} u\varphi\, d\mathcal{H}^{N-1} = 0 \right\}$.

Using (5.1) and (6.1) we obtain

COROLLARY 6.1. *Let Ω be a connected domain satisfying condition 2.1 and $1 \le p < +\infty$. Then there exists a positive constant C, depending only on p and Ω, such that*

$$\|Tu\|_{L^p(\partial\Omega,\gamma)} \le C\, \|\nabla u\|_{L^p(\Omega,\gamma)} \tag{6.2}$$

for any $u \in X$.

We remark that the previous results can be obtained also involving the trace operator from $W^{1,p}(\Omega,\gamma)$ into $L^p(\log L)^{\frac{1}{2p'}}(\partial\Omega,\gamma)$.

REMARK 6.1. (*Application to PDE*) Let consider the eigenvalue problem

$$\begin{cases} -\left(u_{x_i}\varphi\right)_{x_i} = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega, \end{cases} \tag{6.3}$$

where Ω is a connected domain satisfying condition 2.1. Arguing in a classical way using inequality (6.2) and the compactness of the trace operator, it is easy to prove that there exists an increasing sequence of eigenvalues of the problem (6.3) which tends to infinity. Moreover for $\lambda_1 = 0$ the corresponding eigenvalue function $u_1 = const \neq 0$ and the first nontrivial eigenvalue λ_2 has the following characterization

$$\lambda_2 = \min\left\{ \frac{\|\nabla u\|_{L^2(\Omega,\gamma)}}{\|Tu\|_{L^2(\partial\Omega,\gamma)}}, u \in W^{1,2}(\Omega,\gamma) : \int_{\partial\Omega} u\varphi \; d\mathcal{H}^{N-1} = 0 \right\}.$$

REFERENCES

[1] R. A. ADAMS, *Sobolev spaces,* Pure and Applied Mathematics 65. Academic Press, New York-London, (1975).

[2] R. A. ADAMS, *General logarithmic Sobolev inequalities and Orlicz imbeddings*, J. Funct. Anal., 34:2 (1979), pp. 292–303.

[3] A. ALVINO, R. VOLPICELLI, AND A. FERONE, *Sharp Hardy inequalities in the half space with trace remainder term*, Nonlinear Anal., 75:14 (2012), pp. 5466–5472.

[4] B. BRANDOLINI, F. CHIACCHIO, AND C. TROMBETTI, *A sharp lower bound for some Neumann eigenvalues of the Hermite operator*, Differential Integral Equations, to appear.

[5] B. BRANDOLINI, F. CHIACCHIO, A. HENROT, AND C. TROMBETTI, *An optimal Poincaré-Wirtinger inequality in Gauss space*, arXiv:1209.6469.

[6] H. ATTOUCH,G. BUTTAZZO, AND G. MICHAILLE, *Variational analysis in Sobolev and BV spaces*, Applications to PDEs and optimization. MPS/SIAM Series on Optimization 6 SIAM (2006).

[7] C. BENNETT AND K. RUDNICK, *On Lorentz-Zygmund spaces,* Dissertationes Math. (Rozprawy Mat.), 175 (1980).

[8] O. V. BESOV, *On traces on a nonsmooth surface of classes of differentiable functions*, Trudy Mat. Inst. Steklov, 117 (1972), pp. 11–24.

[9] A. CIANCHI AND L. PICK, *Optimal Gaussian Sobolev embeddings*, J. Funct. Anal., 256:11 (2009), pp. 3588–3642.

[10] A. CIANCHI, R. KERMAN, AND L. PICK, *Boundary trace inequalities and rearrangements*, J. Anal. Math., 105 (2008), pp. 241–265.

[11] G. DI BLASIO, F. FEO, AND M. R. POSTERARO, *Regularity results for degenerate elliptic equations related to Gauss measure,* Mathematical Inequalities and Applications, 10:4 (2007), pp. 771–797.

[12] G. DI BLASIO AND F. FEO, *A class of nonlinear degenerate elliptic equations related to the Gauss measure*, J. Math. Anal. Appl., 386:2 (2012), pp. 763–779 .

[13] E. GAGLIARDO, *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni di n variabili*, Rend. Sem. Mat. Univ. Padova, 27 (1957), pp. 284–305.

[14] A. EHRHARD, *Symmétrisation dans l'space de Gauss*, Math. Scand., 53 (1983), pp. 281–301.

[15] A. EHRHARD, *Sur l'inégalité de Sobolev logarithmique de Gross*, (French) Seminar on probability, XVIII, 194–196, Lecture Notes in Math., 1059, Springer (1984).

[16] G. F. FEISSNER, *Hypercontractive semigroups and Sobolev's inequalityhypercontractive semigroup and sobolev's inequality,* Trans. Amer. Math. Soc., 210 (1975), pp. 51–62.

[17] F. FEO, *A comparison result for elliptic equations in the infinite dimensional Gauss space*, Nonlinear Analysis, 73 (2010), pp. 2298–2309.

[18] L. GROSS, *Logarithmic Sobolev inequalities,* Amer. J. Math., 97 (1975), pp. 1061–1083.

[19] A. Kufner, O. John, and S. Fučík, *Function spaces*, Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis, Prague (1977).
[20] M. Ledoux, *Isopérimétrie et inégalitées de Sobolev logarithmiques gaussiennes*, C. R. Acad. Sci. Paris Ser. I Math., 306 (1988), pp. 79–92.
[21] M. Ledoux, *Isoperimetry and Gaussian analysis*, Ecole d' Eté de Probabilités de Saint-Flour - Lecture Notes in Math. Springer, Berlin (1994).
[22] J.-L. Lions and E. Mangenes, *Problèmes aux limites non homogènes et applications III*, Dunod, Paris (1970).
[23] J. Martín and M. Milman, *Isoperimetry and symmetrization for logarithmic Sobolev inequalities*, J. Funct. Anal., 256:1 (2009), pp. 149–178.
[24] Y. J. Park, *Logarithmic Sobolev trace inequality*, Proc. Amer. Math. Soc., 132:7 (2004), pp. 2075–2083.