

SIMULTANEOUS NORMALIZATION AND ALGEBRA HUSKS*

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Abstract. The aim of this note is to use the concept of algebra husks to prove an analog of the flattening decomposition theorem for simultaneous normalizations. Other applications improve earlier flatness criteria.

Key words. Simultaneous normalization, husk, flatness, flattening decomposition.

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Let $X \rightarrow S$ be a morphism with fibers $\{X_s : s \in S\}$. We say that $X \rightarrow S$ has a simultaneous normalization if the normalizations of the fibers $\{\bar{X}_s : s \in S\}$ fit together to form a flat family over S ; see (11) for a precise definition.

The most famous result of this type, due to Hironaka [Hir58], says that if S is regular, the fibers X_s are generically reduced and the reductions of the fibers $\text{red } X_s$ are normal, then $\text{red } X \rightarrow S$ is flat with normal fibers. Several related results are proved in [Kol95].

For projective morphisms, a global condition for the existence of the simultaneous normalization was developed by Chiang-Hsieh and Lipman [CHL06]. They consider the case when S is normal with perfect residue fields and $X \rightarrow S$ has reduced fibers, and prove that a simultaneous normalization exists iff the Hilbert polynomials of the normalizations of the fibers are all the same. We give a proof in (12).

When this condition fails, a simultaneous normalization exists for some subfamilies but not for others. Our main result is an analog of the Flattening decomposition theorem of [Mum66, Lecture 8], giving a precise description of those subfamilies that have a simultaneous normalization:

THEOREM 1. *Let $f : X \rightarrow S$ be a proper morphism whose fibers X_s are generically geometrically reduced.*

Then there is a morphism $\pi : S^n \rightarrow S$ such that for any $g : T \rightarrow S$, the fiber product $X \times_S T \rightarrow T$ has a simultaneous normalization (11) iff g factors through $\pi : S^n \rightarrow S$.

More precisely, we show that $\pi : S^n \rightarrow S$ represents the functor of simultaneous normalizations (11). We discuss in (17) why the analogous result fails if the fibers are not generically reduced. The main result (15) establishes a similar theorem for various partial normalizations.

The key technical step of the proof is to consider not just the normalization of a scheme Y , but all *algebra husks* (2) of \mathcal{O}_Y ; equivalently, all finite birational maps $Y' \rightarrow Y$ as well. More generally, algebra husks make sense for any coherent sheaf of \mathcal{O}_Y -algebras and they lead to a well behaved moduli functor (5). For arbitrary coherent sheaves this was considered in [Kol08a]. It turns out to be easy to derive the variant for \mathcal{O}_Y -algebras from the results in [Kol08a].

In the last section we also settle two of the flatness conjectures from [Kol95] for proper morphisms.

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1. Algebra husks. We start by reviewing the notion of *husks* and their main properties [Kol08a].

DEFINITION 2. Let X be a scheme over a field k and F a quasi coherent sheaf on X . Set $n := \dim \text{Supp } F$. A *husk* of F is a quasi coherent sheaf G together with a homomorphism $q : F \rightarrow G$ such that G has no associated primes of dimension $< n$, and q is an isomorphism at all points of X of dimension $= n$.

Such a G is also an $\mathcal{O}_X/\text{Ann}(F)$ sheaf and so the particular choice of X matters very little.

If, in addition, F is a quasi coherent sheaf of \mathcal{O}_X -algebras, then a husk $q : F \rightarrow G$ is called an *algebra husk* if G is a quasi coherent sheaf of \mathcal{O}_X -algebras and q is an algebra homomorphism.

Assume that X is pure dimensional and generically reduced. Then every coherent algebra husk of \mathcal{O}_X is contained in the structure sheaf $\mathcal{O}_{\bar{X}}$ of the normalization. If X is of finite type over a field k (more generally, if the local rings of X are *Nagata* [Mat86, Sec.31] or *universally Japanese* [Gro67, IV.7.7.2]) then $\mathcal{O}_{\bar{X}}$ is coherent as an \mathcal{O}_X -module. Thus, in these cases, $\mathcal{O}_{\bar{X}}$ is the universal coherent algebra husk of \mathcal{O}_X .

If X is not generically reduced, then there is no universal coherent algebra husk. For instance, for each $m \geq 0$, $k[x, \epsilon] \rightarrow k[x, x^{-m}\epsilon]$ is a coherent algebra husk.

DEFINITION 3. Let $f : X \rightarrow S$ be a morphism and F a quasi coherent sheaf. Let n be the relative dimension of $\text{Supp } F \rightarrow S$. A *husk* of F over S is a quasi coherent sheaf G together with a homomorphism $q : F \rightarrow G$ such that

1. G is flat over S and
2. $q_s : F_s \rightarrow G_s$ is a husk for every $s \in S$.

If, in addition, F is a quasi coherent sheaf of \mathcal{O}_X -algebras, then a husk $q : F \rightarrow G$ is called an *algebra husk over S* if G is a quasi coherent sheaf of \mathcal{O}_X -algebras and q is an algebra homomorphism.

Note that any multiplication map $m_G : G \otimes G \rightarrow G$ extending $m_F : F \otimes F \rightarrow F$ gives an algebra structure on G . That is, m_G is automatically associative (and commutative if m_F is). For example, associativity is equivalent to the vanishing of the difference map

$$m_G(m_G, \text{Id}_G) - m_G(\text{Id}_G, m_G) : G \otimes G \otimes G \rightarrow G.$$

Since the target G has no embedded points and the map vanishes on a dense open set (since m_F is associative), the map is identically zero.

Note that husks and algebra husks are preserved by base change.

DEFINITION 4. Let $f : X \rightarrow S$ be a morphism and F a coherent sheaf on X . Let $\text{Husk}(F)(*)$ be the functor that to a scheme $g : T \rightarrow S$ associates the set of all coherent husks of g_X^*F over T with proper support over T , where $g_X : T \times_S X \rightarrow X$ is the projection.

Assume that H is an f -ample divisor and $p(t)$ a polynomial. Let $\text{Husk}_p(F)(*)$ be the functor that to a scheme $g : T \rightarrow S$ associates the set of all coherent husks of g_X^*F over T with Hilbert polynomial $p(t)$.

If, in addition, F is a coherent sheaf of \mathcal{O}_X -algebras, then $\text{Husk}^{\text{alg}}(F)(*)$ (resp. $\text{Husk}_p^{\text{alg}}(F)(*)$) denotes the functor that to a scheme $g : T \rightarrow S$ associates the set of all coherent algebra husks of g_X^*F (resp. coherent algebra husks over T with Hilbert polynomial $p(t)$).

[Kol08a, Cor.12] shows that $Husk_p(F)$ has a fine moduli space $Husk_p(F)$ which is an algebraic space over S . Our basic existence theorem asserts that similar results hold for algebra husks.

THEOREM 5. *Let $f : X \rightarrow S$ be a projective morphism, H an f -ample divisor, $p(t)$ a polynomial and F a coherent sheaf of \mathcal{O}_X -algebras. Then*

1. $Husk_p^{alg}(F)$ has a fine moduli space $Husk_p^{alg}(F)$ which is an algebraic space of finite type over S .
2. The forgetful map $\sigma : Husk_p^{alg}(F) \rightarrow Husk_p(F)$ is a closed embedding.
3. If F is flat at the generic points of $X_s \cap \text{Supp } F$ for every fiber X_s , then $Husk_p^{alg}(F)$ is proper over S .

Proof. For any $g : T \rightarrow S$, forgetting the algebra structure gives a map

$$\sigma_g : Husk_p^{alg}(F)(T) \rightarrow Husk_p(F)(T).$$

By the first part of (6), σ_g is injective, that is, a husk admits at most one structure as a sheaf of \mathcal{O}_X -algebras such that $F \rightarrow G$ is an algebra homomorphism. Next apply the second part of (6) to $S := Husk_p(F)(T)$ to obtain that $Husk_p^{alg}(F)(T) := S^{alg}$ exists and is a closed subscheme of $Husk_p(F)$, proving (2).

By [Kol08a, Cor.12] this in turn implies (1) and (3) since $Husk_p(F)$ is proper over S if F is flat at the generic points of $X_s \cap \text{Supp } F$ for every $s \in S$. \square

LEMMA 6. *Let $f : X \rightarrow S$ be a proper morphism, F a coherent sheaf of \mathcal{O}_X -algebras and $q : F \rightarrow G$ a coherent husk. Then*

1. G admits at most one structure as an \mathcal{O}_X -algebra husk over S of F .
2. There is a closed subscheme $S^{alg} \subset S$ such that for a morphism $\pi : S' \rightarrow S$, $\pi_X^* G$ is an $\mathcal{O}_{X'}$ -algebra husk of $\pi_X^* F$ over S' iff π factors through S^{alg} , where $\pi_X : S' \times_S X \rightarrow X$ is the projection.

Proof. We may assume that S, S' are affine. The first claim is also local on X ; let $U \subset X$ be affine.

The algebra structures are given by the multiplication maps $m_F : F \otimes F \rightarrow F$ and $m_G : G \otimes G \rightarrow G$. There is an $h \in H^0(U, \mathcal{O}_U)$ which is not a zero divisor on G_s for every $s \in S$ such that $h \cdot G|_U \subset F|_U$. For any sections $g_1, g_2 \in H^0(U, G|_U)$, $h^2 \cdot m_G(g_1 \otimes g_2) = m_F(hg_1 \otimes hg_2)$. Since multiplication by h^2 is injective, the above equality determines $m_G(g_1 \otimes g_2)$ uniquely, proving the first claim.

Next we prove (2) in the projective case. For $r \gg 1$, there is an $h \in H^0(X, \mathcal{O}_X(r))$ that is not a zero divisor on G_s for every $s \in S$ such that $h \cdot G \subset F(r)$. By the above considerations, the multiplication map m_F always extends to a multiplication map $m_G^* : G \otimes G \rightarrow G(2r)$ and G does have an \mathcal{O}_X -algebra structure iff m_G^* actually maps $G \otimes G$ to $G \cong h^2 \cdot G \subset G(2r)$.

Note that since G is flat over S and h^2 is not a zero divisor on G_s for every $s \in S$, $G(2r)/G$ is also flat over S , cf. [Mat86, Thm.22.5]. Hence we can choose $m > 0$ such that $G(m)$ is generated by global sections and $f_*((G(2r)/G)(2m))$ is locally free and commutes with base change.

Pick generating sections $g_i \in H^0(X, G(m))$ and consider the composite

$$g_i \otimes g_j \rightarrow m_G^*(g_i \otimes g_j) \rightarrow (G(2r)/G)(2m).$$

By pushing forward, we obtain global sections

$$\sigma_{ij} \in H^0\left(S, f_*\left((G(2r)/G)(2m)\right)\right)$$

such that for a morphism $\pi : S' \rightarrow S$, π_X^*G is an $\mathcal{O}_{X'}$ -algebra husk of π_X^*F iff the $\pi^*\sigma_{ij}$ are all zero. Thus $S^{\text{alg}} := (\sigma_{ij} = 0 \forall i, j) \subset S$ is the required subscheme.

In the proper but non-projective case, we first prove that $\text{Husk}^{\text{alg}}(F)$ exists and is of finite type over S . To see this, we use the existence of the Hom-schemes of sheaves (7).

First consider $p : \underline{\text{Hom}}(G \otimes G, G) \rightarrow S$ with universal homomorphism $u_p : p^*(G \otimes G) \rightarrow p^*G$. We can compose it with $F \rightarrow G$ to obtain

$$u'_p : p^*(F \otimes F) \rightarrow p^*G.$$

Pulling back the multiplication map m_F and the husk map $F \rightarrow G$ gives another homomorphism

$$p^*m_F : p^*(F \otimes F) \rightarrow p^*F \rightarrow p^*G.$$

We can view both of these maps as sections

$$\underline{\text{Hom}}(G \otimes G, G) \rightrightarrows \underline{\text{Hom}}(p^*(F \otimes F), p^*G).$$

Let $W \subset \underline{\text{Hom}}(G \otimes G, G)$ be the subscheme where these two maps agree. Thus W parametrizes those husks for which the multiplication map $F \otimes F \rightarrow F$ extends to a multiplication map $G \otimes G \rightarrow G$. As noted in (3), these are the algebra husks of F .

Next we prove the valuative criterion of properness for $p : W \rightarrow S$. Let T be the spectrum of a DVR with closed point $0 \in T$ and generic point $t \in T$. Given $g : T \rightarrow S$ we have a husk $q_T : F_T \rightarrow G_T$ which is an algebra husk over t . Set $Z := \text{Supp coker } q_0$. Then $F_0/\text{tors } F_0 \rightarrow G_0$ is an isomorphism over $X_0 \setminus Z$, hence G_T is an algebra husk of F_T over $X \setminus Z$.

Note that G_0 is S_1 and so G_T is S_2 over its support and Z has codimension ≥ 2 in $\text{Supp } G_T$. In particular, every local section of G_T over $X \setminus Z$ extends uniquely to a local section over X by [Gro68, III.3.5]. Therefore, the multiplication map

$$m_{X \setminus Z} : (G_T \otimes G_T)|_{X \setminus Z} \rightarrow G_T|_{X \setminus Z}$$

extends uniquely to a multiplication map $m_X : G_T \otimes G_T \rightarrow G_T$. Thus $W \rightarrow S$ satisfies the valuative criterion of properness.

We have proved that $p : W \rightarrow S$ is a monomorphism of finite type that satisfies the valuative criterion of properness. Thus $p : W \rightarrow S$ is a closed embedding and S^{alg} is its image. \square

7. (see [Gro67, III.7.7.8–9], [LMB00, 4.6.2.1], [Lie06, 2.1.3] or [Kol08a, 33])
 Let $f : X \rightarrow S$ be proper. Let M, N be coherent sheaves on X such that N is flat over S . Then there is a separated S -scheme of finite type $\underline{\text{Hom}}(M, N)$ parametrizing homomorphisms from M to N . That is, for any $g : T \rightarrow S$, there is a natural isomorphism

$$\text{Hom}_T(g_X^*M, g_X^*N) \cong \text{Mor}_S(T, \underline{\text{Hom}}(M, N)),$$

where $g_X : T \times_S X \rightarrow X$ is the fiber product of g with the identity of X .

EXAMPLE 8. This example shows that $\text{Husk}^{\text{alg}}(F)$ is not always a union of connected components of $\text{Husk}(F)$.

Consider the family of plane curves $X := (y^2z - x(x^2 - tz^2)) \subset \mathbb{P}^2 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ over a field k . Let $F := \mathcal{O}_X$ and let G be the subsheaf of rational functions generated by

\mathcal{O}_X and y/x . Since $x \cdot (y/x) = y$ and $y \cdot (y/x) = \frac{1}{z}(x^2 - tz^2)$, we see that $G/F \cong k[t]$ and so G is a husk of F .

Over the central fiber, G_0 is the coordinate ring of the normalization of X_0 , hence G_0 is an algebra husk of F_0 .

For $t \neq 0$, X_t is a smooth elliptic curve and $G_t = \mathcal{O}_{X_t}(P_t)$ where P_t denotes the origin. The multiplication map gives a surjection

$$G_t \otimes G_t \twoheadrightarrow \mathcal{O}_{X_t}(2P_t) \supsetneq \mathcal{O}_{X_t}(P_t) = G_t.$$

Hence there is no algebra structure on G_t extending the algebra \mathcal{O}_{X_t} .

In the proper but non-projective case, we get the following using [Kol08a, Thm.39].

THEOREM 9. *Let $f : X \rightarrow S$ be a proper morphism and F a coherent sheaf of \mathcal{O}_X -algebras. Then $\text{Husk}^{\text{alg}}(F)$ has a fine moduli space $\text{Husk}^{\text{alg}}(F)$ and the forgetful map $\text{Husk}^{\text{alg}}(F) \rightarrow \text{Husk}(F)$ is a closed embedding. \square*

REMARK 10. As in [Kol08a, Defn.9], one can define the the functor $QHusk^{\text{alg}}(F)$ of algebra husks of quotients of F . It also has a fine moduli space $QHusk^{\text{alg}}(F)$ and the forgetful map $QHusk^{\text{alg}}(F) \rightarrow QHusk(F)$ is a closed embedding.

2. Simultaneous normalization.

DEFINITION 11. Let $f : X \rightarrow S$ be a morphism. A *simultaneous normalization* of f is a morphism $n : \bar{X} \rightarrow X$ such that

1. n is finite and an isomorphism at the generic points of the fibers of f , and
2. $\bar{f} := f \circ n : \bar{X} \rightarrow S$ is flat with geometrically normal fibers.

In characteristic 0, and over perfect fields, normal and geometrically normal are the same, but over imperfect fields there are varieties which are normal but not geometrically normal.

Note that, in general, a simultaneous normalization need not be unique (18).

The functor of simultaneous normalizations associates to a scheme $T \rightarrow S$ the set of simultaneous normalizations of $X \times_S T \rightarrow T$.

We start with a proof of the existence criterion [CHL06, Thm.4.2]. The present form is somewhat more general since we allow a semi-normal base and nonreduced fibers as well. (For the definition of semi-normal, see [Kol96, I.7.2].)

THEOREM 12. *Let S be semi-normal with perfect residue fields at closed points. Let $f : X \rightarrow S$ be a projective morphism of pure relative dimension n with generically reduced fibers. The following are equivalent:*

1. X has a simultaneous normalization $n : \bar{X} \rightarrow X$.
2. The Hilbert polynomial of the normalization of the fibers $\chi(\bar{X}_s, \mathcal{O}(tH))$ is locally constant on S .

Proof. The implication (1) \Rightarrow (2) is clear. To see the converse, we may assume that S is connected. Then $\chi(\bar{X}_s, \mathcal{O}(tH))$ is constant; call it $p(t)$. Set $S' := \text{red}(\text{Husk}_p^{\text{alg}}(\mathcal{O}_X))$ with universal family $X' \rightarrow S'$. The structure sheaf $\mathcal{O}_{X'}$ is flat over S' at the generic point of every fiber; see [Gro71, II.2.3] and [Kol95, Thm.8] for the normal case and [Kol96, I.6.5] for the semi-normal case. Thus, by (5.3), $\pi : S' \rightarrow S$ is proper.

Let $s \in S$ be a closed point and $\mathcal{O}_{X_s} \rightarrow F$ any algebra husk with Hilbert polynomial $p(t)$. Since every coherent algebra husk of \mathcal{O}_{X_s} is contained in the structure sheaf of the normalization,

$$p(t) = \chi(X_s, F(tH)) \leq \chi(\bar{X}_s, \mathcal{O}_{\bar{X}_s}(tH)) = p(t).$$

Thus $F = \mathcal{O}_{\bar{X}_s}$ and since the residue field $k(s)$ is perfect, this holds for any field extension of k . Therefore π is one-to-one and surjective on closed geometric points. Furthermore, the closed fibers of $X' \rightarrow S'$ are geometrically normal, hence every fiber is geometrically normal.

Let now $g \in S$ be a generic point. By assumption, $\chi(\bar{X}_g, \mathcal{O}_{\bar{X}_g}(tH)) = p(t)$. Thus $\mathcal{O}_{X_g} \rightarrow \mathcal{O}_{\bar{X}_g}$ is an algebra husk with Hilbert polynomial $p(t)$ and so the injection $g \hookrightarrow S$ lifts to $g \hookrightarrow S'$. Since S is seminormal, these imply that $S' \rightarrow S$ is an isomorphism and $X' \rightarrow S' = S$ gives the simultaneous normalization. \square

EXAMPLE 13. The analog of (12) fails for semi-normalization, even for curves.

As a simple example, start with a flat family of curves $Y \rightarrow C$ whose general fiber is smooth elliptic and Y_c is a cuspidal rational curve for some $c \in C$. Pick 2 smooth points in Y_c and identify them to obtain $X \rightarrow C$. The general fiber is still smooth elliptic but X_c has a cusp and a node plus an embedded point at the node.

The semi-normalization of X_c is a nodal rational curve, yet the semi-normalizations do not form a flat family.

Next we state and prove our main result in a general form.

DEFINITION 14 (Partial normalizations). Let P be a property of schemes or algebraic spaces satisfying the following conditions.

1. P is local, that is, X satisfies P iff an open cover satisfies P .
2. P commutes with smooth morphisms, that is, if $X \rightarrow Y$ is smooth and Y satisfies P then so does X .
3. If the maximal dimensional generic points of X satisfy P then there is a unique smallest algebra husk $\mathcal{O}_X \rightarrow (\mathcal{O}_X)^P$ such that $X^P := \text{Spec}_X(\mathcal{O}_X)^P$ satisfies P . In this case $X^P \rightarrow X$ is called the P -normalization of X . If $X^P = X$ then we say that X is P -normal.
4. P -normalization is open. That is, given $X' \rightarrow X \rightarrow S$ such that the composite $X' \rightarrow S$ is flat, the set of points $x \in X$ such that $X'_s \rightarrow X_s$ is the P -normalization near x is open in X .

Examples of such properties are:

5. S_1 , with $\mathcal{O}_X \rightarrow \mathcal{O}_X/(\text{torsion subsheaf})$ as the P -normalization.
6. S_2 , with $\mathcal{O}_X \rightarrow j_*(\mathcal{O}_{X \setminus Z}/(\text{torsion subsheaf}))$ as the P -normalization (or S_2 -hull) where $Z \subset X$ is a subscheme of codimension ≥ 2 such that $\mathcal{O}_{X \setminus Z}/(\text{torsion subsheaf})$ is S_2 and $j : X \setminus Z \rightarrow X$ is the inclusion.
7. Normal, with the normalization.
8. Semi-normal, with the semi-normalization (cf. [Kol96, Sec.I.7.2]).
9. S_2 and semi-normal, with the S_2 -hull of the semi-normalization.

Note that in cases (5–6), generic points are always P -normal. In cases (7–9) a generic point is P -normal iff it is reduced.

Let $f : X \rightarrow S$ be a morphism. As in (11), a *simultaneous P -normalization* of f is a morphism $n^P : (X/S)^P \rightarrow X$ such that n^P is finite and an isomorphism at the generic points of the fibers of f , the composite $f^P := f \circ n^P : (X/S)^P \rightarrow S$ is flat and for every geometric point $s \rightarrow S$ the induced map $(X/S)_s^P \rightarrow X_s$ is the P -normalization.

As before, the functor of simultaneous P -normalizations associates to a scheme $T \rightarrow S$ the set of simultaneous P -normalizations of $X \times_S T \rightarrow T$.

Our main technical theorem is the following.

THEOREM 15. *Let $f : X \rightarrow S$ be a proper morphism. Let P be a property satisfying (14.1–4) and assume that the fibers X_s are generically geometrically P -normal.*

Then there is a morphism $\pi^P : S^P \rightarrow S$ that represents the functor of simultaneous P -normalizations.

In particular, for any $g : T \rightarrow S$, the fiber product $X \times_S T \rightarrow T$ has a simultaneous P -normalization iff g factors through $\pi^P : S^P \rightarrow S$.

Furthermore, $\pi : S^P \rightarrow S$ is one-to-one and onto on geometric points.

Note that in cases (14.5–6), f can be an arbitrary proper morphism. In cases (14.7–9), we assume that the fibers are generically geometrically reduced. (The necessity of this condition is discussed in (17).) However, f need not be flat nor equidimensional.

Proof. By (9), there is an algebraic space $\text{Husk}^{\text{alg}}(\mathcal{O}_X)$ parametrizing all algebra husks of \mathcal{O}_X . Being a P -normalization is an open condition, hence there is an open subspace

$$S^P := \text{Husk}^{P\text{-n-}\text{alg}}(\mathcal{O}_X) \subset \text{Husk}^{\text{alg}}(\mathcal{O}_X)$$

parametrizing geometric P -normalizations of \mathcal{O}_X .

If Y is an algebraic space over an algebraically closed field then its P -normalization is unique and is geometrically P -normal. This implies that π is one-to-one and onto on geometric points. \square

REMARK 16. In characteristic 0, this implies that for every $s' \in S^P$, $k(s') = k(\pi(s'))$, but in positive characteristic $k(s') \supset k(\pi(s'))$ could be a purely inseparable extension, even for the classical case of normalization. For instance, if k is a function field $K(t)$ of characteristic 3, $S = \text{Spec } K(t)$ and X the plane cubic ($y^2z + x^3 - tz^3 = 0$) then X is regular but not geometrically normal. Over $K(\sqrt[3]{t})$ it becomes singular and its normalization is \mathbb{P}^1 . Thus $S^n = \text{Spec}_{K(t)} K(\sqrt[3]{t})$. (If P =normal, we use S^n to denote S^P .)

As shown by (18), π need not be a locally closed embedding, not even in characteristic 0.

Let Y be a generically geometrically reduced scheme over a field k of positive characteristic. In this case (15) implies that there is a unique purely inseparable extension $k' \supset k$ such that for any extension $L \supset k$, the normalization of X_L is geometrically normal iff $L \supset k'$.

REMARK 17. While the normalization of a nonreduced scheme is well defined, it does not seem possible to define simultaneous normalization for families with generically nonreduced fibers over a nonreduced base.

As a simple example, let $\pi : \mathbb{A}^2 \rightarrow \mathbb{A}^1$ be the projection to the x -axis. Set $X := (y^2 - x^2 = 0)$. Then $\pi : X \rightarrow \mathbb{A}^1$ has a nonreduced fiber over the origin. The simultaneous normalization exists over $\mathbb{A}^1 \setminus \{0\}$ and also over $(x = 0)$ but not over any open neighborhood of $(x = 0)$. What about over the nonreduced scheme $(x^n = 0)$? If we want to get a sensible functor, then there should not be a simultaneous normalization over $(x^n = 0)$ for large n .

On the other hand, consider $Y_n := (y^2 - x^2 = (y - x)^n = 0)$. This is the line $(y - x = 0)$ with some embedded points at the origin. The simultaneous normalization should clearly be the line $(y - x = 0)$. If we want to get a functor, this should hold after base change to any subscheme of \mathbb{A}^1 .

Note, however, that $\pi : X \rightarrow \mathbb{A}^1$ and $\pi : Y_n \rightarrow \mathbb{A}^1$ are isomorphic to each other over $(x^n = 0)$.

EXAMPLE 18 (Simultaneous normalization not unique). Even for flat families with reduced fibers, simultaneous normalization need not be unique. This, however, happens, only when the base is not reduced.

Let k be a field and consider the trivial deformation $k(t)[\epsilon]$. If $D : k(t) \rightarrow k(t)$ is any derivation then

$$k[t]_D := \{f + \epsilon D(f) : f \in k[t]\} + \epsilon k[t] \subset k(t)[\epsilon]$$

is a flat deformation of $k[t]$ over $k[\epsilon]$ which agrees with the trivial deformation iff $D(k[t]) \subset k[t]$.

Consider the case $D(f(t)) := t^{-1}f'(t)$. The deformation $k[t]_D$ is nontrivial since $D(t) = t^{-1}$. On the other hand,

$$t^2 = (t^2 + \epsilon D(t^2)) - \epsilon \cdot 2, \quad \text{and} \quad t^3 = (t^3 + \epsilon D(t^3)) - \epsilon \cdot 3t$$

are both in $k[t]_D$, thus $k[t]_D$ contains the trivial deformation

$$k[t^2, t^3] + \epsilon k[t^2, t^3]$$

of $k[t^2, t^3]$. Hence both $k[t]_D$ and $k[t] + \epsilon k[t]$ are simultaneous normalizations of the trivial deformation of $k[t^2, t^3]$ over $k[\epsilon]/(\epsilon^2)$.

It is easy to see that $k[t]_D$ cannot be extended to deformations over $k[\epsilon]/(\epsilon^3)$, save in characteristic 3, where, for any $b \in k$,

$$x + \epsilon \frac{1}{x} + \epsilon^2 \left(\frac{1}{x^3} + \frac{b}{x} \right)$$

generates an extension as a $k[\epsilon]/(\epsilon^3)$ -algebra.

EXAMPLE 19. We give an example of surface $f : X \rightarrow \text{Spec } k[t^2, t^3]$ such that

1. X is reduced and S_2 ,
2. f is flat except at a single point,
3. \mathcal{O}_X has no hull,
4. over $\text{Spec } k[t]$, the hull is the structure sheaf of $\mathbb{P}^1 \times \text{Spec } k[t]$ and
5. for $\text{char } k \neq 3$, $\text{Spec } k[t] \rightarrow \text{Spec } k[t^2, t^3]$ represents the simultaneous normalization functor.

We start with the normalization of X , which is $\mathbb{P}^1_{x:y} \times \mathbb{A}^1_t$. The map $\mathbb{P}^1_{x:y} \times \mathbb{A}^1_t \rightarrow X$ will be a homeomorphism. On the $y = 1$ chart, X is the spectrum of the ring

$$R := k[x^n + nx^{n-2}t : n \geq 2, x^nt^m : n \geq 0, m \geq 2] \subset k[x, t].$$

Note that $f(x) + g(x)t \in R$ iff $g(x) = \frac{1}{x}f'(x)$.

R is finitely generated; one generating set is given by

$$x^2 + 2t, x^3 + 3xt, t^2, xt^2, t^3, xt^3.$$

Indeed, this set gives all the monomials t^m and xt^m for $m \geq 2$. Now

$$x^n t^m = x^{n-2} t^m (x^2 + 2t) - 2x^{n-2} t^{m+1}$$

gives all other monomials $x^n t^m$ for all $m \geq 2$. Finally products of $x^2 + 2t$ and $x^3 + 3xt$ give all the $x^n + nx^{n-2}t$ modulo t^2 .

Consider $x^{-1}k[x]$ with the usual $k[x, t]$ -module structure. One easily checks that

$$k[x, t] \rightarrow x^{-1}k[x] \quad \text{given by} \quad f_0(x) + f_1(x)t + \dots \mapsto f_1(x) - \frac{1}{x}f'_0(x)$$

is an R -module homomorphism whose kernel is R . Since $x^{-1}k[x]$ has no embedded points, we see that R is S_2 .

By explicit computation, the fiber of R over the origin is the cuspidal curve $k[x^2, x^3]$ with 2 embedded points at the origin. Thus R is generically flat over $k[t^2, t^3]$ but it is not flat at the origin.

The $(x = 1)$ chart is easier. It is given by the spectrum of the ring

$$Q := k[y^n - ny^{n+2}t : n \geq 1, y^n t^m : n \geq 0, m \geq 2] \subset k[y, t].$$

Note that $f(y) + g(y)t \in Q$ iff $g(y) = -y^3 f'(y)$ and Q is flat over $k[t^2, t^3]$.

The next result shows that the above problems with simultaneous normalization only appear in codimension 1 on the fibers.

PROPOSITION 20. *Let $f : X \rightarrow S$ be a proper and equidimensional morphism. Assume that there is a closed subscheme $Z \subset X$ such that*

1. $\text{codim}(X_s, Z \cap X_s) \geq 2$ for every $s \in S$ and
2. $X \setminus Z$ is flat over S with geometrically normal fibers.

Then $\pi : S^n \rightarrow S$ as in (15) is a monomorphism. If f is projective, then $\pi : S^n \rightarrow S$ is a locally closed decomposition, that is, a locally closed embedding and a bijection on geometric points.

Proof. First we show that for any $T \rightarrow S$, a simultaneous normalization of $X_T := X \times_S T \rightarrow T$ is unique. To see this, let $h : Y_T \rightarrow X_T$ be a simultaneous normalization and $j : X_T \setminus Z_T \hookrightarrow X_T$ the open embedding. Then $h_*(\mathcal{O}_{Y_T})$ is an coherent sheaf on X_T which has depth 2 along Z_T and which agrees with $\mathcal{O}_{X_T}/(\text{torsion})$ outside Z_T . Thus

$$h_*(\mathcal{O}_{Y_T}) = j_*(\mathcal{O}_{X_T \setminus Z_T}), \tag{20.3}$$

which shows that Y_T is unique. Thus $\pi : S^n \rightarrow S$ is a monomorphism.

In the projective case, let $p(t)$ be the largest polynomial that occurs as a Hilbert polynomial of the normalization of a geometric fiber of f (21) and let $S_p^n \subset S^n$ denote the open subscheme of normalizations with Hilbert polynomial p . We prove that $S_p^n \rightarrow S$ is a proper monomorphism.

Consider $\text{Husk}_p^{\text{alg}}(\mathcal{O}_X) \rightarrow S$. It parametrizes partial normalizations of the fibers with Hilbert polynomial $p(t)$. Since $p(t)$ is the largest Hilbert polynomial, this implies that $\text{Husk}_p^{\text{alg}}(\mathcal{O}_X)$ parametrizes normalizations of fibers that are geometrically normal. Thus $S_p^n = \text{Husk}_p^{\text{alg}}(\mathcal{O}_X)$ and so $S_p^n \rightarrow S$ is proper.

A proper monomorphism is a closed embedding, hence $\pi : S_p^n \rightarrow S$ is a closed embedding.

Finally, we replace S by $S \setminus \pi(S_p^n)$ and conclude by Noetherian induction. \square

LEMMA 21. *(cf. [CHL06, Sec.3]) Let $f : X \rightarrow S$ be a projective morphism and H an f -ample divisor. For $s \in S$, let $\chi(\bar{X}_{k(\bar{s})}, \mathcal{O}(tH))$ denote the Hilbert polynomial of the normalization of the geometric fiber of f over s . Then*

1. $s \mapsto \chi(\bar{X}_{k(\bar{s})}, \mathcal{O}(tH))$ is constructible.
2. If f has pure relative dimension n with generically geometrically reduced fibers then $s \mapsto \chi(\bar{X}_{k(\bar{s})}, \mathcal{O}(tH))$ is upper semi continuous.

Proof. We may assume that S is reduced. Let $s \in S$ be a generic point, $K \supset k(s)$ an algebraic closure and $\bar{X}_K \rightarrow X_K$ the normalization. There is a finite extension $L \supset k(s)$ such that $\bar{X}_L \rightarrow X_L$ is geometrically normal. Let $S_L \rightarrow S$ be a quasi-finite morphism whose generic fiber is $\text{Spec } L \rightarrow s$. Let $n_L : X_L^n \rightarrow \text{red}(X \times_S S_L)$ be the normalization. The generic fiber of $f \circ n_L$ is geometrically normal. Thus, by shrinking S_L if necessary, we may assume that $f \circ n_L$ is flat with geometrically normal fibers. In particular, the Hilbert polynomials of normalizations of geometric fibers of f are the same for every point of the open set $\text{im}(S_L \rightarrow S)$. The first part follows by Noetherian induction.

In order to prove upper semi continuity, it is enough to deal with the case when S is the spectrum of a DVR, X is normal and the generic fiber X_g is geometrically normal. Let $\bar{X}_0 \rightarrow X_0$ denote the normalization of the geometric special fiber. Since X is normal, X_0 has no embedded points and the same holds for \bar{X}_0 . We assumed that X_0 is generically geometrically reduced, thus \bar{X}_0 is generically reduced. Thus $\mathcal{O}_{X_0} \rightarrow \mathcal{O}_{\bar{X}_0}$ is an injection and

$$\chi(\bar{X}_0, \mathcal{O}(tH)) \geq \chi(X_0, \mathcal{O}(tH)) = \chi(X_0, \mathcal{O}(tH)) = \chi(X_g, \mathcal{O}(tH)). \quad \square$$

3. Other applications. As another application, we prove the flatness conjecture [Kol95, 6.2.2] and a generalization of the conjecture [Kol95, 6.2.1]. The original conjectures are about arbitrary morphisms, but here we have to restrict ourselves to the proper case. The example [Kol95, 15.5] shows that in (22) the rational singularity assumption is necessary.

COROLLARY 22. *Let S be a reduced scheme over a field of characteristic 0 and $f : X \rightarrow S$ a proper morphism. Assume that there is a closed subscheme $Z \subset X$ such that*

1. $\text{codim}(X_s, Z \cap X_s) \geq 2$ for every $s \in S$,
2. $X \setminus Z$ is flat over S with normal fibers, and
3. the normalization \bar{X}_s has rational singularities for every $s \in S$.

Let $j : X \setminus Z \hookrightarrow X$ be the injection and $\bar{X} := \text{Spec}_X j_(\mathcal{O}_{X \setminus Z})$.*

Then $\bar{f} : \bar{X} \rightarrow S$ is flat and its fibers are normal with only rational singularities.

Proof. The case when S is the spectrum of a DVR is in [Kol95, 14.2].

For P -normalization, let $\pi^n : S^n \rightarrow S$ be as in (15). By (20), π^n is a monomorphism and by the above cited [Kol95, 14.2], π^n satisfies the valuative criterion of properness, hence it is proper. Therefore, π^n is an isomorphism. The rest follows from (20.3). \square

REMARK 23. The proof of (22) in fact shows that if a result of this type holds for a certain class of singularities (instead of rational ones) when the base is the spectrum of a DVR, then it also holds for an arbitrary reduced base.

In particular, it also applies when the fibers have normal crossing singularities in codimension one and their S_2 -hulls are semi-rational. The proof of [Kol95, 14.2] works in this case, using the semi-resolution theorem of [Kol08b].

COROLLARY 24. *Let $0 \in S$ be a normal, local scheme and $f : X \rightarrow S$ a proper morphism of relative dimension 1. Assume that*

1. f is smooth at the generic points of X_0 ,
2. the generic fiber of f is either smooth or defined over a field of characteristic 0 and
3. the reduced fiber $\text{red } X_0$ has only finitely many partial normalizations (25).

Let $n : \bar{X} \rightarrow X$ denote the normalization. Then $\bar{f} := f \circ n : \bar{X} \rightarrow S$ is flat with reduced fibers.

Proof. Let $g \in S$ be the generic point. Then \mathcal{O}_{X_g} is reduced, hence $\mathcal{O}_{\bar{X}_g}$ is a husk of \mathcal{O}_{X_g} which gives an isolated point $P_g \in \text{Husk}^{\text{alg}}(\mathcal{O}_{X_g})$. Let $S'_g \subset \text{Husk}^{\text{alg}}(\mathcal{O}_{X_g})$ denote the irreducible component containing P_g . Then S'_g is 0-dimensional and reduced if $\bar{X}_g = X_g$. Thus its reduced closure $S' \subset \text{red}(\text{Husk}^{\text{alg}}(\mathcal{O}_X))$ is an irreducible component such that the induced map $\pi : S' \rightarrow S$ is an isomorphism near s if X_g is smooth and a monomorphism in general. Thus $\pi : S' \rightarrow S$ is birational if the generic fiber of f is either smooth or defined over a field of characteristic 0.

By (5.3), π is proper. The fiber $\pi^{-1}(0)$ parametrizes partial normalizations of X_0 , hence it is finite by assumption. Therefore, by Zariski's main theorem, π is an isomorphism.

Let $u : Y' \rightarrow S'$ be the universal flat family of husks. There is a natural morphism $Y' \rightarrow X$ which is finite and birational.

Since S and the fibers of u are S_2 , so is Y' (cf. [Mat86, Thm.23.3]). Moreover, u is smooth along the generic fiber and along the generic points of the special fiber, hence Y' is regular in codimension 1. By Serre's criterion Y' is normal, hence $\bar{X} \cong Y$. \square

25 (Curves with finitely many partial normalizations). *We are interested in reduced curves C over a field k such that only finitely many curves sit between C and its normalization, even after base change. That is, up to isomorphisms which are the identity on C , there are only finitely many diagrams $\bar{C}_{\bar{k}} \rightarrow C_i \rightarrow C_{\bar{k}}$. This condition depends on the singularities of C only, and there are only few singularities with this property. By the results of [GK85, KS85, GK90], the only such plane curve singularities are the simple singularities A_n, D_n, E_6, E_7, E_8 .*

Another series is given by the semi-normal curve singularities. Over \bar{k} these are analytically isomorphic to the coordinate axes in \mathbb{A}^n for some n .

The example (27) shows that ordinary quadruple points have infinitely many partial normalizations and the conclusion of (24) also fails for them.

26. (A correction to [Kol95, 15.5].) *Let X be a smooth projective variety of dimension n with $H^1(X, \mathcal{O}_X) \neq 0$ and L an ample line bundle on X . Let $C(X) := \text{Spec} \sum_m H^0(X, L^m)$ be the corresponding cone over X with vertex $v \in C(X)$. Let $\pi : C(X) \rightarrow \mathbb{A}^n$ be a projection with 1-dimensional fibers.*

The second part of [Kol95, 15.5] asserts that π is not flat at v for $n \geq 2$. However, this holds only when $n > 2$. If $n = 2$ then π is flat but the fiber through v has embedded points at v .

These also show that in (24) some strong restrictions on the singularities are necessary.

EXAMPLE 27. Another interesting example is given by the deformations of the plane quartic with an ordinary quadruple point

$$C_0 := (xy(x^2 - y^2) = 0) \subset \mathbb{P}^2.$$

Let $\mathbf{C}_4 \rightarrow \mathbb{P}^{14}$ be the universal family of degree 4 plane curves and $\mathbf{C}_{4,1} \rightarrow S^{12}$ the 12-dimensional subfamily whose general members are elliptic curves with 2 nodes. S^{12} is not normal, thus, to put ourselves in the settings of (24), we normalize S^{12} and pull back the family.

We claim that if we take the normalization of the total space $\bar{\mathbf{C}}_{4,1} \rightarrow \bar{S}^{12}$, we get a family of curves whose fiber over $[C_0]$ has embedded points. Most likely, the family is not even flat, but I have not checked this.

We prove this by showing that in different families of curves through $[C_0] \in S^{12}$ we get different flat limits.

To see this, note that the semi normalization C_0^{sn} of C_0 can be thought of as 4 general lines through a point in \mathbb{P}^4 . In suitable affine coordinates, its coordinate ring is

$$k[u_1, \dots, u_4]/(u_i u_j : i \neq j) \supset k[u_1 + u_3 + u_4, u_2 + u_3 - u_4].$$

There is a 1-parameter family of partial semi normalizations of C_0 corresponding to the 3-dimensional linear subspaces

$$\langle u_1, \dots, u_4 \rangle \supset W_\lambda \supset \langle u_1 + u_3 + u_4, u_2 + u_3 - u_4 \rangle.$$

Each W_λ corresponds to a projection of C_0^{sn} to \mathbb{P}^3 ; call the image $C_\lambda \subset \mathbb{P}^3$. Then C_λ is 4 general lines through a point in \mathbb{P}^3 ; thus it is a (2, 2)-complete intersection curve of arithmetic genus 1. (Note that the C_λ are isomorphic to each other, but the isomorphism will not commute with the map to C_0 in general.) Every C_λ can be realized as the special fiber in a family $S_\lambda \rightarrow B_\lambda$ of (2, 2)-complete intersection curves in \mathbb{P}^3 whose general fiber is a smooth elliptic curve.

By projecting these families to \mathbb{P}^2 , we get a 1-parameter family $S'_\lambda \rightarrow B_\lambda$ of curves in S^{12} whose special fiber is C_0 .

Let now $\bar{S}'_\lambda \subset \bar{\mathbf{C}}_{4,1}$ be the preimage of this family in the normalization. Then \bar{S}'_λ is dominated by the surface S_λ .

There are two possibilities. First, if \bar{S}'_λ is isomorphic to S_λ , then the fiber of $\bar{\mathbf{C}}_{4,1} \rightarrow \bar{S}^{12}$ over $[C_0]$ is C_λ . This, however, depends on λ , a contradiction. Second, if \bar{S}'_λ is not isomorphic to S_λ , then the fiber of $\bar{S}'_\lambda \rightarrow B_\lambda$ over the origin is C_0 with some embedded points. Since C_0 has arithmetic genus 3, we must have at least 2 embedded points.

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