

THE CUT LOCI ON ELLIPSOIDS AND CERTAIN LIOUVILLE MANIFOLDS*

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Abstract. We show that some riemannian manifolds diffeomorphic to the sphere have the property that the cut loci of general points are smoothly embedded closed disks of codimension one. Ellipsoids with distinct axes are typical examples of such manifolds.

Key words. Cut locus, ellipsoid, Liouville manifold, integrable geodesic flow.

AMS subject classifications. Primary 53C22, Secondary 53A05

1. Introduction. On a complete riemannian manifold, any geodesic $\gamma(t)$ starting at a point $\gamma(0) = p$ has the property that any segment $\{\gamma(t) \mid 0 \leq t \leq T\}$ is minimal, i.e., the length of the segment is equal to the distance between the points p and $\gamma(T)$, if $T > 0$ is small. If the supremum t_0 of the set of such T is finite, then the point $\gamma(t_0)$ is called the *cut point* of p along the geodesic $\gamma(t)$ ($t \geq 0$). The *cut locus* of the point p is then defined as the set of all cut points of p along the geodesics starting at p . For the general properties of cut loci, we refer to [19], [26].

The study of cut locus was started at 1905 by H. Poincaré [22] in the case of convex surfaces, and there are several classical results, for example, [21], [35], [36]. From its definition, the cut locus of a point p on a compact manifold M is homotopically equivalent to $M - \{p\}$, but it can be very complicated, see [5], [9]. The structure of cut locus was studied in connection with the singularity theory, see [2], [3], [34]. Recently, a property of cut locus was used to solve Ambrose's problem on surfaces [8], [9], and it was proved that the distance function to the cut locus has Lipschitz continuity [13], [20]. Other applications of cut locus are found in [4], [20] also.

It is well known that the cut locus of any point on the sphere of constant curvature consists of a single point, and it is also known that this property characterizes the sphere of constant curvature (an affirmatively solved case of the Blaschke conjecture, see [1]). However, in most cases, to determine cut loci are quite difficult problems. There are only a few cases where the cut loci are well understood; for example, analytic surfaces [21], symmetric spaces and some homogeneous spaces [7], [23], [24], [25], [31], certain surfaces of revolution [6], [30], [32], [33], Alexandrov surfaces [27], tri-axial ellipsoids and some Liouville surfaces [10], [11], [29] ([29] is an experimental work). Especially in higher dimensional case there are not many results without symmetric spaces and some singular spaces [14], even if using computational approximations.

In the earlier paper [10], we proved that the cut locus of a non-umbilic point on a tri-axial ellipsoid is a segment of the curvature line containing the antipodal point, inspired by an experimental work [12]. Also, we gave the complete proof of Jacobi's last geometric statement ([15], [16], see also [28], which contains historical remarks). Furthermore, we have seen in [11] that there are many surfaces possessing such simple cut loci. Surfaces we considered in [11] are so-called Liouville surfaces, i.e., surfaces whose geodesic flows possess first integrals which are fiberwise quadratic

*Received October 14, 2009; accepted for publication March 29, 2010.

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forms. In such cases the geodesic equations are explicitly solved by quadratures. But, to determine cut loci we needed some additional conditions, which is satisfied in the case of ellipsoid.

In the present paper, we shall give a higher dimensional version of the above-mentioned results. We shall consider cut loci of points on certain Liouville manifolds diffeomorphic to n -sphere, and prove that the cut locus of any point is a smoothly embedded, closed $(n - 1)$ -disk, if the point does not belong to a certain submanifold of codimension two. We shall also prove that the cut locus of a point on that submanifold is a closed $(n - 2)$ -disk. The n -dimensional ellipsoids with $n + 1$ distinct axes will be shown to possess such properties. Here, “Liouville manifold” is a higher dimensional version of Liouville surface, which we shall explain in the next section.

Now, taking the ellipsoid $M : \sum_{i=0}^n u_i^2/a_i = 1$ ($0 < a_n < \dots < a_0$) as an example, let us illustrate our results in detail. The elliptic coordinate system $(\lambda_1, \dots, \lambda_n)$ on M ($\lambda_n \leq \dots \leq \lambda_1$) is defined by the following identity in λ :

$$\sum_{i=0}^n \frac{u_i^2}{a_i - \lambda} - 1 = \frac{\lambda \prod_{k=1}^n (\lambda_k - \lambda)}{\prod_i (a_i - \lambda)}.$$

For a fixed $u \in M$, λ_k are determined by n “confocal quadrics” passing through u . From λ_k 's, u_i are explicitly described as:

$$u_i^2 = \frac{a_i \prod_{k=1}^n (\lambda_k - a_i)}{\prod_{j \neq i} (a_j - a_i)}.$$

Let N_k and J_k be the submanifolds of M defined by

$$N_k = \{u = (u_0, \dots, u_n) \in M \mid u_k = 0\} \quad (0 \leq k \leq n),$$

$$J_k = \{u \in M \mid u_k = 0, \sum_{i \neq k} \frac{u_i^2}{a_i - a_k} = 1\} \quad (1 \leq k \leq n - 1).$$

They have the following properties: N_k is totally geodesic, codimension 1; $J_k \subset N_k$, J_k is diffeomorphic to $S^{k-1} \times S^{n-k-1}$; $\bigcup_k J_k$ is the set of points where some principal curvature with respect to the inclusion $M \subset \mathbb{R}^{n+1}$ has multiplicity ≥ 2 ; and

$$N_k = \{\lambda_k = a_k \text{ or } \lambda_{k+1} = a_k\}, \quad J_k = \{\lambda_k = \lambda_{k+1} = a_k\}.$$

Then our theorem in this case may be stated as follows (see Theorem 7.1 for the general setting).

THEOREM 1.1. *Let us denote by $C(p)$ the cut locus of a point $p \in M$. Let $(\lambda_1^0, \dots, \lambda_n^0)$ be the elliptic coordinates of p . Then:*

- (1) *If $p \notin J_{n-1}$, then $C(p)$ is an $(n - 1)$ -dimensional closed disk which is contained in a submanifold (possibly with boundary) defined by $\lambda_n = \lambda_n^0$. Also, $C(p)$ contains the antipodal point of p in its interior. For each interior point q of $C(p)$ there are exactly two minimal geodesics joining p and q ; the tangent vectors of those geodesics at p are symmetric with respect to the hyperplane $d\lambda_n = 0$. For each boundary point q of $C(p)$, there is a unique minimal geodesic from p to q , along which q is the first conjugate point of p with multiplicity one.*

- (2) If $p \in J_{n-1}$, then $C(p)$ is an $(n-2)$ -dimensional closed disk contained in J_{n-1} . It is identical with the cut locus of p in the $(n-1)$ -dimensional ellipsoid N_{n-1} . For each interior point q of $C(p)$ there is an S^1 -family of minimal geodesics joining p and q ; the tangent vectors of those geodesics at p form a cone whose orthogonal projection to $T_p J_{n-1}$ is one-dimensional. For each boundary point q of $C(p)$, there is a unique minimal geodesic from p to q , and along it q is the first conjugate point of p ; but the multiplicity is two in this case.

The organization of the paper is as follows. In §2 we shall briefly explain Liouville manifolds in the form what we need. In §3 we shall illustrate how to solve geodesic equations on a Liouville manifold. Since the geodesic flow is completely integrable in this case, solutions are given by integrating a system of closed 1-forms. In this particular case, a natural coordinate system provides “separation of variables”. This coordinate system is analogous to the elliptic coordinate system on ellipsoids. In §4 we shall give an assumption under which the results on cut loci are obtained. Some useful inequalities are proved there.

In §5 basic properties of Jacobi fields and their zeros are investigated, which are crucial in the arguments of the following sections. In §6 we define a value $t_0(\eta)$ to each unit covector η , which will indicate the cut point of the geodesic with initial covector η . Then, we prove some preliminary facts on the behavior of geodesics starting at a fixed point. The main theorem, Theorem 7.1, will be stated in §7 and proved in §§7-9.

In the forthcoming paper, we shall clarify the structures of conjugate loci of general points on certain Liouville manifolds, which will be a higher dimensional version of “the last geometric statement of Jacobi” explained in [10], [28].

Preliminary remarks and notations. We shall consider geodesic equations in the hamiltonian formulation. Let M be a riemannian manifold and g its riemannian metric. By $\flat : TM \rightarrow T^*M$ we denote the bundle isomorphism determined by g (Legendre transformation). We also use the symbol $\sharp = \flat^{-1}$. The canonical 1-form on T^*M is denoted by α . For a canonical coordinate system (x, ξ) on T^*M (x being a coordinate system on M), α is expressed as $\sum_i \xi_i dx_i$. Then the 2-form $d\alpha$ represents the standard symplectic structure on T^*M .

Let E be the function on T^*M defined by

$$E(\lambda) = \frac{1}{2}g(\sharp(\lambda), \sharp(\lambda)) = \frac{1}{2} \sum_{i,j} g^{ij}(x)\xi_i\xi_j.$$

We call it the (kinetic) energy function of M . For a function F, H on T^*M , we define a vector field X_F and the Poisson bracket $\{F, H\}$ by

$$X_F = \sum_i \left(\frac{\partial F}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i} \frac{\partial}{\partial \xi_i} \right), \quad \{F, H\} = X_F H.$$

Then X_E generates the geodesic flow, i.e., the projection of each integral curve of X_E to M is a geodesic of the riemannian manifold M .

2. Liouville manifolds. By definition, Liouville manifold (M, \mathcal{F}) is a pair of an n -dimensional riemannian manifold M and an n -dimensional vector space \mathcal{F} of functions on T^*M such that i) each $F \in \mathcal{F}$ is fiberwise a quadratic polynomial; ii) those quadratic forms are simultaneously normalizable on each fiber; iii) \mathcal{F} is commutative with respect to the Poisson bracket; and, iv) \mathcal{F} contains the hamiltonian of the geodesic flow. For the general theory of Liouville manifolds, we refer to [18]. In

this paper we only need a subclass of “compact Liouville manifolds of rank one and type (A)”, described in [18]. So, in this section, we shall briefly explain about it.

Each Liouville manifold treated here is constructed from $n + 1$ constants $a_0 > \dots > a_n > 0$ and a positive C^∞ function $A(\lambda)$ on the closed interval $a_n \leq \lambda \leq a_0$. Let $\alpha_1, \dots, \alpha_n$ be positive numbers defined by

$$\alpha_i = 2 \int_{a_i}^{a_{i-1}} \frac{A(\lambda) d\lambda}{\sqrt{(-1)^i \prod_{j=0}^n (\lambda - a_j)}} \quad (i = 1, \dots, n).$$

Define the function f_i on the circle $\mathbb{R}/\alpha_i\mathbb{Z} = \{x_i\}$ ($1 \leq i \leq n$) by the conditions:

$$(2.1) \quad \left(\frac{df_i}{dx_i}\right)^2 = \frac{(-1)^i 4 \prod_{j=0}^n (f_i - a_j)}{A(f_i)^2}$$

$$(2.2) \quad f_i(0) = a_i, \quad f_i\left(\frac{\alpha_i}{4}\right) = a_{i-1}, \quad f_i(-x_i) = f_i(x_i) = f_i\left(\frac{\alpha_i}{2} - x_i\right).$$

Then the range of f_i is $[a_i, a_{i-1}]$.

Put

$$R = \prod_{i=1}^n (\mathbb{R}/\alpha_i\mathbb{Z}).$$

Let τ_i ($1 \leq i \leq n - 1$) be the involutions on the torus R defined by

$$\tau_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, -x_i, \frac{\alpha_{i+1}}{2} - x_{i+1}, x_{i+2}, \dots, x_n),$$

and let $G (\simeq (\mathbb{Z}/2\mathbb{Z})^{n-1})$ be the group of transformations generated by $\tau_1, \dots, \tau_{n-1}$. Then it turns out that the quotient space $M = R/G$ is homeomorphic to the n -sphere. Moreover, let $p \in R$ be a ramification point of the branched covering $R \rightarrow R/G$. Suppose p is fixed by $\tau_{i_1}, \dots, \tau_{i_k}$, and is not fixed by other τ_j 's. Taking a suitable coordinate system (y_1, \dots, y_n) obtained from (x) by transpositions ($x_i \leftrightarrow x_j$) and translations ($x_i \rightarrow x_i + c$), it may be supposed that p is represented by $y = 0$ and τ_{i_l} is given by

$$(y_1, \dots, y_n) \mapsto (y_1, \dots, y_{2l-2}, -y_{2l-1}, -y_{2l}, y_{2l+1}, \dots, y_n).$$

Then we can define a differentiable structure on M so that

$$(y_1^2 - y_2^2, 2y_1y_2, \dots, y_{2k-1}^2 - y_{2k}^2, 2y_{2k-1}y_{2k}, y_{2k+1}, \dots, y_n)$$

is a smooth coordinate system around the image of p . With this M is diffeomorphic to the standard n -sphere. One can prove those facts by comparing the branched covering $R \rightarrow R/G$ with the standard case; see [18, p.73].

Now, put

$$b_{ij}(x_i) = \begin{cases} (-1)^i \prod_{\substack{1 \leq k \leq n-1 \\ k \neq j}} (f_i(x_i) - a_k) & (1 \leq j \leq n-1) \\ (-1)^{i+1} \prod_{k=1}^{n-1} (f_i(x_i) - a_k) & (j = n) \end{cases},$$

and define functions $F_1, \dots, F_{n-1}, F_n = 2E$ on the cotangent bundle by

$$(2.3) \quad \sum_{j=1}^n b_{ij}(x_i) F_j(x, \xi) = \xi_i^2 \quad (1 \leq i \leq n),$$

where ξ_i are the fiber coordinates with respect to the base coordinates (x_1, \dots, x_n) . Although there are points on T^*R where F_i are not well-defined, it turns out that F_i represent well-defined smooth functions on T^*M . Computing the inverse matrix of (b_{ij}) explicitly, we have

$$2E = \sum_{i=1}^n \frac{(-1)^{n-i} \xi_i^2}{\prod_{\substack{1 \leq l \leq n \\ l \neq i}} (f_l(x_l) - f_i(x_i))}$$

$$F_j = \frac{1}{\prod_{\substack{1 \leq k \leq n-1 \\ k \neq j}} (a_k - a_j)} \sum_{i=1}^n \frac{(-1)^{n-i} \prod_{\substack{1 \leq l \leq n \\ l \neq i}} (f_l(x_l) - a_j)}{\prod_{\substack{1 \leq l \leq n \\ l \neq i}} (f_l(x_l) - f_i(x_i))} \xi_i^2$$

$(1 \leq j \leq n - 1) .$

One can also see that E , restricted to each cotangent space of M , is a positive definite quadratic form. Therefore

$$(2.4) \quad g = \sum_i (-1)^{n-i} \left(\prod_{l \neq i} (f_l - f_i) \right) dx_i^2$$

is a well-defined riemannian metric on M , and E is the hamiltonian of the associated geodesic flow. We call E the energy function of the riemannian manifold M . From the formula (2.3) one can easily see that

$$\{F_i, F_j\} = 0 \quad (1 \leq i, j \leq n) ,$$

where $\{, \}$ denotes the Poisson bracket (see [18, Prop. 1.1.3]). In particular, the geodesic flow is completely integrable in the sense of hamiltonian mechanics.

As examples, if $A(\lambda)$ is a constant function, then M is the sphere of constant curvature. This case is explained in detail in [18, pp.71-74]. If $A(\lambda) = \sqrt{\lambda}$, then M is isometric to the ellipsoid $\sum_{i=0}^n u_i^2/a_i = 1$. In this case, the system of functions $(f_1(x_1), \dots, f_n(x_n))$ is nothing but the elliptic coordinate system (see Introduction), i.e., $f_i(x_i) = \lambda_i$. One can easily check that the induced metric $\sum_i du_i^2$ coincides with the formula (2.4) when f_i satisfy the equations (2.1) and $A(\lambda) = \sqrt{\lambda}$.

Finally, let us define certain submanifolds of M which are analogous to those for the ellipsoid stated in Introduction: Put

$$N_k = \{x \in M \mid f_k(x_k) = a_k \text{ or } f_{k+1}(x_{k+1}) = a_k\} \quad (0 \leq k \leq n),$$

$$J_k = \{x \in M \mid f_k(x_k) = f_{k+1}(x_{k+1}) = a_k\} \quad (1 \leq k \leq n - 1).$$

Then we have, putting $(F_k)_p = F_k|_{T_p^*M}$,

LEMMA 2.1.

- (1) $J_k = \{p \in M \mid (F_k)_p = 0\}$.
- (2) $N_k = \{p \in M \mid \text{rank } (F_k)_p \leq 1\} \quad (1 \leq k \leq n - 1)$.
- (3) $\bigcup_k J_k$ is identical with the branch locus of the covering $R \rightarrow M = R/G$.
- (4) N_k is a totally geodesic submanifold of codimension one $(0 \leq k \leq n)$.
- (5) $J_k \subset N_k$, and J_k is diffeomorphic to $S^{k-1} \times S^{n-k-1}$.

Proof. For (1) and (2), see [18, pp.52-56]. (3) is obvious. (4) follows from the fact that N_k is the fixed point-set of the involutive isometry $(x_1, \dots, x_n) \mapsto (x_1, \dots, -x_k, \dots, x_n)$. (5) is easily seen by comparing the branched covering with the standard one, [18, p.73]. \square

3. Geodesic equations. The geodesic equations are generally written as

$$\frac{dx_i}{dt} = \frac{\partial E}{\partial \xi_i}, \quad \frac{d\xi_i}{dt} = -\frac{\partial E}{\partial x_i}.$$

However, since our geodesic flow is completely integrable, it is better to consider the equation of geodesics with $F_j = c_j$ ($1 \leq j \leq n-1$) and $2E = 1$. If $c = (c_1, \dots, c_{n-1}, 1)$ is a regular value of the map

$$F = (F_1, \dots, F_{n-1}, 2E) : T^*M \rightarrow \mathbb{R}^n,$$

then its inverse image is a disjoint union of tori, and the vector fields X_{F_j}, X_E on it are mutually commutative and linearly independent everywhere. Here X_f denotes the hamiltonian vector field determined by a function f ;

$$X_f = \sum_i \left(\frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial}{\partial \xi_i} \right).$$

Let ω_j ($1 \leq j \leq n$) be the dual 1-forms of $\{\pi_* X_{F_j}\}$, where $\pi : T^*M \rightarrow M$ is the bundle projection. Then, by (2.3) we have

$$\omega_l = \sum_i \frac{b_{il}}{2\xi_i} dx_i \quad (1 \leq l \leq n).$$

They are closed 1-forms, and the geodesic orbits are determined by

$$(3.1) \quad \omega_l = 0 \quad (1 \leq l \leq n-1),$$

and the length parameter t on an orbit is given by

$$(3.2) \quad dt = 2\omega_n.$$

Putting

$$\Theta(\lambda) = \sum_{j=1}^{n-1} \left(\prod_{\substack{1 \leq k \leq n-1 \\ k \neq j}} (\lambda - a_k) \right) c_j - \prod_{k=1}^{n-1} (\lambda - a_k),$$

we have from (2.3)

$$\xi_i = \epsilon_i \sqrt{\sum_j b_{ij}(x_i)c_j} = \epsilon_i \sqrt{(-1)^i \Theta(f_i(x_i))} \quad (1 \leq i \leq n),$$

where $\epsilon_i = \text{sgn } \xi_i = \text{sgn} \left(\frac{dx_i}{dt} \right) = \pm 1$. If a covector (x, ξ) with $F_1 = c_1, \dots, F_{n-1} = c_{n-1}, 2E = 1$ satisfies $\xi_i \neq 0$ for any $1 \leq i \leq n$, then we have

$$(-1)^i \Theta(f_i(x_i)) > 0.$$

Therefore for such c_1, \dots, c_{n-1} , the equation $\Theta(\lambda) = 0$ has $n-1$ distinct real roots $b_1 > b_2 > \dots > b_{n-1}$, and they satisfy

$$f_1(x_1) > b_1 > f_2(x_2) > b_2 > \dots > f_{n-1}(x_{n-1}) > b_{n-1} > f_n(x_n).$$

Thus we have the identity

$$\Theta(\lambda) = - \prod_{l=1}^{n-1} (\lambda - b_l),$$

and c_j are expressed by b_l 's as

$$(3.3) \quad c_j = \frac{- \prod_{l=1}^{n-1} (a_j - b_l)}{\prod_{\substack{1 \leq k \leq n-1 \\ k \neq j}} (a_j - a_k)} \quad (1 \leq j \leq n-1).$$

Conversely, let b_1, \dots, b_{n-1} be any real numbers satisfying

$$(3.4) \quad a_{i+1} \leq b_i \leq a_{i-1}, \quad b_{i+1} \leq b_i$$

for any i , and define c_1, \dots, c_{n-1} by (3.3). Then there is a covector (x, ξ) with $F_1 = c_1, \dots, F_{n-1} = c_{n-1}, 2E = 1$. It can be verified that if b_1, \dots, b_{n-1} satisfy

$$(3.5) \quad a_{i+1} < b_i < a_{i-1}, \quad b_i \neq a_i, \quad b_{i+1} < b_i \quad \text{for any } i$$

then the corresponding $c = (c_1, \dots, c_{n-1}, 1)$ is a regular value of \mathbf{F} .

To describe the behavior of the geodesics it is more convenient to use the values (b_1, \dots, b_{n-1}) rather than using (c_1, \dots, c_{n-1}) directly. So, we shall mainly use (b_1, \dots, b_{n-1}) as the values of first integrals which determine the Lagrange tori $\mathbf{F}^{-1}(c)$. Also, we shall denote by H_1, \dots, H_{n-1} the functions on the unit cotangent bundle U^*M whose values are b_1, \dots, b_{n-1} . Namely, H_i 's are determined by

$$F_j(\mu) = \frac{- \prod_{l=1}^{n-1} (a_j - H_l(\mu))}{\prod_{\substack{1 \leq k \leq n-1 \\ k \neq j}} (a_j - a_k)} \quad (1 \leq j \leq n-1),$$

$$H_1(\mu) \geq \dots \geq H_{n-1}(\mu), \quad \mu \in U^*M.$$

The range of H_i are given by (3.4).

Now, put

$$a_i^+ = \max\{a_i, b_i\} \quad (1 \leq i \leq n-1), \quad a_n^+ = a_n$$

$$a_i^- = \min\{a_i, b_i\} \quad (1 \leq i \leq n-1), \quad a_0^- = a_0.$$

If b_1, \dots, b_{n-1} satisfy the condition (3.5), then the π -image of a connected component of $\mathbf{F}^{-1}(c)$ (a Lagrange torus) is of the form

$$L_1 \times \dots \times L_n \subset M,$$

where each L_i is a connected component of the inverse image of $[a_i^+, a_{i-1}^-]$ by the map

$$f_i : \mathbb{R}/\alpha_i\mathbb{Z} \rightarrow [a_i, a_{i-1}].$$

(Observe that the ‘‘generalized band’’ $L_1 \times \dots \times L_n \subset R$ is injectively mapped to M by the branched covering $R \rightarrow M$.)

Along a geodesic $(x_1(t), \dots, x_n(t))$, the coordinate function $x_i(t)$ oscillates on L_i if L_i is an interval, or $x_i(t)$ moves monotonously if L_i is the whole circle. Also, the function $f_i(x_i(t))$ oscillates on the interval $[a_i^+, a_{i-1}^-]$.

After all, the equations of geodesic orbits

$$\omega_l = 0 \quad (1 \leq l \leq n - 1)$$

are described as

$$\sum_{i=1}^n \frac{\epsilon_i (-1)^i \prod_{\substack{1 \leq k \leq n-1 \\ k \neq l}} (f_i(x_i) - a_k) dx_i}{\sqrt{(-1)^{i-1} \prod_{k=1}^{n-1} (f_i(x_i) - b_k)}} = 0 \quad (1 \leq l \leq n - 1).$$

Note that this system of equations is equivalent to

$$\sum_{i=1}^n \frac{\epsilon_i (-1)^i G(f_i) dx_i}{\sqrt{(-1)^{i-1} \prod_{k=1}^{n-1} (f_i - b_k)}} = 0$$

for any polynomial $G(\lambda)$ of degree $\leq n - 2$. By (2.1) those equations are also described as

$$(3.6) \quad \sum_{i=1}^n \frac{\epsilon'_i (-1)^i G(f_i) A(f_i) df_i}{\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} = 0,$$

where $\epsilon'_i = \text{sgn of } df_i(x_i(t))/dt$.

By (3.6) we have

$$\sum_{i=1}^n \int_s^t \frac{(-1)^i G(f_i) A(f_i)}{\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} \left| \frac{df_i(x_i(t))}{dt} \right| dt = 0$$

for any polynomial $G(\lambda)$ of degree $\leq n - 2$ and for a fixed $s \in \mathbb{R}$. By using the variables σ_i defined by

$$\sigma_i(t) = \int_0^t \left| \frac{df_i(x_i(t))}{dt} \right| dt,$$

this formula is rewritten as

$$(3.7) \quad \sum_{i=1}^n \int_{\sigma_i(s)}^{\sigma_i(t)} \frac{(-1)^i G(f_i) A(f_i)}{\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} d\sigma_i = 0.$$

Here, f_i is regarded as a function of σ_i , i.e., putting $\phi_i(t) = a_i + |t|$ for $|t| \leq a_{i-1} - a_i$ and extending it to \mathbb{R} as a periodic function with the period $2(a_{i-1} - a_i)$, we have

$$f_i = \phi_i(\sigma_i + \epsilon_i(f_i(x_i(0)) - a_i)),$$

where $\epsilon_i = \pm 1$ is the sign of $df_i(x_i(t))/dt$ at $t = 0$. Also, integrating $dt = 2\omega_n = \sum_i (b_{in}/\xi_i) dx_i$, we have

$$(3.8) \quad \sum_{i=1}^n \int_{\sigma_i(s)}^{\sigma_i(t)} \frac{(-1)^{i+1} \tilde{G}(f_i) A(f_i)}{2\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} d\sigma_i = t - s,$$

where $\tilde{G}(\lambda)$ is any monic polynomial in λ of degree $n - 1$.

4. A monotonicity condition for $A(\lambda)$. We put the following conditions on the function $A(\lambda)$:

$$(4.1) \quad (-1)^{k-1}A^{(k)}(\lambda) > 0 \quad \text{on } [a_n, a_0] \quad (1 \leq k \leq n - 1)$$

for $n \geq 3$, where $A^{(k)}$ denotes the k -th derivative of A . For the case $n = \dim M = 2$, we need (4.1) for $1 \leq k \leq 2$, as described in our earlier paper [11]. A typical example satisfying the condition (4.1) is the ellipsoid, in which case $A(\lambda) = \sqrt{\lambda}$. Since the condition (4.1) is C^{n-1} -open, there are surely many $A(\lambda)$ satisfying it.

In the rest of this section, we shall prove some inequalities which are obtained under the condition (4.1). Put

$$G_l(\lambda) = \prod_{\substack{1 \leq k \leq n-1 \\ k \neq l}} (\lambda - b_k) \quad (1 \leq l \leq n - 1).$$

PROPOSITION 4.1. *If $A(\lambda)$ satisfies the condition (4.1), and if b_1, \dots, b_{n-1} and a_0, \dots, a_n are all distinct, then the following inequalities hold:*

(1)

$$\sum_{i=1}^n \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^{n-i+\#I} A(\lambda) \prod_{j \in I} (\lambda - b_j)}{\sqrt{-\prod_{k=1}^{n-1} (\lambda - b_k) \cdot \prod_{k=0}^n (\lambda - a_k)}} d\lambda < 0,$$

where I is any (possibly empty) subset of $\{1, \dots, n - 1\}$ such that $\#I \leq n - 2$;

(2)

$$\frac{\partial}{\partial b_l} \sum_{i=1}^n \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^i G_l(\lambda) A(\lambda) d\lambda}{\sqrt{-\prod_{k=1}^{n-1} (\lambda - b_k) \cdot \prod_{k=0}^n (\lambda - a_k)}} > 0,$$

where $1 \leq l \leq n - 1$.

The inequality (1) is still valid if b_j 's ($j \notin I$) are mutually distinct. Precisely speaking, when a sequence of b_j 's with b_j 's and a_k 's being all distinct converges to some b_j 's which satisfy $b_k \neq b_l$ for any $k, l \in J, k \neq l$, then the formula in (1) has a limit and the limit is still negative.

In the following two lemmas, we shall assume that b_1, \dots, b_{n-1} and a_0, \dots, a_n are all distinct.

LEMMA 4.2.

$$\sum_{i=1}^n \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^i G(\lambda) d\lambda}{\sqrt{-\prod_{k=1}^{n-1} (\lambda - b_k) \cdot \prod_{k=0}^n (\lambda - a_k)}} = 0$$

for any polynomial $G(\lambda)$ of degree $\leq n - 2$.

Proof. Let $W = \{\lambda\}$ be the region $\mathbb{C} \cup \{\infty\} - \bigcup_{i=1}^n [a_i^+, a_{i-1}^-]$. Then there are a meromorphic function μ on W such that

$$\mu^2 = - \prod_{k=1}^{n-1} (\lambda - b_k) \cdot \prod_{k=0}^n (\lambda - a_k),$$

and the holomorphic 1-form $(G(\lambda)/\mu)d\lambda$ on W . Taking the sum of contour integrals around the intervals $[a_i^+, a_{i-1}^-]$, one obtains the desired formula. \square

LEMMA 4.3. *Let J be any nonempty subset of $\{1, \dots, n-1\}$, and let $B(\lambda)$ be the function defined by*

$$(4.2) \quad \frac{A(\lambda)}{\prod_{k \in J} (\lambda - b_k)} = \sum_{k \in J} \frac{e_k}{\lambda - b_k} + B(\lambda), \quad e_k = \frac{A(b_k)}{\prod_{\substack{l \in J \\ l \neq k}} (b_k - b_l)}.$$

Suppose $A(\lambda)$ satisfies the condition (4.1). Then $B(\lambda)$ satisfies

$$(-1)^{\#J+m} B^{(m)}(\lambda) < 0 \quad \text{for } a_n \leq \lambda \leq a_0 \quad \text{and } 0 \leq m \leq n-1-\#J.$$

Proof. We shall prove this by an induction on $\#J$. When $J = \{k\}$, then

$$(4.3) \quad B(\lambda) = \frac{A(\lambda) - A(b_k)}{\lambda - b_k} = \int_0^1 A'(t(\lambda - b_k) + b_k) dt,$$

and we have $(-1)^{1+m} B^{(m)}(\lambda) < 0$ by the assumption on $A(\lambda)$.

Now suppose $\#J \geq 1$, $l \notin J$ and let $J_1 = J \cup \{l\}$. Then

$$\begin{aligned} \frac{A(\lambda)}{\prod_{k \in J_1} (\lambda - b_k)} &= \sum_{k \in J} \frac{e_k}{(\lambda - b_k)(\lambda - b_l)} + \frac{B(\lambda)}{\lambda - b_l} \\ &= \sum_{k \in J} \frac{1}{b_k - b_l} \left(\frac{e_k}{\lambda - b_k} - \frac{e_k}{\lambda - b_l} \right) + \frac{B(b_l)}{\lambda - b_l} + \frac{B(\lambda) - B(b_l)}{\lambda - b_l}. \end{aligned}$$

Let us denote the last term in the right-hand side by $B_1(\lambda)$. Since it is written as

$$\int_0^1 B'(t(\lambda - b_l) + b_l) dt,$$

we have $(-1)^{\#J+1+m} B_1^{(m)}(\lambda) < 0$ by the induction assumption. \square

Proof of Proposition 4.1. First, suppose that b_1, \dots, b_{n-1} and a_0, \dots, a_n are all distinct. Let $A(\lambda)$ be a positive function on $[a_n, a_0]$ satisfying the condition (4.1). Let I be as in Proposition 4.1 (1) and let J be its complement in $\{1, \dots, n-1\}$. Define the function $B(\lambda)$ by the formula (4.2). Then, by Lemmas 4.3 and 4.2 we have

$$(4.4) \quad \begin{aligned} &\sum_{i=1}^n \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^{n-i+\#I} A(\lambda) \prod_{l \in I} (\lambda - b_l)}{\sqrt{-\prod_{k=1}^{n-1} (\lambda - b_k) \cdot \prod_{k=0}^n (\lambda - a_k)}} d\lambda \\ &= \sum_{i=1}^n \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^{n-i+\#I} B(\lambda) \prod_{l=1}^{n-1} (\lambda - b_l)}{\sqrt{-\prod_{k=1}^{n-1} (\lambda - b_k) \cdot \prod_{k=0}^n (\lambda - a_k)}} d\lambda. \end{aligned}$$

Since $(-1)^{i-1} \prod_{j=1}^{n-1} (\lambda - b_j) > 0$ on (a_i^+, a_{i-1}^-) , and since

$$(-1)^{n-1-\#I} B(\lambda) < 0$$

by Lemma 4.3, we have the inequality (1) in this case.

Next, let us consider the limit case. The limit b_j 's are assumed to satisfy $b_k \neq b_l$ for any $k, l \in J, k \neq l$. Note that the function $B(\lambda)$ is defined by the formula (4.2) and it only depends on $A(\lambda)$ and b_j 's ($j \in J$). Since the limit b_j 's ($j \in J$) are mutually distinct, it follows that the function $B(\lambda)$ has a limit. Therefore the right-hand side of the formula (4.4) has a finite limit and it is still negative by the same reason as above.

To prove (2), we put

$$\frac{A(\lambda)}{\lambda - b_l} = \frac{A(b_l)}{\lambda - b_l} + B(\lambda, b_l).$$

Then the left-hand side of (2) is equal to

$$\begin{aligned} & \frac{\partial}{\partial b_l} \sum_{i=1}^n \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^i B(\lambda, b_l) \prod_{j=1}^{n-1} (\lambda - b_j)}{\sqrt{-\prod_{k=1}^{n-1} (\lambda - b_k) \cdot \prod_{k=0}^n (\lambda - a_k)}} d\lambda \\ (4.5) \quad &= \sum_{i=1}^n \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^i \left(\frac{\partial}{\partial b_l} B(\lambda, b_l) \right) \prod_{j=1}^{n-1} (\lambda - b_j)}{\sqrt{-\prod_{k=1}^{n-1} (\lambda - b_k) \cdot \prod_{k=0}^n (\lambda - a_k)}} d\lambda \\ & - \frac{1}{2} \sum_{i=1}^n \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^i B(\lambda, b_l) \prod_{\substack{1 \leq j \leq n-1 \\ j \neq l}} (\lambda - b_j)}{\sqrt{-\prod_{k=1}^{n-1} (\lambda - b_k) \cdot \prod_{k=0}^n (\lambda - a_k)}} d\lambda. \end{aligned}$$

The second line of the right-hand side is equal to

$$-\frac{1}{2} \sum_{i=1}^n \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^i B_1(\lambda, b_l) \prod_{1 \leq j \leq n-1} (\lambda - b_j)}{\sqrt{-\prod_{k=1}^{n-1} (\lambda - b_k) \cdot \prod_{k=0}^n (\lambda - a_k)}},$$

where

$$B_1(\lambda, b_l) = \frac{B(\lambda, b_l) - A'(b_l)}{\lambda - b_l} = \frac{\partial}{\partial b_l} B(\lambda, b_l).$$

Since $B_1(\lambda, b_l) < 0$, it follows that the right-hand side of the formula (4.5) is positive. \square

5. Jacobi fields. In this section we shall consider Jacobi fields along a geodesic which is not totally contained in the submanifold N_i for any $0 \leq i \leq n$. Let $\gamma(t) = (x_1(t), \dots, x_n(t))$ be such a geodesic. In this case, the corresponding values b_i of the first integrals H_i satisfy $b_i \neq a_{i+1}$ and $b_i \neq a_{i-1}$ for any $1 \leq i \leq n - 1$. We shall consider the following three cases separately: (i) b_1, \dots, b_{n-1} and a_0, \dots, a_n are all distinct; (ii) there are some i such that $b_i = a_i$, but other b_j 's are not equal to any a_k nor b_k ; (iii) there are some j such that $b_j = b_{j-1}$, and there may be some i such that $b_i = a_i$, but there is no l such that $b_l = a_{l+1}$ or $b_l = a_{l-1}$.

First, let us consider the case where b_1, \dots, b_{n-1} and a_0, \dots, a_n are all distinct. For each $1 \leq i \leq n - 1$, let $S_i \subset \mathbb{R}$ be the set of the time s such that $f_i(x_i(s)) = b_i$ ($b_i = a_i^+$) or $f_{i+1}(x_{i+1}(s)) = b_i$ ($b_i = a_i^-$). Then S_i are discrete subsets of \mathbb{R} . At each point $\gamma(s)$ where $s \notin S_i$ for any i , the system of functions (H_1, \dots, H_{n-1}) can be used as a coordinate system on the unit cotangent space $U_{\gamma(s)}^* M$ around the covector $(x(s), \xi(s)) = b(\dot{\gamma}(s))$. Then, identifying $\partial/\partial H_i \in T_{b(\dot{\gamma}(s))}(U_{\gamma(s)}^* M)$ with a covector in

$T_{\gamma(s)}^*M$ in a natural manner, we put $\tilde{V}_i(s) = \sharp(\frac{\partial}{\partial H_i} / |\frac{\partial}{\partial H_i}|) \in T_{\gamma(s)}M$ at $\gamma(s)$. As is easily seen, the norm $|\partial/\partial H_i|$ is equal to

$$\frac{1}{2} \sqrt{\frac{(-1)^{n-1} G_i(b_i)}{\prod_{m=1}^n (f_m(x_m) - b_i)}} .$$

At the point $\gamma(s)$ where $s \in S_i$, we put $\nu_i^2 = f_i(x_i(s)) - H_i$ if $b_i = a_i^+$ (resp. $\nu_i^2 = H_i - f_{i+1}(x_{i+1}(s))$ if $b_i = a_i^-$), and use ν_i as a coordinate function on $U_{\gamma(s)}^*M$ instead of H_i . We choose the sign of ν_i so that it is equal to the sign of ξ_i (resp. ξ_{i+1}). Then we put $\tilde{V}_i(s) = \sharp(\frac{\partial}{\partial \nu_i} / |\frac{\partial}{\partial \nu_i}|)$ in this case. It is easy to see that $\mathbb{R} \ni s \mapsto \tilde{V}_i(s)$ is smooth up to the sign. Therefore we can take a smooth vector field $V_i(t)$ along the geodesic $\gamma(t)$ such that $V_i(t) = \pm \tilde{V}_i(t)$ for any $t \in \mathbb{R}$. We now define the Jacobi field $Y_{i,s}(t)$ along the geodesic $\gamma(t)$ by the initial conditions $Y_{i,s}(s) = 0$ and $Y'_{i,s}(s) = V_i(s)$ for any $s \in \mathbb{R}$, where $Y'_{i,s}(t)$ denotes the covariant derivative of $Y_{i,s}(t)$ with respect to $\partial/\partial t$.

Let us denote by $\Omega(Y, Z)$ the symplectic inner product of two Jacobi fields along $\gamma(t)$ which are orthogonal to $\dot{\gamma}(t)$ for any t :

$$\Omega(Y, Z) = g(Y(t), Z'(t)) - g(Y'(t), Z(t)) ,$$

which is constant in t . Let \mathcal{Y}_i ($1 \leq i \leq n - 1$) be the vector space of Jacobi fields along $\gamma(t)$ spanned by $\{Y_{i,s}(t) \mid s \in \mathbb{R}\}$.

PROPOSITION 5.1. *Along the geodesic $\gamma(t)$ such that b_1, \dots, b_{n-1} and a_0, \dots, a_n are all distinct, the Jacobi fields defined above have the following properties.*

- (1) $Y_{i,s}(t) \in \mathbb{R}V_i(t)$ for any $1 \leq i \leq n - 1$ and $s, t \in \mathbb{R}$. Also, $V_1(t), \dots, V_{n-1}(t), \dot{\gamma}(t)$ are mutually orthogonal for any $t \in \mathbb{R}$.
- (2) \mathcal{Y}_i and \mathcal{Y}_j ($i \neq j$) are mutually orthogonal with respect to the symplectic inner product Ω , i.e., $\Omega(Y_i, Y_j) = 0$ for any $Y_i \in \mathcal{Y}_i$ and $Y_j \in \mathcal{Y}_j$.
- (3) Each $V_i(t)$ is parallel along the geodesic $\gamma(t)$.
- (4) Each \mathcal{Y}_i is two-dimensional.
- (5) If $\gamma(s_1)$ and $\gamma(s_2)$ ($s_1 < s_2$) are mutually conjugate along the geodesic $\gamma(t)$, then there is an i ($1 \leq i \leq n - 1$) and a nonzero Jacobi field $Y \in \mathcal{Y}_i$ such that $Y(s_1) = Y(s_2) = 0$.
- (6) $Y_{i,s_1}(s_2) \neq 0$ if $s_1 \notin S_i$, $s_2 \neq s_1$, and either $[s_1, s_2) \cap S_i = \emptyset, s_1 < s_2$ or $(s_2, s_1] \cap S_i = \emptyset, s_2 < s_1$.
- (7) The Jacobi field $Y_{i,s_1}(t)$ ($s_1 \in S_i$) vanishes at $t = s_2$ if and only if $s_2 \in S_i$.

Proof. Let $\gamma(u, t) = (\dots, x_k(u, t), \dots)$ be a one-parameter family of geodesics such that $x_k(0, t) = x_k(t)$ and $(\partial x_k / \partial u)|_{u=0}$ represents the Jacobi field $Y_{i,s_1}(t)$. Suppose that $G = G_j, j \neq i$, and $s = s_1$ and $t = s_2$ do not belong to $S_i \cup S_j$ in the formula (3.7). We then differentiate the formula in u . Since

$$\frac{\partial H_i}{\partial u} \Big|_{u=0} \neq 0 ; \quad \frac{\partial H_k}{\partial u} \Big|_{u=0} = 0 \quad (k \neq i) ,$$

we have

$$(5.1) \quad \sum_{l=1}^n \frac{\epsilon_l' (-1)^l G_j(f_l) A(f_l)}{\sqrt{-\prod_{k=1}^{n-1} (f_l - b_k) \cdot \prod_{k=0}^n (f_l - a_k)}} d(f_l(x_l))(Y_{i,s_1}(s_2)) - \frac{1}{2c} \sum_{l=1}^n \int_{\sigma_l(s_1)}^{\sigma_l(s_2)} \frac{(-1)^l G_{i,j}(f_l) A(f_l)}{\sqrt{-\prod_{k=1}^{n-1} (f_l - b_k) \cdot \prod_{k=0}^n (f_l - a_k)}} d\sigma_l = 0 ,$$

where $c = \pm$ (the norm of $\partial/\partial H_i$ at $\gamma(s_1)$) and $f_l = f_l(x_l(s_2))$ in the first line, and $G_{i,j}(\lambda) = \prod_{k \neq i,j} (\lambda - b_k)$. Observe that the second line in the above formula vanishes by the formula (3.7). Moreover, the covector

$$\frac{1}{4} \sum_{l=1}^n \frac{\epsilon'_l(-1)^l G_j(f_l) A(f_l)}{\sqrt{-\prod_{k=1}^{n-1} (f_l - b_k) \cdot \prod_{k=0}^n (f_l - a_k)}} d(f_l(x_l)) \Big|_{f_l=f_l(x_l(s_2))}$$

is equal to the one represented by $\partial/\partial H_j$ at $\gamma(s_2)$, which is a nonzero scalar multiple of $b(Y'_{j,s_2}(s_2))$. Thus we have

$$\Omega(Y_{i,s_1}, Y_{j,s_2}) = g(Y_{i,s_1}(s_2), Y'_{j,s_2}(s_2)) = 0,$$

which is valid for any $s_1, s_2 \in \mathbb{R}$ by continuity. In particular, we have $g(Y_{i,s_1}(s_2), V_j(s_2)) = 0$ for any $j \neq i$, and also $g(V_i(s_1), V_j(s_1)) = 0$ by differentiating it at $s_2 = s_1$. Thus we have (1) and (2).

(3) and (4) follow immediately from (1) and (2). The assertion (5) is also obvious. Next, we shall prove (6). First, we assume $s_1 < s_2$ and $s_2 \notin S_i$. In the same way as above, we have

$$(5.2) \quad \sum_{l=1}^n \frac{\epsilon'_l(-1)^l G_i(f_l) A(f_l)}{\sqrt{-\prod_{k=1}^{n-1} (f_l - b_k) \cdot \prod_{k=0}^n (f_l - a_k)}} d(f_l(x_l))(Y_{i,s_1}(s_2)) + \frac{1}{2c} \sum_{l=1}^n \int_{\sigma_l(s_1)}^{\sigma_l(s_2)} \frac{(-1)^l G_i(f_l) A(f_l)}{(f_l - b_i) \sqrt{-\prod_{k=1}^{n-1} (f_l - b_k) \cdot \prod_{k=0}^n (f_l - a_k)}} d\sigma_l = 0.$$

Note that, since $[s_1, s_2] \cap S_i = \emptyset$, $f_l - b_i$ never vanish on the interval $[\sigma_l(s_1), \sigma_l(s_2)]$. The second line in the above formula being negative, we have $g(Y_{i,s_1}(s_2), Y'_{i,s_2}(s_2)) \neq 0$. Thus $Y_{i,s_1}(s_2) \neq 0$.

Next, let us take $s_3 \in S_i$ such that $s_1 < s_3$ and $[s_1, s_3] \cap S_i = \emptyset$. As proved above,

$$\left| \frac{\partial}{\partial H_i} \right|_{\gamma(s_1)} \left| \frac{\partial}{\partial H_i} \right|_{\gamma(s_2)} g(Y_{i,s_1}(s_2), Y'_{i,s_2}(s_2)) = -\frac{1}{8} \sum_{l=1}^n \int_{\sigma_l(s_1)}^{\sigma_l(s_2)} \frac{(-1)^i G_i(f_l) A(f_l)}{(f_l - b_i) \sqrt{-\prod_{k=1}^{n-1} (f_l - b_k) \cdot \prod_{k=0}^n (f_l - a_k)}} d\sigma_l$$

for any s_2 such that $s_1 < s_2 < s_3$. Suppose $b_i = a_i^+$. Since

$$g(Y_{i,s_1}(s_2), Y'_{i,s_2}(s_2)) = \Omega(Y_{s_1}, Y_{s_2}) = -g(Y'_{i,s_1}(s_1), Y_{i,s_2}(s_1)),$$

multiplying both sides by $2|\nu_i| = 2\sqrt{f_i(x_i(s_2)) - b_i}$, and taking a limit $s_2 \rightarrow s_3$, we have

$$(5.3) \quad -c' g(Y'_{i,s_1}(s_1), Y_{i,s_3}(s_1)) = \frac{1}{2} \frac{(-1)^{i+1} G_i(b_i) A(b_i)}{\sqrt{-\prod_{k \neq i} (b_i - b_k) \cdot \prod_{k=0}^n (b_i - a_k)}},$$

where $c' = |\partial/\partial H_i|_{\gamma(s_1)} |\partial/\partial \nu_i|_{\gamma(s_3)}$. Since the left-hand side of the above formula is equal to

$$c' g(Y_{i,s_1}(s_3), Y'_{i,s_3}(s_3)),$$

and since the right-hand side does not vanish, we have

$$(5.4) \quad Y_{i,s_1}(s_3) \neq 0, \quad Y_{i,s_3}(s_1) \neq 0.$$

The case where $s_2 < s_1$ is similar. Therefore the assertion (6) follows.

Now, in the situation of (6), take $s_0 \in S_i$ such that $s_0 < s_1$ and $(s_0, s_1] \cap S_i = \emptyset$. Then, again multiplying both sides of the formula (5.3) by $|\nu_i| = \sqrt{f_i(x_i(s_1)) - b_i}$ and taking a limit $s_1 \rightarrow s_0$, we have

$$g(Y_{i,s_0}(s_3), Y'_{i,s_3}(s_3)) = 0.$$

Thus it follows that $Y_{i,s_0}(s_3) = 0$, and combined with (5.4) we have (7). \square

The following corollary is immediate.

COROLLARY 5.2. *Fix t_0 and let $t_0 < t_1^i < t_2^i < \dots$ be the zeros of the Jacobi field $Y_{i,t_0}(t)$ for $t \geq t_0$. Then:*

- (1) *If $t_0 \in S_i$, then the set $\{t_k^i\}$ coincides with $\{t \in S_i \mid t > t_0\}$*
- (2) *If $t_0 \notin S_i$, then every $t_k^i \notin S_i$, and there is just one element of S_i in the interval (t_k^i, t_{k+1}^i) for each k .*
- (3) *The set of conjugate points of $\gamma(t_0)$ along $\gamma(t)$ ($t > t_0$) is equal to $\{\gamma(t_k^i) \mid k \geq 1, 1 \leq i \leq n-1\}$.*

We shall prove one more result on the zeros of Jacobi fields in this case, which needs the assumption (4.1).

PROPOSITION 5.3. *Fix i and take s_1 and s_2 such that $s_1 \notin S_i$, $s_1 < s_2$, and $\sigma_l(s_2) - \sigma_l(s_1) \leq 2(a_{l-1}^- - a_l^+)$ for any l . Then $Y_{i,s_1}(s_2) \neq 0$.*

Proof. Let $s_3 \in S_i$ such that $s_1 < s_3$ and $[s_1, s_3] \cap S_i = \emptyset$. If $s_2 \leq s_3$, then the assertion follows from (6) of the previous proposition. Now suppose $s_3 < s_2$. As above, we shall compute $g(Y_{i,s_1}(s_2), Y'_{i,s_2}(s_2))$. In this case, however, the formula (5.2) is invalid, because the integral diverge at $t = s_3$. So, instead, we differentiate the formula

$$(5.5) \quad - \sum_{l=1}^n \int_{\sigma_l(s_2)}^{2(a_{l-1}^- - a_l^+) + \sigma_l(s_1)} \frac{(-1)^l G_i(f_l) A(f_l) d\sigma_l}{\sqrt{-\prod_{k=1}^{n-1} (f_l - b_k) \cdot \prod_{k=0}^n (f_l - a_k)}} \\ + 2 \sum_{l=1}^n \int_{a_l^+}^{a_{l-1}^-} \frac{(-1)^l G_i(\lambda) A(\lambda) d\lambda}{\sqrt{-\prod_{k=1}^{n-1} (\lambda - b_k) \cdot \prod_{k=0}^n (\lambda - a_k)}} = 0$$

in terms of the deformation parameter defining cY_{i,s_1} , c being \pm (the norm of $\partial/\partial H_i$ at $\gamma(s_1)$):

$$(5.6) \quad \sum_{l=1}^n \frac{\epsilon'_l (-1)^l G_i(f_l) A(f_l)}{\sqrt{-\prod_{k=1}^{n-1} (f_l - b_k) \cdot \prod_{k=0}^n (f_l - a_k)}} d(f_l(x_l))(cY_{i,s_1}(s_2)) \\ - \frac{1}{2} \sum_{l=1}^n \int_{\sigma_l(s_2)}^{2(a_{l-1}^- - a_l^+) + \sigma_l(s_1)} \frac{(-1)^l G_i(f_l) A(f_l) d\sigma_l}{(f_l - b_i) \sqrt{-\prod_{k=1}^{n-1} (f_l - b_k) \cdot \prod_{k=0}^n (f_l - a_k)}} \\ + 2 \frac{\partial}{\partial b_i} \sum_{l=1}^n \int_{a_l^+}^{a_{l-1}^-} \frac{(-1)^l G_i(\lambda) A(\lambda) d\lambda}{\sqrt{-\prod_{k=1}^{n-1} (\lambda - b_k) \cdot \prod_{k=0}^n (\lambda - a_k)}} = 0.$$

Note that b_i is not contained in the range of f_l while σ_l moves in the interval $[\sigma_l(s_2), 2(a_{l-1}^- - a_l^+) + \sigma_l(s_1)]$ ($l = i, i + 1$). Since the second line of the formula (5.6) is positive or zero, and since the third line is positive by Proposition 4.1 (2), it therefore follows that $g(Y_{i,s_1}(s_2), Y'_{i,s_2}(s_2)) \neq 0$. \square

Next, we shall consider Jacobi fields along the geodesic $\gamma(t)$ for which some b_i is equal to a_i , but other b_j 's are not equal to any a_k nor b_k . For i with $b_i = a_i$, let S_i be the set of $s \in \mathbb{R}$ where $f_i(x_i(s)) = b_i$. One can see from the formula (3.7) that S_i is also the set of $s \in \mathbb{R}$ where $f_{i+1}(x_{i+1}(s)) = b_i$, i.e., $s \in S_i$ if and only if $\gamma(s) \in J_i$. For such i and $s \in S_i$, we define $\tilde{Y}_{i,s}(t)$ as the Jacobi field $\pi_*(X_{F_i})$ along the geodesic $\gamma(t)$. For $s \notin S_i$, $Y_{i,s}(t)$ is defined as before. Also, for j with $b_j \neq a_j$, the set S_j and the Jacobi fields $Y_{j,s}(t)$ are defined as before.

PROPOSITION 5.4. *For a geodesic $\gamma(t)$ stated above, the statements in Propositions 5.1, 5.3 and Corollary 5.2 equally hold.*

Proof. Only the parts related to the Jacobi field $\tilde{Y}_{i,s}(t) = \pi_*(X_{F_i})$ would be nontrivial. Suppose $b_i = a_i$ and $s_1 \notin S_j$, $s_2 \in S_i$. Considering the symplectic inner product of two Jacobi fields $Y_{j,s_1}(t)$ and $\tilde{Y}_{i,s_2}(t)$, we have

$$\begin{aligned} \Omega(Y_{j,s_1}, \tilde{Y}_{i,s_2}) &= c\omega\left(\frac{\partial}{\partial H_j}, X_{F_i}\right)_{\mathfrak{b}(\dot{\gamma}(s_1))} \\ &= c \frac{\partial c_i}{\partial b_j} = \frac{c \prod_{m \neq j} (a_i - b_m)}{\prod_{\substack{1 \leq k \leq n-1 \\ k \neq i}} (a_i - a_k)} \begin{cases} = 0 & (j \neq i) \\ \neq 0 & (j = i) \end{cases}, \end{aligned}$$

where ω is the symplectic 2-form $\sum_k d\xi_k \wedge dx_k$, $\partial/\partial H_j$ is the tangent vector to $U_{\gamma(s_1)}^*M$ at $\mathfrak{b}(\dot{\gamma}(s_1))$ defined as before, and $c = 1/|\partial/\partial H_j|$. The proposition follows from this formula. \square

Next, we shall consider Jacobi fields along a geodesic for which there are some j such that $b_j = b_{j-1}$ and there may be some i such that $b_i = a_i$, but there is no l such that $b_l = a_{l+1}$ or $b_l = a_{l-1}$. In this case, $f_j(x_j(t)) (= b_j = b_{j-1})$ remains constant along the geodesic $\gamma(t)$. We put this value λ_j^0 for convenience. For each point $\gamma(s)$ on the geodesic, we adopt μ_j, μ_{j-1} as the coordinate functions on the unit cotangent space $U_{\gamma(s)}^*M$, around the covector $\mathfrak{b}(\dot{\gamma}(s))$, instead of H_j, H_{j-1} , defined by the formula:

$$\mu_{j-1} = H_{j-1} + H_j - 2\lambda_j^0, \quad \mu_j^2 = 4(H_{j-1} - \lambda_j^0)(\lambda_j^0 - H_j).$$

We choose the sign of μ_j so that it is equal to that of ξ_j . Let us denote by $Z_{j,s}(t)$, $Z_{j-1,s}(t)$ the Jacobi fields along the geodesic $\gamma(t)$ with the initial conditions

$$Z_{k,s}(s) = 0, \quad Z'_{k,s}(s) = \sharp(\partial/\partial \mu_k)/|\partial/\partial \mu_k| \quad (k = j, j - 1).$$

Note that

$$\left| \frac{\partial}{\partial \mu_{j-1}} \right| = \left| \frac{\partial}{\partial \mu_j} \right| = \frac{1}{2} \sqrt{\frac{(-1)^n G_{j,j-1}(\lambda_j^0)}{\prod_{m \neq j} (f_m - \lambda_j^0)}}, \quad \left\langle \frac{\partial}{\partial \mu_{j-1}}, \frac{\partial}{\partial \mu_j} \right\rangle = 0$$

at each covector $\mathfrak{b}(\dot{\gamma}(s))$.

Define the real number $\theta_{s_1}(s_2)$ by the formula

$$(5.7) \quad \sum_{\substack{1 \leq l \leq n \\ l \neq j}} \int_{\sigma_l(s_1)}^{\sigma_l(s_2)} \frac{(-1)^l G_{j,j-1}(f_l) A(f_l) d\sigma_l}{|f_l - \lambda_j^0| \sqrt{-\prod_{k \neq j, j-1} (f_l - b_k) \cdot \prod_{k=0}^n (f_l - a_k)}} + 2\theta_{s_1}(s_2) \frac{(-1)^j G_{j,j-1}(\lambda_j^0) A(\lambda_j^0)}{\sqrt{\prod_{k \neq j, j-1} (\lambda_j^0 - b_k) \prod_k (\lambda_j^0 - a_k)}} = 0 .$$

We then have the following proposition.

PROPOSITION 5.5.

- (1) $Z_{k,s_1}(s_2) = 0$ for $k = j, j - 1$ and any s_1, s_2 such that $\theta_{s_1}(s_2) = \pi$.
- (2) $Z_{j,s_1}(s_2)$ and $Z_{j-1,s_1}(s_2)$ are linearly independent for any s_1 and s_2 such that $0 < \theta_{s_1}(s_2) < \pi$.

Proof. We consider a one-parameter family of geodesics $t \rightarrow \gamma(u, t)$ such that $\gamma(0, t) = \gamma(t)$, $\gamma(u, s_1) = \gamma(s_1)$, and the values b_i of the first integrals H_i for $\gamma(u, t)$ are the same as those for $\gamma(t)$ except that $b_{j-1}(u) = H_{j-1}(b(\dot{\gamma}(u, t))) = \lambda_j^0 + u^2$. Since $b_j = \lambda_j^0 = f_j(x_j(u, s_1))$ for any u , it follows that the Jacobi fields $Y_{j,s_1}(t)$ and $Y_{j-1,s_1}(t)$ are defined along the geodesic $\gamma(u, t)$ for $u \neq 0$. Observe that on the unit cotangent space $U_{\gamma(s_1)}^* M$, $(\partial/\partial\nu_j)/|\partial/\partial\nu_j|$ tends to $\pm(\partial/\partial\mu_j)/|\partial/\partial\mu_j|$ and $(\partial/\partial H_{j-1})/|\partial/\partial H_{j-1}|$ tends to $(\partial/\partial\mu_{j-1})/|\partial/\partial\mu_{j-1}|$ as $u \rightarrow 0$. Thus the Jacobi fields $Y_{j,s_1}(t)$ and $Y_{j-1,s_1}(t)$ along the geodesic $\gamma(u, t)$ converge to Jacobi fields $Z_{j,s_1}(t)$ and $Z_{j-1,s_1}(t)$ up to the sign along the geodesic $\gamma(t)$ as $u \rightarrow 0$, respectively.

Moreover, with this procedure of taking the limit, we claim that the Jacobi fields $Y_{j,s_2}(t)$ and $Y_{j-1,s_2}(t)$ along the geodesic $\gamma(u, t)$ tend to

$$\epsilon (\cos \theta Z_{j,s_2}(t) + \sin \theta Z_{j-1,s_2}(t)) \quad \text{and} \quad \epsilon (-\sin \theta Z_{j,s_2}(t) + \cos \theta Z_{j-1,s_2}(t))$$

respectively, where $\epsilon = \pm 1$ and $\theta = \theta_{s_1}(s_2)$. To see this, we begin with the formula before taking the limit:

$$(5.8) \quad \sum_{i=1}^n \int_{\sigma_i(s_1)}^{\sigma_i(s_2)} \frac{(-1)^i G_{j,j-1}(f_i) A(f_i)}{\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} d\sigma_i = 0 .$$

Define the function $\theta(u, t)$ by

$$f_j(x_j(u, t)) = b_j(\cos \theta(u, t))^2 + b_{j-1}(u)(\sin \theta(u, t))^2 , \\ \theta(u, s_1) = 0, \quad (\partial/\partial t)\theta \geq 0 .$$

Then, taking the limit $u \rightarrow 0$, we see that

$$\int_{\sigma_j(s_1)}^{\sigma_j(s_2)} \frac{(-1)^j G_{j,j-1}(f_j) A(f_j)}{\sqrt{-\prod_{k=1}^{n-1} (f_j - b_k) \cdot \prod_{k=0}^n (f_j - a_k)}} d\sigma_j$$

tends to

$$2\theta(0, s_2) \frac{(-1)^j G_{j,j-1}(\lambda_j^0) A(\lambda_j^0)}{\sqrt{\prod_{k \neq j, j-1} (\lambda_j^0 - b_k) \prod_k (\lambda_j^0 - a_k)}} .$$

Thus we have $\theta(0, t) = \theta_{s_1}(t)$ by (5.7). The covector $\partial/\partial H_j$ at the point $\gamma(u, s_2)$ is equal to

$$\frac{1}{4} \sum_{i=1}^n \frac{\epsilon'_i(-1)^i G_j(f_i) A(f_i) df_i}{\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}},$$

which tends to, as $u \rightarrow 0$,

$$\begin{aligned} & \frac{1}{4} \sum_{i \neq j} \frac{f_i - \lambda_j^0}{|f_i - \lambda_j^0|} \frac{\epsilon'_i(-1)^i G_{j,j-1}(f_i) A(f_i) df_i}{\sqrt{-\prod_{k \neq j, j-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} \\ & + \frac{1}{4} \frac{(-1)^{j+1} \cot \theta G_{j,j-1}(\lambda_j^0) A(\lambda_j^0) df_j}{\sqrt{\prod_{k \neq j, j-1} (\lambda_j^0 - b_k) \cdot \prod_{k=0}^n (\lambda_j^0 - a_k)}}, \end{aligned}$$

where $\theta = \theta_{s_1}(s_2)$. Also, $\partial/\partial H_{j-1}$ tends to

$$\begin{aligned} & \frac{1}{4} \sum_{i \neq j} \frac{f_i - \lambda_j^0}{|f_i - \lambda_j^0|} \frac{\epsilon'_i(-1)^i G_{j,j-1}(f_i) A(f_i) df_i}{\sqrt{-\prod_{k \neq j, j-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} \\ & + \frac{1}{4} \frac{(-1)^j \tan \theta G_{j,j-1}(\lambda_j^0) A(\lambda_j^0) df_j}{\sqrt{\prod_{k \neq j, j-1} (\lambda_j^0 - b_k) \cdot \prod_{k=0}^n (\lambda_j^0 - a_k)}}. \end{aligned}$$

As is easily seen, we have

$$\begin{aligned} b(Z'_{j-1, s_2}(s_2)) &= \frac{c}{4} \sum_{i \neq j} \frac{f_i - \lambda_j^0}{|f_i - \lambda_j^0|} \frac{\epsilon'_i(-1)^i G_{j,j-1}(f_i) A(f_i) df_i}{\sqrt{-\prod_{k \neq j, j-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} \\ b(Z'_{j, s_2}(s_2)) &= \frac{c}{4} \frac{(-1)^{j+1} G_{j,j-1}(\lambda_j^0) A(\lambda_j^0) df_j}{\sqrt{\prod_{k \neq j, j-1} (\lambda_j^0 - b_k) \cdot \prod_{k=0}^n (\lambda_j^0 - a_k)}}, \end{aligned}$$

where $c = 1/|\partial/\partial \mu_{j-1}| = 1/|\partial/\partial \mu_j|$ at $\gamma(s_2)$. Therefore the claim follows.

From the formulas obtained above and (5.3), we thus have

$$\begin{aligned} & g(Z_{j-1, s_1}(s_2), \cos \theta Z'_{j, s_2}(s_2) + \sin \theta Z'_{j-1, s_2}(s_2)) = 0, \\ & g(Z_{j, s_1}(s_2), -\sin \theta Z'_{j, s_2}(s_2) + \cos \theta Z'_{j-1, s_2}(s_2)) = 0, \\ (5.9) \quad & g(Z_{j, s_1}(s_2), \cos \theta Z'_{j, s_2}(s_2) + \sin \theta Z'_{j-1, s_2}(s_2)) \\ &= \frac{\sin \theta}{4cc'} \frac{(-1)^j G_{j,j-1}(\lambda_j^0) A(\lambda_j^0)}{\sqrt{-\prod_{k \neq j, j-1} (\lambda_j^0 - b_k) \cdot \prod_{k=0}^n (\lambda_j^0 - a_k)}}, \end{aligned}$$

where c and c' are the norms of $\partial/\partial \mu_j$ at $\gamma(s_1)$ and $\gamma(s_2)$ respectively. In particular, we have:

$$\begin{aligned} & \cos \theta \Omega(Z_{j-1, s_1}, Z_{j, s_2}) + \sin \theta \Omega(Z_{j-1, s_1}, Z_{j-1, s_2}) = 0 \\ & -\sin \theta \Omega(Z_{j, s_1}, Z_{j, s_2}) + \cos \theta \Omega(Z_{j, s_1}, Z_{j-1, s_2}) = 0, \end{aligned}$$

where $\theta = \theta_{s_1}(s_2)$. As is easily seen, the above formula is also valid when $s_2 < s_1$, in which case $\theta_{s_1}(s_2) = -\theta_{s_2}(s_1) < 0$. Therefore, exchanging s_1 and s_2 in the above

formula, we have

$$(5.10) \quad \begin{aligned} \Omega(Z_{j,s_1}, Z_{j,s_2}) &= \Omega(Z_{j-1,s_1}, Z_{j-1,s_2}) \\ \Omega(Z_{j-1,s_1}, Z_{j,s_2}) &= -\Omega(Z_{j,s_1}, Z_{j-1,s_2}) . \end{aligned}$$

By (5.9) and (5.10) we also have

$$(5.11) \quad \begin{aligned} &g(Z_{j-1,s_1}(s_2), -\sin \theta Z'_{j,s_2}(s_2) + \cos \theta Z'_{j-1,s_2}(s_2)) \\ &= \frac{\sin \theta}{4cc'} \frac{(-1)^j G_{j,j-1}(\lambda_j^0) A(\lambda_j^0)}{\sqrt{-\prod_{k \neq j, j-1} (\lambda_j^0 - b_k) \cdot \prod_{k=0}^n (\lambda_j^0 - a_k)}} . \end{aligned}$$

Now the assertion (2) easily follows from (5.9) and (5.11). Also, from those formulas we have

$$\begin{aligned} g(Z_{j,s_1}(s_2), Z'_{j,s_2}(s_2)) &= g(Z_{j,s_1}(s_2), Z'_{j-1,s_2}(s_2)) = 0 \\ g(Z_{j-1,s_1}(s_2), Z'_{j,s_2}(s_2)) &= g(Z_{j-1,s_1}(s_2), Z'_{j-1,s_2}(s_2)) = 0 , \end{aligned}$$

provided $\theta_{s_1}(s_2) = \pi$. Since the Jacobi fields $Z_{j,s}, Z_{j-1,s}$ belong to the limit of the vector space $\mathcal{Y}_j + \mathcal{Y}_{j-1}$, and since it is orthogonal to the limit of $\sum_{k \neq j, j-1} \mathcal{Y}_k$ with respect to the symplectic inner product Ω , it therefore follows that $Z_{j,s_1}(s_2) = Z_{j-1,s_1}(s_2) = 0$. This finishes the proof of the proposition. \square

REMARK 5.6. For i with $b_i \neq b_{i-1}$ and $b_i \neq b_{i+1}$, Propositions 5.1, 5.3 and Corollary 5.2 equally hold for the Jacobi field $Y_{i,s}(t)$.

6. Geodesics starting at a one point. In this and the subsequent sections we shall assume that the conditions (4.1) are satisfied. Let $p_0 \in M$ be an arbitrary point. We may assume without loss of generality that p_0 is represented by $(x_1, \dots, x_n) = (x_1^0, \dots, x_n^0)$, where $0 \leq x_i^0 \leq \alpha_i/4$ ($1 \leq i \leq n$). Let $U_{p_0}^* M$ be the sphere of unit covectors at p_0 . We denote by

$$t \mapsto \gamma(t, \eta) = (x_1(t, \eta), \dots, x_n(t, \eta))$$

the geodesic with the initial covector $\eta \in U_{p_0}^* M$ at $t = 0$. The function $x_i(t, \eta)$ is uniquely determined as a smooth function when $b_i \neq a_i$ and $b_{i-1} \neq a_{i-1}$ for each i . In this case, the geodesic does not meet $J_i \cup J_{i-1}$, a part of the branch locus. If $b_i = a_i$, then the geodesic meets J_i and one gets more than one representations for $x_i(t, \eta)$ and $x_{i+1}(t, \eta)$ that are continuous at the branch point and smooth elsewhere. Note that $t \mapsto f_i(x_i(t, \eta))$ is uniquely determined in any case.

As before, we put

$$\sigma_i(t, \eta) = \int_0^t \left| \frac{df_i(x_i(t, \eta))}{dt} \right| dt .$$

We shall assign a real number $t_0(\eta) > 0$ to each $\eta \in U_{p_0}^* M$. First we consider the case which is *not* equal to any one of the following three cases: (i) the geodesic $\gamma(t, \eta)$ is totally contained in the submanifold N_n , i.e., $b_{n-1} = a_n$; (ii) $\gamma(t, \eta)$ is totally contained in the submanifold N_{n-1} and $f_n(x_n^0) = a_{n-1} = b_{n-1} < f_{n-1}(x_{n-1}^0)$; and (iii) $\gamma(t, \eta)$ is totally contained in the submanifold N_{n-1} and $p_0 \in J_{n-1}$, in particular, $f_n(x_n^0) = a_{n-1} = b_{n-1} = f_{n-1}(x_{n-1}^0)$. Then, define $t_0(\eta)$ by the formula

$$\sigma_n(t_0(\eta), \eta) = 2(a_{n-1}^- - a_n^+) .$$

In the cases (i) and (ii) listed above, we define $t_0(\eta)$ as follows: Let $Y(t)$ be the Jacobi field along the geodesic $\gamma(t, \eta)$ such that $Y(0) = 0$ and $Y'(0) = (\partial/\partial x_n)/|\partial/\partial x_n|$. Then $t = t_0(\eta)$ is the first positive time such that $Y(t) = 0$. In the case (iii) we define the Jacobi field $Y(t)$ along the geodesic $\gamma(t, \eta)$ such that $Y(0) = 0$ and $Y'(0)$ is the unit normal vector to N_{n-1} . Then $t = t_0(\eta)$ is the first positive time such that $Y(t) = 0$. It is easily seen that $x_n(t_0(\eta), \eta) = -x_n^0$, or $\frac{\alpha_n}{2} + x_n^0$ in any case.

It will be proved in Theorem 7.1 that the time $t = t_0(\eta)$ gives the cut point of p_0 along the geodesic $\gamma(t, \eta)$. In particular, it will become clear that $t_0(\eta)$ is a continuous function of $\eta \in U_{p_0}^* M$ and $p_0 \in M$. At this stage, we shall only prove a partial result.

PROPOSITION 6.1. *For any $\eta \in U_{p_0}^* M$ and $p_0 \in M$, there is a sequence η_k ($k = 1, 2, \dots$) of unit covectors such that the corresponding values b_1, \dots, b_{n-1} of H_1, \dots, H_{n-1} at η_k and a_0, \dots, a_n are all distinct for each k , and*

$$\lim_{k \rightarrow \infty} \eta_k = \eta, \quad \lim_{k \rightarrow \infty} t_0(\eta_k) = t_0(\eta) .$$

Proof. At each covector η which is not of the cases (i), (ii), (iii), the function $t_0(\eta)$ is clearly continuous, and we can find such $\{\eta_k\}$. For η of the cases (i) or (ii) we note that $t_0(\eta)$ is equal to the limit $\lim_{s \rightarrow 0} t_0(\eta_s)$, where $\eta_s \in U_{p_0}^*$ is a one-parameter family of covectors such that (i) $b_{n-1} = a_n + s^2$, (ii) $b_{n-1} = a_{n-1} + s^2$, and other b_j 's are the same value as those for $\eta = \eta_0$.

Now, for $\eta \in U_{p_0}^* M$ of the cases (ii), (iii), we first choose $\{\tilde{\eta}_k\} \in U_{p_k}^* M$ such that each $\tilde{\eta}_k$ is of the case (ii), $\tilde{\eta}_k \rightarrow \eta$ ($k \rightarrow \infty$), and the values b_1, \dots, b_{n-2} for each $\tilde{\eta}_k$ and a_0, \dots, a_n are all distinct. Then, for each k we choose $\eta_k \in U_{p_k}^* M$ in the one-parameter family of covectors given above whose limit is $\tilde{\eta}_k$ so that $\eta_k \rightarrow \eta$ as $k \rightarrow \infty$. The case (i) is similar. \square

For a while, we shall assume that $p_0 \notin J_{n-1}$. Put

$$U_+ = \{\eta \in U_{p_0}^* M \mid \xi_n(\eta) > 0\}$$

$$U_- = \{\eta \in U_{p_0}^* M \mid \xi_n(\eta) < 0\} .$$

Note that they are well-defined hemispheres under the assumption $p_0 \notin J_{n-1}$. Let $\eta' \in U_{p_0}^* M$ be the reflection image of $\eta \in U_{p_0}^* M$ with respect to the hyperplane H_n in $T_{p_0}^* M$ defined by $\xi_n = 0$, i.e., $\xi_n(\eta') = -\xi_n(\eta)$, $\xi_i(\eta') = \xi_i(\eta)$ ($1 \leq i \leq n-1$).

PROPOSITION 6.2. $\gamma(t_0(\eta'), \eta') = \gamma(t_0(\eta), \eta)$ for any $\eta \in U_+$.

Proof. It is enough to show this for covectors η such that b_i 's and a_j 's are all distinct. By (3.6) we have

$$\sum_{i=1}^n \int_0^{t_0(\eta)} \frac{(-1)^i G(f_i) A(f_i)}{\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} \left| \frac{df_i(x_i(t, \eta))}{dt} \right| dt = 0$$

for any polynomial $G(\lambda)$ of degree $\leq n-2$. By using the variables σ_i given above, this formula is rewritten as

$$(6.1) \quad \sum_{i=1}^n \int_0^{\sigma_i(t_0(\eta), \eta)} \frac{(-1)^i G(f_i) A(f_i)}{\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} d\sigma_i = 0 .$$

Note that

$$\begin{aligned}
 (6.2) \quad & \int_0^{\sigma_n(t_0(\eta), \eta)} \frac{(-1)^n G(f_n) A(f_n)}{\sqrt{-\prod_{k=1}^{n-1} (f_n - b_k) \cdot \prod_{k=0}^n (f_n - a_k)}} d\sigma_n \\
 &= 2 \int_{a_n^+}^{a_{n-1}^-} \frac{(-1)^n G(\lambda) A(\lambda)}{\sqrt{-\prod_{k=1}^{n-1} (\lambda - b_k) \cdot \prod_{k=0}^n (\lambda - a_k)}} d\lambda .
 \end{aligned}$$

Since the values of each b_i are the same for the two covectors η and η' , and since $\sigma_n(t_0(\eta), \eta) = 2(a_{n-1}^- - a_n^+) = \sigma_n(t_0(\eta'), \eta')$, we then have

$$\begin{aligned}
 (6.3) \quad & \sum_{i=1}^{n-1} \int_0^{\sigma_i(t_0(\eta), \eta)} \frac{(-1)^i G(f_i) A(f_i)}{\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} d\sigma_i \\
 &= \sum_{i=1}^{n-1} \int_0^{\sigma_i(t_0(\eta'), \eta')} \frac{(-1)^i G(f_i) A(f_i)}{\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} d\sigma_i .
 \end{aligned}$$

Now, let I be the set of $i \in \{1, \dots, n-1\}$ such that

$$\sigma_i(t_0(\eta), \eta) > \sigma_i(t_0(\eta'), \eta') .$$

Then, as we shall prove in the next lemma, there is a polynomial $G(\lambda)$ of degree $\leq n-2$ such that $(-1)^i G(\lambda) > 0$ for $\lambda \in (a_i^+, a_{i-1}^-)$, $i \in I$, and $(-1)^i G(\lambda) < 0$ for $\lambda \in (a_i^+, a_{i-1}^-)$, $i \notin I$, if $I \neq \emptyset$. With such $G(\lambda)$, the formula (6.3) clearly yields a contradiction. Therefore, $I = \emptyset$ and

$$\sigma_i(t_0(\eta), \eta) = \sigma_i(t_0(\eta'), \eta') .$$

for every $1 \leq i \leq n-1$. This indicates

$$x_i(t_0(\eta), \eta) = x_i(t_0(\eta'), \eta') .$$

for any $1 \leq i \leq n$, and therefore $\gamma(t_0(\eta'), \eta') = \gamma(t_0(\eta), \eta)$. \square

LEMMA 6.3. *Suppose b_i 's and a_i 's are all distinct. Let I_1 be a subset of $\{1, \dots, n\}$ and let I_2 be its complement. Assume both I_1 and I_2 are nonempty. Then there is a polynomial $G(\lambda)$ of degree $\leq n-2$ such that*

$$(-1)^i G(\lambda) \begin{cases} > 0 & \text{for } \lambda \in (a_i^+, a_{i-1}^-), i \in I_1 \\ < 0 & \text{for } \lambda \in (a_i^+, a_{i-1}^-), i \in I_2 \end{cases} .$$

Proof. Assume $1 \in I_1$. We put

$$G(\lambda) = -\prod (\lambda - b_k) ,$$

where the product are taken over all such $k \in \{1, \dots, n-1\}$ that both k and $k+1$ belongs to I_1 or that both k and $k+1$ belongs to I_2 . Since both I_1 and I_2 are nonempty, it follows that $\deg G \leq n-2$. Also, it is clear that the signs of the function $G(\lambda)$ is different on the two intervals (a_k^+, a_{k-1}^-) and (a_{k+1}^+, a_k^-) if and only if $\lambda - b_k$ is a factor of $G(\lambda)$, i.e., k and $k+1$ belong to the same group. Since $-G(\lambda) > 0$

on (a_1^+, a_0^-) , it follows that this $G(\lambda)$ has the desired property. In case $1 \in I_2$, then $-G(\lambda)$ possesses the desired property. \square

PROPOSITION 6.4. $t_0(\eta) = t_0(\eta')$ for any $\eta \in U_{p_0}^* M$.

Proof. By (3.8) we have

$$(6.4) \quad t_0(\eta) = \sum_{i=1}^n \int_0^{\sigma_i(t_0(\eta), \eta)} \frac{(-1)^{i+1} A(f_i) \prod_{k=1}^{n-1} (f_i - a_k)}{2\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} d\sigma_i.$$

Since $\sigma_i(t_0(\eta), \eta) = \sigma_i(t_0(\eta'), \eta')$ for any i by Proposition 6.2, it therefore follows that $t_0(\eta) = t_0(\eta')$. \square

PROPOSITION 6.5. *Suppose that the geodesic $\gamma(t, \eta)$ does not totally contained in any N_j for any $0 \leq j \leq n$. Then, $\sigma_i(t_0(\eta), \eta) < 2(a_{i-1}^- - a_i^+)$ for any $i \leq n - 1$ such that $b_i \neq b_{i-1}$.*

Proof. The assumption implies that there is no i such that $b_i = a_{i+1}$ or $b_{i+1} = a_i$. First, suppose that b_1, \dots, b_{n-1} and a_0, \dots, a_n are all distinct. Let I_1 be the set of $i \in \{1, \dots, n - 1\}$ such that $\sigma_i(t_0(\eta), \eta) \geq 2(a_{i-1}^- - a_i^+)$. Assume that $I_1 \neq \emptyset$. Put $I_2 = \{1, \dots, n\} - I_1$. Note that $n \in I_2$. For these I_1 and I_2 , let $G(\lambda)$ be the polynomial given in the proof of Lemma 6.3. Then we have

$$(6.5) \quad \begin{aligned} & 2 \sum_{i=1}^n \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^i G(\lambda) A(\lambda) d\lambda}{\sqrt{-\prod_{k=1}^{n-1} (\lambda - b_k) \cdot \prod_{k=0}^n (\lambda - a_k)}} \\ &= - \sum_{i \in I_1} \int_{2(a_{i-1}^- - a_i^+)}^{\sigma_i(t_0(\eta), \eta)} \frac{(-1)^i G(f_i) A(f_i)}{\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} d\sigma_i \\ &+ \sum_{i \in I_2 - \{n\}} \int_{\sigma_i(t_0(\eta), \eta)}^{2(a_{i-1}^- - a_i^+)} \frac{(-1)^i G(f_i) A(f_i)}{\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} d\sigma_i. \end{aligned}$$

Here, the polynomial $G(\lambda)$ is of the form

$$G(\lambda) = \begin{cases} -\prod_{k \in K} (\lambda - b_k) & (\text{if } 1 \in I_1) \\ \prod_{k \in K} (\lambda - b_k) & (\text{if } 1 \in I_2) \end{cases},$$

where K is the subset of $\{1, \dots, n - 1\}$ such that $k \in K$ means k and $k + 1$ belong to the same group, i.e., $k, k + 1 \in I_1$, or $k, k + 1 \in I_2$. Therefore, $n - 1 - \#K$ is the number of such $k \in \{1, \dots, n - 1\}$ that k and $k + 1$ belong to the different groups. Since $n \in I_2$, it follows that

$$n - 1 - \#K \text{ is } \begin{cases} \text{odd} & \text{if } 1 \in I_1 \\ \text{even} & \text{if } 1 \in I_2. \end{cases}$$

Therefore, by Proposition 4.1 (1) it follows that the first line in the formulas (6.5) is positive, while the second and the third lines are nonpositive, which is a contradiction. Thus I_1 must be empty, and the proposition follows.

Next, we shall consider the case where $b_{j-1} = b_j$ for several j , but other b_k and a_k are all distinct. In this case, we define the subset I_1 of $\{1, \dots, n - 1\}$ as follows:

For k with $b_{k-1} \neq b_k$, $k \in I_1$ if and only if $\sigma_k(t_0(\eta), \eta) \geq 2(a_{k-1}^- - a_k^+)$; for k with $b_{k-1} = b_k$, $k \in I_1$ if and only if $k-1 \in I_1$ or $k+1 \in I_1$. Note that $b_{k-1} < b_{k-2}$ and $b_{k+1} < b_k$ if $b_k = b_{k-1}$.

Then, by the same way as above, we define the sets I_2 , K and the polynomial $G(\lambda)$. Put

$$J = \{j \mid b_j < b_{j-1}, 1 \leq j \leq n-1\}.$$

Since $k-1 \in K$ or $k \in K$ if $b_k = b_{k-1}$, we then have, instead of (6.5), the following formula:

$$(6.6) \quad \begin{aligned} & 2 \sum_{i \in J} \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^i G(\lambda) A(\lambda) d\lambda}{\sqrt{-\prod_{k=1}^{n-1} (\lambda - b_k) \cdot \prod_{k=0}^n (\lambda - a_k)}} \\ &= - \sum_{i \in I_1 \cap J} \int_{2(a_{i-1}^- - a_i^+)}^{\sigma_i(t_0(\eta), \eta)} \frac{(-1)^i G(f_i) A(f_i)}{\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} d\sigma_i \\ &+ \sum_{i \in I_2 \cap J} \int_{\sigma_i(t_0(\eta), \eta)}^{2(a_{i-1}^- - a_i^+)} \frac{(-1)^i G(f_i) A(f_i)}{\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} d\sigma_i. \end{aligned}$$

If $I_1 \cap J \neq \emptyset$, then we have a contradiction by the same reason as above.

Finally, let us further assume that $b_i = a_i$ for some i . In this case, the times t such that $f_i(x_i(t, \eta)) = a_i$ and those such that $f_{i+1}(x_{i+1}(t, \eta)) = a_i$ coincide. Therefore, in each side of the formula (6.5) or (6.6), the sum of the integrals in σ_i and σ_{i+1} remains finite, and the arguments above are also effective in this case. \square

PROPOSITION 6.6. *Suppose that the geodesic $\gamma(t, \eta)$ does not totally contained in any N_k . For a fixed j with $b_j = b_{j-1}$, let $\theta_{s_1}(s_2)$ be the value defined in the formula (5.7) in the previous section. Then, $\theta_0(t_0(\eta)) < \pi$ for such j .*

Proof. By (5.7) we have

$$\begin{aligned} & \sum_{\substack{1 \leq l \leq n \\ l \neq j}} \int_0^{\sigma_l(s)} \frac{(-1)^l G_{j,j-1}(f_l) A(f_l) d\sigma_l}{|f_l - \lambda_j^0| \sqrt{-\prod_{k \neq j, j-1} (f_l - b_k) \cdot \prod_{k=0}^n (f_l - a_k)}} \\ &+ 2\theta_0(s) \frac{(-1)^j G_{j,j-1}(\lambda_j^0) A(\lambda_j^0)}{\sqrt{\prod_{k \neq j, j-1} (\lambda_j^0 - b_k) \prod_k (\lambda_j^0 - a_k)}} = 0. \end{aligned}$$

Also, taking a limit $a_j^+, a_{j-1}^- \rightarrow \lambda_j^0$ in Lemma 4.2, we have

$$\begin{aligned} & \sum_{\substack{1 \leq l \leq n \\ l \neq j}} \int_0^{2(a_{i-1}^- - a_i^+)} \frac{(-1)^l G_{j,j-1}(f_l) A(\lambda_j^0) d\sigma_l}{|f_l - \lambda_j^0| \sqrt{-\prod_{k \neq j, j-1} (f_l - b_k) \cdot \prod_{k=0}^n (f_l - a_k)}} \\ &+ 2\pi \frac{(-1)^j G_{j,j-1}(\lambda_j^0) A(\lambda_j^0)}{\sqrt{\prod_{k \neq j, j-1} (\lambda_j^0 - b_k) \prod_k (\lambda_j^0 - a_k)}} = 0. \end{aligned}$$

Therefore we obtain the following formula:

$$\begin{aligned} & \sum_{\substack{1 \leq l \leq n \\ l \neq j}} \int_{\sigma_l(s)}^{2(a_{l-1}^- - a_l^+)} \frac{(-1)^l G_{j,j-1}(f_l) A(f_l) d\sigma_l}{|f_l - \lambda_j^0| \sqrt{-\prod_{k \neq j, j-1} (f_l - b_k) \cdot \prod_{k=0}^n (f_l - a_k)}} \\ & - \sum_{\substack{1 \leq l \leq n \\ l \neq j}} \int_0^{2(a_{l-1}^- - a_l^+)} \frac{(A(f_l) - A(\lambda_j^0)) (-1)^l G_{j,j-1}(f_l) d\sigma_l}{|f_l - \lambda_j^0| \sqrt{-\prod_{k \neq j, j-1} (f_l - b_k) \cdot \prod_{k=0}^n (f_l - a_k)}} \\ & + 2(\pi - \theta_0(s)) \frac{(-1)^j G_{j,j-1}(\lambda_j^0) A(\lambda_j^0)}{\sqrt{\prod_{k \neq j, j-1} (\lambda_j^0 - b_k) \prod_k (\lambda_j^0 - a_k)}} = 0 . \end{aligned}$$

We put $s = t_0(\eta)$. The first line of this formula is nonpositive by the previous proposition. Also, applying the $(n - 1)$ -dimensional version of Proposition 4.1 (1) to the positive function

$$(A(\lambda) - A(\lambda_j^0)) / (\lambda - \lambda_j^0) ,$$

the second line is negative. Since $(-1)^j G_{j,j-1}(\lambda_j^0) > 0$, it thus follows that $\theta_0(t_0(\eta)) < \pi$. \square

As a consequence, we have the following proposition.

PROPOSITION 6.7. *Suppose that the geodesic $\gamma(t, \eta)$ does not totally contained in any N_k . Then:*

- (1) *There is no conjugate point of p_0 along the geodesic $\gamma(t, \eta)$ in the interval $0 < t < t_0(\eta)$.*
- (2) *$\gamma(t_0(\eta), \eta)$ is not a conjugate point of p_0 along the geodesic $\gamma(t, \eta)$, unless $b_{n-1}(= H_{n-1}(\eta)) = f_n(x_n^0)$.*
- (3) *If $b_{n-1} = f_n(x_n^0)$, then $\gamma(t_0(\eta), \eta)$ is a conjugate point of p_0 along the geodesic $\gamma(t, \eta)$ with multiplicity one.*

Proof. (1) and (2) follow from all results in §5 and Propositions 6.5 and 6.6. Now, let us prove (3). Since $f_n(x_n^0) = b_{n-1}$, it follows from Corollary 5.2 (1) that $Y_{n-1,0}(t_0(\eta)) = 0$. Hence $\gamma(t_0(\eta), \eta)$ is a conjugate point of p_0 along the geodesic $\gamma(t, \eta)$. Now we show that $Y_{j,0}(t_0(\eta)) \neq 0$ (or, $Z_{j,0}(t_0(\eta)) \neq 0$) for any $j \leq n - 2$. First, suppose that $b_j \neq b_{j-1}$ for any j . For $k \leq n - 2$ with $b_k \neq f_k(x_k^0)$, $f_{k+1}(x_{k+1}^0)$, we have $Y_{k,0}(t_0(\eta)) \neq 0$ by Propositions 6.5 and 5.3. If $b_k = f_k(x_k^0)$ or $f_{k+1}(x_{k+1}^0)$, then again we have $Y_{k,0}(t_0(\eta)) \neq 0$ by Proposition 6.5 and Corollary 5.2 (1). In case $b_j = b_{j-1}$ for some j , we also have $Z_{j,0}(t_0(\eta)) \neq 0$ and $Z_{j-1,0}(t_0(\eta)) \neq 0$ in the same way as above by Proposition 6.6. \square

7. Cut locus (1). Let M be a Liouville manifold (diffeomorphic to S^n) constructed from constants $a_0 > \dots > a_n > 0$ and a positive function $A(\lambda)$ on the interval $a_n \leq \lambda \leq a_0$ as explained in §2. We assume that the function $A(\lambda)$ satisfies the conditions (4.1), i.e.,

$$(-1)^{k-1} A^{(k)}(\lambda) > 0 \quad \text{on } [a_n, a_0] \quad (1 \leq k \leq n - 1) .$$

We also assume that $n = \dim M \geq 3$ in the following theorem, since it is necessary in some part of our proof, and since the two-dimensional case is treated in another paper [11].

Let p_0 be a point as in §6. Let N be the subset of M represented by $x_n = \frac{\alpha_n}{2} + x_n^0$ or $-x_n^0$, which is a submanifold of M diffeomorphic to the $(n-1)$ -sphere if $0 \leq x_n^0 < \alpha_n/4$, and which is a submanifold with boundary diffeomorphic to closed $(n-1)$ -disk if $x_n^0 = \alpha_n/4$. Let $t_0(\eta)$ be the value defined in the previous section.

THEOREM 7.1.

- (1) *The cut point of p_0 along the geodesic $\gamma(t, \eta)$ is given by $t = t_0(\eta)$ for any $p_0 \in M$ and $\eta \in U_{p_0}^* M$.*
- (2) *Suppose $p_0 \notin J_{n-1}$. Then, the assignment $\eta \mapsto \gamma(t_0(\eta), \eta)$ gives a homeomorphism from $\overline{U_+}$ to its image $C(p_0)$, the cut locus of p_0 , and it gives C^∞ embeddings of U_+ and $\partial\overline{U_+}$ respectively. In particular, $C(p_0)$ is diffeomorphic to an $(n-1)$ -closed disk, and it is contained in (the interior of) N . Also, for each $\eta \in \partial\overline{U_+}$, $\gamma(t_0(\eta), \eta)$ is the first conjugate point of p_0 of multiplicity one along the geodesic $t \mapsto \gamma(t, \eta)$.*
- (3) *Suppose $p_0 \in J_{n-1}$. Then the cut locus $C(p_0)$ coincides with the cut locus of p_0 in the totally geodesic submanifold N_{n-1} , which is a smoothly embedded $(n-2)$ -disk in J_{n-1} . For each interior point q of $C(p_0)$ there is an S^1 -family of minimal geodesics joining p_0 and q ; the tangent vectors of those geodesics at p_0 form a cone whose orthogonal projection to $T_{p_0} J_{n-1}$ is one-dimensional. For each boundary point q of $C(p_0)$, there is a unique minimal geodesic from p_0 to q , and along it q is the first conjugate point of p_0 of multiplicity two.*

In this and the next two sections, we shall prove this theorem. The proof will be divided into five cases: (I) $p_0 \notin N_k$ for any k ; (II) $0 < x_n^0 < \alpha_n/4$, but $p_0 \in N_l$ for some l ; (III) $x_n^0 = 0$; (IV) $x_n^0 = \alpha_n/4$, and $p_0 \notin J_{n-1}$; (V) $p_0 \in J_{n-1}$. In this section we shall consider the case (I) and prove (1) and (2) of the theorem in this case. The proofs for the cases (II) \sim (V) will be given in the next two sections.

For each $\eta \in U_-$, let $t_-(\eta)$ be the first positive time t such that $x_n(t, \eta) = -x_n^0$. Define the mapping $\Phi : U_{p_0}^* M \rightarrow N$ by

$$\Phi(\eta) = \begin{cases} \gamma(t_0(\eta), \eta) & (\eta \in \overline{U_+}) \\ \gamma(t_-(\eta), \eta) & (\eta \in \overline{U_-}) \end{cases}.$$

Then, $\Phi(\eta) \in N$ is the first point where the geodesic $\gamma(t, \eta)$ meets N for any η . We shall prove that Φ is a homeomorphism. To do so, we need several lemmas.

Take a point p'_0 represented as $(x_1^0, \dots, x_{n-1}^0, x_n^1)$, where $0 \leq x_n^1 < x_n^0 < \alpha_n/4$. Let U'_+ be the hemisphere of $U_{p'_0}^* M$ defined by $\xi_n > 0$. We define the mapping $\psi : \overline{U_+} \rightarrow U'_+$ so that it preserves the values b_i of H_i ($1 \leq i \leq n-1$), i.e., by $\psi(p_0; \xi_1, \dots, \xi_n) = (p'_0; \tilde{\xi}_1, \dots, \tilde{\xi}_n)$, where

$$\tilde{\xi}_i = \xi_i \quad (1 \leq i \leq n-1), \quad \tilde{\xi}_n = \sqrt{(-1)^{n-1} \prod_{k=1}^{n-1} (f_n(x_n^1) - b_k)}.$$

Note that b_k 's are functions of $(p_0; \xi_1, \dots, \xi_n) \in \overline{U_+}$. Since $b_{n-1} \geq f_n(x_n^0) > f_n(x_n^1)$, the image $\psi(\overline{U_+})$ is contained in the interior U'_+ . Let N' be the submanifold of M defined by $x_n = -x_n^1$, and define the diffeomorphism $\Psi : N \rightarrow N'$ by

$$\Psi(x_1, \dots, x_{n-1}, -x_n^0) = (x_1, \dots, x_{n-1}, -x_n^1).$$

We also define $\tilde{\Phi} : U'_+ \rightarrow N'$ in the same way as $\Phi|_{\overline{U_+}}$.

LEMMA 7.2. $\Psi(\Phi(\eta)) = \tilde{\Phi}(\psi(\eta))$ for any $\eta \in \overline{U}_+$.

Proof. We write $\psi(\eta) = \tilde{\eta}$ for simplicity. For the geodesics $\gamma(t, \eta)$ and $\gamma(t, \tilde{\eta})$, we have the equality (6.1) and the similar one. Taking the equality (6.2) into account, we have the similar formula as (6.3):

$$\begin{aligned} & \sum_{i=1}^{n-1} \int_0^{\sigma_i(t_0(\eta), \eta)} \frac{(-1)^i G(f_i) A(f_i)}{\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} d\sigma_i \\ &= \sum_{i=1}^{n-1} \int_0^{\sigma_i(t_0(\tilde{\eta}), \tilde{\eta})} \frac{(-1)^i G(f_i) A(f_i)}{\sqrt{-\prod_{k=1}^{n-1} (f_i - b_k) \cdot \prod_{k=0}^n (f_i - a_k)}} d\sigma_i . \end{aligned}$$

Therefore, in the same way as the proof of Proposition 6.2, we have $\sigma_i(t_0(\tilde{\eta}), \tilde{\eta}) = \sigma_i(t_0(\eta), \eta)$ and hence $x_i(t_0(\tilde{\eta}), \tilde{\eta}) = x_i(t_0(\eta), \eta)$ for any $i \leq n - 1$. Thus we have $\gamma(t_0(\tilde{\eta}), \tilde{\eta}) = \Psi(\gamma(t_0(\eta), \eta))$. By the formula (6.4) we also have $t_0(\tilde{\eta}) = t_0(\eta)$. \square

By Proposition 6.7, we know that $\Phi|_{U_+}$ is a local diffeomorphism and so is true for the initial point p'_0 . Therefore it follows from the above lemma that $\Phi|_{\overline{U}_+}$ is a local homeomorphism and $\Phi|_{\partial\overline{U}_+}$ is a local diffeomorphism. For the mapping Φ on \overline{U}_- , we have the following

LEMMA 7.3. $\Phi|_{\overline{U}_-}$ is a C^1 local diffeomorphism.

Proof. By Proposition 6.7 and by the above observation, we know that $\Phi|_{U_-}$ and $\Phi|_{\partial\overline{U}_-}$ ($= \Phi|_{\partial\overline{U}_+}$) are C^∞ immersions. Let $\{\eta_s\}$ be a one-parameter family of unit covectors at p_0 such that $\eta_s \in U_-$ ($s > 0$), $\eta_0 \in \partial\overline{U}_-$, and $\dot{\eta}_s = (\partial/\partial\nu_{n-1})/|\partial/\partial\nu_{n-1}|$, where the variable ν_{n-1} is the one defined in §5. We shall show that $\Phi|_{\overline{U}_-}$ is of class C^1 and a local diffeomorphism at η_0 .

Differentiating the equality

$$\sum_{l=1}^n \int_0^{\sigma_l(t_-(\eta_s), \eta_s)} \frac{(-1)^l G_{n-1}(f_l) A(f_l) d\sigma_l}{\sqrt{-\prod_{k=1}^{n-1} (f_l - b_k) \cdot \prod_{k=0}^n (f_l - a_k)}} = 0$$

in s , one obtains

$$(7.1) \quad \begin{aligned} 0 &= \beta(c_s Y_{n-1,0}(t_-(\eta_s)) + \frac{\partial}{\partial s} t_-(\eta_s) \cdot \dot{\gamma}(t_-(\eta_s), \eta_s)) \\ &- \nu_{n-1} \sum_{l=1}^n \int_0^{\sigma_l(t_-(\eta_s), \eta_s)} \frac{(-1)^l G_{n-1}(f_l) A(f_l) d\sigma_l}{(f_l - b_{n-1}) \sqrt{-\prod_{k=1}^{n-1} (f_l - b_k) \cdot \prod_{k=0}^n (f_l - a_k)}} , \end{aligned}$$

where $c_s = \pm|\partial/\partial\nu_{n-1}|$ at η_s and β is the 1-form;

$$\beta = \sum_{l=1}^{n-1} \frac{\epsilon'_l (-1)^l G_{n-1}(f_l(x_l)) A(f_l(x_l))}{\sqrt{-\prod_{k=1}^{n-1} (f_l(x_l) - b_k) \cdot \prod_{k=0}^n (f_l(x_l) - a_k)}} d(f_l(x_l)) .$$

Then, taking the limit $s \searrow 0$, we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial s} t_-(\eta_s)|_{s=0} \beta(\dot{\gamma}(t_-(\eta_0), \eta_0)) \\ &+ \frac{4\epsilon'_n (-1)^n G_{n-1}(b_{n-1}) A(b_{n-1})}{\sqrt{-\prod_{k \neq n-1} (b_{n-1} - b_k) \cdot \prod_{k=0}^n (b_{n-1} - a_k)}} . \end{aligned}$$

Noting that the covector $\flat(\dot{\gamma}(t_-(\eta_0), \eta_0))$ is equal to

$$\frac{1}{2} \sum_{l=1}^{n-1} \frac{\epsilon_l' (-1)^{l+1} A(f_l(x_l)) \prod_{k=1}^{n-1} (f_l(x_l) - b_k)}{\sqrt{-\prod_{k=1}^{n-1} (f_l(x_l) - b_k) \cdot \prod_{k=0}^n (f_l(x_l) - a_k)}} d(f_l(x_l))$$

at $\gamma(t_-(\eta_0), \eta_0)$, we see that

$$\frac{1}{b_1 - b_{n-1}} < -\beta(\dot{\gamma}(t_-(\eta_0), \eta_0)) < \frac{1}{b_{n-2} - b_{n-1}}.$$

This indicates that $(\partial/\partial s)t_-(\eta_s)|_{s=0}$ is finite and nonzero.

Also, by similar formulas to (7.1), the derivatives of $\gamma(t_-(\eta), \eta)$ by the normalized $\partial/\partial H_j$ ($j \leq n - 2$) are of the form $Y_{j,0}(t_-(\eta)) + c_\eta \dot{\gamma}(t_-(\eta), \eta)$ (or $Z_{j,0}(t_-(\eta)) + c_\eta \dot{\gamma}(t_-(\eta), \eta)$) $\in T_{\gamma(t_-(\eta), \eta)}N$, which are continuous in η near the boundary $\partial\overline{U_-}$. Therefore the mapping $\Phi|_{\overline{U_-}}$ is of class C_1 and the lemma follows. \square

COROLLARY 7.4. $\Phi : U_{p_0}^*M \rightarrow N$ is a homeomorphism.

Proof. The above lemma implies that $\Phi|_{\overline{U_-}}$ is a local homeomorphism. Thus, combined with the above result, we see that $\Phi : U_{p_0}^*M \rightarrow N$ is a local homeomorphism. Since both $U_{p_0}^*M$ and N are homeomorphic to the $(n - 1)$ -sphere, and since $n \geq 3$, it therefore follows that Φ is really a homeomorphism. \square

We shall prove that the image of the map $\overline{U_+} \ni \eta \mapsto \gamma(t_0(\eta), \eta)$ is just the cut locus of p_0 . Let us temporarily denote this image by \mathcal{C} . Note that, for any $\eta \in U_{p_0}^*M$, the cut point of p_0 along the geodesic $\gamma(t, \eta)$ will appear at $t \leq t_0(\eta)$, because of Propositions 6.4 and 6.2. In particular, putting

$$V = \{t\eta \in T_{p_0}^*M \mid \eta \in U_{p_0}^*M, 0 \leq t < t_0(\eta)\},$$

we have the following lemma. Put $\text{Exp}_{p_0}(t\eta) = \gamma(t, \eta)$.

LEMMA 7.5.

- (1) $\text{Exp}_{p_0} : \overline{V} \rightarrow M$ is surjective.
- (2) $\text{Exp}_{p_0}(V) \cap \mathcal{C} = \emptyset$.

Proof. Let $q \in M$ be any point ($\neq p_0$) and let $\gamma(t, \eta)$ ($0 \leq t \leq T$) be a minimal geodesic joining p_0 and q ($\eta \in U_{p_0}^*M$). Since $T \leq t_0(\eta)$, (1) follows. Next, assume that there is some $\eta \in U_{p_0}^*M$ and $0 < T < t_0(\eta)$ such that $\gamma(T, \eta) \in \mathcal{C}$. Then, $x_n(T, \eta) = -x_n^0$ or $\frac{\alpha_n}{2} + x_n^0$. Note that, if $\eta \in \overline{U_+}$, then $t = t_0(\eta)$ is the first positive time when $x_n(T, \eta) = -x_n^0$ or $\frac{\alpha_n}{2} + x_n^0$. Thus we have $\eta \in U_-$ and $T = t_-(\eta)$. But, as we have proved in the previous corollary, $\gamma(T, \eta) \notin \mathcal{C}$ in this case, a contradiction. Thus (2) follows. \square

Fix $\eta \in U_{p_0}^*M$ and suppose that the cut point of p_0 along the geodesic $\gamma(t, \eta)$ appear before $t = t_0(\eta)$, i.e., the geodesic segment $\gamma(t, \eta)$ ($0 \leq t \leq t_0(\eta)$) is no longer minimal. Then there is another minimal geodesic $\gamma(t, \bar{\eta})$ ($0 \leq t \leq T$) joining p_0 and $q = \gamma(t_0(\eta), \eta)$, $\bar{\eta} \in U_{p_0}^*M$.

Since the geodesic segment $\gamma(t, \bar{\eta})$ ($0 \leq t \leq T$) is minimal, we have $T \leq t_0(\bar{\eta})$. Also, since $\gamma(T, \bar{\eta}) = q \in \mathcal{C}$, we have $T = t_0(\bar{\eta})$ by Lemma 7.5 (2). Then, by the injectivity of Φ we have $\bar{\eta} = \eta$ or η' . But this implies that the geodesic segment $\gamma(t, \eta)$ ($0 \leq t \leq t_0(\eta)$) is minimal, a contradiction. Thus $t = t_0(\eta)$ gives the cut point of p_0 along the geodesic $\gamma(t, \eta)$. This completes the proof of (1) and (2) of the theorem in the case where $0 < x_i^0 < \alpha_n/4$ for any i .

8. Cut locus (2). In this section, we shall give a proof of Theorem 7.1 for the case (II) described in the previous section. The cases (III) ~ (V) will be considered in the next section. Note that the statement (1) of the theorem holds for any p_0 and any $\eta \in U_{p_0}^*M$, which is a consequence of the results in the previous section, Proposition 6.1, and the continuous dependence of cut points on the initial covectors. Thus we shall prove (2) for the cases (II) ~ (IV) and (3) for the case (V).

Now, let us consider the case (II); $0 < x_n^0 < \alpha_n/4$ and $p_0 \in N_l$ for some $l \leq n-1$. As in the previous section, we shall show that $\Phi : U_{p_0}^*M \rightarrow N$ is a homeomorphism.

PROPOSITION 8.1. *Suppose $p_0 \in N_l$ and let $\eta \in U_{p_0}^*M$ be a covector such that the geodesic $\gamma(t, \eta)$ is totally contained in N_l . Let $Y_l(t)$ be a nonzero Jacobi field along the geodesic $\gamma(t, \eta)$ such that $Y_l(0) = 0$ and $Y_l(t)$ is orthogonal to N_l everywhere. Then, $Y_l(t_0(\eta)) \neq 0$.*

The proof will be given below. This proposition together with Proposition 6.7 applied to the intersection of the Liouville manifolds N_l in which the geodesic is contained show that the mapping $\Phi|_{U_+}$ and $\Phi|_{\partial\overline{U_+}}$ are immersions. Then, in the same way as the previous section, we see that $\Phi|_{\overline{U_+}}$ is a local homeomorphism. On the other hand, since $t_0(\eta)$ represents the cut point, and since $t_-(\eta) < t_0(\eta)$, the mapping $\Phi|_{U_-}$ is a C^∞ embedding and $\Phi(U_-) \cap \Phi(\overline{U_+}) = \emptyset$. Also $\Phi(U_{p_0}^*M) = N$ by continuity. Therefore it follows that $\Phi : U_{p_0}^*M \rightarrow N$ is a homeomorphism. This indicates (2) of the theorem in this case.

In the rest of this section we shall prove Proposition 8.1. We may assume that there is only one such l that the geodesic is totally contained in N_l . According to the position of the geodesic $\gamma(t, \eta)$, there are four different cases: (i) the geodesic $\gamma(t, \eta)$ intersects J_l transversally; (ii) $\gamma(t, \eta)$ does not meet J_l ; (iii) $\gamma(t, \eta)$ is tangent to J_l , but not contained in it; (iv) $\gamma(t, \eta)$ is contained in J_l .

Let us begin with the case (i), and first assume $p_0 \notin J_l$. We may also assume $f_{l+1}(x_{l+1}^0) < b_l = a_l = f_l(x_l^0)$; the case where $f_{l+1}(x_{l+1}^0) = b_l = a_l < f_l(x_l^0)$ is similar. Note that $f_l(x_l^0) < b_{l-1}$ in this case, since the intersection of $\gamma(t, \eta)$ and J_l is transversal in N_l . Then the Jacobi field $Y_l(t)$ is given by the one-parameter family of geodesics $\{\gamma(t, \eta_s)\}$, where $\eta_s \in U_{p_0}^*M$ satisfies $\eta_0 = \eta$ and $H_l(\eta_s) = b_l - s^2$, $H_j(\eta_s) = b_j$ for $j \neq l$.

To show the proposition in this case, we use a technique similar to the one used in the proof of Lemma 7.2, which is as follows. Take a point p'_0 represented as $(x_1^0, \dots, x_l^1, \dots, x_n^0)$, where $0 = x_l^0 < x_l^1 < \alpha_l/4$ and $f_l(x_l^1) < b_{l-1}, a_{l-1}$. Let U'_{l-} be the hemisphere of $U_{p'_0}^*M$ defined by $\xi_l < 0$ and so be U_{l-} in $U_{p_0}^*M$. Taking a sufficiently small neighborhood W of η in $U_{p_0}^*M$, we define the mapping $\psi : \overline{U_{l-}} \cap W \rightarrow U'_{l-}$ so that it preserves the values of H_i ($1 \leq i \leq n-1$), i.e., by $\psi(p_0; \xi_1, \dots, \xi_n) = (p'_0; \tilde{\xi}_1, \dots, \tilde{\xi}_n)$, where

$$\tilde{\xi}_i = \xi_i \quad (i \neq l), \quad \tilde{\xi}_l = \sqrt{(-1)^{l-1} \prod_{k \neq l} (f_l(x_l^1) - H_k)}.$$

Note that H_k 's are functions of $(p_0; \xi_1, \dots, \xi_n) \in \overline{U_{l-}}$.

Let \tilde{x}_l^1 be the value of $x_l(t, \psi(\eta_s))$ at the time when $\sigma_l(t, \psi(\eta_s)) = 2(a_{l-1}^- - a_l^+)$, which is $-x_l^1$ or $x_l^1 + \alpha_l/2$. Also, \tilde{x}_l^0 is similarly defined. Let N' be the submanifold of M defined by $x_l = \tilde{x}_l^1$, and define the diffeomorphism $\Psi : N' \rightarrow N_l$ by

$$\Psi(x_1, \dots, \tilde{x}_l^1, \dots, x_n) = (x_1, \dots, \tilde{x}_l^0, \dots, x_n).$$

Then we have the following lemma. The proof being similar to that for Lemma 7.2, we omit.

LEMMA 8.2. $\Psi(\gamma(t_2(\psi(\eta_s)), \psi(\eta_s))) = \gamma(t_2(\eta_s), \eta_s)$ for any $s > 0$, where $t_2(\eta_s)$ denotes the time when $\sigma_l(t_2(\eta_s), \eta_s) = 2(a_{l-1}^- - a_l^+)$.

Since $t = t_2(\eta_s)$ is the first positive time when the geodesic $\gamma(t, \eta_s)$ reaches N_l again, it follows that $t_2(\eta_0) = \lim_{s \rightarrow 0} t_2(\eta_s)$ is the first positive time when the Jacobi field $Y_l(t)$ vanishes. Applying Proposition 6.7 to the geodesic $\gamma(t, \psi(\eta_0))$, we have $t_0(\psi(\eta_0)) < t_2(\psi(\eta_0))$. Since

$$\sigma_n(t_2(\psi(\eta_s)), \psi(\eta_s)) = \sigma_n(t_2(\eta_s), \eta_s),$$

we then have $\sigma_n(t_2(\eta_0), \eta_0) > 2(a_{n-1}^- - a_n^+)$, which implies $t_0(\eta_0) < t_2(\eta_0)$, and hence $Y_l(t_0(\eta_0)) \neq 0$.

Next, let us consider the case (i) with the condition $p_0 \in J_l$. Let $\eta_s \in U_{p_0}^* M$ be as above so that the geodesic $\gamma(t, \eta_0)$ is transversal to J_l in N_l . Then the family of geodesics $\{\gamma(t, \eta_s)\}_{s>0}$ coincides with the family $\{\gamma(t, \zeta_r(\eta_{s_0}))\}$ for a fixed $s_0 > 0$, where $\{\zeta_r\}$ is the one-parameter group of diffeomorphisms of $U^* M$ generated by X_{F_l} . Thus, in this case, the first positive time $t_2(\eta_0)$ when the Jacobi field $Y_l(t)$ vanishes has the property that

$$\gamma(t_2(\eta_0), \eta_s) = \gamma(t_2(\eta_0), \eta_0) \in J_l, \quad \sigma_l(t_2(\eta_0), \eta_s) = 2(a_{l-1}^- - a_l^+).$$

Now, let us consider N_l as an $(n - 1)$ -dimensional Liouville manifold constructed from the constants a_j ($j \neq l$) and the function $A(\lambda)$. Then the variables $f_l(x_l)$ and $f_{l+1}(x_{l+1})$ are connected to a single variable whose range is $[a_{l+1}, a_{l-1}]$, and the total variation of this variable along the geodesic $\gamma(t, \eta_0)$ ($0 \leq t \leq t_2(\eta_0)$) is equal to $2(a_{l-1}^- - a_{l+1}^+)$. Hence by Proposition 6.5 for the $(n - 1)$ -dimensional manifold N_l , we have $t_0(\eta_0) < t_2(\eta_0)$, and thus $Y_l(t_0(\eta_0)) \neq 0$.

Next, we shall consider the case (ii); the geodesic $\gamma(t, \eta)$ does not intersect J_l . There are two cases: $a_l = f_l(x_l(t, \eta)) = b_{l-1}$; $b_{l+1} = f_{l+1}(x_{l+1}(t, \eta)) = a_l$. The proofs for them are similar, so we may assume $a_l = b_{l-1}$. Note that $b_l < a_l$ in this case, since $\gamma(t, \eta)$ does not meet J_l . The Jacobi field $Y_l(t)$ is given by the one-parameter family of geodesics $\{\gamma(t, \eta_s)\}$, where $\eta_s \in U_{p_0}^* M$ satisfies $\eta_0 = \eta$ and $H_{l-1}(\eta_s) = a_l + s^2$, $H_j(\eta_s) = b_j$ for $j \neq l - 1$. Define $\theta_s(t)$ by the formula

$$f_l(x_l(t, \eta_s)) = a_l(\cos \theta_s(t))^2 + H_{l-1}(\eta_s)(\sin \theta_s(t))^2, \quad \theta_s(0) = 0$$

and put $\theta_0(t) = \lim_{s \rightarrow 0} \theta_s(t)$. Let $t_2(\eta)$ be the time such that $\theta_0(t_2(\eta)) = \pi$. Then $t = t_2(\eta)$ is the first positive time when $Y_l(t) = 0$. We shall show that $t_0(\eta) < t_2(\eta)$. We have

$$\begin{aligned} & \sum_{i \neq l} \int_0^{\sigma_i(t_2(\eta), \eta)} \frac{(-1)^i G_{l,l-1}(f_i) A(f_i) d\sigma_i}{|f_i - a_l| \sqrt{-\prod_{k \neq l-1} (f_i - b_k) \cdot \prod_{k \neq l} (f_i - a_k)}} \\ & + \frac{(-1)^l 2\pi G_{l,l-1}(a_l) A(a_l)}{\sqrt{-\prod_{k \neq l-1} (a_l - b_k) \cdot \prod_{k \neq l} (a_l - a_k)}} = 0. \end{aligned}$$

Also, a similar observation as in the proof of Lemma 4.2 indicates

$$\begin{aligned} & -2 \sum_{i \neq l} \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^i G_{l,l-1}(\lambda) A(a_l) d\lambda}{|\lambda - a_l| \sqrt{-\prod_{k \neq l-1} (\lambda - b_k) \cdot \prod_{k \neq l} (\lambda - a_k)}} \\ &= \frac{(-1)^l 2\pi G_{l,l-1}(a_l) A(a_l)}{\sqrt{-\prod_{k \neq l-1} (a_l - b_k) \cdot \prod_{k \neq l} (a_l - a_k)}}. \end{aligned}$$

Thus we have the formula:

$$\begin{aligned} (8.1) \quad & 2 \sum_{i \neq l} \int_{a_i^+}^{a_{i-1}^-} \frac{A(\lambda) - A(a_l)}{|\lambda - a_l|} \frac{(-1)^i G_{l,l-1}(\lambda) d\lambda}{\sqrt{-\prod_{k \neq l-1} (\lambda - b_k) \cdot \prod_{k \neq l} (\lambda - a_k)}} \\ &= \sum_{i \neq l} \int_{\sigma_i(t_2(\eta), \eta)}^{2(a_{i-1}^- - a_i^+)} \frac{(-1)^i G_{l,l-1}(f_i) A(f_i) d\sigma_i}{|f_i - a_l| \sqrt{-\prod_{k \neq l-1} (f_i - b_k) \cdot \prod_{k \neq l} (f_i - a_k)}}. \end{aligned}$$

Take a sufficiently large constant $c > 0$ and put

$$B(\lambda) = c - \frac{A(\lambda) - A(a_l)}{\lambda - a_l}, \quad [i] = i \ (i < l), \quad [i] = i - 1 \ (i > l).$$

Then, by Lemma 4.2 ($(n - 1)$ -dimensional case), the left-hand side of the formula (8.1) is rewritten as

$$2 \sum_{[i]=1}^{n-1} \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^{[i]+1} G_{l,l-1}(\lambda) B(\lambda) d\lambda}{\sqrt{-\prod_{[k]=1}^{n-1} ((\lambda - a_k^-)(\lambda - a_k^+))}}.$$

Since $B(\lambda)$ satisfies the condition (4.1), the above value is positive by Proposition 4.1 (1) ($(n - 1)$ -dimensional case). If $t_0(\eta) \geq t_2(\eta)$, then, applying Proposition 6.5 to the Liouville manifold N_l , we have $\sigma_i(t_2(\eta), \eta) \leq 2(a_{i-1}^- - a_i^+)$ for any $i \neq l$. This indicates that the right-hand side of the formula (8.1) is nonpositive, a contradiction. Therefore, it follows that $t_0(\eta) < t_2(\eta)$, and $Y_l(t_0(\eta)) \neq 0$.

Next, we shall consider the case (iii); $\gamma(t, \eta)$ is tangent to J_l , but not contained in it. First, we assume $p_0 \notin J_l$. In this case, it holds that either $f_{l+1}(x_{l+1}^0) < b_l = a_l = f_l(x_l^0) = b_{l-1}$ or $b_{l+1} = f_{l+1}(x_{l+1}^0) = a_l = b_l < f_l(x_l^0)$. Since the proofs are similar, we may assume

$$f_{l+1}(x_{l+1}^0) < b_l = a_l = f_l(x_l^0) = b_{l-1}.$$

Define a one-parameter family of unit covectors η_s at p_0 such that $\eta_0 = \eta$, $H_l(\eta_s) = a_l - s^2$, and $H_j(\eta_s) = b_j$ for $j \neq l$. Then, the geodesics $\gamma(t, \eta_s)$ ($s \neq 0$) are still on N_l , but do not meet J_l . Since the zeros of a family of Jacobi fields are continuously depending on the parameter, it follows that $\lim_{s \rightarrow 0} t_2(\eta_s) = t_2(\eta)$ represents the first positive time t such that $Y_l(t) = 0$. Now, substitute $\eta = \eta_s$ in the formula (8.1) and take a limit $s \rightarrow 0$. Then, if $t_0(\eta) \geq t_2(\eta)$, one gets a similar contradiction as above. Thus we have $t_0(\eta) < t_2(\eta)$, and $Y_l(t_0(\eta)) \neq 0$ in this case.

Next, we assume that $p_0 \in J_l$. Let $\eta_s \in U_{p_0}^* M$ ($\eta_0 = \eta$) be a one-parameter family of covectors such that the infinitesimal variation of the geodesics $\{\gamma(t, \eta_s)\}$ at $s = 0$ is equal to $Y_l(t)$. Let $t_2(\eta_s)$ be the first positive time such that $\gamma(t, \eta_s) \in N_l$. Then,

$t_2(\eta) = \lim_{s \rightarrow 0} t_2(\eta_s)$ is the first positive time such that $Y_l(t) = 0$. Also, by the same reason as in the case (i), we have $\gamma(t_2(\eta_s), \eta_s) \in J_l$ and so does for $s = 0$. Hence we have $\sigma_{l+1}(t_2(\eta), \eta) = 2(a_l^- - a_{l+1}^+)$, and thus $t_0(\eta) < t_2(\eta)$ by Proposition 6.5.

Finally, let us consider the case (iv); $\gamma(t, \eta)$ is contained in J_l . In this case, we have

$$b_{l+1} = f_{l+1}(x_{l+1}^0) = b_l = a_l = f_l(x_l^0) = b_{l-1} .$$

Define the one-parameter family of the initial points $p_0(s)$ and the initial covectors $\eta_s \in U_{p_0(s)}^* M$ so that $H_{l+1}(\eta_s) = H_l(\eta_s) = b_l - s^2$ and $H_i(\eta_s) = b_i$ ($i \neq l, l+1$). Then the formula (8.1) is valid for η_s . Taking a limit $s \rightarrow 0$, we have:

$$\begin{aligned} & 2 \sum_{\substack{1 \leq [i] \leq n-1 \\ [i] \neq l}} \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^{[i]+1} G_{l,l-1}(\lambda) B(\lambda) d\lambda}{\sqrt{-\prod_{[k]=1}^{n-1} ((\lambda - a_k^-)(\lambda - a_k^+))}} \\ &= \sum_{i \neq l, l+1} \int_{\sigma_i(t_2(\eta), \eta)}^{2(a_{i-1}^- - a_i^+)} \frac{(-1)^i G_{l,l-1}(f_i) A(f_i) d\sigma_i}{|f_i - a_l| \sqrt{-\prod_{k \neq l-1} (f_i - b_k) \cdot \prod_{k \neq l} (f_i - a_k)}} . \end{aligned}$$

Since the left-hand side of the above formula is positive by Proposition 4.1, we have $t_0(\eta) < t_2(\eta)$ as before. This completes the proof of Proposition 8.1.

9. Cut locus (3). In this section, we shall give a proof of Theorem 7.1 (2) for the cases (III) and (IV), and (3) for the case (V). First, we shall consider the case (III); $p_0 \in N_n$.

We use Lemma 7.2 in the case where $x_n^1 = 0$ and use it by exchanging p_0 and p'_0 . As a consequence, we see that the mapping

$$(U_{p_0}^* M \supset) U_+ \ni \eta \longmapsto \gamma(t_0(\eta), \eta) \in N_n$$

is a C^∞ embedding. Therefore, to prove (2) in this case it is enough to show that the mapping

$$(9.1) \quad \partial \overline{U_+} \ni \eta \longmapsto \gamma(t_0(\eta), \eta) \in N_n$$

is an embedding.

For $p_0 \in N_n$ and $\eta \in U_{p_0}^* N_n$, let $\tilde{t}_0(\eta)$ denotes the value defined in the same way as $t_0(\eta)$ for the $(n-1)$ -dimensional Liouville manifold N_n . (Note that N_n is constructed from the constants $0 < a_{n-1} < \dots < a_0$ and the function $A(\lambda)$ as in §2.) As we have proved in (1), $t = \tilde{t}_0(\eta)$ gives the cut point of p_0 along the geodesic $\gamma(t, \eta)$ in N_n . In particular, we have $t_0(\eta) \leq \tilde{t}_0(\eta)$. Therefore, the following proposition will indicate that the mapping (9.1) is an embedding.

PROPOSITION 9.1. $t_0(\eta) < \tilde{t}_0(\eta)$ for any $p_0 \in N_n$ and $\eta \in U_{p_0}^* N_n$.

Proof. We use the formula

$$\begin{aligned} & \sum_{i=1}^{n-1} \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^{i+1} G_{n-1,n-2}(\lambda) B(\lambda) d\lambda}{\sqrt{-\prod_{k=1}^{n-2} (\lambda - b_k) \prod_{k=0}^{n-1} (\lambda - a_k)}} \\ &= \sum_{i=1}^{n-1} \int_{\sigma_i(t_0(\eta), \eta)}^{2(a_{i-1}^- - a_i^+)} \frac{(-1)^i G_{n-1,n-2}(f_i) A(f_i) d\sigma_i}{(f_i - a_n) \sqrt{-\prod_{k=1}^{n-2} (f_i - b_k) \prod_{k=0}^{n-1} (f_i - a_k)}} , \end{aligned}$$

where

$$B(\lambda) = c - \frac{A(\lambda) - A(a_n)}{\lambda - a_n}$$

and $c > 0$ is a sufficiently large constant. As before, the left-hand side of the above formula is positive, whereas each integrand of the right-hand side is negative for $i \leq n - 2$. Thus, if $t_0(\eta) = \tilde{t}_0(\eta)$, then

$$2(a_{n-2}^- - a_{n-1}^+) = \sigma_{n-1}(\tilde{t}_0(\eta), \eta) = \sigma_{n-1}(t_0(\eta), \eta),$$

and we have a contradiction. Therefore it follows that $t_0(\eta) < \tilde{t}_0(\eta)$. \square

Next, we shall consider the case (IV); $x_n^0 = \alpha_n/4$ and $p_0 \notin J_{n-1}$. By the fact similar to Lemma 7.2 and by the proved cases, we see that the map $\eta \mapsto \gamma(t_0(\eta), \eta)$ gives C^∞ embeddings $U_+ \rightarrow N$ and $\partial\overline{U_+} \rightarrow N$, where N is the subset of N_{n-1} given by $x_n = -\alpha_n/4$. To see that the cut locus $C(p_0)$, the union of the images of those embeddings, is in the interior of N , it is enough to show that $C(p_0)$ does not meet J_{n-1} , a connected component of which is equal to the boundary of N . Assume that $\gamma(t_0(\eta), \eta) \in J_{n-1}$ for some $\eta \in \overline{U_+}$. By Lemma 2.1 we see that $F_{n-1}(\eta) = 0$. Since $p_0 \notin J_{n-1}$ and $p_0 \in N_{n-1}$, it thus follows that $\eta \in U_{p_0}^* N_{n-1}$, i.e., $\eta \in \partial\overline{U_+}$. Now put

$$\gamma(t) = \gamma(t_0(\eta) - t, \eta)$$

Then, $\gamma(t)$ is a geodesic starting at $\gamma(t_0(\eta), \eta) \in J_{n-1}$ and its first conjugate point is $p_0 = \gamma(t_0(\eta))$. But, as we shall see just below, the first conjugate point of any geodesic starting at a point in J_{n-1} also belongs to J_{n-1} , which is a contradiction. Thus $C(p_0)$ is contained in the interior of N . This finishes the proof of (2) of the theorem in this case.

Finally we prove the statement (3) of the theorem for the case (V); $p_0 \in J_{n-1}$. Note that $t = t_0(\eta)$ gives the cut point of p_0 along the geodesic $\gamma(t, \eta)$ for any $\eta \in U_{p_0}^* M$. We apply the results proved above to the $(n-1)$ -dimensional Liouville manifold N_{n-1} , which is constructed from the constants $0 < a_n < a_{n-2} < \dots < a_0$ and the function $A(\lambda)$. Noting the fact $J_{n-1} \cap J_{n-2} = \emptyset$, we see that the cut locus $\tilde{C}(p_0)$ of p_0 in N_{n-1} is an $(n-2)$ -closed disk, and it is the image of the map

$$\overline{U_+} \cap T_{p_0}^* N_{n-1} \rightarrow J_{n-1}, \quad \eta \mapsto \gamma(\tilde{t}_0(\eta), \eta),$$

where $\tilde{t}_0(\eta)$ is the value which is defined in the same way as $t_0(\eta)$ for the $(n-1)$ -dimensional Liouville manifold N_{n-1} . It has also been proved that the above map is an embedding on the interior and on the boundary.

Let $\tilde{\eta}$ be a unit covector such that $\tilde{\eta} \notin T_{p_0}^* N_{n-1}$. Let $\{\zeta_s\}$ be the one-parameter transformation group of T^*M generated by $X_{F_{n-1}}$. Then $\tilde{\eta}_s = \zeta_s(\tilde{\eta}) \in U_{p_0}^* M$ whose orthogonal projection to $T_{p_0}^* J_{n-1}$ does not depend on s , and $\tilde{\eta}_{\pm\infty} = \lim_{s \rightarrow \pm\infty} \tilde{\eta}_s \in T_{p_0}^* N_{n-1}$. By the definition of $t_0(\tilde{\eta}_s)$ we have $\gamma(t_0(\tilde{\eta}_s), \tilde{\eta}_s) \in J_{n-1}$. Therefore the Jacobi field $\pi_* X_{F_{n-1}}$ along the geodesic $\gamma(t, \tilde{\eta}_s)$ also vanish at $t = t_0(\tilde{\eta}_s)$. Thus we have

$$\gamma(t_0(\tilde{\eta}_s), \tilde{\eta}_s) = \gamma(t_0(\tilde{\eta}_{\pm\infty}), \tilde{\eta}_{\pm\infty}), \quad t_0(\tilde{\eta}_s) = t_0(\tilde{\eta}_{\pm\infty})$$

for any $s \in \mathbb{R}$. Since $t = t_0(\tilde{\eta}_s)$ gives the cut point of p_0 along the geodesic $\gamma(t, \tilde{\eta}_s)$, and since $\tilde{\eta}_{+\infty} \in U^* N_{n-1}$ and $\tilde{\eta}_{-\infty} \in U^* N_{n-1}$ are symmetric with respect to the hyperplane $T_{p_0}^* J_{n-1} \subset T_{p_0}^* N_{n-1}$, it follows that $\tilde{t}_0(\eta_{\pm\infty}) = t_0(\eta_{\pm\infty})$. Thus we have

proved that the cut locus $C(p_0)$ of p_0 in M coincides with $\tilde{C}(p_0)$ and that if $\eta_1, \eta_2 \in U_{p_0}^* M$ have the same $T_{p_0}^* J_{n-1}$ -components, then $\gamma(t_0(\eta_1), \eta_1) = \gamma(t_0(\eta_2), \eta_2)$. From these it also follows that for $\eta \in U_{p_0}^* J_{n-1}$, $t = t_0(\eta)$ gives the first conjugate point of p_0 with multiplicity two along the geodesic $\gamma(t, \eta)$. This finishes the proof of Theorem 7.1.

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