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# LOCALIZATION, HURWITZ NUMBERS AND THE WITTEN CONJECTURE*

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**Abstract.** In this note, we use the combinatorial method of Goulden-Jackson-Vakil to give a simple proof of Witten conjecture-Kontsevich theorem.

**Key words.** Moduli space, intersection number.

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## 1. Introduction.

The well-known Witten conjecture states that the intersection theory of the $\psi$ classes on the moduli spaces of Riemann surfaces is equivalent to the "Hermitian matrix model" of two-dimensional gravity. All $\psi$-integrals can be efficiently computed by using the Witten conjecture [13], first proved by Kontsevich [6]. Today, there are many different approach to prove this conjecture, see [4], [5], [11] and [12] . For convenience, we use Witten's natation

$$\langle \tau_{\beta_1} \cdots \tau_{\beta_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\beta_1} \cdots \psi_n^{\beta_n}. \tag{1.1}$$

The natural generating function for the $\psi$-integrals described above is

$$F_g(t) := \sum_{n\geq 0} \frac{1}{n!} \sum_{k_1,\cdots,k_n} t_{k_1} \cdots t_{k_n} \langle \tau_{\beta_1} \cdots \tau_{\beta_n} \rangle_g, \quad F(t,\lambda) := \sum_{g\geq 0} F_g \lambda^{2g-2}. \tag{1.2}$$

For example, the first system of differential equations conjectured by Witten are the KDV equations. Let $F(t) := F(t,1)$, define

$$\langle\langle \tau_{\beta_1} \cdots \tau_{\beta_n} \rangle\rangle := \frac{\partial}{\partial t_{k_1}} \cdots \frac{\partial}{\partial t_{k_n}} F(t), \tag{1.3}$$

then the KDV equations for $F(t)$ are equivalent to a sequence of recursive relations for $n \geq 1$:

$$(2n+1)\langle\langle \tau_n \tau_0^2 \rangle\rangle = \langle\langle \tau_{n-1}\tau_0 \rangle\rangle \langle\langle \tau_0^3 \rangle\rangle + 2\langle\langle \tau_{n-1}\tau_0^2 \rangle\rangle \langle\langle \tau_0^2 \rangle\rangle + \frac{1}{4}\langle\langle \tau_{n-1}\tau_0^4 \rangle\rangle. \tag{1.4}$$

In [5] the authors give a simple proof of the Witten conjecture by first proving a recursion formula conjectured by Dijkgraaf-Verlinde-Verlinde in [1], and as corollary they are able to give a simple proof of the Witten conjecture by using asymptotic analysis. In this note, we use the method in [3] to prove the recursion formula in [1], therefore the Witten conjecture without using the asymptotic analysis. Combining the coefficients derived in our note and the approach in [3], we can derive more recursion formulas of Hodge integrals.

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**2. Localization and the Hurwitz Numbers.** Denote by $\mu$ a partition of $d > 0$. Let $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$ be the moduli space of relative stable morphism to $\mathbb{P}^1$, which is a Deligne-Mumford stack of virtual dimension $r = 2g - 2 + d + l(\mu)$ constructed in [8]. We refer readers to [9] for the property of $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$. The $\mathbb{C}^*$-action on $\mathbb{P}^1$

$$t \cdot [z^0 : z^1] = [tz^0 : z^1],$$

induces an $\mathbb{C}^*$-action on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$. There is a branching morphism

$$\mathrm{Br} : \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu) \longrightarrow \mathbb{P}^r, \tag{2.1}$$

with this action, the branching morphism is $\mathbb{C}^*$-equivariant. The Hurwitz numbers can be defined by

$$H_{g,\mu} := \int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)]^{\mathrm{virt}}} \mathrm{Br}^* H^r \tag{2.2}$$

with the hyperplane class $H \in H^2(\mathbb{P}^r; \mathbb{Z})$.

**2.1. Localization and Hurwitz Numbers.** From the localization formula in [9], we have

$$H_{g,\mu} = (-1)^k k! \widetilde{I}^k_{g,\mu}, \tag{2.3}$$

where $\widetilde{I}^k_{g,\mu}$ are the contributions of graphs of $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$. Taking $k = 0$, it implies the well-known ELSV formula [2]:

$$H_{g,\mu} = \frac{r!}{|\mathrm{Aut}(\mu)|} \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)}. \tag{2.4}$$

For $k = 1$, it becomes the cut-and-join equation

$$\begin{aligned} H_{g,\mu} = {} & \sum_{\nu \in J(\mu)} I_1(\nu) H_{g,\nu} + \sum_{\nu \in C(\mu)} I_2(\nu) H_{g-1,\nu} \\ & + \sum_{g_1 + g_2 = g} \sum_{\nu^1 \cup \nu^2 \in C(\mu)} \binom{r-1}{2g_1 - 2 + |\nu^1| + l(\nu^1)} I_3(\nu^1, \nu^2) H_{g_1,\nu^1} H_{g_2,\nu^2}. \end{aligned} \tag{2.5}$$

**2.2. Notations.** In this subsection, we explain some notations appeared in the above subsection. Let $\mu : \mu_1 \geq \cdots \geq \mu_n > 0$, and for each positive integer $i$, denote $m_i(\mu)$ the number of the integers $i$ appear in $\mu$. Recall the definitions of $J_\mu$ and $C_\mu$ (see [5] or [7])

$$J^{i,j}(\mu) = \{(\mu_1, \cdots, \widehat{\mu}_i, \cdots, \widehat{\mu}_j, \cdots, \mu_n, \mu_i + \mu_j)\}, \ J(\mu) = \cup_{i=1}^n \cup_{j=i+1}^n J^{i,j}(\mu);$$

$$C^{i,p}(\mu) = \{(\mu_1, \cdots, \widehat{\mu}_i, \cdots, \mu_n, p, \mu_i - p)\}, \ C^i(\mu) = \cup_{p=1}^{\mu_i} C^{i,p}(\mu), \ C(\mu) = \cup_{i=1}^n C^i(\mu).$$

If $\nu \in J^{i,j}(\mu)$, then write $\nu := \mu^{i,j}$, and the $I_1(\nu)$ is given by

$$I_1(\nu) = \frac{\mu_i + \mu_j}{1 + \delta^{\mu_i}_{\mu_j}} m_{\mu_i + \mu_j}(\mu^{i,j}). \tag{2.6}$$

For $\nu \in C^{i,p}(\mu)$, then write $\nu = \mu^{i,p}$, and the $I_2(\nu)$ is defined by

$$I_2(\nu) = \frac{p(\mu_i - p)}{1 + \delta^p_{\mu_i - p}} m_p(\nu)(m_{\mu_i - p}(\nu) - \delta^p_{\mu_i - p}). \tag{2.7}$$

If $\nu \in C^{i,p}(\mu)$, then let $\nu^1 \cup \nu^2 = \nu$, and $I_3(\nu^1, \nu^2)$ is defined by

$$I_3(\nu^1,\nu^2) = \frac{p(\mu_i - p)}{1+\delta^p_{\mu_i - p}} m_p(\nu^1) m_{\mu_i - p}(\nu^2). \tag{2.8}$$

Define the formal power series

$$\Phi(\lambda, p) = \sum_{\mu} \sum_{g \geq 0} H_{g,\mu} \frac{\lambda^{2g-2+|\mu|+l(\mu)}}{(2g-2+|\mu|+l(\mu))!} p_\mu. \tag{2.9}$$

It is well known that $\Phi(\lambda, p)$ satisfies the following version of cut-and-join equation [10]

$$\frac{\partial \Phi}{\partial \lambda} = \frac{1}{2} \sum_{i.j \geq 1} \left( ijp_{i+j} \frac{\partial^2 \Phi}{\partial p_i \partial p_j} + ijp_{i+j} \frac{\partial \Phi}{\partial p_i} \frac{\partial \Phi}{\partial p_j} + (i+j) p_i p_j \frac{\partial \Phi}{\partial p_{i+j}} \right). \tag{2.10}$$

Define

$$\Phi_{g,n}(z,p) = \sum_{d \geq 1} \sum_{\mu \vdash d, l(\alpha) = n} \frac{H_{g,\mu}}{r!} p_\mu z^d,$$

which can be written into the following form by Equation 2.4

$$\Phi_{g,n}(z;p) = \frac{1}{n!} \sum_{b_1,\cdots,b_n \geq 0, 0 \leq k \leq g} (-1)^k \langle \tau_{b_1} \cdots \tau_{b_n} \lambda_k \rangle_g \prod_{i=1}^n \phi_{b_i}(z;p), \tag{2.11}$$

where

$$\phi_i(z;p) = \sum_{m \geq 0} \frac{m^{m+i}}{m!} p_m z^m, \quad i \geq 0. \tag{2.12}$$

**3. Symmetrization Operator and Rooted Tree Series.** In this section, we use the method of Goulden-Jackson-Vakil in [3] to prove the recursion formula, which implies the Witten conjecture/Kontsevich theorem. The proof consists of the following steps: (1) introduce three operators to change the variables as in [3]; (2) compare the leading coefficients of both sides of the cut-and-join equation to derive the recursion formula. Kim-Liu's proof of this recursion formula is via the asymptotic analysis. They write each $\mu_i = x_i N$ for some $x_i \in \mathbb{Q}$ and let $N \in \mathbb{N}$ goes to infinity. The main technic in [5] is the asymptotic estimate of series

$$\sum_{p=1}^n \frac{p^{p+i}}{p!}, \quad \sum_{p+q=n} \frac{p^{p+i+1} q^{q+j+1}}{p! q!}$$

for any $i, j \in \mathbb{N}$. The idea here is that by applying the transcendental changing of variable formula in [3], the cut-and-join equation becomes a system of polynomial equalities, which avoid the technical asymptotic estimate.

**3.1. Symmetrization Operator.** First, we symmetrize $\Phi_{g,n}(z,p)$ by using the linear symmetrization operator $\Xi_n$ defined by

$$\Xi_n(p_\alpha z^{|\alpha|}) = \delta_{l(\alpha),n} \sum_{\sigma \in S_n} x^{\alpha_1}_{\sigma(1)} \cdots x^{\alpha_n}_{\sigma(n)}. \tag{3.1}$$

The following lemma is elementary, see [3].

LEMMA 3.1. *For $n, g \geq 0$ and assume $n \geq 3$ if $g = 0$, then we have*

$$\begin{aligned} \Xi_n(\Phi_{g,n}(z,p))(x_1, \cdots, x_n) \\ = \frac{1}{n!} \sum_{b_1,\cdots,b_n \geq 0, 0 \leq k \leq g} (-1)^k \langle \tau_{b_1} \cdots \tau_{b_n} \lambda_k \rangle_g \sum_{\sigma \in S_n} \prod_{i=1}^{n} \phi_{b_i}(x_{\sigma(i)}), \end{aligned} \tag{3.2}$$

*where*

$$\phi_i(x) := \phi(x; 1) = \sum_{m \geq 1} \frac{m^{m+i}}{m!} x^m. \tag{3.3}$$

**3.2. Rooted Tree Series.** The *rooted tree series* $w(x)$ is introduced in [3]:

$$w(x) = \sum_{m \geq 1} \frac{m^{m-1}}{m!} x^m, \tag{3.4}$$

which is the unique formal power series solution of the functional equation

$$w = xe^w. \tag{3.5}$$

Thus we have

$$\phi_i(x) = \left(x \frac{d}{dx}\right)^{i+1} w(x) := \nabla_x^{i+1} w(x) \tag{3.6}$$

with $\nabla_x := x \frac{d}{dx}$.

Let $y(x) = \frac{1}{1-w(x)}$ and $y_j = y(x_j)$, then $y - 1$ is a uniformizer in the ring $\mathbb{Q}[[x]]$. Consider the changing of variables operator $L : \mathbb{Q}[[x]] \to \mathbb{Q}[[y-1]]$, which send a formal power series in the variable $x$ into a formal power series in the variable $y - 1$. In this paper, apply the operator $L$ to formal power series $\phi_i(x)$, the formal power series we obtain are in fact polynomials in $y$. We prove this fact in the following lemma.

LEMMA 3.2. *Denote $w_j = w(x_j)$, then*

$$L\nabla_{x_j} = (y_j^2 - y_j)\nabla_{y_j} L, \quad L\nabla_{w_j} = (y_j - 1)\nabla_{y_j} L, \tag{3.7}$$

$$L(\phi_i(x_j)) = [(y_j^2 - y_j)\nabla_{y_j}]^i (y_j - 1), \quad i \geq 0. \tag{3.8}$$

*Proof.* Differentiating the Equation 3.5, we obtain

$$\nabla_{x_j} = \frac{1}{1 - w_j} \nabla_{w_j}.$$

Note that $dy_j = y_j^2 dw$, then $\nabla_{w_j} = (y_j - 1)\nabla_{y_j}$ and

$$L\nabla_{x_j} = L\left(\frac{1}{1 - w_j}\nabla_{w_j}\right) = y_j(y_j - 1)\nabla_{y_j} L.$$

The rest identities are left to the readers. □

**4. Proof of the Dijkgraaf-Verlinde-Verlinde Conjecture.** For $i, j \geq 0$, $i + j \leq n$, let $\overset{x}{\underset{i,j}{\Xi}}$ be the mapping, applied to a series in $x_1, \cdots, x_n$, given by

$$\overset{x}{\underset{i,j}{\Xi}} f(x_1, \cdots, x_n) = \sum_{\mathcal{R},\mathcal{S},\mathcal{T}} f(x_{\mathcal{R}}, x_{\mathcal{S}}, x_{\mathcal{T}}), \tag{4.1}$$

where the summation is over all ordered partitions $(\mathcal{R}, \mathcal{S}, \mathcal{T})$ of $\{1, \cdots, n\}$, where $\mathcal{R} = \{x_{r_1}, \cdots, x_{r_i}\}$, $\mathcal{S} = \{x_{s_1}, \cdots, x_{s_j}\}$, $\mathcal{T} = \{x_{t_1}, \cdots, x_{t_{n-i-j}}\}$ and

$$(x_{\mathcal{R}}, x_{\mathcal{S}}, x_{\mathcal{T}}) = (x_{r_1}, \cdots, x_{r_i}, x_{s_1}, \cdots, x_{s_j}, x_{t_1}, \cdots, x_{t_{n-i-j}}),$$

and where $r_1 < \cdots < r_i$, $s_1 < \cdots < s_j$, and $t_1 < \cdots < t_{n-i-j}$. The following result gives an expression for the result of applying the symmetrization operator $\Xi_n$ to the cut-and-join equation for $\Phi_{g,n}(z, p)$. Denote $\triangle_{y_j} := (y_j^2 - y_j)\nabla_{y_j}$. Applying the symmetrization operator $\Xi_n$ to the cut-and-join Equation, Goulden-Jackson-Vakil prove the following version of cut-and-join equation [3]

$$\left(\sum_{i=1}^{n}(y_i - 1)\nabla_{y_i} + n + 2g - 2\right) L\Xi_n\Phi_{g,n}(y_1, \cdots, y_n) = T_1' + T_2' + T_3' + T_4', \tag{4.2}$$

where

$$T_1' = \frac{1}{2}\sum_{i=1}^{n}\left(\triangle_{y_i}\triangle_{y_{n+1}} L\Xi_{n+1}\Phi_{g-1,n+1}(y_1, \cdots, y_{n+1})\right)|_{y_{n+1}=y_i},$$

$$T_2' = \overset{y}{\underset{1,1}{\Xi}} y_1^2 \frac{y_2 - 1}{y_1 - y_2}\triangle_{y_1} L\Xi_{n-1}\Phi_{g,n-1}(y_1, y_3, \cdots, y_n),$$

$$T_3' = \sum_{k=3}^{n}\overset{y}{\underset{1,k-1}{\Xi}}\left(\triangle_{y_1} L\Xi_k\Phi_{0,k}(y_1, \cdots, y_k)\right)\left(\triangle_{y_1} L\Xi_{n-k+1}\Phi_{g,n-k+1}(y_1, y_{k+1}, \cdots, y_n)\right),$$

$$T_4' = \frac{1}{2}\sum_{1\leq k\leq n, 1\leq a\leq g-1}\overset{y}{\underset{1,k-1}{\Xi}}\left(\triangle_{y_1} L\Xi_k\Phi_{a,k}(y_1, \cdots, y_k)\right) \cdot \left(\triangle_{y_1} L\Xi_{n-k+1}\Phi_{g-a,n-k+1}(y_1, y_{k+1}, \cdots, y_n)\right).$$

**4.1. Expansions.** We have the following expansion

$$L\left(\prod_{i=1}^{n}\phi_{b_i}(x_{\sigma(i)})\right) = \prod_{i=1}^{n}(2b_i - 1)!!y_{\sigma(i)}^{2b_i+1} + \text{lower terms}. \tag{4.3}$$

From this point, we see that the polynomial $L\Xi_n H_n^g(y_1, \cdots, y_n)$ can be written as

$$L\Xi_n\Phi_{g,n}(y_1, \cdots, y_n) = \sum_{b_1+\cdots+b_n=3g-3+n}\langle\tau_{b_1}\cdots\tau_{b_n}\rangle_g\prod_{i=1}^{n}(2b_i - 1)!!y_i^{2b_i+1} + \text{l.t.}$$

where l.t. denote lower order terms. We write the left hand side of Equation 4.2 by LHS while another side by $\text{RHS}_1, \text{RHS}_2, \text{RHS}_3$ and $\text{RHS}_4$, then

$$\begin{aligned} LHS &= \sum_{i=1}^{n} y_i\nabla_{y_i}\sum_{b_1+\cdots+b_n=3g-3+n}[\langle\tau_{b_1}\cdots\tau_{b_n}\rangle_g(2b_1 - 1)!!\cdots(2b_n - 1)!!]\prod_{l=1}^{n} y_l^{2b_l+1} + \text{l.t.} \\ &= \sum_{b_1+\cdots+b_n=3g-3+n}[\langle\tau_{b_1}\cdots\tau_{b_n}\rangle_g(2b_1 - 1)!!\cdots(2b_n - 1)!!]\sum_{i=1}^{n}(2b_i + 1)y_i\prod_{l=1}^{n} y_l^{2b_l+1} + \text{l.t.} \end{aligned}$$

$$RHS_1 = \frac{1}{2} \sum_{b_1+\cdots+b_{n+1}=3g-5+n} \left[\langle \tau_{b_1} \cdots \tau_{b_{n+1}} \rangle_{g-1} (2b_1-1)!! \cdots (2b_{n+1}-1)!!\right]$$
$$\cdot \sum_{i=1}^{n} \left( (2b_i+1)(2b_{n+1}+1) y_i^2 y_{n+1}^2 \prod_{l=1}^{n+1} y_l^{2b_l+1} \right) |_{y_i=y_{n+1}} + \text{l.t.}$$
$$RHS_2 = \Xi_{1,1}^{y} \left( \sum_{b_1+b_3+\cdots+b_n=3g-4+n} [(2b_1+1)!!(2b_3-1)!! \cdots (2b_n-1)!! \langle \tau_{b_1}\tau_{b_3} \cdots \tau_{b_n} \rangle_g] \right.$$
$$\left. \cdot \sum_{m\geq 0} \left(\frac{y_2}{y_1}\right)^m y_2 y_1^3 y_1^{2b_1+1} \prod_{l=2}^{n} y_l^{2b_l+1} \right) + \text{l.t.}$$
$$RHS_3 = \sum_{k=3}^{n} \Xi_{1,k-1}^{y} \left( \sum_{b_1+\cdots+b_k=k-3} (2b_1-1)!! \cdots (2b_k-1)!! \langle \tau_{b_1} \cdots \tau_{b_k} \rangle_0 (2b_1+1) y_1^2 \prod_{l=1}^{k} y_l^{2b_l+1} \right)$$
$$\cdot \left( \sum_{\overline{b}_1+b_{k+1}+\cdots+b_n=3g-k-2+n} [(2\overline{b}_1-1)!!(2b_{k+1}-1)!! \cdots (2b_n-1)!! \langle \tau_{\overline{b}_1}\tau_{b_{k+1}} \cdots \tau_{b_n} \rangle_g] \right.$$
$$\left. \cdot \; (2\overline{b}_1+1) y_1^2 \prod_{l=k+1}^{n} y_l^{2b_l+1} y_1^{2\overline{b}_1+1} \right) + \text{l.t.}$$
$$RHS_4 = \frac{1}{2} \sum_{1\leq k\leq n, 1\leq a\leq g-1} \Xi_{1,k-1}^{y}$$
$$\cdot \left( \sum_{b_1+\cdots+b_k=3a-3+k} (2b_1-1)!! \cdots (2b_k-1)!! \langle \tau_{b_1} \cdots \tau_{b_k} \rangle_a (2b_1+1) y_1^2 \prod_{l=1}^{k} y_l^{2b_l+1} \right)$$
$$\cdot \left( \sum_{\overline{b}_1+b_{k+1}+\cdots+b_n=3g-k-2+n-3a} [(2\overline{b}_1-1)!!(2b_{k+1}-1)!! \cdots (2b_n-1)!! \right.$$
$$\left. \cdot \; \langle \tau_{\overline{b}_1}\tau_{b_{k+1}} \cdots \tau_{b_n} \rangle_{g-a}] \; (2\overline{b}_1+1) y_1^2 \prod_{l=k+1}^{n} y_l^{2b_l+1} y_1^{2\overline{b}_1+1} \right) + \text{l.t.}$$

**4.2. Picking Out the Coefficients.** Now, we only consider the coefficients of monomial $y_1^{2(b_1+1)} y_2^{2b_2+1} \cdots y_n^{2b_n+1}$ for $b_1+\cdots+b_n = 3g-3+n$ on both sides of Equation 4.2. By simply calculating, these coefficients are given by

$$LHS = (2b_1+1)!!(2b_2-1)!! \cdots (2b_n-1)!! \langle \tau_{b_1} \cdots \tau_{b_n} \rangle_g$$
$$RHS_1 = \frac{1}{2} \sum_{a+b=b_1-2} (2a+1)!!(2b+1)!! \prod_{l=2}^{n} (2b_l-1)!! \langle \tau_a\tau_b\tau_{b_2} \cdots \tau_{b_n} \rangle_{g-1}$$
$$RHS_2 = \sum_{l=2}^{n} (2(b_1+b_l-1)+1)!!(2b_2-1)!! \cdots (2b_{l-1}-1)!!(2b_{l+1}-1)!! \cdots (2b_n-1)!!$$
$$\cdot \; \langle \sigma_{b_1+b_l-1}\sigma_{b_2} \cdots \sigma_{b_{l-1}}\sigma_{b_{l+1}} \cdots \sigma_{b_n} \rangle_g$$
$$RHS_{3,4} = \frac{1}{2} \sum_{X\cup Y=S} \sum_{a+b=b_1-2} \sum_{g_1+g_2=g} (2a+1)!!(2b+1)!! \prod_{l=2}^{n} (2b_l-1)!!$$
$$\cdot \; \langle \tau_a \prod_{\alpha\in X} \tau_\alpha \rangle_{g_1} \langle \tau_b \prod_{\beta\in Y} \tau_\beta \rangle_{g_2},$$

where $S = \{b_2, \cdots, b_n\}$. Multiplying the constant $(2b_2+1)\cdots(2b_n+1)$, we obtain the recursion formula of Dijkgraaf-Verlinde-Verlinde, which implies the Witten conjecture

$$\langle\widetilde{\tau}_{b_1}\prod_{l=2}^{n}\widetilde{\tau}_{b_l}\rangle_g = \sum_{l=2}^{n}(2b_l+1)\langle\widetilde{\tau}_{b_1+b_l-1}\prod_{k=2,k\neq l}^{n}\widetilde{\tau}_{b_k}\rangle_g + \frac{1}{2}\sum_{a+b=b_1-2}\langle\widetilde{\tau}_a\widetilde{\tau}_b\prod_{l=2}^{n}\widetilde{\tau}_{b_l}\rangle_{g-1}$$
$$\frac{1}{2}\sum_{X\cup Y=\{b_2,\cdots,b_n\}}\sum_{\sum_{a+b=b_1-2},g_1+g_2=g}\langle\widetilde{\tau}_a\prod_{\alpha\in X}\widetilde{\tau}_\alpha\rangle_{g_1}\langle\widetilde{\tau}_b\prod_{\beta\in Y}\widetilde{\tau}_\beta\rangle_{g_2}.$$

where $\widetilde{\tau}_{b_l} = [(2b_l+1)!!]\tau_{b_l}$.

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