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ON THE FIXED LOCI OF THE CANONICAL SYSTEMS OVER NORMAL SURFACE SINGULARITIES*

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Abstract. It is shown that the relative canonical linear system over a normal surface singularity has at most exceptional sets of rational singular points as its fixed part.

Key words. Surface singularity, fixed component, reducible curve.

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Introduction. In the study of algebraic surfaces of general type, it is often important to consider the canonical linear system and the rational map associated to it. In this sense, the fixed part of the canonical system can be regarded as the "worst" curve on the surface, and one may naturally ask what is its feature and how to control it. However, not too much is known so far. This is an experiment for a better understanding of the fixed part, and we consider here the local version of the problem.

Let (V, o) be a germ of a normal surface singularity and $\pi : X \to V$ its minimal resolution. For a line bundle L on X with $H^0(X, L) \neq 0$, the fixed part of the linear system $|L|$ is the biggest effective divisor F supported in $\pi^{-1}(o)$ such that the restriction map $H^0(X, L) \to H^0(F, L)$ is the zero map. The fixed part of the canonical linear system $|K_X|$ will be sometimes called the canonical fixed part in this paper.

The purpose of the present article is to show the following:

MAIN THEOREM. *Let (V, o) be a normal surface singularity and $\pi : X \to V$ the minimal resolution. If L is a line bundle on X such that $L - K_X$ is nef, then the fixed part of $|L|$ supports at most exceptional sets of rational singular points. Furthermore, if (U, p) denotes the rational singular point obtained by contracting a connected component of the fixed part of $|L|$, then the multiplicity* $\mathrm{mult}(U, p)$ *and the embedding dimension* $\mathrm{embdim}(U, p)$ *satisfy*

$$\mathrm{mult}(U, p) \leq 2p_f(V, o), \quad \mathrm{embdim}(U, p) \leq 2p_f(V, o) + 1$$

where $p_f(V, o)$ denotes the fundamental genus of (V, o), that is, the arithmetic genus of the fundamental cycle on $\pi^{-1}(o)$.

When the fixed part supports exceptional sets of rational double points, we can show that they are necessarily of type A (Corollary 3.3). This suggests that the singular point obtained by contracting a connected component of the canonical fixed part is rather special among rational singular points, though we do not know how to characterize them. We remark that $|L|$ is free from base points if (V, o) itself is a rational singular point (see, Proposition 2.7 for a slightly stronger assertion). So our result applies essentially to singularities "of general type".

On the technical side, one may see that an easy lemma [4, Lemma 2.2.1] plays a very important rôle throughout the paper. In fact, we give in Sect. 1 a decomposition

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of a numerically 1-connected curve, Theorem 1.1, as an application of it. In Sect. 2, we prove the Main Theorem in Theorem 2.4 and Proposition 2.6. Our strategy here is to associate a particular curve, called a *loupe*, for each fixed component of $|L|$. It is the fundamental cycle on its support, with self-intersection number -1, containing the fixed component as a non-multiple component. In order to find the loupe, we again use [4, Lemma 2.2.1]. Its decomposition detected by Theorem 1.1 enables us to argue inductively on the number of fixed components. The proof of Proposition 2.7 referred above is also based on [4, Lemma 2.2.1]. In Sect. 3, we state some further properties of the loupes and show Corollary 3.3 as an application. In Sect. 4, we restrict ourselves to (weakly) elliptic singularities [11] in order to clarify, to some extent, how our method relates to Yau's elliptic sequence [13]. When the fixed part corresponds to a rational double point of type A and the biggest loupe contracts to an elliptic singularity, Theorem 4.1 shows that the associated sequence of loupes is nothing more than the elliptic sequence.

In a forthcoming paper, we shall study numerically connected curves and treat the semi-global case, that is, fibers in relatively minimal fibred surfaces.

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1. A decomposition theorem. Let $D = \sum_{i=1}^{N} m_i A_i$ be a divisor on a non-singular surface X, where A_i is a compact irreducible curve and $m_i \in \mathbb{Z}$. If all the m_i's are non-negative, D is called effective. For two divisors D_1 and D_2, $D_2 \preceq D_1$ means that $D_1 - D_2$ is effective.

Let D be a non-zero effective divisor. We usually identify it with the corresponding 1-dimensional subscheme of X. We put $p_a(D) = 1 - \chi(D, \mathcal{O}_D)$ and call it the arithmetic genus of D. By the adjunction formula, we have $2p_a(D)-2 = D(K_X+D)$. We say that D is numerically 1-connected if $D_1D_2 > 0$ holds for any effective decomposition $D = D_1 + D_2$ with $0 \prec D_1$, $0 \prec D_2$. A line bundle on D is *nef* if it is of non-negative degree on any irreducible components. It is well-known that, if D is 1-connected, we have $h^0(D, \mathcal{O}_D) = 1$ and for a nef line bundle L we have $h^0(D, -L) \neq 0$ if and only if $\mathcal{O}_D(L) \simeq \mathcal{O}_D$. Furthermore, if we have an effective decomposition $D = D_1 + D_2$ with $D_1D_2 = 1$ for a numerically 1-connected divisor D, then D_1 and D_2 are both numerically 1-connected. For these facts and further properties of 1-connected curves, we refer the readers to [3, Appendix].

Let L be a line bundle on D. A point on D is called a base point of the linear system $|L|$ if every section in $H^0(D, L)$ vanishes at that point. We denote by $\mathrm{Bs}|L|$ the set of all base points of $|L|$.

The purpose of the section is to show the following theorem. One can find a similar result in [8, Theorem 4.1].

Theorem 1.1. *Let D be a numerically 1-connected divisor on a non-singular surface. Assume that an irreducible component A of D is a fixed component of $|K_D|$ and put $n = A(D - A)$. Then $A \simeq \mathbb{P}^1$ and, either $D = A$ or there are effective subdivisors Γ_i of D ($1 \leq i \leq n$) and a decomposition $D = A + \Gamma_1 + \cdots + \Gamma_n$ enjoying the following properties:*

(1) *Γ_i is numerically 1-connected and $A\Gamma_i = 1$ holds for each $i \in \{1, \ldots, n\}$.*
(2) *A is not a component of Γ_i when $i \geq 2$.*
(3) *$\mathcal{O}_{\Gamma_i+\cdots+\Gamma_n}(\Gamma_{i-1}) \simeq \mathcal{O}_{\Gamma_i+\cdots+\Gamma_n}$ holds for any i with $2 \leq i \leq n$.*

Proof. We may assume $n > 0$, since we clearly have $D = A \simeq \mathbb{P}^1$ when $n = 0$. We let D_1 be a minimal effective divisor such that $A \preceq D_1 \preceq D$ and the restriction map $H^0(D, K_D) \to H^0(D_1, K_D)$ is surjective.

We first claim that $D_1 \neq D$. This can be seen as follows. Take any irreducible component $B \preceq D - A$ and consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_B(K_B) \to \mathcal{O}_D(K_D) \to \mathcal{O}_{D-B}(K_D) \to 0.$$

Since $H^1(B, K_B) \to H^1(D, K_D)$ is dual of the restriction map $H^0(D, \mathcal{O}_D) \to H^0(B, \mathcal{O}_B)$, it is an isomorphism, hence $H^0(D, K_D) \to H^0(D - B, K_D)$ is surjective. In particular, we can assume that $D_1 \preceq D - B$ by the minimality of D_1. Hence $D_1 \neq D$.

We have $K_{D_1} - K_D = -(D - D_1)$ on D_1. Since D is 1-connected, we have $(D - D_1)D_1 > 0$. In particular, we see that $K_{D_1} - K_D$ is not nef on D_1. Then it follows from [4, Lemma 2.2.1] that

(a) A is of multiplicity one in D_1,

(b) $-(D - D_1)$ is nef on $D_1 - A$, and

(c) the image of $H^0(D_1, K_D) \to H^0(A, K_D)$ contains the image of the natural map $H^0(A, K_D - (D_1 - A)) \hookrightarrow H^0(A, K_D)$.

Since we have assumed that $A \subset \mathrm{Bs}|K_D|$, the restriction map $H^0(D, K_D) \to H^0(A, K_D)$ is the zero map. Hence so is $H^0(D_1, K_D) \to H^0(A, K_D)$ by the choice of D_1. It follows from (c) that $H^0(A, K_D - (D_1 - A)) = 0$. By the adjunction formula, we have $\mathcal{O}_A(K_D - (D_1 - A)) \simeq \mathcal{O}_A(K_A + (D - D_1))$. Since $-(D - D_1)$ is nef on $D_1 - A$ by (b), we have $(D - D_1)(D_1 - A) \leq 0$. So, $(D - D_1)A > 0$, because we have $(D - D_1)D_1 > 0$ by the 1-connectedness of D. Therefore, $\mathcal{O}_A(K_D - (D_1 - A))$ is non-special and we get $h^0(A, K_D - (D_1 - A)) = p_a(A) - 1 + A(D - D_1)$ by the Riemann-Roch theorem. Since this has to be zero, we get $p_a(A) = 0$ and $A(D - D_1) = 1$. Then, $(D - D_1)(D_1 - A) = 0$ and it follows from (b) that $\mathcal{O}_{D_1 - A}(D - D_1)$ is numerically trivial. We have shown that $A \simeq \mathbb{P}^1$ and $D_1(D - D_1) = 1$. Note that the last equality implies that D_1 and $D - D_1$ are also numerically 1-connected.

Take any point $p \in A$ not lying on $D_1 - A$. Since $A \subset \mathrm{Bs}|K_D|$, it follows from the cohomology long exact sequence for

$$0 \to \mathcal{O}_{D_1}(K_D - p) \to \mathcal{O}_{D_1}(K_D) \to \mathcal{O}_p \to 0$$

that $H^0(D_1, K_D) \to \mathcal{O}_p$ is zero, and hence $H^1(D_1, K_D - p) \neq 0$. By the Serre duality theorem, we get $H^0(D_1, p - (D - D_1)) \neq 0$. Since $\mathcal{O}_{D_1}(p - (D - D_1))$ is numerically trivial and D_1 is 1-connected, we conclude that $\mathcal{O}_{D_1}(D - D_1) \simeq \mathcal{O}_{D_1}(p)$. In particular, this shows that $\mathcal{O}_{D_1 - A}(D - D_1) \simeq \mathcal{O}_{D_1 - A}$. Furthermore, we have $h^0(D_1, \mathcal{O}_{D_1}(p)) \geq 2$, since the same argument as above shows $\mathcal{O}_{D_1}(D - D_1) \simeq \mathcal{O}_{D_1}(p')$ and, hence, $\mathcal{O}_{D_1}(p) \simeq \mathcal{O}_{D_1}(p')$ for any other point $p' \in A \setminus \mathrm{Supp}(D_1 - A)$. Then we get $h^0(D_1, K_{D_1} - p) = h^0(D_1, \mathcal{O}_{D_1}(p)) - 2 + p_a(D_1) \geq p_a(D_1)$ by the Riemann-Roch and the Serre duality theorems. On the other hand, we clearly have $h^0(D_1, K_{D_1} - p) \leq h^0(D_1, K_{D_1}) = p_a(D_1)$. In sum, $h^0(D_1, K_{D_1} - p) = h^0(D_1, K_{D_1}) = p_a(D_1)$. This shows that p is a base point of $|K_{D_1}|$. Since $p \in A$ is general, we conclude that A is a fixed component of $|K_{D_1}|$.

Since $A \subset \mathrm{Bs}|K_{D_1}|$ and D_1 is numerically 1-connected, we can repeat the above argument with the pair (D_1, A) instead of (D, A) noting that we have $A(D_1 - A) = A(D - A) - A(D - D_1) = n - 1$. In this way, we can find a sequence of numerically 1-connected divisors

$$A = D_n \preceq D_{n-1} \preceq \cdots \preceq D_1 \preceq D_0 = D$$

such that $D_{i-1} - D_i$ is 1-connected, $\mathcal{O}_{D_i-A}(D_{i-1} - D_i) \simeq \mathcal{O}_{D_i-A}$, $A(D_{i-1} - D_i) = 1$ and $A \subset \mathrm{Bs}|K_{D_i}|$ for any i, $1 \le i \le n$. If we put $\Gamma_i = D_{i-1} - D_i$, then $D = A + \Gamma_1 + \cdots + \Gamma_n$. Note that we have $D_i - A = \Gamma_{i+1} + \cdots + \Gamma_n$. Hence (3) is nothing more than $\mathcal{O}_{D_i-A}(\Gamma_i) \simeq \mathcal{O}_{D_i-A}$. Since A is of multiplicity one in D_1 by (a), we get (2). $\square$

Remark 1.2. (1) The converse holds in the following sense: If $A \simeq \mathbb{P}^1$ and D decomposes as above, then $A \subset \mathrm{Bs}|K_D|$. See, the proof of [8, Theorem 4.1].

(2) The assertion (3) in Theorem 1.1 implies that $\mathcal{O}_{\Gamma_j}(\Gamma_i) \simeq \mathcal{O}_{\Gamma_j}$ when $i < j$. Therefore, (1) and (3) of Theorem 1.1 (without any assumption regarding D) imply that either $\mathrm{Supp}(\Gamma_i) \cap \mathrm{Supp}(\Gamma_j) = \emptyset$ or $\Gamma_j \preceq \Gamma_i$ when $i < j$. Indeed, suppose that $\mathrm{Supp}(\Gamma_i) \cap \mathrm{Supp}(\Gamma_j) \neq \emptyset$. Since $\Gamma_i\Gamma_j = 0$, Γ_i and Γ_j have a common component. Put $B = \gcd(\Gamma_i, \Gamma_j)$, $C_i = \Gamma_i - B$ and $C_j = \Gamma_j - B$. We have $C_iC_j \ge 0$, since they have no common components. Assume that $C_j \neq 0$. Since $\mathcal{O}_{\Gamma_j}(\Gamma_i) \simeq \mathcal{O}_{\Gamma_j}$, we have $\mathcal{O}_{C_j}(\Gamma_i) \simeq \mathcal{O}_{C_j}$ and it follows that $C_iC_j + BC_j = 0$. Hence $BC_j \le 0$, which is impossible since $\Gamma_j = C_j + B$ is 1-connected. Therefore, we have $C_j = 0$ and $\Gamma_j = B \preceq \Gamma_i$. This remark will be used in several places in what follows.

Corollary 1.3. *Let D and A be as in Theorem 1.1. Assume furthermore that K_D is nef. Then $(D - A)A \le p_a(D)$.*

Proof. We know $A \simeq \mathbb{P}^1$. Put $n = A(D - A)$ and let $D = A + \Gamma_1 + \cdots + \Gamma_n$ be the decomposition as in Theorem 1.1. Then it is easy to see that $p_a(D) = p_a(\Gamma_1) + \cdots + p_a(\Gamma_n)$ by using the numerical properties $A\Gamma_i = 1$ and $\Gamma_i\Gamma_j = 0$ when $i \neq j$. Since K_D is nef, we have $0 \le \deg K_D|_{\Gamma_i} = \deg K_{\Gamma_i} + (D - \Gamma_i)\Gamma_i = \deg K_{\Gamma_i} + 1$. Hence $\deg K_{\Gamma_i} = 2p_a(\Gamma_i) - 2$ is non-negative, implying $p_a(\Gamma_i) > 0$. Then we get $p_a(D) \ge n = A(D - A)$ as desired. $\square$

Let $\mathcal{A} = \bigcup_{i=1}^N A_i$ be a connected bunch of irreducible curves A_i. The intersection form is negative semi-definite on $\mathcal{A}$ if and only if there exists an effective (non-zero) divisor Z supported on $\mathcal{A}$ such that $-Z$ is nef on $\mathcal{A}$. The smallest curve among such Z's exists and is called the *numerical cycle* [10]. When the intersection form is negative definite, it is usually called the *fundamental cycle* ([1], [2]). It is easy to see that a numerically 1-connected divisor Z is the numerical cycle (on its support) if $-Z$ is nef on Z.

Lemma 1.4. *Let D be an effective non-zero divisor on a non-singular surface such that $-D$ is nef on D. Assume that D decomposes as $D = A + \Gamma_1 + \cdots + \Gamma_n$ for some positive integer n, where $A \prec D$ is an effective divisor, the Γ_i's are 1-connected divisors satisfying $A\Gamma_i = 1$ for $i \ge 1$ and $\mathcal{O}_{\Gamma_j}(\Gamma_i)$ is numerically trivial for $i < j$. If A does not have a common component with Γ_i, then $-\Gamma_i$ and $-\tilde{\Gamma}_i$ are both nef on Γ_i, where $\tilde{\Gamma}_i = \sum_{j \ge i,\, \Gamma_j \preceq \Gamma_i} \Gamma_j$. In particular, Γ_i is the numerical cycle on its support provided that A does not have a common component with Γ_i.*

Proof. We remark that $\Gamma_i\Gamma_j = 0$ and, either $\Gamma_j \preceq \Gamma_i$ or $\mathrm{Supp}(\Gamma_i) \cap \mathrm{Supp}(\Gamma_j) = \emptyset$ when $i < j$. Let B be an arbitrary irreducible component of Γ_i. Note that any Γ_j not appearing in $\tilde{\Gamma}_i$ is either disjoint from Γ_i or bigger than Γ_i, and we have $B\Gamma_j = 0$ for such Γ_j. Then we have $DB = AB + \tilde{\Gamma}_iB$. Since $DB \le 0$ and $AB \ge 0$ by $\gcd(A, \Gamma_i) = 0$, we get $\tilde{\Gamma}_iB \le 0$ as wished.

If B is a component of Γ_j for some $j > i$, then we have $\Gamma_iB = 0$, because $\mathcal{O}_{\Gamma_j}(\Gamma_i)$ is numerically trivial. Hence we may assume that B is not a component of Γ_j for any

$j > i$. Then $\Gamma_j B \geq 0$ for any $j > i$. We have $0 \geq \tilde{\Gamma}_i B = \Gamma_i B + B \sum_{j>i, \Gamma_j \preceq \Gamma_i} \Gamma_j \geq \Gamma_i B$. Therefore, $-\Gamma_i$ is nef on Γ_i. It follows that Γ_i is the numerical cycle on its support, since Γ_i is 1-connected. □

Proposition 1.5. *Let Z be the fundamental cycle on the exceptional set of a rational normal surface singularity. If $A \prec Z$ is an irreducible component, then Z decomposes as*

$$Z = A + Z_1 + \cdots + Z_n,$$

where $n = A(Z - A)$, each Z_i is a 1-connected divisor, $AZ_i = 1$ and, $\mathcal{O}_{Z_i+\cdots+Z_n}(-Z_{i-1}) \simeq \mathcal{O}_{Z_i+\cdots+Z_n}$ when $2 \leq i \leq n$. Furthermore, Z_i is the fundamental cycle on its support when $i \geq 2$, and the same is valid for Z_1 provided that A is a non-multiple component of Z.

Proof. We have $H^1(Z, \mathcal{O}_Z) = 0$ and $h^0(Z, \mathcal{O}_Z) = 1$. Hence Z is 1-connected. The restriction map $H^0(Z, K_Z) \to H^0(A, K_Z)$ is the zero map for the trivial reason. Hence, by Theorem 1.1, we get the decomposition of Z as wished. The fact that Z_i is the fundamental cycle follows from Lemma 1.4, since A is not a component of Z_i when $i \geq 2$, by Theorem 1.1, (2). □

Let the situation be as above. We remark that, if $A \not\preceq Z_i$ (which holds at least for $i \geq 2$), then Z_i has a non-multiple component A_i with $AA_i = 1$ by $AZ_i = 1$ and, therefore, Z_i also decomposes into a sum of A_i and several numerically disjoint fundamental cycles similarly as in the statement of Proposition 1.5.

2. Proof of Main Theorem. We return to the situation we are interested in. Let (V, o) be a germ of a normal surface singularity and $\pi : X \to V$ the minimal resolution. A non-zero effective (integral) divisor on X whose support is contained in $\pi^{-1}(o)$ will be simply called a *curve* in what follows. Since the intersection form is negative definite on the exceptional set, we have $A^2 < 0$ and $h^0(A, \mathcal{O}_A(A - L)) = 0$ for any curve A and a nef line bundle L. The latter implies, via the Serre duality theorem, that $H^1(A, L) = 0$ for any line bundle L such that $L - K_X$ is nef. We denote by Z the fundamental cycle on $\pi^{-1}(o)$, that is, the smallest curve such that $-Z$ is nef on $\pi^{-1}(o)$. Recall that the restriction map $H^0(X, L) \to H^0(Z, L)$ is surjective when $L - K_X$ is a nef line bundle on X, because we have $H^1(X, L - Z) = 0$ by the Kodaira-type vanishing theorem (see, e.g., [6]). Therefore, when $L - K_X$ is nef, an irreducible curve E is contained in $\mathrm{Bs}|L|$ if and only if the restriction map $H^0(Z, L) \to H^0(E, L)$ is the zero map.

We sometimes need the following easy lemma (compare this with [7, Lemma 3.2]):

Lemma 2.1. *Let Δ be a curve with $\Delta^2 = -1$. Then it is numerically 1-connected. If Δ' is another curve with $(\Delta')^2 = -1$, then either Δ and Δ' are disjoint or one is a subcurve of the other.*

Proof. Let $\Delta = \Delta_1 + \Delta_2$ be any effective decomposition, that is, a decomposition of Δ into a sum of two curves Δ_1, Δ_2. Then $\Delta^2 = \Delta_1^2 + \Delta_2^2 + 2\Delta_1\Delta_2$. Since we have $\Delta_1^2 < 0$, $\Delta_2^2 < 0$ and $\Delta^2 = -1$, we get $\Delta_1\Delta_2 > 0$. Therefore, Δ is numerically 1-connected.

In order to show the second part, we assume that $\mathrm{Supp}(\Delta) \cap \mathrm{Supp}(\Delta') \neq \emptyset$. Since $0 > (\Delta + \Delta')^2 = -2 + 2\Delta\Delta'$, we get $\Delta\Delta' \leq 0$. It follows that Δ and Δ' have a common component. Put $C = \gcd(\Delta, \Delta')$, $\Delta = A + C$ and $\Delta' = B + C$. We assume that $A \neq 0$, $B \neq 0$ and show that this leads us to a contradiction. Since

$$0 > (A + B + C)^2 = (A + C)^2 + (B + C)^2 - C^2 + 2AB = -2 - C^2 + 2AB$$

we have $C^2 > 2AB - 2$. It follows from $C^2 < 0$ and $AB \geq 0$ that $C^2 = -1$ and $AB = 0$DThen we get $AC + BC \leq 1$ by

$$0 \geq \Delta\Delta' = C^2 + (A + B)C + AB = -1 + AC + BC.$$

On the other hand, it follows from $\Delta^2 = \Delta'^2 = -1$ that $A^2 + 2AC = B^2 + 2BC = 0$. Since $A \neq 0$ and $B \neq 0$, we have $A^2 < 0$ and $B^2 < 0$. Hence $AC > 0$ and $BC > 0$, which contradicts $AC + BC \leq 1$. Therefore, either A or B must be zero. □

The following is the heart of our arguments.

Proposition 2.2. *Let L be a line bundle on Z such that $L - K_X$ is nef. Let E be an irreducible curve contained in* $\mathrm{Bs}|L|$. *Then $E \simeq \mathbb{P}^1$ and there exists the smallest reducible subcurve $\Delta = \Delta(E, L) \preceq Z$ with the following properties.*

(1) *Δ contains E as a component of multiplicity one and the restriction map $H^0(Z, L) \to H^0(\Delta, L)$ is surjective.*

(2) *Δ is the fundamental cycle on its support and $\mathcal{O}_\Delta(L - K_X)$ is numerically trivial.*

(3) *$E\Delta = \Delta^2 = -1$ and $\mathcal{O}_\Delta(L - K_X - \Delta) \simeq \mathcal{O}_\Delta(p)$ for any point $p \in E \setminus$* $\mathrm{Supp}(\Delta - E)$.

(4) *$E \subset \mathrm{Bs}|K_\Delta|$ and Δ decomposes as $\Delta = E + \Gamma_1 + \cdots + \Gamma_{n-1}$, where $n = -E^2$ and the Γ_i's are curves with $E\Gamma_i = -\Gamma_i^2 = 1$ for $i \geq 1$ and $\mathcal{O}_{\Gamma_i + \cdots + \Gamma_{n-1}}(\Gamma_{i-1}) \simeq \mathcal{O}_{\Gamma_i + \cdots + \Gamma_{n-1}}$ for $i \geq 2$.*

Proof. This is an analogue of Theorem 1.1. We let Δ be a minimal curve such that $E \preceq \Delta \preceq Z$ and the restriction map $H^0(Z, L) \to H^0(\Delta, L)$ is surjective. If $\Delta = E$, then we must have $H^0(E, L) = 0$, since $E \subset \mathrm{Bs}|L|$. Then the Riemann-Roch theorem shows that $-h^1(E, L) = \deg L|_E + 1 - p_a(E) = (1/2)(K_X E - E^2) + \deg(L - K_X)|_E$, which is impossible because $E^2 < 0$, $K_X E \geq 0$ and $\deg(L - K_X)|_E \geq 0$ by the nefness of $L - K_X$ and K_X. Therefore, Δ is reducible. Since $K_\Delta - L = \Delta - (L - K_X)$ on Δ and $\Delta^2 < 0$, we see that $K_\Delta - L$ is not nef on Δ. Then it follows from [4, Lemma 2.2.1] that E is of multiplicity one in Δ, $\mathcal{O}_{\Delta - E}(\Delta - (L - K_X))$ is nef and the image of $H^0(\Delta, L) \to H^0(E, L)$ contains the subspace coming from $H^0(E, L - (\Delta - E))$. Since $\mathcal{O}_{\Delta - E}(\Delta - (L - K_X))$ and $L - K_X$ are nef, we obtain $(\Delta - E)\Delta \geq \deg(L - K_X)|_{\Delta - E} \geq 0$. Since $E \subset \mathrm{Bs}|L|$, we must have $h^0(E, L - (\Delta - E)) = 0$. We have $\mathcal{O}_E(L - (\Delta - E)) \simeq \mathcal{O}_E(K_E + (L - K_X) - \Delta)$ and see that it is non-special, because $E\Delta < 0$ and $\deg(L - K_X)|_E \geq 0$. Then we get $E \simeq \mathbb{P}^1$, $E\Delta = -1$ and $\deg(L - K_X)|_E = 0$ similarly as in the proof of Theorem 1.1. Using $\Delta^2 < 0$, we get $\Delta^2 = -1$ by $E\Delta = -1$ and $(\Delta - E)\Delta \geq \deg(L - K_X)|_{\Delta - E} \geq 0$. Furthermore, we see that $\mathcal{O}_\Delta(L - K_X)$ and $\mathcal{O}_{\Delta - E}(\Delta)$ are both numerically trivial. Hence Δ is the fundamental cycle on its support, because Δ is 1-connected by Lemma 2.1 and $-\Delta$ is nef on $\mathrm{Supp}(\Delta)$.

We now let $p \in E$ be any non-singular point of Δ. By using $p \in \mathrm{Bs}|L|$ and

$$0 \to \mathcal{O}_\Delta(L - p) \to \mathcal{O}_\Delta(L) \to \mathcal{O}_p \to 0,$$

we get $H^1(\Delta, L - p) \neq 0$. Since $\mathcal{O}_\Delta(L - p) \simeq \mathcal{O}_\Delta(K_\Delta - \Delta + (L - K_X) - p)$ which is numerically equivalent to K_Δ, we get $\mathcal{O}_\Delta(-\Delta + L - K_X) \simeq \mathcal{O}_\Delta(p)$ by the 1-connectedness of Δ. Since $p \in E$ can move, we have $h^0(\Delta, \mathcal{O}_\Delta(p)) \geq 2$ which enables us to conclude that $p \in \mathrm{Bs}|K_\Delta|$ similarly as in the proof of Theorem 1.1. Therefore, $E \subset \mathrm{Bs}|K_\Delta|$. Then we have the decomposition $\Delta = E + \Gamma_1 + \cdots + \Gamma_{n-1}$ as in Theorem 1.1. Since $E\Gamma_i = 1$, $\Delta\Gamma_i = 0$ and $\Gamma_i\Gamma_j = 0$ when $j \neq i$, we get $\Gamma_i^2 = -1$. By Lemma 2.1, Γ_i is numerically 1-connected.

By virtue of Lemma 2.1 and the fact $\Delta^2 = -1$, we see that Δ is not only a minimal but also the smallest curve with the desired properties. □

DEFINITION 2.3. Let L be a line bundle on the fundamental cycle Z with $L - K_X$ nef and E an irreducible curve in $\mathrm{Bs}|L|$. We call the curve Δ as in Proposition 2.2 the *loupe* for E (with respect to L).

Now, we are going to show the following theorem which covers the first half of the Main Theorem in Introduction.

THEOREM 2.4. *Let (V, o) be a normal surface singular point and $\pi : X \to V$ the minimal resolution. If L is a line bundle on X such that $L - K_X$ is nef on $\pi^{-1}(o)$, then the fixed part of $|L|$ supports at most exceptional sets of rational singular points.*

Proof. Let $\mathcal{E} = \bigcup_i E_i$ be a connected bunch of irreducible curves E_i contained in $\mathrm{Bs}|L|$. We sometimes regard $\mathcal{E}$ as a reduced divisor. For each i, we take the loupe Δ_i for E_i with respect to L. Since $\mathcal{E}$ is connected and $\Delta_i^2 = -1$, the set $\{\Delta_i\}$ of loupes has the biggest element Δ by Lemma 2.1. Then $\mathcal{E} \preceq \Delta$. We let $E \preceq \mathcal{E}$ be the component whose loupe is Δ. Put $E^2 = -n$. By (4) of Proposition 2.2, Δ decomposes as $\Delta = E + \Gamma_1 + \cdots + \Gamma_{n-1}$, where Γ_α is a 1-connected curve satisfying $E\Gamma_\alpha = -\Gamma_\alpha^2 = 1$ and $\mathcal{O}_{\Gamma_\alpha + \cdots + \Gamma_{n-1}}(\Gamma_{\alpha-1}) \simeq \mathcal{O}_{\Gamma_\alpha + \cdots + \Gamma_{n-1}}$. Recall that the last condition shows either $\mathrm{Supp}(\Gamma_\alpha) \cap \mathrm{Supp}(\Gamma_\beta) = \emptyset$ or $\Gamma_\beta \preceq \Gamma_\alpha$ when $\alpha < \beta$ (see, Remark 1.2). Note that we cannot have $\Gamma_\alpha = \Gamma_\beta$ when $\alpha < \beta$, because $\Gamma_\alpha\Gamma_\beta = 0$ and $\Gamma_\alpha^2 = -1$. After changing the labeling if necessary, we may assume that $\{E_1, \ldots, E_\ell\}$ is the set of all irreducible components of $\mathcal{E} - E$ with $EE_i > 0$. For each $i \in \{1, \ldots, \ell\}$, we denote by $\alpha(i)$ the smallest one among those indices α's with $E_i \preceq \Gamma_\alpha$. Then every other Γ_β containing E_i is a subcurve of $\Gamma_{\alpha(i)}$. Since $E \not\preceq \Gamma_{\alpha(i)}$ and $E\Gamma_{\alpha(i)} = 1$, we get $EE_i = 1$ and see that E_i is the unique component of multiplicity one in $\Gamma_{\alpha(i)}$ which meets E. Furthermore, we know that $\Gamma_{\alpha(1)}, \ldots, \Gamma_{\alpha(\ell)}$ are mutually disjoint, because, otherwise, there would be two distinct indices i, j with $\Gamma_{\alpha(i)} \prec \Gamma_{\alpha(j)}$ which would imply $E_i = E_j$ by what we have just seen. We also remark that, for any $E_j \preceq \mathcal{E}$, $E_j \neq E$, we can find a unique $\Gamma_{\alpha(i)}$ such that $E_j \preceq \Gamma_{\alpha(i)}$, because E_j should be connected to E by a path consisting of curves in $\mathcal{E}$; we would immediately get a contradiction to that $\mathcal{E}$ is connected if there were no such i. Put $\mathcal{E}_i = (\mathcal{E} - E) \cap \mathrm{Supp}(\Gamma_{\alpha(i)})$ for $i = 1, \ldots, \ell$. Then $\mathcal{E}_i$ is connected and we get the decomposition of $\mathcal{E} - E$ into the connected components: $\mathcal{E} - E = \bigsqcup_{i=1}^{\ell} \mathcal{E}_i$.

We denote by $Z_\mathcal{E}$ the fundamental cycle on $\mathcal{E}$. Let G be the biggest subcurve of Δ with $\mathrm{Supp}(G) = \mathcal{E}$. Then $0 \geq E_j\Delta = E_jG + E_j(\Delta - G) \geq E_jG$ for any $E_j \preceq \mathcal{E}$. This shows that $-G$ is nef on $\mathcal{E}$. It follows $Z_\mathcal{E} \preceq G \preceq \Delta$. In particular, we know that E is of multiplicity one in $Z_\mathcal{E}$.

Now, we claim that $H^1(Z_\mathcal{E}, \mathcal{O}_{Z_\mathcal{E}}) = 0$. We argue by induction on the number of irreducible components. If $\mathcal{E}$ is a single curve, then we clearly have $Z_\mathcal{E} = \mathcal{E} \simeq \mathbb{P}^1$ and the assertion follows. We assume that $\mathcal{E}$ consists of several irreducible components. Let $\mathcal{E}_i$ be as above. Since the number of irreducible components of $\mathcal{E}_i$ is strictly smaller than that of $\mathcal{E}$, by the hypothesis of the induction, we have $H^1(Z_{\mathcal{E}_i}, \mathcal{O}_{Z_{\mathcal{E}_i}}) = 0$ for the fundamental cycle $Z_{\mathcal{E}_i}$ on $\mathcal{E}_i$. This implies that we obtain a rational singular point by contracting $\mathcal{E}_i$. Hence, if G_i denotes the biggest subcurve of $Z_\mathcal{E}$ supported on $\mathcal{E}_i$, then we also have $H^1(G_i, \mathcal{O}_{G_i}) = 0$ (see [1], [2]). Recall that E is of multiplicity one in $Z_{\mathcal{E}}$. Since $Z_{\mathcal{E}} - E = G_1 + \cdots + G_\ell$ and $\mathrm{Supp}(G_i) \cap \mathrm{Supp}(G_j) = \emptyset$ when $i \neq j$, we get $H^1(Z_{\mathcal{E}} - E, \mathcal{O}_{Z_{\mathcal{E}}-E}) = \bigoplus_{i=1}^{\ell} H^1(G_i, \mathcal{O}_{G_i}) = 0$. Let η be a section of the line bundle $[\Delta - Z_{\mathcal{E}}]$ defining the curve $\Delta - Z_{\mathcal{E}}$. Since E is not a component of $\Delta - Z_{\mathcal{E}}$, η induces an effective divisor on E. The restriction maps and the natural injections induced by η give us the following commutative diagram.

$$\begin{array}{ccc} H^0(Z_{\mathcal{E}}, K_{Z_{\mathcal{E}}}) & \longrightarrow & H^0(E, K_{Z_{\mathcal{E}}}) \\ {\scriptstyle \cdot\eta}\downarrow & & \downarrow{\scriptstyle \cdot\eta|_E} \\ H^0(\Delta, K_\Delta) & \longrightarrow & H^0(E, K_\Delta) \end{array}$$

Since $E \subset \mathrm{Bs}|K_\Delta|$, the map at the bottom row is the zero map. Hence $H^0(Z_{\mathcal{E}}, K_{Z_{\mathcal{E}}}) \to H^0(E, K_{Z_{\mathcal{E}}})$ is also the zero map. Note that its kernel is isomorphic to $H^0(Z_{\mathcal{E}} - E, K_{Z_{\mathcal{E}}-E})$ whose dual is $H^1(Z_{\mathcal{E}} - E, \mathcal{O}_{Z_{\mathcal{E}}-E})$ which is zero as we saw above. It follows that $H^0(Z_{\mathcal{E}}, K_{Z_{\mathcal{E}}}) = 0$. Then we get $H^1(Z_{\mathcal{E}}, \mathcal{O}_{Z_{\mathcal{E}}}) = 0$ as desired, by the Serre duality theorem.

This shows that $Z_{\mathcal{E}}$ is the fundamental cycle of a rational singular point and completes the proof of Theorem 2.4. □

Remark 2.5. (1) We do not know whether the restriction map $H^0(Z, L) \to H^0(Z_{\mathcal{E}}, L)$ is the zero map or not. This explains a reason why we need a round-about argument as above.
(2) Δ and $Z_{\mathcal{E}}$ are the fundamental cycles on their respective supports. Since $\Delta^2 = -1$ and K_X is nef, we have $p_a(\Delta) > 0$ while we know $p_a(Z_{\mathcal{E}}) = 0$. It follows that we have not only $Z_{\mathcal{E}} \preceq \Delta$ but also that $\mathcal{E}$ is strictly smaller than $\mathrm{Supp}(\Delta)$. This also shows that if (V, o) is rational, then $|L|$ has no fixed components.
(3) By Proposition 1.5, $Z_{\mathcal{E}}$ decomposes as

$$Z_{\mathcal{E}} = E + Z_1 + \cdots + Z_k,$$

where $k = E(Z_{\mathcal{E}} - E)$, each Z_i is the fundamental cycle on its support with $EZ_i = 1$ and $\mathcal{O}_{Z_j+\cdots+Z_k}(-Z_{j-1}) \simeq \mathcal{O}_{Z_j+\cdots+Z_k}$ for $2 \leq j \leq k$. This may be useful to study the configuration of $\mathcal{E}$. We also remark that $EZ_{\mathcal{E}} \leq EZ_{\mathcal{E}} + E(\Delta - Z_{\mathcal{E}}) = E\Delta = -1$.

There are three basic invariants of (V, o) (cf. [11]): The geometric genus $p_g(V, o) = \dim\ (R^1\pi_*\mathcal{O}_X)_o$, the arithmetic genus $p_a(V, o) = \sup\{p_a(D) \mid 0 \prec D,\ \mathrm{Supp}(D) \subset \pi^{-1}(o)\}$ and the fundamental genus $p_f(V, o) = p_a(Z)$. We have $p_f(V, o) \leq p_a(V, o) \leq p_g(V, o)$. Note that the inequalities are strict in many cases. For example, if (V, o) is a hypersurface singularity defined by $x^3 + y^4 + z^{12} = 0$, then $p_f = 3$, $p_a = 4$ and $p_g = 8$.

The following completes the proof of the Main Theorem.

Proposition 2.6. *Let (V, o) be a normal surface singularity and $\pi : X \to V$ the minimal resolution. Let L be a line bundle on X such that $L - K_X$ is nef and assume that $|L|$ has a fixed component.*

(1) *If (U, p) denotes the rational singular point obtained by contracting a connected component of the fixed part of $|L|$, then the multiplicity* $\mathrm{mult}(U, p)$ *and the embedding dimension* $\mathrm{embdim}(U, p)$ *satisfy*

$$\mathrm{mult}(U, p) \leq 2p_f(V, o), \quad \mathrm{embdim}(U, p) \leq 2p_f(V, o) + 1.$$

(2) *If an irreducible curve E is a fixed component of $|L|$, then $-E^2 \leq p_f(V,o)+1$.*

Proof. (1) We retain the notation and assumptions as in the proof of Theorem 2.4. $\mathcal{E}$ is now the connected component of $\mathrm{Bs}|L|$ which produces (U,p). Since $Z_{\mathcal{E}}$ is the fundamental cycle of a rational singularity, it is numerically 1-connected and we have $-Z_{\mathcal{E}}^2 = K_X Z_{\mathcal{E}} + 2$ by the adjunction formula. Since K_X is nef and $Z_{\mathcal{E}} \prec \Delta$, we have $K_X Z_{\mathcal{E}} \leq K_X\Delta = 2p_a(\Delta) - 1$ by $\Delta^2 = -1$. Assume that $K_X Z_{\mathcal{E}} = K_X\Delta$. Then $\Delta - Z_{\mathcal{E}}$ consists of (-2)-curves. Since Δ is numerically 1-connected, $\mathrm{Supp}(\Delta)$ is a connected set. Furthermore, we know that $\mathcal{E}$ is a proper subset of $\mathrm{Supp}(\Delta)$ (see, Remark 2.5, (2)). It follows that there exists an irreducible curve A contained in the closure of $\mathrm{Supp}(\Delta) \setminus \mathcal{E}$ that meets $\mathcal{E}$. On the other hand, since A is a (-2)-curve and $\mathcal{O}_A(K_X) \simeq \mathcal{O}_A(L)$ by (2) of Proposition 2.2, any section of L is constant on A, which should be zero because A meets $\mathcal{E}$. Hence $A \subset \mathrm{Bs}|L|$. This contradicts that $\mathcal{E}$ is a connected component of $\mathrm{Bs}|L|$. Therefore, $K_X Z_{\mathcal{E}}$ is strictly smaller than $K_X\Delta$. Then we get $K_X Z_{\mathcal{E}} \leq 2p_a(\Delta) - 2$ which implies $-Z_{\mathcal{E}}^2 \leq 2p_a(\Delta)$. Since $\Delta \preceq Z$, we have $p_a(\Delta) \leq p_a(Z) = p_f(V,o)$. In sum, we have shown $-Z_{\mathcal{E}}^2 \leq 2p_f(V,o)$. Now, the assertion follows from M. Artin's formulas: $\mathrm{mult}(U,p) = -Z_{\mathcal{E}}^2$ and $\mathrm{embdim}(U,p) = -Z_{\mathcal{E}}^2 + 1$.

(2) We already know that $E \simeq \mathbb{P}^1$. Let Δ be the loupe for E with respect to L. By Proposition 2.2, it is a numerically 1-connected curve with $\Delta^2 = E\Delta = -1$ and $E \subset \mathrm{Bs}|K_\Delta|$. Furthermore, we know that $\mathcal{O}_{\Delta - E}(\Delta)$ is numerically trivial. Then $\deg K_\Delta|_A = \deg K_X|_A + A\Delta = \deg K_X|_A \geq 0$ for any irreducible component A of $\Delta - E$, that is, K_Δ is nef on $\Delta - E$. Recall that E is of multiplicity one in Δ. Then we can show that $(\Delta - E)E \leq p_a(\Delta)$ holds similarly as in the proof of Corollary 1.3. Therefore, $-E^2 \leq p_a(\Delta) + 1 \leq p_f(V,o) + 1$. ∎

The bound in (2) is sharp, while (1) may be rather weak when $p_f(V,o) > 1$.

When (V,o) is a rational singular point, Proposition 2.6 implies that $|L|$ is free from fixed components whenever $L - K_X$ is nef. Since K_X is nef and $H^1(Z, \mathcal{O}_Z) = 0$, the following slightly more general result shows that, in fact, we have $\mathrm{Bs}|L| = \emptyset$ in this case.

Proposition 2.7. *Let D be an effective non-zero divisor with $H^1(D, \mathcal{O}_D) = 0$ on a smooth surface. If M is a nef line bundle on D, then the restriction map $H^0(D,M) \to H^0(A,M)$ is surjective for any irreducible component A of D. In particular,* $\mathrm{Bs}|M| = \emptyset$ *when M is nef.*

Proof. First, we notice that $H^1(D', \mathcal{O}_{D'}) = 0$ holds for any effective divisor D' with $D' \preceq D$, since $H^1(D, \mathcal{O}_D) = 0$. In particular, we have $A \simeq \mathbb{P}^1$. Let Δ be a minimal effective divisor with $A \preceq \Delta \preceq D$ such that the restriction map $H^0(D,M) \to H^0(\Delta, M)$ is surjective. Since $H^1(\Delta, \mathcal{O}_\Delta) = 0$, we have $p_a(\Delta) \leq 0$ and, hence, $\deg K_\Delta = 2p_a(\Delta) - 2 \leq -2$. It follows that $K_\Delta - M$ is not nef on Δ. By [4, Lemma 2.2.1], A is of multiplicity one in Δ and $K_\Delta - M$ is nef on $\Delta - A$. Then

$$\deg K_\Delta = \deg K_\Delta|_{\Delta - A} + \deg K_\Delta|_A \geq \deg M|_{\Delta - A} + \deg K_A + (\Delta - A)A$$

Notice that we have $\deg K_\Delta \leq -2$, $\deg K_A = -2$ and $\deg M|_{\Delta - A} \geq 0$. Then $(\Delta - A)A \leq 0$. On the other hand, since A is of multiplicity one in Δ, we have $(\Delta - A)A \geq 0$. Hence $(\Delta - A)A = 0$ and it follows $A \cap (\Delta - A) = \emptyset$. Then $H^0(\Delta, M) = H^0(A, M) \oplus H^0(\Delta - A, M)$ and the restriction $H^0(\Delta, M) \to H^0(A, M)$ is surjective. Hence we must have $\Delta = A$ by the minimality of Δ. ∎

3. Further remarks. Here, we state some properties of loupes not needed for the proof of Theorem 2.4 for the later use.

Let L be as before a line bundle with $L - K_X$ nef. Take an irreducible curve $E \subset \mathrm{Bs}|L|$ and let Δ be the loupe for E with respect to L. Put $E^2 = -n$ and let

$$\Delta = E + \Gamma_1 + \cdots + \Gamma_{n-1}$$

be the decomposition of Δ as in (4) of Proposition 2.2. Then we have $\mathcal{O}_{\Gamma_i+\cdots+\Gamma_{n-1}}(\Gamma_{i-1}) \simeq \mathcal{O}_{\Gamma_i+\cdots+\Gamma_{n-1}}$ for $i \geq 2$ which implies either $\Gamma_j \prec \Gamma_i$ or $\mathrm{Supp}(\Gamma_i) \cap \mathrm{Supp}(\Gamma_j) = \emptyset$ when $i < j$. Since E is of multiplicity one in Δ and $E\Gamma_i = 1$, each Γ_i has a unique irreducible component of multiplicity one which intersects E at a point.

Lemma 3.1. *Each Γ_i is the fundamental cycle on its support and $-\Gamma_i - \sum_{\Gamma_j \prec \Gamma_i} \Gamma_j$ is nef on Γ_i. There exists a unique irreducible component $A_i \preceq \Gamma_i$ of multiplicity one such that $A_i\Gamma_i = -1$ and $C\Gamma_i = 0$ for any curve with $C \preceq \Gamma_i - A_i$. Furthermore, $A_i \neq A_j$ for $j \neq i$, and $A_iE = 1$ holds if and only if Γ_i is minimal in $\{\Gamma_\nu\}_{\nu=1}^{n-1}$.*

Proof. The first assertion follows from Lemma 1.4. Since $\Gamma_i^2 = -1$ and $\mathcal{O}_{\Gamma_i}(-\Gamma_i)$ is nef, one immediately sees that A_i as in the statement exists. When $i < j$, $\mathcal{O}_{\Gamma_j}(\Gamma_i)$ is numerically trivial by Proposition 2.2, (4), and we get $A_j\Gamma_i = 0$. Similarly, we get $A_i\Gamma_j = 0$ when $j < i$. Hence $A_j \neq A_i$ for $j \neq i$. Let k be the biggest index such that $\Gamma_k \preceq \Gamma_i$. Since $\mathcal{O}_{\Delta - E}(\Delta)$ is numerically trivial by Proposition 2.2, we obtain $A_kE = 1$ from $0 = A_k\Delta = A_kE + A_k\Gamma_k$. It follows that A_k is the unique component of Γ_i which meets E. □

The following will be useful when we study the configuration of the fixed part by an inductive argument.

Proposition 3.2. *Assume that Γ_i is minimal in $\{\Gamma_1, \dots, \Gamma_{n-1}\}$ and let $A_i \preceq \Gamma_i$ be the curve as in the previous lemma. Then $\mathcal{O}_{\Gamma_i}(L - K_X - \Gamma_i) \simeq \mathcal{O}_{\Gamma_i}(p_i)$, where $p_i = A_i \cap E$. Furthermore, Γ_i is a minimal curve among those curves C with $A_i \preceq C \preceq Z$ such that $H^0(X, L) \to H^0(C, L)$ is surjective. In particular, Γ_i is the loupe for A_i with respect to L, when $A_i \subset \mathrm{Bs}|L|$.*

Proof. Note that we have $\mathcal{O}_{\Gamma_i}(\Gamma_j) \simeq \mathcal{O}_{\Gamma_i}$ for any $j \neq i$ by the choice of Γ_i. It follows $\mathcal{O}_{\Gamma_i} \simeq \mathcal{O}_{\Gamma_i}(K_X - L + \Delta) \simeq \mathcal{O}_{\Gamma_i}(K_X - L + \Gamma_i + E)$. Hence we get $\mathcal{O}_{\Gamma_i}(L - K_X - \Gamma_i) \simeq \mathcal{O}_{\Gamma_i}(p_i)$.

Before showing the second assertion, we notice that the restriction map $H^0(\Delta, L) \to H^0(\Gamma_\alpha, L)$ is surjective for any $\alpha \in \{1, \dots, n-1\}$. To see this, consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_{\Delta - \Gamma_\alpha}(L - \Gamma_\alpha) \to \mathcal{O}_\Delta(L) \to \mathcal{O}_{\Gamma_\alpha}(L) \to 0.$$

We know that $\Delta - \Gamma_\alpha$ is numerically 1-connected, because $(\Delta - \Gamma_\alpha)\Gamma_\alpha = 1$. We have $\mathcal{O}_{\Delta-\Gamma_\alpha}(L - \Gamma_\alpha) \simeq \mathcal{O}_{\Delta-\Gamma_\alpha}(K_{\Delta-\Gamma_\alpha} + L - K_X - \Delta)$. Since $\mathcal{O}_{\Delta-\Gamma_\alpha}(L - K_X - \Delta)$ is nef of degree one, we get $H^1(\Delta - \Gamma_\alpha, L - \Gamma_\alpha) = 0$. Hence $H^0(\Delta, L) \to H^0(\Gamma_\alpha, L)$ is surjective.

Let C be any proper subcurve of Γ_i with $A_i \preceq C$. Then $\mathcal{O}_{\Gamma_i - C}(L - K_X - \Gamma_i) \simeq \mathcal{O}_{\Gamma_i - C}$ by $\mathcal{O}_{\Gamma_i}(L - K_X - \Gamma_i) \simeq \mathcal{O}_{\Gamma_i}(p_i)$. It follows that $\mathcal{O}_{\Gamma_i - C}(L - C) \simeq \omega_{\Gamma_i - C}$. Consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_{\Gamma_i - C}(L - C) \to \mathcal{O}_{\Gamma_i}(L) \to \mathcal{O}_C(L) \to 0.$$

We have $h^1(\Gamma_i - C, L - C) = h^1(\Gamma_i - C, \omega_{\Gamma_i - C}) \neq 0$ and $H^1(\Gamma_i, L) = 0$. Hence the restriction map $H^0(\Gamma_i, L) \to H^0(C, L)$ cannot be surjective. This means that Γ_i is a minimal curve with $A_i \preceq \Gamma_i \preceq Z$ such that $H^0(X, L) \to H^0(\Gamma_i, L)$ is surjective. ∎

Though the following can be shown by using the A-D-E classification and the fact given in Remark 2.5 that there is a non-multiple component $E \preceq Z_{\mathcal{E}}$ with $EZ_{\mathcal{E}} < 0$, we present the proof as an application of Proposition 3.2 for the use in the next section.

Corollary 3.3. *Let (V, o) be a normal surface singularity and $\pi : X \to V$ the minimal resolution. Take a line bundle L on X such that $L - K_X$ is nef. Let $\mathcal{E} = \bigcup_{i=0}^{m-1} E_i$ be a connected bunch of irreducible curves $E_i \subset \mathrm{Bs}|L|$. If the singular point obtained by contracting $\mathcal{E}$ is a rational double point, then it is of type $\mathbf{A}_m$.*

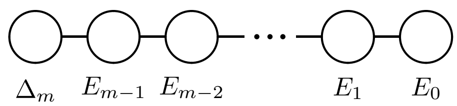

Fig. 1.

Proof. We know that all the E_i's are (-2)-curves. Take the loupe Δ_i for E_i with respect to L. Since $E_i^2 = -2$, the decomposition of Δ_i as in Proposition 2.2 is of the form $E_i + \Gamma_{i,1}$. By Proposition 3.2, we can assume that $\Gamma_{i,1} = \Delta_{i+1}$ for $i \in \{0, \dots, m-2\}$ after changing the ordering of the E_j's if necessary. Put $\Delta_m := \Gamma_{m-1,1}$. Then $\Delta_m \prec \cdots \prec \Delta_1 \prec \Delta_0$ and $E_i = \Delta_i - \Delta_{i+1}$ for $i \in \{0, \dots, m-1\}$. Furthermore, we have $E_i E_{i+1} = 1$ for $i \in \{0, \dots, m-2\}$ and $E_i E_j = 0$ when $|i - j| > 1$. Hence $\mathcal{E}$ corresponds to the rational double point of type $\mathbf{A}_m$, and (a part of) the dual graph of Δ_0 is as in Fig. 1. ∎

In particular, this suggests that we cannot produce an arbitrary rational singular point by contracting the canonical fixed part.

Finally in this section, we want to emphasize the importance of the study of $|K_\Delta|$ by showing that $|K_\Delta|$ faithfully inherits information on the base locus of $|L|$.

Lemma 3.4. *For any subcurve $C \prec \Delta$, the restriction map $H^0(X, L) \to H^0(C, L)$ is the zero map if and only if so is the map $H^0(\Delta, K_\Delta) \to H^0(C, K_\Delta)$. Furthermore,* $\mathrm{Bs}|K_\Delta| = \mathrm{Bs}|L| \cap \mathrm{Supp}(\Delta)$.

Proof. Recall that the restriction map $H^0(X, L) \to H^0(\Delta, L)$ is surjective and $\mathcal{O}_\Delta(K_\Delta) \simeq \mathcal{O}_\Delta(L - p)$ for a general point $p \in E$ by Proposition 2.2. Consider the commutative diagram

$$\begin{array}{ccc} H^0(\Delta, K_\Delta) & \longrightarrow & H^0(C, K_\Delta) \\ \big\downarrow & & \big\downarrow \\ H^0(\Delta, L) & \longrightarrow & H^0(C, L), \end{array}$$

where the horizontal maps are restrictions and the vertical maps are natural inclusions induced by $\mathcal{O}_\Delta(L - p) \hookrightarrow \mathcal{O}_\Delta(L)$. Since $p \in E \subset \mathrm{Bs}|L|$, $H^0(\Delta, K_\Delta) \to H^0(\Delta, L)$ is an isomorphism. Hence the first assertion follows from the above diagram. As to the second assertion, we only have to note that an isolated base point, if exists, is on $\Delta - E$. ∎

4. Weakly elliptic singularities. A normal surface singularity (V,o) is called a numerically Gorenstein singularity if there exists a curve Z_K such that $-Z_K$ is numerically equivalent to K_X on $\pi^{-1}(o)$. Such a curve Z_K is called the *canonical cycle* and the geometric genus of (V,o) is given by $p_g(V,o)=h^1(Z_K,\mathcal{O}_{Z_K})$ (see e.g., [10]). A normal surface singularity (V,o) is called an elliptic singularity [11] if $p_a(V,o)=1$. If an elliptic singularity is numerically Gorenstein, then Z_K is a numerically 0-connected curve, and S.S.T. Yau [13] introduced a sequence $\{Z_i\}_{i=0}^n$ of curves with $Z_K=\sum_{i=0}^n Z_i$, called the *elliptic sequence*, where the Z_i's are the fundamental cycles on their respective supports (since we are on the minimal resolution), $Z_n \prec Z_{n-1} \prec \cdots \prec Z_1 \prec Z_0 = Z$ and Z_n is contracted to an elliptic Gorenstein singularity with $p_g=1$ (a minimally elliptic singularity). It is shown in [13] that $p_g(V,o)$ is at most $n+1$, that is, the length of the elliptic sequence.

THEOREM 4.1. *Let (V,o) be a normal surface singularity and $\pi: X\to V$ the minimal resolution. Let L be a line bundle on X such that $L-K_X$ is nef, and suppose that $|L|$ has a fixed component. Then the following hold.*

(1) *If (V,o) is an elliptic singularity, then the fixed part of $|L|$ supports at most exceptional sets of rational double points of type* $\mathbf{A}$.

(2) *Let $\bigcup_{i=0}^{m-1}E_i$ be a connected component of the fixed part of $|L|$ which supports the exceptional set of the rational double point of type $\mathbf{A}_m$. Take the loupe Δ_i for E_i with respect to L, and change the indices so that $\Delta_{m-1}\prec\cdots\prec\Delta_1\prec\Delta_0$ if necessary. Suppose that $p_a(\Delta_0)=1$. Then Δ_0 is contracted to an elliptic numerically Gorenstein singularity and $\{\Delta_i\}_{i=0}^m$ forms its elliptic sequence, where $\Delta_m=\Delta_0-\sum_{i=0}^{m-1}E_i$. Furthermore, $\mathcal{O}_{\Delta_m}(L-K_X)\simeq\mathcal{O}_{\Delta_m}(\Delta_{m-1})$.*

Proof. The assertion (1) follows from Corollary 3.3, because (U,p) as in Proposition 2.6 should be a rational double point if $p_a(V,o)=1$. In order to show (2), we employ the same notation and conventions as in the proof of Corollary 3.3, assuming now that $\mathcal{E}$ is a connected component of the fixed part. Recall that $\mathcal{O}_{\Delta_0}(L-K_X)$ is numerically trivial by Proposition 2.2, (2). By Proposition 3.2, $\Delta_m=\Delta_0-\sum_{i=0}^{m-1}E_i$ inherits nice properties from the preceding Δ_i's: We have $\Delta_m^2=-1$ and there exists a unique irreducible component E_m of multiplicity one in Δ_m with $E_{m-1}E_m=1$, $E_m\Delta_m=-1$ and $\mathcal{O}_{\Delta_m}(L-K_X-\Delta_m)\simeq\mathcal{O}_{\Delta_m}(p)$, where $p=E_{m-1}\cap E_m$. We also know that Δ_m is the fundamental cycle on its support.

Suppose that $p_a(\Delta_0)=1$. By [11] and [5], we obtain an elliptic singularity by contracting Δ_0, since $p_a(\Delta_0)=1$ and Δ_0 is the fundamental cycle. We know that Δ_i is the fundamental cycle on its support and $\mathcal{O}_{\Delta_i-E_i}(L-K_X-\Delta_i)\simeq\mathcal{O}_{\Delta_i-E_i}$ for any $i\in\{0,\dots,m\}$. Since $\Delta_i^2=-1$ and K_X is nef, we get $p_a(\Delta_i)\geq 1$ by $0\leq K_X\Delta_i=2p_a(\Delta_i)-1$. On the other hand, since Δ_i is a 1-connected curve, we have $p_a(\Delta_i)=h^1(\Delta_i,\mathcal{O}_{\Delta_i})\leq h^1(\Delta_0,\mathcal{O}_{\Delta_0})=1$. Hence $p_a(\Delta_i)=1$. Since $\Delta_i=\Delta_1-E_1-\cdots-E_{i-1}$ and Δ_i does not meet $E_0+\cdots+E_{i-2}$, we have the following:

CLAIM 4.2. *$\mathcal{O}_{\Delta_i}(L-K_X-\Delta_j)\simeq\mathcal{O}_{\Delta_i}$ when $i>j$ and $\mathcal{O}_{\Delta_i}(L-K_X-\Delta_i)\simeq\mathcal{O}_{\Delta_i}(E_{i-1})$ when $i>0$.*

We remark that E_m is not a (-2)-curve, because, otherwise, we have $\deg L|_{E_m}=\deg K_X|_{E_m}=0$ and the property $E_{m-1}E_m=1$ would imply that E_m has to be a fixed component of $|L|$, contradicting that $\mathcal{E}$ is a connected component. Then E_m is the unique irreducible component of Δ_0 which has positive intersection with K_X. In fact, we have $K_XE_m=1$ by $K_X\Delta_0=1$.

Claim 4.3. *Δ_m is numerically 2-connected.*

Proof. Let $\Delta_m = A + B$ be any effective decomposition. We may assume that $E_m \preceq A$. Then $B\Delta_m = 0$. Furthermore, we have $K_X B = 0$ which implies that B^2 is a negative even integer. Then $0 = \Delta_m B = AB + B^2 \leq AB - 2$. Hence Δ_m is 2-connected. ∎

Since Δ_m is a 2-connected curve with $p_a(\Delta_m) = 1$, we have $\omega_{\Delta_m} \simeq \mathcal{O}_{\Delta_m}$ by [3, Proposition (A.7)]. This implies that Δ_m is the fundamental cycle of an elliptic Gorenstein singularity with $p_g = 1$ (a minimally elliptic singularity). Now, put $\tilde{\Delta}_i = \Delta_i + \Delta_{i+1} + \cdots + \Delta_m$ for $i \in \{0, \dots, m\}$.

Claim 4.4. *$-\tilde{\Delta}_i$ is numerically equivalent to K_X on Δ_i for any $i \in \{0, \dots, m\}$. Furthermore, $\mathcal{O}_{\Delta_i}(K_X) \simeq \mathcal{O}_{\Delta_i}(-\tilde{\Delta}_i)$ holds if and only if $\mathcal{O}_{\Delta_m}((m-i)(L-K_X)) \simeq \mathcal{O}_{\Delta_m}$.*

Proof. We already know that $\omega_{\Delta_m} \simeq \mathcal{O}_{\Delta_m}$, that is, $\mathcal{O}_{\Delta_m}(K_X) \simeq \mathcal{O}_{\Delta_m}(-\Delta_m)$. This gives us the assertion for $i = m$.

We may assume that $i < m$. The fact that $-\tilde{\Delta}_i$ is numerically equivalent to K_X on Δ_i easily follows from $K_X\Delta_i = 1$ and Claim 4.2, if we note that $L - K_X$ is numerically equivalent to zero on Δ_0.

Since Δ_i is numerically 1-connected, $\mathcal{O}_{\Delta_i}(K_X) \simeq \mathcal{O}_{\Delta_i}(-\tilde{\Delta}_i)$ is equivalent to $H^0(\Delta_i, K_X + \tilde{\Delta}_i) \neq 0$. We consider the exact sequence

$$0 \to \mathcal{O}_{E_i+\cdots+E_{m-1}}(K_X + \tilde{\Delta}_i - \Delta_m) \to \mathcal{O}_{\Delta_i}(K_X + \tilde{\Delta}_i) \to \mathcal{O}_{\Delta_m}(K_X + \tilde{\Delta}_i) \to 0.$$

Since $K_X + \tilde{\Delta}_i - \Delta_m$ is anti-nef of degree -1 on the numerically 1-connected rational curve $E_i + \cdots + E_{m-1}$, we get $H^q(E_i + \cdots + E_{m-1}, K_X + \tilde{\Delta}_i - \Delta_m) = 0$ for $q = 0, 1$. Hence $H^0(\Delta_i, K_X + \tilde{\Delta}_i) \simeq H^0(\Delta_m, \mathcal{O}_{\Delta_m}(K_X + \tilde{\Delta}_i))$. It follows from Claim 4.2 that

$$\mathcal{O}_{\Delta_m}(K_X + \tilde{\Delta}_i) \simeq \mathcal{O}_{\Delta_m}((m-i)(L-K_X) + K_X + \Delta_m) \simeq \mathcal{O}_{\Delta_m}((m-i)(L-K_X)).$$

Therefore, we have $\mathcal{O}_{\Delta_i}(K_X) \simeq \mathcal{O}_{\Delta_i}(-\tilde{\Delta}_i)$ if and only if $\mathcal{O}_{\Delta_m}((m-i)(L-K_X)) \simeq \mathcal{O}_{\Delta_m}$. ∎

We have shown that $\tilde{\Delta}_i$ is the canonical cycle for the elliptic numerically Gorenstein singularity (V_i, p_i) obtained by contracting Δ_i and that $\{\Delta_j\}_{j=i}^m$ is the elliptic sequence for each i with $0 \leq i \leq m$. We have $\mathcal{O}_{\Delta_m}(L) \simeq \mathcal{O}_{\Delta_m}(K_X + \Delta_{m-1})$ by Claim 4.2. Therefore, we get (2) of Theorem 4.1. ∎

The converse of Theorem 4.1, (2) holds in the following sense.

Proposition 4.5. *Let (V, o) be an elliptic numerically Gorenstein singularity with $p_g(V, o) \geq 2$ whose fundamental cycle satisfies $Z^2 = -1$ on the minimal resolution X. Let $\{Z_i\}_{i=0}^n$, $Z_0 = Z$, be the elliptic sequence. If L is a line bundle on X numerically equivalent to K_X, then either* $\mathrm{Bs}|L|$ *is one point* (*which is a non-singular point of Z_n*) *or it supports the exceptional set of the rational double point of type $\mathbf{A}_n$. The latter happens if and only if $\mathcal{O}_{Z_n}(L - K_X) \simeq \mathcal{O}_{Z_n}(Z_{n-1})$.*

Proof. Since $Z^2 = -1$, the minimal resolution dual graph is classified (see e.g., [12] and [9, Proposition 5.13]) which is much similar to one in Fig. 1. We see that $C_i = Z_i - Z_{i+1}$ is a (-2)-curve for $0 \leq i \leq n-1$ and that Z_n is the fundamental cycle of an elliptic Gorenstein singularity with $p_g = 1$. Furthermore, the dual graph of $C_0 + \cdots + C_{n-1}$ is of Dynkin type $\mathbf{A}_n$ with $C_{j-1}C_j = 1$ for $1 \leq j \leq n-1$, $C_iC_j = 0$

when $|i-j|>1$. We have $C_{n-1}Z_n=1$ and the intersection point $p=C_{n-1}\cap\mathrm{Supp}(Z_n)$ is a non-singular point of Z_n. Since $\omega_{Z_n}\simeq\mathcal{O}_{Z_n}$, we have $\mathcal{O}_{Z_n}(K_X)\simeq\mathcal{O}_{Z_n}(-Z_n)\simeq\mathcal{O}_{Z_n}(-Z_{n-1}+C_{n-1})$. Therefore, $\mathcal{O}_{Z_n}(p)\simeq\mathcal{O}_{Z_n}(K_X+Z_{n-1})$.

Note that we have $Z_i^2=-1$ and $p_a(Z_i)=1$ for any i, $0\le i\le n$. We know that the restriction map $H^0(X,L)\to H^0(Z,L)$ is surjective and that $h^0(Z_i,L)=1$ for each i with $0\le i\le n$, since L is numerically equivalent to K_X. Consider the cohomology long exact sequence for

$$0\to\mathcal{O}_{C_0+\cdots+C_{n-1}}(L-Z_n)\to\mathcal{O}_Z(L)\to\mathcal{O}_{Z_n}(L)\to 0.$$

We have $H^0(C_0+\cdots+C_{n-1},L-Z_n)=0$, since $L-Z_n$ is anti-nef of degree -1 on $C_0+\cdots+C_{n-1}$. Hence the restriction map $H^0(Z,L)\to H^0(Z_n,L)$ is an isomorphism. Let $s\in H^0(Z,L)$ be a non-zero element. Then $s|_{Z_n}$ vanishes exactly at one point $q\in Z_n$ which is a non-singular point of Z_n, because Z_n is numerically 2-connected. We have $\mathcal{O}_{Z_n}(q)\simeq\mathcal{O}_{Z_n}(L)$. Note that s is constant on $C_0+\cdots+C_{n-1}$. It follows that $s\in H^0(Z,L)$, which vanishes at q, also vanishes on $C_0+\cdots+C_{n-1}$ if and only if $p=q$. By what shown above, we have $p=q$ if and only if $\mathcal{O}_{Z_n}(L)\simeq\mathcal{O}_{Z_n}(K_X+Z_{n-1})$. □

When $L=K_X$ on Z_n, the condition $\mathcal{O}_{Z_n}(L)\simeq\mathcal{O}_{Z_n}(K_X+Z_{n-1})$ becomes $\mathcal{O}_{Z_n}(Z_{n-1})\simeq\mathcal{O}_{Z_n}$ which is equivalent to $p_g(V_{n-1},p_{n-1})=2$ by [9, 2.20], where (V_{n-1},p_{n-1}) denotes the singularity obtained by contracting Z_{n-1}, and then (V_{n-1},p_{n-1}) is Gorenstein by [9, 3.5]. We have $\mathrm{mult}(V_{n-1},p_{n-1})=2$ and $\mathrm{embdim}(V_{n-1},p_{n-1})=3$ by [9, 5.4].

Corollary 4.6. *Let (V,o) be an elliptic numerically Gorenstein singularity with $p_g(V,o)\ge 2$ whose fundamental cycle satisfies $Z^2=-1$ on the minimal resolution X. Then the fixed part of $|K_X|$ supports the exceptional set of the rational double point of type $\mathbf{A}_n$ if and only if the length of the elliptic sequence is $n+1$ and $p_g(V_{n-1},p_{n-1})=2$ holds for the singularity obtained by contracting Z_{n-1}, where $\{Z_i\}_{i=0}^n$ denotes the elliptic sequence.*

Example 4.7. Let n be a positive integer and consider two hypersurface singularities respectively defined by the following equations:

(I) $z^2=x(x^{4n+2}+y^4)$, (II) $z^2=x^{6(2n+1)}+y^3$

Both are elliptic singularities and have the same minimal resolution dual graph as in Fig. 1 with $m=2n$ and Δ_m being a (-1)-elliptic curve. We have $p_g(V,o)=n+1$ for (I) and $p_g(V,o)=2n+1$ for (II). It can be checked directly that $\mathrm{Bs}|K_X|$ is one point when (I) is the case, while it consists of $2n$ (-2)-curves forming the Dynkin diagram of type $\mathbf{A}_{2n}$ when (II).

5. Appendix (Isolated base points). We have ignored isolated base points so far. We can say at least the following about them. Although this is very similar to Theorem 1.1 and Proposition 2.2, we give a proof for the readers' convenience.

Proposition 5.1. *Let (V,o) be a normal surface singularity and $\pi:X\to V$ the minimal resolution. Let L be a line bundle on X such that $L-K_X$ is nef. If $x\in\mathrm{Bs}|L|$, then there exists a subcurve Δ of Z satisfying:*

(1) *The restriction $H^0(Z,L)\to H^0(\Delta,L)$ is surjective.*

(2) *Δ is the fundamental cycle on its support, $\Delta^2=-1$ and x is a non-singular point of Δ.*

(3) *$L - K_X$ is numerically trivial on Δ and $\mathcal{O}_\Delta(L) \simeq \omega_\Delta \otimes \mathcal{O}_\Delta(x)$.*

Proof. We take an irreducible component A through x and let $\Delta = \Delta(A)$ be a minimal curve such that $A \preceq \Delta \preceq Z$ and the restriction map $H^0(Z, L) \to H^0(\Delta, L)$ is surjective. We see that $K_\Delta - L = \Delta - (L - K_X)$ is not nef on Δ, since $L - K_X$ is nef and $\Delta^2 < 0$. It follows from [4, Lemma 2.2.1] that A is of multiplicity one in Δ, $K_\Delta - L$ is nef on $\Delta - A$ and the image of $H^0(Z, L) \to H^0(A, L)$ contains the image of the injection $H^0(A, L - (\Delta - A)) \to H^0(A, L)$. Since $K_\Delta - L$ is not nef on Δ, we have $\deg(L - K_\Delta)|_A = \deg(L - (\Delta - A))|_A - \deg K_A > 0$. From this, we infer that $H^0(\Delta, L) \to H^0(\Delta - A, L)$ is surjective. This allows us to assume that there are no components of $\Delta - A$ containing x. In fact, if there is another component $B \prec \Delta$ with $x \in B$, then we can argue with B instead of A, since what we have shown implies that $\Delta(B) \preceq \Delta - A \prec \Delta = \Delta(A)$.

Then, since $x \in \mathrm{Bs}|L|$, we must have $x \in \mathrm{Bs}|\mathcal{O}_A(L - (\Delta - A))|$. Since $\deg \mathcal{O}_A(L - (\Delta - A)) > \deg K_A$, we see that x is a non-singular point of A and $\mathcal{O}_A(L - (\Delta - A)) \simeq \mathcal{O}_A(K_A + x)$ (see, e.g., [10, p. 113]). The last isomorphism shows that $A\Delta + 1 = \deg(L - K_X)|_A \geq 0$. Then $0 > \Delta^2 = (\Delta - A)\Delta + A\Delta \geq \deg(L - K_X)|_{\Delta - A} - 1 \geq -1$. In sum, we get $A\Delta = \Delta^2 = -1$ and see that $\mathcal{O}_\Delta(L - K_X)$ and $\mathcal{O}_\Delta(\Delta + x)$ are both numerically trivial. Since $H^0(\Delta, L) \to \mathcal{O}_x$ is zero, we have $H^1(\Delta, L - x) \neq 0$ which is equivalent to $H^0(\Delta, K_\Delta - L + x) \neq 0$. Since $\Delta^2 = -1$, Δ is numerically 1-connected by Lemma 2.1. Since $\mathcal{O}_\Delta(K_\Delta - L + x)$ is numerically trivial, we conclude that $\mathcal{O}_\Delta(L) \simeq \mathcal{O}_\Delta(K_\Delta + x)$. We know that Δ is the fundamental cycle on its support, because Δ is numerically 1-connected and $\mathcal{O}_\Delta(-\Delta)$ is nef. □

LEMMA 5.2. *Let the notation and assumptions be as above. Assume furthermore that x is an isolated base point of $|L|$. If $p_a(\Delta) = 1$, then Δ is numerically 2-connected.*

Proof. Recall that Δ is a numerically 1-connected curve that is the fundamental cycle on its support. If Δ is not 2-connected, then we have an effective decomposition $\Delta = \Delta_1 + \Delta_2$, $\Delta_1\Delta_2 = 1$. In particular, Δ_1 and Δ_2 are both numerically 1-connected. Since $\Delta^2 = -1$ and $p_a(\Delta) = 1$, we have $K_X\Delta = 1$ which enables us to assume that $K_X\Delta_1 = 1$, $K_X\Delta_2 = 0$. Since $\Delta_1^2 + \Delta_2^2 = -3$, we have $\Delta_1^2 = -1$ and $\Delta_2^2 = -2$ for the reason of parity. Hence Δ_2 is a connected curve consisting of (-2)-curves, all with multiplicity one. Recall that L and K_X are numerically equivalent on Δ by Proposition 5.1. It follows that we have $\mathcal{O}_{\Delta_2}(L) \simeq \mathcal{O}_{\Delta_2}$. This implies that $x \notin \mathrm{Supp}(\Delta_2)$, because x is an isolated base point of $|L|$. Then $\mathcal{O}_{\Delta_2}(\Delta)$ is numerically trivial, because so is $\mathcal{O}_\Delta(\Delta + x)$. Hence $0 = \Delta_2\Delta = \Delta_2(\Delta_1 + \Delta_2) = 1 - 2 = -1$, a contradiction. Therefore, Δ is numerically 2-connected. □

We obtain an elliptic Gorenstein singularity with $p_g = 1$ by contracting Δ as above. So, Lemma 5.2 gives us a clearer picture of $\mathrm{Bs}|L|$, when (V, o) is an elliptic singularity and $L - K_X$ is nef: $\mathrm{Bs}|L|$ consists of (-2)-curves forming configurations of type **A** and several isolated points lying on the fundamental cycles (with self-intersection numbers -1) of elliptic Gorenstein singularities with $p_g = 1$.

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