ASIAN J. MATH.
Vol. 12, No. 3, pp. 285–288, September 2008

001

IMPOSSIBLE METRIC CONDITIONS ON EXOTIC $\mathbb{R}^4$'S*

LAURENCE R. TAYLOR†

Abstract. There are many theorems in the differential geometry literature of the following sort.

> *Let M be a complete Riemannian manifold with some conditions on various curvatures, diameters, volumes, etc. Then M is homotopy equivalent to a finite CW complex, or M is the interior of a compact, topological manifold with boundary.*

At first glance it seems unlikely that such theorems have anything to say about smooth manifolds homeomorphic to $\mathbb{R}^4$. However, there is a common theme to all the proofs which forbids the existence of such metrics on most (and possibly all) exotic $\mathbb{R}^4$'s.

Key words. Exotic 4-spaces, vector fields.

AMS subject classifications. Primary 57N13; Secondary 57R70

1. Definitions and the main result. We say a smooth manifold E homeomorphic to $\mathbb{R}^4$ satisfies the *DFT* condition (for De Michelis, Freedman and Taubes) provided that, for every compact subset $K \subset E$, there exists an open neighborhood U of K such that

1. $K \subset U$
2. the closure of U, $\overline{U}$, is homeomorphic to D^4
3. U can be engulfed by itself rel K

Precisely, condition (3) means that there exists a smooth, ambient, isotopy of E from the identity to ι such that $\iota\colon U \to E$ satisfies $\overline{U} \subset \iota(U)$ and the isotopy is the identity on K.

As discussed in section 2, all *exotic* $\mathbb{R}^4$'s (smoothings of $\mathbb{R}^4$ not diffeomorphic to the standard smoothing) known to the author do not have the *DFT*-property.

Differential geometry enters the picture via critical points of functions related to the distance function. Such functions are not necessarily smooth so the notion of critical point needs to be interpreted and there is a standard way to do this going back at least to Grove and Shiohama [8]. The idea involves constructing vector fields and using differential geometry to get enough similarity to the gradient-like vector fields of Morse theory to prove results. Here is the main observation of this note.

THEOREM 1.1. *Let E be a smooth, manifold homeomorphic to $\mathbb{R}^4$ with a proper Lipschitz function which has bounded critical values. Then E satisfies the DFT-condition.*

2. Remarks on the *DFT* property. Taubes [11, Thm. 1.4, p. 366] proves that many exotic $\mathbb{R}^4$'s have the property that neighborhoods of large compact sets can not be embedded smoothly in exotic $\mathbb{R}^4$'s with periodic ends. Hence these $\mathbb{R}^4$'s are not *DFT* since ι can be used to construct a periodic end smoothing of $\mathbb{R}^4$ containing a neighborhood of any compact set K.

De Michelis and Freedman [4] state in the first line of the last paragraph on page 220 that the family of $\mathbb{R}^4$'s they construct do not satisfy the *DFT*-property even if the ι is not required to be isotopic to the identity.

*Received September 8, 2007; accepted for publication April 10, 2008.
†Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA (taylor.2@nd.edu). Partially supported by the N.S.F.

Gompf [6] argues that none of the smoothings of $\mathbb{R}^4$ in his menagerie have the DFT property. Certainly the universal $\mathbb{R}^4$ of Freedman and Taylor [5] is not DFT.

On the other hand, if the smooth Schoenflies conjecture is true but the smooth Poincaré conjecture is false, there will be exotic DFT $\mathbb{R}^4$'s.

3. Patterns of application. Rather than produce a long list of theorems, here is a meta-principle for generating theorems.

Meta-Principle. *Take any theorem in differential geometry regarding the existence of complete metrics with special properties. If the proof shows that a distance function or some other proper Lipschitz function has bounded critical values then such metrics do not exist on a non-DFT $\mathbb{R}^4$.*

Remark. The distance function from any point $p \in E$ is proper Lipschitz if the Riemannian metric on E is complete.

Here are three examples.

Example 3.1. Reading the paper of Lott and Shen [9] gave the author the idea for this note. The first paragraph on page 281 shows that an exotic $\mathbb{R}^4$ with lower quadratic curvature decay, quadratic volume growth and which does not collapse at infinity is DFT.

Example 3.2. An early finite-type theorem is by Abresch [1] which says that if the curvature decays faster than quadratic on some complete Riemannian exotic $\mathbb{R}^4$, then it satisfies the DFT-property.

Example 3.3. An example involving mostly Ricci curvature and diameter is [2, Thm. B, p. 356].

4. The proof of Theorem 1.1. Let E be any smoothing of $\mathbb{R}^4$. Say that a flat topological embedding $e\colon S^3 \subset E$ is *not a barrier to isotopy* provided there is a smooth vector field on E with compact support so that if $\iota\colon E \to E$ is the isotopy at time 1 generated by the vector field, then $\overline{U} \subset \iota(U)$, where U is the bounded component of $E - e(S^3)$.

Say that E has *no barrier to isotopy to* ∞ if, for every compact set $K \subset E$, there is a flat topological embedding $e\colon S^3 \subset E$ such that K lies in the bounded component of $E - e_K(S^3)$ and such that e_K is not a barrier to isotopy .

Proposition 4.1. *Let E be a smooth, manifold homeomorphic to $\mathbb{R}^4$. If E has no barrier to isotopy to ∞ then E satisfies the DFT-condition.*

Proof. Given K compact, pick an $e_K\colon S^3 \to E$ with K in the bounded component of $E - e_K(S^3)$ such that e_K is not a barrier to isotopy. Let χ_1 be the smooth vector field promised by the definition. Let U be the bounded component of $E - e_K(S^3)$ and notice U satisfies (1) and (2). Since $K \subset U$, there is another smooth vector field χ such that χ vanishes on K and agrees with χ_1 on a neighborhood of $e(S^3)$. Let $I\colon E \times [0,1] \to E$ be the isotopy generated by the flow for χ. Since χ vanishes on K, the isotopy I fixes K and since χ agrees with χ_1 on $e(S^3)$, $I_1\big(e(S^3), 1\big) \subset E - \overline{U}$. Hence I is the isotopy required for (3). $\square$

Proof of Theorem 1.1. Let $\rho\colon E \to [0, \infty)$ be a proper Lipschitz function. Since the critical values are bounded by hypothesis, there is an r_0 such that for any critical point $x \in E$, $\rho(x) < r_0$. Since ρ is proper, $\rho^{-1}\big([0, r_0]\big)$ is compact. Let $e\colon S^3 \to E$ be any flat embedding with $\rho^{-1}\big([0, r_0]\big)$ in the bounded component of $E - e(S^3)$. It will be shown that e is not a barrier to isotopy, from which it follows that E has no barrier to isotopy to ∞, from which it follows that E satisfies the DFT-property using Proposition 4.1.

Since S^3 is compact, $\rho\big(e(S^3)\big) \subset [r_1, r_2]$ with $r_0 < r_1$. The idea is contained in the proofs of [7, Lemma 3.1, p. 108] or [3, Lemma 1.4, p. 2]. They start by constructing a vector field locally on $\rho^{-1}\big([r_1, r_2]\big)$ and patching it together using a smooth partition of unity. They observe that the conditions they need are open conditions so the field can be taken to be smooth. Then the proofs show that the resulting flow (or the flow for the negative of the constructed field) moves $\rho^{-1}(r_1)$ out past $\rho^{-1}(r_2)$ and so $e(S^3)$ ends up in $\rho^{-1}(r_2, \infty) = E - \rho^{-1}\big([0, r_2]\big) \subset E - \overline{U}$ since $\rho\big(\overline{U}\big) \subset [0, r_2]$. □

5. Concluding remarks. Differential geometry can show that two smooth 4-manifolds are diffeomorphic, avoiding the "greater than or equal to 5" hypothesis of differential topology. As examples, the Cartan-Hadamard Theorem [10, Thm. 4.1, p. 221] and the Cheeger-Gromoll Soul Theorem [10, Thm. 3.4, p. 215] both prove that a 4-manifold with very restrictive curvature conditions is *diffeomorphic* to the standard $\mathbb{R}^4$.

The results presented here start with much weaker hypotheses than the Cartan-Hadamard or the Cheeger-Gromoll Soul Theorems, but the conclusions are also weaker. One question would be whether some of these theorems could be strengthened to show that the manifold was diffeomorphic to the standard $\mathbb{R}^4$. A second question would be to use the DFT property to produce interesting metrics, perhaps metrics strong enough to prove that a DFT $\mathbb{R}^4$ is standard.

REFERENCES

[1] U. Abresch, *Lower curvature bounds, Toponogov's theorem, and bounded topology. II*, Ann. Sci. École Norm. Sup. (4), 20 (1987), pp. 475–502.

[2] U. Abresch and D. Gromoll, *On complete manifolds with nonnegative Ricci curvature*, J. Amer. Math. Soc., 3 (1990), pp. 355–374.

[3] J. Cheeger, *Critical points of distance functions and applications to geometry*, Geometric topology: recent developments (Montecatini Terme, 1990), Springer, Berlin, 1991, pp. 1–38.

[4] S. De Michelis and M. H. Freedman, *Uncountably many exotic* $\mathbf{R}^4$*'s in standard* 4*-space*, J. Differential Geom., 35 (1992), pp. 219–254.

[5] M. H. Freedman and L. R. Taylor, *A universal smoothing of four-space*, J. Differential Geom., 24 (1986), pp. 69–78.

[6] R. E. Gompf, *An exotic menagerie*, J. Differential Geom., 37 (1993), pp. 199–223.

[7] R. E. Greene, *A genealogy of noncompact manifolds of nonnegative curvature: history and logic*, Comparison geometry (Berkeley, CA, 1993–94), Cambridge Univ. Press, Cambridge, 1997, pp. 99–134.

[8] K. Grove and K. Shiohama, *A generalized sphere theorem*, Ann. Math. (2), 106 (1977), pp. 201–211.

[9] J. Lott and Z. Shen, *Manifolds with quadratic curvature decay and slow volume growth*, Ann. Sci. École Norm. Sup. (4), 33 (2000), pp. 275–290 (English, with English and French summaries).

[10] T. Sakai, *Riemannian geometry*, Translations of Mathematical Monographs, vol. 149, American Mathematical Society, Providence, RI, 1996. Translated from the 1992 Japanese original by the author.

[11] C. H. Taubes, *Gauge theory on asymptotically periodic* 4*-manifolds*, J. Differential Geom., 25 (1987), pp. 363–430.