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RIGIDITY OF CYLINDERS WITHOUT CONJUGATE POINTS*

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Abstract. During the last decades, several investigations were concerned with rigidity statements for manifolds without conjugate points (some results can be found in the references). Based on an idea by E.Hopf [H], K.Burns and G.Knieper proved in [BK] that cylinders without conjugate points and with a lower sectional curvature bound must be flat if the length of the shortest loop at every point is globally bounded.

The present article reduces the last condition to a limit for the asymptotic growth of loop-length as the basepoint approaches the ends of the cylinder (Thm. 18). Along the way, the shape of cylinders without conjugate points is characterized: The loop-length must be strictly monotone increasing to both ends outside a – possibly empty – tube consisting of closed geodesics (Thm. 10).

Key words. Global Riemannian geometry, rigidity results, curvature bounds

AMS subject classifications. 53C21, 53C24

1. Preliminaries.

1.1. Conjugate points, Riccati equation. Let M be a smooth, complete surface with a Riemannian metric $\langle\,,\rangle$ and sectional curvature K; furthermore TM the tangent bundle and $\pi : TM \to M$ the footpoint-projection, $\widetilde{M}$ the universal Riemannian covering of M and $\widetilde{\pi} : \widetilde{M} \to M$ the projection.

Given $X \subseteq M$ note by $SX := \{v \in TM \mid \pi(v) \in X; \|v\| = 1\}$ the unit vectors with footpoint in X; let λ for every p denote the Lebesgue-measure on S_pM, $\mu = vol_M \times \lambda$ the Liouville-measure on SM and $g^t : SM \to SM, v \mapsto \frac{d}{ds}\big|_{s=t} \exp_{\pi(v)}(sv)$ the geodesic flow at time t.

For $v \in SM$ regard the geodesic $\gamma_v(t) := \exp_{\pi(v)}(tv)$, parameterized by arclength, with sectional curvature $K(t) := K(\gamma_v(t))$; the *Jacobi equation* related to γ_v is then

$$(J_v) \qquad y''(t) + K(t)y(t) = 0.$$

DEFINITION 1. M is called *without conjugate points,* if for any $v \in SM$, every non-trivial solution of (J_v) vanishes once at most.

If M is a surface without conjugate points, then for all $v \in SM, s \in \mathbb{R}$ there exists a solution $y(v, s, t)$ of (J_v) with boundary values $y(v, s, 0) = 1$ and $y(v, s, s) = 0$. The next theorem characterizes this property (for the 3rd part, see [H]).

THEOREM 2. *The following criteria are equivalent:*
1. *M has no conjugate points;*
2. *any two geodesics in $\widetilde{M}$ can intersect once at most, in particular all geodesics in $\widetilde{M}$ are minimal;*
3. *the stable resp. unstable solutions of (J_v), defined by $y_-(v, t) := \lim_{s\to\infty} y(v, s, t)$ and $y_+(v, t) := \lim_{s\to-\infty} y(v, s, t)$ respectively, exist $\forall\ v \in SM$ on the entire $\mathbb{R}$.*

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The *Riccati equation* related to γ_v is

$$(R_v) \qquad u'(t) + u^2(t) + K(t) = 0;$$

it is obtained from (J_v) by transformation $u = y'/y$. In general, the zero locus of y must be excepted. In absence of conjugate points $y_\pm > 0$, and $u_-(v,t) := \frac{y'_-(v,t)}{y_-(v,t)}$ and $u_+(v,t) := \frac{y'_+(v,t)}{y_+(v,t)}$ solve (R_v) on $\mathbb{R}$ for every $v \in SM$. Set $U(v) := u_+(v,0)$.

1.2. Comparison theorems. The existence of a lower curvature boundary allows us to compare M with surfaces of constant curvature:

Proposition 3. *Suppose M is free of conjugate points and $K \geq -b^2$ for some $b > 0$. For $A, B, C \in M$ let $\triangle \subset M$ denote the triangle with vertices A, B, C, whereas the edges are minimal geodesic segments. If M' is the plane with constant curvature $K' = -b^2$ and $\triangle' \subset M'$ is the geodesic triangle spanned by A', B', C', and if $d'(A', C') = d(A, C), d'(A', B') = d(A, B)$ and $\angle(C'A'B') = \angle(CAB)$, then $d'(B', C') \geq d(B, C)$.*

Proof. This is an application of a triangle-comparison-theorem.

Lemma 4. *For $v \in SM$, $b > 0$ and $r < s$ let $K(t) \geq -b^2\ \forall t \in [r, s]$. If y is a solution of (J_v) with $0 < y(t)\ \forall t \in [r, s]$, then $y(t) \leq y(r)\cosh(b(t-r)) + y'(r)\sinh(b(t-r))/b$ and $\frac{y'(t)}{y(t)} < b\coth(b(t-r))$ hold for all $t \in [r, s]$.*

Proof. Set $z(t) := y(r)\cosh(b(t-r)) + y'(r)\sinh(b(t-r))/b$ and $w(t) := y'(t)z(t) - y(t)z'(t)$; remark, that z cannot vanish twice and $w(r) = 0$. Also set $\hat{s} := \sup\{t \in [r, s] \mid z(t) > 0\}$; then for all $t \in [r, \hat{s}]$

$$\begin{aligned} & w'(t) = y''(t)z(t) - y(t)z''(t) = (-K(t) - b^2)y(t)z(t) \leq 0 \\ \Rightarrow \quad & w(t) = \int_r^t w'(u)\,du \leq 0 \quad \Rightarrow \quad \frac{w(t)}{y(t)z(t)} = \frac{y'(t)}{y(t)} - \frac{z'(t)}{z(t)} \leq 0 \\ \Rightarrow \quad & y(t) = y(r)\exp\left(\int_r^t \frac{y'(u)\,du}{y(u)}\right) \leq y(r)\exp\left(\int_r^t \frac{z'(u)\,du}{z(u)}\right) = z(t). \end{aligned}$$

Therefore $\hat{s} = s$, as otherwise $0 < y(\hat{s}) \leq z(\hat{s}) = 0$. The second inequality now results from

$$\frac{y'(t)}{y(t)} \leq \frac{z'(t)}{z(t)} = \frac{by(r)\sinh(b(t-r)) + y'(r)\cosh(b(t-r))}{y(r)\cosh(b(t-r)) + y'(r)\sinh(b(t-r))/b} < b\coth(b(t-r)).$$

Using a similar method, Hopf [H] showed:

Corollary 5. *Let M be free of conjugate points, then U is $\mu-$measurable. If in addition there is a $b > 0$ with $K \geq -b^2$, then $|U| \leq b$.*

The flatness-condition is mainly based on [BK], Lem. 1.3:

Lemma 6. *Let M without conjugate points and Q a compact subset of M with ∂Q piecewise smooth. Then*

$$\int_{SQ} U^2(v)\ d\mu(v) \leq -2\pi \int_Q K(p)\ d\,vol_M(p) + 2\int_{\partial Q}\int_{S_pM} |U(v)|\ d\lambda(v)\ dL(p).$$

2. Cylinders.

2.1. Geodesic loops & closed geodesics. Consider a smooth cylinder C (i.e. a complete surface diffeomorphic to $\mathbb{R} \times S^1$) equipped with a Riemannian metric $\langle \cdot\,, \cdot \rangle$ without conjugate points and curvature K. Denote by $\widetilde{C} \simeq \mathbb{R}^2$ the universal Riemannian covering for C. The fundamental group is $\pi_1(C) \simeq \mathbb{Z}$; let $\varphi : \widetilde{C} \to \widetilde{C}$ be a generator of the deck transformation group of C.

Definition 7.
1. For $l > 0$, an arclength-parameterized geodesic segment $c : [0, l] \to C$ with $c(0) = c(l)$ is called *geodesic loop with basepoint* $c(0)$.
2. If further $c'(0) = c'(l)$ (and so $c(t + l) = c(t)\ \forall t$), c is a *closed geodesic*.

Remark that closed geodesics cannot have transversal self-intersections: If $c(u) = c(v)$ for some $u < v \in [0, l]$ and $\widetilde{c} : \mathbb{R} \to \widetilde{C}$ denotes a lift of c, there would be $m, z \in \mathbb{Z} \setminus \{0\}$ such that $\varphi^z \widetilde{c}(t) = \widetilde{c}(t + l)\ \forall t$ and $\widetilde{c}(v) = \varphi^m \widetilde{c}(u)$

$$\begin{aligned}&\Rightarrow \widetilde{c}(v + nl) = \varphi^{nz}\widetilde{c}(v) = \varphi^{nz+m}\widetilde{c}(u) = \varphi^m \widetilde{c}(u + nl)\ \forall n \in \mathbb{Z}\\ &\Rightarrow \widetilde{c}(t + v - u) = \varphi^m \widetilde{c}(t)\ \forall t \in \mathbb{R}\end{aligned}$$

by Thm. 2; hence $c(t + v - u) = c(t)\ \forall t$. Therefore we may always assume closed geodesics to be simple, for l should be the (least) period of c.

Proposition 8. *A geodesic loop is a closed geodesic, iff it has minimal length in the set of non-contractible loops in C.*

Proof. Take $c : [0; l] \to C$ to be a simple geodesic loop of length l. If c is minimal, $c'(l) = c'(0)$; as it could be shortened by variation if it would contain a vertex at $c(0)$.

On the other hand, if c is a closed geodesic, let $\sigma : \mathbb{R} \to \widetilde{C}$ be a lift; w.l.o.g. suppose $\sigma(l) = \varphi\sigma(0)$. Also take an arbitrary non-contractible loop $a : [0; \lambda] \to C$ of length λ with a lift $\alpha : [0; \lambda] \to \widetilde{C}$. Then $\alpha(\lambda) = \varphi^z \alpha(0)$ for some $z \in \mathbb{Z} \setminus \{0\}$.

As φ operates isometrically on $\widetilde{C}$, the triangle-inequality implies

$$\begin{aligned}nl|z| &= d(\sigma(0), \sigma(nzl))\\ &\le d(\sigma(0), \alpha(0)) + d(\varphi^{nz}\alpha(0), \varphi^{nz}\sigma(0)) + \sum_{j=0}^{n-1} d(\varphi^{jz}\alpha(0), \varphi^{(j+1)z}\alpha(0))\\ &= 2d(\sigma(0), \alpha(0)) + \sum_{j=0}^{n-1} d(\varphi^{j}\alpha(0), \varphi^{j}\alpha(\lambda))\\ &= 2d(\sigma(0), \alpha(0)) + nd(\alpha(0), \alpha(\lambda)) \le 2d(\sigma(0), \alpha(0)) + n\lambda \quad \forall\, n \in \mathbb{N},\end{aligned}$$

which proves $\lambda \ge |z|l \ge l$ as $n \to \infty$.

Let $\gamma : \mathbb{R} \to C$ be an arclength-parameterized geodesic s.th. $C \setminus \gamma$ is simply-connected, with γ_1 a lift to $\widetilde{C}$ and $\gamma_2 = \varphi\gamma_1$. γ_1 and γ_2 cannot intersect, because then γ would contain self-intersections and $C \setminus \gamma$ could not be connected.

Set $l(s) := d(\gamma_1(s), \gamma_2(s))$ and denote by σ_s the arclength-parameterized geodesic through $\sigma_s(0) = \gamma_1(s)$ and $\sigma_s(l(s)) = \gamma_2(s)$ and by $c_s := \widetilde{\pi} \circ \sigma_s$ the projection of σ_s onto C; then $c_s\big|[0; l(s)]$ is a geodesic loop with basepoint $\gamma(s)$.

Let $\alpha_s := \angle(\sigma_s'(0), \gamma_1'(s))$ and $\beta_s := \angle(\gamma_2'(s), -\sigma_s'(l(s)))$ denote the angles between γ_1 resp. γ_2 and σ_s; obviously $0 < \alpha_s, \beta_s < \pi \quad \forall\, s \in \mathbb{R}$.

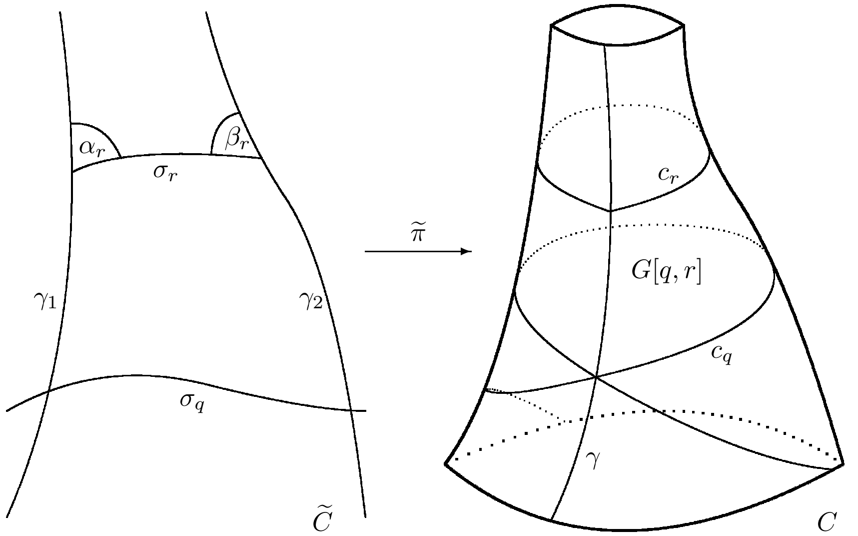

FIG. 1. *Geodesic loops and their liftings*

For each interval $I = [q,r]$ or $I =]q,r[\subseteq \mathbb{R}$ with $-\infty \le q \le r \le \infty$ define $GI := \{c_s(t) \mid s \in I, t \in [0; l(s)]\} \subseteq C$.

For fixed s consider the geodesic variation

$$H :]s-\varepsilon, s+\varepsilon[\times [0; l(s)] \to \widetilde{C},\ H(r,t) = \sigma_r(tl(r)/l(s)).$$

The related Jacobi-vectorfield is $Y_s(t) := \frac{\partial}{\partial r}\Big|_{r=s} H(r,t)$, and its normal component $y_s(t) := \|Y_s(t) - \langle Y_s(t), \sigma_s'(t)\rangle \sigma_s'(t)\|$. y_s is strictly positive since it could vanish at most for a single $t \in [0, l(s)]$ – while $y_s(0) = \sin\alpha_s(t) > 0$ and $y_s(l(s)) = \sin\beta_s(t) > 0$.

The *1st variation formula* claims

$$l'(s) = \langle Y_s(t), \sigma_s'(t)\rangle\Big|_0^{l(s)} = -\cos\beta_s - \cos\alpha_s = -2\cos\frac{\alpha_s - \beta_s}{2}\cos\frac{\alpha_s+\beta_s}{2}.$$

REMARK 9. Because of $-\pi < \alpha_s - \beta_s < \pi$, the following holds:
$l'(s) = 0 \Leftrightarrow \alpha_s + \beta_s = \pi \Leftrightarrow c_s'(l(s)) = c_s'(0) \Leftrightarrow c_s$ is a closed geodesic.

THEOREM 10. *There exist* $-\infty \le q \le r \le \infty$*, such that all geodesic loops in* $G[q,r]$ *are closed geodesics of constant length* $l \equiv l(q)$*, and* $l'(s) > 0$ *for* $s \in]r, \infty[$ *and* $l'(s) < 0$ *for* $s \in]-\infty, q[$.

Proof. Since $C \setminus \gamma$ is contractible, every closed geodesic must be intersected by γ in some point and is thus a loop to this basepoint.

If there don't exist any closed geodesics, l' has the same sign everywhere according to Rem. 9; in this case set $q = r = \pm\infty$ depending on whether $l' < 0$ or $l' > 0$.

In the other case, take c_a and c_b to be closed geodesics for some $a \le b$. Due to Prop. 8, $l(a) \le l(b) \le l(a) \Rightarrow l(a) = l(b)$.

Furthermore $l \ge l(a)$ on the entire $\mathbb{R}$; let $m \in [a,b]$ be a maximum locus for l on $[a,b]$. Then $l'(m) = 0$, i.e. c_m is a closed geodesic. Thus the same argument states

$l(m) = l(a)$, so l must be constant on $[a, b]$. As $l' \equiv 0$ on $[a, b]$, all geodesic loops $G\{s\}$ with $s \in [a, b]$ are closed geodesics.

The claim now follows by setting $q := \inf\{s \in \mathbb{R} \mid l'(s) = 0\}$ and $r := \sup\{s \in \mathbb{R} \mid l'(s) = 0\}$.

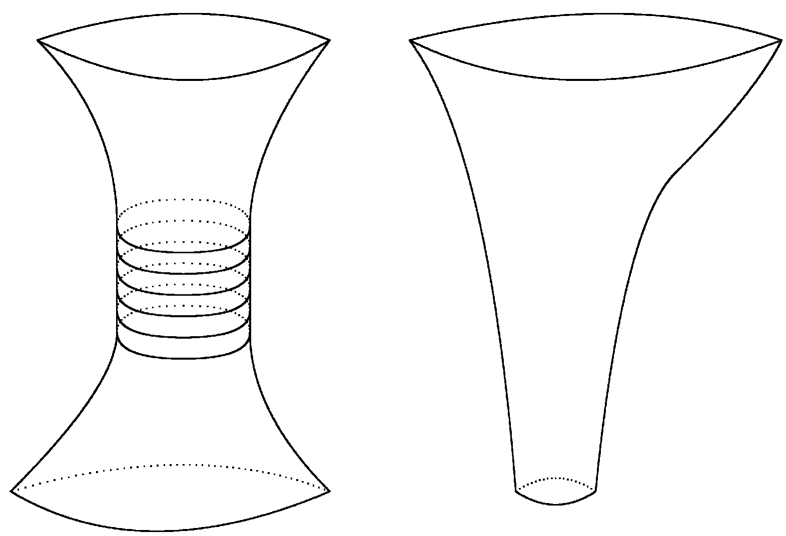

FIG. 2. *Two types of cylinders: with and without closed geodesics*

REMARK 11. Thm. 10 provides a classification of cylinders without conjugate points in types with resp. without closed geodesics.

To simplify the notation, mainly in section 2.3, assume that the choice of the parameterization for γ complies with either of these conditions:

1. If C possesses closed geodesics, c_0 shall be one of them. This effects $l' \leq 0$ on $\mathbb{R}_-$ and $l' \geq 0$ on $\mathbb{R}_+$.
2. If C doesn't contain closed geodesics, suppose that $l' > 0$ everywhere.

2.2. An integral inequality for U^2.

LEMMA 12.

1. *The geodesic γ can be chosen minimal in C.*
2. *If $\liminf_{s \to \pm\infty} l(s)/|s| < 2$ then $G\mathbb{R} = C$.*

Proof. The 1st claim is proved in [BK], p. 630. For the 2nd part, suppose that there exists some $p \in C \setminus G\mathbb{R}$ with lift $\widetilde{p} \in \widetilde{C}$, which is w.l.o.g. situated in the half-strip between $\gamma_1|\mathbb{R}_+$, $\gamma_2|\mathbb{R}_+$ and σ_0. Let ψ_1 and $\psi_2 \subset \widetilde{C}$ be the geodesic segments from $\widetilde{p}$ to $\gamma_1(0)$ and $\gamma_2(0)$ respectively. σ_s varies continuously in s, thus (as it does near $s = 0$) for every $s \geq 0$ it intersects ψ_1 in some point $p_1(s)$ and ψ_2 in another point $p_2(s)$.

The triangle-inequality states

$$\begin{aligned}
2s &= d(\gamma_1(0),\gamma_1(s)) + d(\gamma_2(0),\gamma_2(s)) \\
&\le d(\gamma_1(0),p_1(s)) + d(p_1(s),\gamma_1(s)) + d(\gamma_2(0),p_2(s)) + d(p_2(s),\gamma_2(s)) \\
&\le L(\psi_1) + L(\psi_2) + l(s) \\
\Rightarrow\ 2 &\le \liminf_{s\to\infty} l(s)/s
\end{aligned}$$

– the 2nd claim is just the negation.

REMARK 13. For every $s \in \mathbb{R}, t \in [0, l(s)]$, $y_s(t)$ is the density of the Riemannian volume with respect to the product measure of the length on c_s and that on γ.

To prove this, set $\partial_s(s,t) := \frac{\partial}{\partial s}\sigma_s(t)$ and $\partial_t(s,t) := \frac{\partial}{\partial t}\sigma_s(t) = \sigma_s'(t)$. Using $\|\sigma_s'(t)\| \equiv 1$ and $\frac{\partial}{\partial s}\sigma_s(t) = Y_s(t) - \sigma_s'(t)l'(s)t/l(s)$ compute

$$\begin{aligned}
\frac{d^2 vol_C(s,t)}{ds\,dt} &= \sqrt{\det\begin{pmatrix} \langle\partial_s,\partial_s\rangle & \langle\partial_s,\partial_t\rangle \\ \langle\partial_t,\partial_s\rangle & \langle\partial_t,\partial_t\rangle \end{pmatrix}(s,t)} \\
&= \|\partial_s(s,t) - \langle\partial_s(s,t),\sigma_s'(t)\rangle\sigma_s'(t)\| \\
&= \|Y_s(t) - \langle Y_s(t),\sigma_s'(t)\rangle\sigma_s'(t)\| = y_s(t).
\end{aligned}$$

In the sequel, abbreviate

$$V(s) := \int_0^{l(s)} \int_{Sc_s(t)} U^2(v)\,d\lambda(v)\,y_s(t)\,dt \ge 0.$$

LEMMA 14. *For fixed $q < r \in \mathbb{R}$,*

$$\left(\int_q^r V(s)\,ds\right)^2 \le 32\pi\left(V(q)\int_0^{l(q)}\frac{dt}{y_q(t)} + V(r)\int_0^{l(r)}\frac{dt}{y_r(t)}\right) + 8\pi^2(\alpha_r+\beta_r-\alpha_q-\beta_q)^2.$$

Proof. Lem. 6 gives

$$\begin{aligned}
\int_{SG[q,r]} U^2(v)\,d\mu(v) \le -2\pi\int_{G[q,r]} K(p)\,d\,vol_C(p) &+ 2\int_0^{l(q)}\int_{Sc_q(t)} |U(v)|\,d\lambda(v)\,dt \\
&+ 2\int_0^{l(r)}\int_{Sc_r(t)} |U(v)|\,d\lambda(v)\,dt, \qquad (1)
\end{aligned}$$

wherein the curvature-integral is

$$\int_{G[q,r]} K(p)\,d\,vol_C(p) = \alpha_q + \beta_q - \alpha_r - \beta_r \qquad (2)$$

due to Gauss-Bonnet. Applying the Cauchy-Schwarz-inequality twice, the other integrals can be estimated by

$$\begin{aligned}
\left(\int_0^{l(s)}\int_{Sc_s(t)} |U(v)|\,d\lambda(v)\,dt\right)^2 &\le \int_0^{l(s)}\left(\int_{Sc_s(t)} |U(v)|\,d\lambda(v)\right)^2 y_s(t)\,dt \int_0^{l(s)}\frac{dt}{y_s(t)} \\
&\le \int_0^{l(s)} 2\pi\int_{Sc_s(t)} U^2(v)\,d\lambda(v) y_s(t)\,dt \int_0^{l(s)}\frac{dt}{y_s(t)} \\
&= 2\pi V(s)\int_0^{l(s)}\frac{dt}{y_s(t)} \qquad (3)
\end{aligned}$$

On the other hand, Rem. 13 allows to write

$$\int_{SG[q,r]} U^2(v)\, d\mu(v) = \int_q^r \int_0^{l(s)} \int_{Sc_s(t)} U^2(v)\, d\lambda(v)\, y_s(t)\, dt\, ds = \int_q^r V(s)\, ds. \tag{4}$$

The ineqs. (1) to (4) gather to

$$\int_q^r V(s)\, ds \le \sqrt{8\pi V(q) \int_0^{l(q)} \frac{dt}{y_q(t)}} + \sqrt{8\pi V(r) \int_0^{l(r)} \frac{dt}{y_r(t)}} + 2\pi(\alpha_r + \beta_r - \alpha_q - \beta_q). \tag{5}$$

Since $0 \le (\sqrt{a} - \sqrt{c})^2 \Rightarrow a + c + 2\sqrt{ac} \le 2a + 2c \Rightarrow \sqrt{a} + \sqrt{c} \le \sqrt{2(a+c)}$ for arbitrary $a, c \ge 0$ the right-hand side of (5) can be estimated again by

$$\begin{aligned} &\sqrt{8\pi V(q) \int_0^{l(q)} \frac{dt}{y_q(t)}} + \sqrt{8\pi V(r) \int_0^{l(r)} \frac{dt}{y_r(t)}} + 2\pi(\alpha_r + \beta_r - \alpha_q - \beta_q) \\ \le &\sqrt{16\pi V(q) \int_0^{l(q)} \frac{dt}{y_q(t)} + 16\pi V(r) \int_0^{l(r)} \frac{dt}{y_r(t)}} + 2\pi(\alpha_r + \beta_r - \alpha_q - \beta_q) \\ \le &\sqrt{32\pi V(q) \int_0^{l(q)} \frac{dt}{y_q(t)} + 32\pi V(r) \int_0^{l(r)} \frac{dt}{y_r(t)} + 8\pi^2(\alpha_r + \beta_r - \alpha_q - \beta_q)^2}, \end{aligned}$$

which leads to the claimed inequality.

2.3. Flatness condition in case of bounded curvature. During this section, suppose that $K > -b^2$ for some $b > 0$ and that γ is minimal (cf. Lem. 12).

LEMMA 15. $|\cos\alpha_s|, |\cos\beta_s| \le \tanh(bl(s)/2) \quad \forall\ s \in \mathbb{R}$.

Proof. For every r, the geodesic segment from $\gamma_2(s) = \sigma_s(l(s))$ to $\gamma_1(s+r)$ is longer than $d_C(\gamma(s), \gamma(s+r)) = r$, because it is a lift of a geodesic segment in C between $\gamma(s)$ and $\gamma(s+r)$, and γ is minimal.

In a plane of constant curvature $-b^2$, consider a geodesic triangle, where two edges, one of length $l(s)$ and one of length r, span an angle of α_s. The length of the edge on the opposite side shall be a. Comparing this triangle with the geodesic triangle in $\widetilde{C}$ with vertices $\gamma_1(s), \gamma_1(s+r)$ and $\gamma_2(s)$, Prop. 3 implies $a \ge d(\gamma_2(s), \gamma_1(s+r)) > r$.

Hence the hyperbolic cosine-theorem holds for any $r > 0$

$$\begin{aligned} &\cos\alpha_s = \frac{\cosh(bl(s))\cosh(br) - \cosh(ba)}{\sinh(bl(s))\sinh(br)} < \frac{(\cosh(bl(s)) - 1)\cosh(br)}{\sinh(bl(s))\sinh(br)} \\ \Rightarrow \quad &\cos\alpha_s \le \frac{\cosh(bl(s)) - 1}{\sinh(bl(s))} = \tanh(bl(s)/2), \end{aligned}$$

as $r \to \infty$. The same estimation, applied to β_s and the opponent angles $\pi - \alpha_s, \pi - \beta_s$ proves the claim.

COROLLARY 16. $\int_0^{l(s)} dt/y_s(t) < \frac{\pi}{b}\cosh^2(bl(s)/2) \quad \forall\ s \in \mathbb{R}$.

Proof. First, claim

$$y_s(t) \ge x_s(t) := \frac{\cosh(b(t - l(s)/2))}{\cosh^2(bl(s)/2)} \qquad \forall\, s \in \mathbb{R}, t \in [0, l(s)].$$

In accordance with Lem. 15,

$$y_s(0) = \sin \alpha_s = \sqrt{1 - \cos^2 \alpha_s} \geq \sqrt{1 - \tanh^2 \frac{bl(s)}{2}} = \frac{1}{\cosh(bl(s)/2)} = x_s(0)$$

$\forall\ s \in \mathbb{R}$ and as well $y_s(l(s)) \geq 1/\cosh(bl(s)/2) = x_s(l(s))$ (cf. [BK] Lem. 2.4). Fix $0 < \delta < 1$ and assume that there are s, t s.th. $y_s(t) < \delta x_s(t)$. Then define $\tau := \inf\{t \in [0, l(s)] \mid y_s(t) < \delta x_s(t)\}$; obviously $\tau > 0$, $y_s(\tau) = \delta x_s(\tau)$ and $y_s'(\tau) \leq \delta x_s'(\tau)$. As $y_s > 0$ on $[0, l(s)]$, by Lem. 4 get for $\tau \leq t \leq l(s)$

$$\begin{aligned} y_s(t) &\leq y_s(\tau) \cosh(b(t-\tau)) + y_s'(\tau) \frac{\sinh(b(t-\tau))}{b} \\ &\leq \delta x_s(\tau) \cosh(b(t-\tau)) + \delta x_s'(\tau) \frac{\sinh(b(t-\tau))}{b} = \delta x_s(t) \end{aligned}$$

– where the last equality refers to the fact, that both sides solve the Jacobi equation with $K \equiv -b^2$ and coincide in τ in their values and 1st derivatives.

But that leads to the contradiction $y_s(l(s)) \leq \delta x_s(l(s)) < x_s(l(s)) \leq y_s(l(s))$, which shows $y_s \geq \delta x_s$. Since δ can be chosen arbitrarily, $y_s \geq \sup_{\delta < 1} \delta x_s = x_s$.

Thus

$$\begin{aligned} \int_0^{l(s)} \frac{dt}{y_s(t)} &\leq \int_0^{l(s)} \frac{\cosh^2(bl(s)/2)\, dt}{\cosh(b(t - l(s)/2))} = \int_{-l(s)/2}^{l(s)/2} \frac{\cosh^2(bl(s)/2)\, dt}{\cosh(bt)} \\ &= \cosh^2 \frac{bl(s)}{2} \int_{-l(s)/2}^{l(s)/2} \frac{2e^{bt}\, dt}{e^{2bt} + 1} = \cosh^2 \frac{bl(s)}{2} \int_{e^{-bl(s)/2}}^{e^{bl(s)/2}} \frac{2\, dx}{b(x^2+1)} \\ &= \cosh^2(bl(s)/2) \frac{2 \arctan(e^{bt})}{b} \Bigg|_{-l(s)/2}^{l(s)/2} < \frac{\pi}{b} \cosh^2(bl(s)/2). \end{aligned}$$

Lemma 17.

$$\int_{-\infty}^{\infty} \frac{|\alpha_s + \beta_s - \pi|\, ds}{\cosh^2(bl(s)/2)} < \frac{2\pi^2}{b}.$$

Proof. Since $|\alpha_s - \beta_s| < \pi$, the 1st variation formula acquires the form

$$\frac{l'(s)}{2\cos((\alpha_s - \beta_s)/2)} = -\cos \frac{\alpha_s + \beta_s}{2} = \sin \frac{\alpha_s + \beta_s - \pi}{2} \quad \forall\ s \in \mathbb{R}.$$

Here, $\cos \frac{\alpha_s - \beta_s}{2}$ becomes minimal, when $|\alpha_s - \beta_s|$ is maximal; meanwhile due to Lem. 15 $\arccos \tanh(bl(s)/2) \leq \alpha_s, \beta_s \leq \pi - \arccos \tanh(bl(s)/2)$ and so

$$\begin{aligned} \cos \frac{\alpha_s - \beta_s}{2} &\geq \cos \frac{\pi - 2 \arccos \tanh(bl(s)/2)}{2} = \sin \arccos \tanh(bl(s)/2) \\ &= \sqrt{1 - \tanh^2(bl(s)/2)} = 1/\cosh(bl(s)/2). \end{aligned}$$

Now if $l' \geq 0$ on $[q, r]$ then

$$\begin{aligned} 0 \leq \alpha_s + \beta_s - \pi \leq \pi \sin \frac{\alpha_s + \beta_s - \pi}{2} &= \frac{\pi l'(s)}{2 \cos((\alpha_s - \beta_s)/2)} \\ &\leq \frac{\pi l'(s) \cosh(bl(s)/2)}{2} \qquad \forall\, q \leq s \leq r \\ \Rightarrow \quad \int_q^r \frac{|\alpha_s + \beta_s - \pi|\, ds}{\cosh^2(bl(s)/2)} \leq \int_q^r \frac{\pi l'(s)\, ds}{2\cosh(bl(s)/2)} &= \int_{l(q)}^{l(r)} \frac{\pi\, dl}{2\cosh(bl/2)} \\ &= \frac{2\pi \arctan \mathrm{e}^{bl(s)/2}}{b}\bigg|_q^r \end{aligned}$$

just as computed in the proof of Cor. 16. In case that $l' \leq 0$ on $[q, r]$, deduce analogously

$$\begin{aligned} 0 \geq \alpha_s + \beta_s - \pi \geq \pi \sin \frac{\alpha_s + \beta_s - \pi}{2} &\geq \frac{\pi l'(s) \cosh(bl(s)/2)}{2} \\ \Rightarrow \quad \int_q^r \frac{|\alpha_s + \beta_s - \pi|\, ds}{\cosh^2(bl(s)/2)} \leq \int_q^r \frac{-\pi l'(s)\, ds}{2\cosh(bl(s)/2)} &= \frac{-2\pi \arctan \mathrm{e}^{bl(s)/2}}{b}\bigg|_q^r . \end{aligned}$$

In light of Rem. 11,

$$\int_{-\infty}^{\infty} \frac{|\alpha_s + \beta_s - \pi|\, ds}{\cosh^2(bl(s)/2)} \leq \frac{2\pi \arctan \mathrm{e}^{bl(s)/2}}{b}\bigg|_0^{\infty} - \frac{2\pi \arctan \mathrm{e}^{bl(s)/2}}{b}\bigg|_{-\infty}^{0} < \frac{2\pi^2}{b}$$

if C contains closed geodesics; while for cylinders without closed geodesics even

$$\int_{-\infty}^{\infty} \frac{|\alpha_s + \beta_s - \pi|\, ds}{\cosh^2(bl(s)/2)} \leq \frac{2\pi \arctan \mathrm{e}^{bl(s)/2}}{b}\bigg|_{-\infty}^{\infty} < \frac{\pi^2}{b}$$

holds.

THEOREM 18. *Let C be a cylinder free of conjugate points and $K \geq -b^2$. If $\limsup_{s\to\pm\infty} \frac{l(s)}{\ln|s|} < 1/b$, then C is flat.*

Proof. For $r \geq 0$ define $L(r) := \max(l(r), l(-r))$ and $W(r) := \int_{-r}^{r} V(s)\, ds$. Using Cor. 16, Lem. 14 states

$$\begin{aligned} W^2(r) &\leq 32\pi \left(V(-r) \int_0^{l(-r)} \frac{dt}{y_{-r}(t)} + V(r) \int_0^{l(r)} \frac{dt}{y_r(t)} \right) + 8\pi^2(\alpha_r + \beta_r - \alpha_{-r} - \beta_{-r})^2 \\ &\leq \frac{32\pi^2 \cosh^2(bL(r)/2)}{b} \big(V(-r) + V(r)\big) + 8\pi^2(\alpha_r + \beta_r - \alpha_{-r} - \beta_{-r})^2. \end{aligned}$$

The triangle-inequality yields

$$\begin{aligned} (\alpha_r + \beta_r - \alpha_{-r} - \beta_{-r})^2 &\leq 2\pi|\alpha_r + \beta_r - \alpha_{-r} - \beta_{-r}| \\ &\leq 2\pi\big(|\alpha_r + \beta_r - \pi| + |\alpha_{-r} + \beta_{-r} - \pi|\big) \\ &\leq 2\pi \left(\frac{|\alpha_r + \beta_r - \pi|}{\cosh^2(bl(r)/2)} + \frac{|\alpha_{-r} + \beta_{-r} - \pi|}{\cosh^2(bl(-r)/2)} \right) \cosh^2 \frac{bL(r)}{2} \end{aligned}$$

– which together with $V(-r)+V(r)=W'(r)$ implies

$$W^2(r) \le \frac{32\pi^2}{b}\left(W'(r)+\frac{b\pi|\alpha_r+\beta_r-\pi|}{2\cosh^2(bl(r)/2)}+\frac{b\pi|\alpha_{-r}+\beta_{-r}-\pi|}{2\cosh^2(bl(-r)/2)}\right)\cosh^2\frac{bL(r)}{2}.$$

Now assume that $W(R)>0$ for some $R>0$ – so by the monotonicity of W also $W(r)>0\ \forall\, r\ge R$. Then

$$\frac{W'(r)}{W^2(r)} \ge \frac{b}{32\pi^2\cosh^2(bL(r)/2)} - \frac{b\pi}{2W^2(r)}\left(\frac{|\alpha_r+\beta_r-\pi|}{\cosh^2(bl(r)/2)}+\frac{|\alpha_{-r}+\beta_{-r}-\pi|}{\cosh^2(bl(-r)/2)}\right)$$

for all $r\ge R$; and integration leads to (cf. [BK], Lem 3.12)

$$\begin{aligned}
\frac{1}{W(R)} \ge \left.\frac{-1}{W(r)}\right|_R^\infty &= \int_R^\infty \frac{W'(r)\,dr}{W^2(r)}\\
&\ge \int_R^\infty \frac{b\,dr}{32\pi^2\cosh^2(bL(r)/2)}\\
&\quad -\frac{b\pi}{2W^2(R)}\int_R^\infty\left(\frac{|\alpha_r+\beta_r-\pi|}{\cosh^2(bl(r)/2)}+\frac{|\alpha_{-r}+\beta_{-r}-\pi|}{\cosh^2(bl(-r)/2)}\right)dr\\
&\ge \int_R^\infty \frac{b\,dr}{32\pi^2\cosh^2(bL(r)/2)} - \frac{b\pi}{2W^2(R)}\int_{-\infty}^\infty\frac{|\alpha_s+\beta_s-\pi|\,ds}{\cosh^2(bl(s)/2)}\\
&> \int_R^\infty \frac{b\,dr}{32\pi^2\cosh^2(bL(r)/2)} - \frac{\pi^3}{W^2(R)}
\end{aligned}$$

according Lem. 17.

But by the assumption, $bL(r)<\ln r$ for $r>R, R$ sufficiently large, so

$$\int_R^\infty \frac{dr}{\cosh^2(bL(r)/2)} > \int_R^\infty \frac{4\,dr}{e^{bL(r)}+3} \ge \int_R^\infty\frac{4\,dr}{r+3}=\infty$$

– a contradiction. So $W\equiv 0$. Using Lem. 12, this proves

$$\int_{SC} U^2(v)\,d\mu(v) = \int_{-\infty}^\infty V(s)\,ds = \limsup_{r\to\infty} W(r) = 0.$$

Hence $U=0$ μ-a.e. and therefore $K\equiv 0$ by Riccati equation, as K is continuous.

REFERENCES

[B] K. Burns, *The flat strip theorem fails for surfaces with no conjugate points*, Proc. Amer. Math. Soc., 115 (1992), pp. 199–206.

[BBB] W. Ballmann, M. Brin and K. Burns, *On surfaces with no conjugate points*, J. Differential Geometry, 25 (1987), pp. 249–273.

[BI] D. Burago and S. Ivanov, *Riemannian tori without conjugate points are flat* Geom. Funct. Anal., 4 (1995), pp. 259–269.

[BK] K. Burns and G. Knieper, *Rigidity of surfaces with no conjugate points*, J. Differential Geometry, 34 (1991), pp. 623–650.

[CK] C. B. Croke and B. Kleiner, *A rigidity theorem for manifolds without conjugate points*, Ergod. Th. Dyn. Sys., 18 (1998), pp. 813–829.

[Gr] L. W. Green, *Surfaces without conjugate points*, Trans. Amer. Math. Soc., 76 (1954), pp. 529–546.

[Gu] R. Gulliver, *On the variety of manifolds without conjugate points*, Trans. Amer. Math. Soc., 210 (1975), pp. 185–201.

[H] E. Hopf, *Closed surfaces without conjugate points*, Proc. Nat. Acad. Sci., 34 (1948), pp. 47–51.

[I] N. Innami, *Euclidean metric and flat metric outside a compact set*, Proc. Amer. Math. Soc., 105 (1989), pp. 701–705.

[MZ] S. Mendonca and D. Zhou, *Maximal ergodic theorems and applications to Riemannian geometry*, Isr. J. Math., 139 (2004), pp. 319–335.