Asian J. Math.
Vol. 11, No. 4, pp. 555–566, December 2007

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VOLUME AND ANGLE STRUCTURES ON 3-MANIFOLDS*

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Abstract. We propose an approach to find constant curvature metrics on triangulated closed 3-manifolds using a finite dimensional variational method whose energy function is the volume. The concept of an angle structure on a tetrahedron and on a triangulated closed 3-manifold is introduced following the work of Casson, Murakami and Rivin. It is proved by A. Kitaev and the author that any closed 3-manifold has a triangulation supporting an angle structure. The moduli space of all angle structures on a triangulated 3-manifold is a bounded open convex polytope in a Euclidean space. The volume of an angle structure is defined. Both the angle structure and the volume are natural generalizations of tetrahedra in the constant sectional curvature spaces and their volume. It is shown that the volume functional can be extended continuously to the compact closure of the moduli space. In particular, the maximum point of the volume functional always exists in the compactification. The main result shows that for a 1-vertex triangulation of a closed 3-manifold if the volume function on the moduli space has a local maximum point, then either the manifold admits a constant curvature Riemannian metric or the manifold contains a non-separating 2-sphere or real projective plane.

Key words. 3-manifolds, constant curvature metrics, triangulations, volume, angle structures, normal surfaces, Schlaefli formula

AMS subject classifications. 57M30

1. Introduction.

1.1. The purpose of the announcement is to propose an approach to find constant curvature metrics on triangulated closed 3-manifolds using a finite dimensional variational method whose energy function is the volume. It is a generalization of the corresponding program introduced by Casson [La] and Rivin [Ri1] for finding hyperbolic metrics on compact ideally triangulated 3-manifolds with torus boundary. It is also motivated by the work of Murakami [Mu]. Very recently, Casson and Rivin's approach was successfully carried out by Francois Gueritaud [Gu] to give a new proof of the existence of hyperbolic structures on 1-holed torus bundles over the circle with Anosov monodromy.

By an *angle structure* on a 3-simplex we mean an assignment of a number, called the *dihedral angle*, to each edge of the 3-simplex so that dihedral angles at three edges sharing a common vertex are the inner angles of a spherical triangle. (Note that the similar concept introduced by Casson and Rivin requires that the vertex triangle be a Euclidean triangle). Since a spherical triangle is determined by its inner angles subject to four linear inequalities, the moduli space of all angle structures on a 3-simplex, denoted by $AS(3)$, is an open bounded convex polytope in $\mathbf{R}^6$. Examples of angle structures on a 3-simplex are classical geometric tetrahedra, i.e., Euclidean, hyperbolic and spherical tetrahedra measured by dihedral angles. However, not every angle structure is of this form. We define the *generalized volume* (or *volume* for short) of an angle structure on a 3-simplex by generalizing the Schlaefli formula. To be more precise, the Schlaefli formula for volume of spherical or hyperbolic tetrahedra says that volume can be defined by integrating the Schlaefli 1-form which depends on the

* Received February 17, 2006; accepted for publication August 28, 2006.
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dihedral angles and the edge lengths. Furthermore, one finds the edge length from the dihedral angles by using the Cosine Law twice. We follow this path to define the volume of an angle structure by first defining the edge lengths using the Cosine Law and then verifying the resulting Schlaefli 1-form is closed in the moduli space $AS(3)$. The generalized volume is the integration of the Schlaefli 1-form. In the case of classical geometric tetrahedra, the generalized volume coincides with classical geometric volume for spherical and hyperbolic tetrahedra and is zero for Euclidean tetrahedra.

By an *angle structure* on a triangulated closed 3-manifold (M, T) we mean an assignment of a number, called the dihedral angle, to each edge of each 3-simplex in the triangulation T, so that (1) the assignment is an angle structure on each 3-simplex in T, (2) the sum of dihedral angles at each edge in T is 2π. The basic examples of angle structures are totally geodesic triangulations in a 3-manifold with a constant curvature metric. The *volume* of an angle structure on a triangulated manifold is defined to be the sum of the volume of its 3-simplexes. From the definition, it is clear that the moduli space of all angle structures on a fixed triangulated 3-manifold (M, T), denoted by $AS(M, T)$, forms a bounded convex polytope in a Euclidean space. The main theorem is the following.

THEOREM 1.1. *Suppose (M, T) is a closed 3-manifold with a triangulation T so that the moduli space of all angle structures $AS(M, T)$ is non-empty. Then volume function can be extended continuously to the compact closure of the space $AS(M, T)$. If the volume functional has a local maximum point in $AS(M, T)$, then,*

(a) the manifold M supports a constant sectional curvature Riemannian metric, or

(b) there is a normal surface of positive Euler characteristic in the triangulation T so that it intersects each 3-simplex in at most one normal disk.

In particular, if the triangulation T has only one vertex, then the normal surface in case (b) is non-separating.

The existence of angle structures on a triangulation is a linear programming problem.

PROBLEM 1.2. *Does every closed irreducible non-Haken non Seifert-fibered 3-manifold have a 1-vertex triangulation supporting an angle structure?*

We expect the problem has an affirmative solution. See [JR] for more information on efficient triangulaions. In [LT], a relationship between the existence of angle structure and the normal surface theory is established.

If we do not assume 1-vertex condition, then the following has been proved by A. Kitaev and myself.

THEOREM 1.3. *For any closed 3-manifold M^3, there is a triangulation of M^3 supporting an angle structure.*

By theorem 1.1, the maximal point of the volume function on the compact closure of $AS(M, T)$ always exists. If the maximum point is in $AS(M, T)$, then we conclude either the manifold M is geometric, or the triangulation admits a special normal surface of positive Euler characteristic. It is expected ([Lu3]) that the maximum point in the boundary $\partial AS(M, T)$ will give rise either a geometric structure on the manifold, or a special normal surface of non-negative Euler characteristic in the triangulation.

1.2. Using dihedral angles instead of edge lengths to parameterize a classical geometric tetrahedra seems to have some advantages. First of all, the Schlaefli formula suggests that dihedral angles are natural variables with respect to the volume. Second, dihedral angle parameterization puts all hyperbolic, spherical and Euclidean tetrahedra in one frame work. Third, volume considered as a function of dihedral angles can be extended continuously to the degenerated tetrahedra. This was conjectured by John Milnor [Mi] and was established recently in [Lu1] and [GL] ([Ri2] has a new proof together with some generalizations). On the other hand, volume considered as a function of edge lengths cannot be extended continuously to degenerated tetrahedra. The use of dihedral angle parameterization has also appeared in the work of Murakami [Mu].

1.3. In this subsection, we briefly sketch the main ideas used in the definition of volume and the proof of theorem 1.1.

By a *2-dimensional angle structure* on a triangle, we mean an assignment of a number in $(0, \pi)$, called the *angle,* to each vertex of the triangle. Classical geometric triangles are examples of 2-dimensional angle structures. It is known that a 2-dimensional angle structure is the same as a Moebius triangle, i.e., a triangle of inner angles in $(0, \pi)$ in the Riemann sphere bounded by circles and lines (see [Lu2]). We will interchange the use of terminology Moebius triangle and 2-dimensional angle structure in the rest of the paper. For a classical geometric tetrahedron, the codimension-1 faces of them are classical geometric triangles of the same type. Similarly, for an angle structure on a tetrahedron, the codimension-1 face of it is a 2-dimensional angle structure. Namely, the inner angle of a vertex of a codimension-1 face is the spherical edge length in the spherical vertex triangle. The edge length can be calculated by the Cosine Law for the spherical vertex triangle. To define the Schlaefli 1-form, we have to define the edge length of a Moebius triangle. This is achieved by generalizing the Cosine Law for classical geometric triangles. The main observation is that the side of the Cosine Law involving inner angles is still valid for 2-dimensional angle structures. We define the length of an edge in a Moebius triangle by using the Cosine Law. Having defined the lengths of edges of a Moebius triangles, we are able to define the edge length of an angle structure on a tetrahedron by verifying the compatibility condition. Namely, although each edge of a tetrahedron is adjacent to two triangular faces, the length of the edge is independent of the choice two Moebius triangles adjacent it. From this, one forms the schlaefli 1-form defined on the space $AS(3)$ of all angle structures and shows that the 1-form is closed. We define the volume to be the integration of this 1-form.

The volume function defined in this way automatically satisfies the Schlaefli formula. By the Schlaefli formula, we are able to identify the critical points of the volume in the space of all angle structures $AS(M, T)$. By the Lagrangian multiplier, a critical point $p \in AS(M, T)$ of the volume is the same as the following. If e is an edge in T adjacent to two tetrahedra A and B, then the lengths of e in A and B (in their angle structures in p) are the same. This produces a condition for gluing angle structures on 3-simplexes along their codimension-1 faces. One the other hand, both 2-dimensional and 3-dimensional angle structures can be classified into three types: Euclidean, hyperbolic and spherical. The type of an angle structure depends only on the length of an edge. As a consequence, at the critical point, all 3-simplices have the same type. If each 3-simplex in p is a classical geometric tetrahedron, then condition (a) in theorem 1.1 holds. If there is a 3-simplex in p which is not a classical geometric tetrahedron, we are able to produce a normal surface so that it cuts each 3-simplex in at most one normal disk and its Euler characteristic is positive. The positivity of the Euler characteristic is a consequence of the following observation. Namely, the sum of the dihedral angles at two pairs of opposite edges of a Euclidean or a hyperbolic tetrahedron is less than 2π. Finally, if the triangulation has only one vertex, then all edges become loops. Since the normal surface intersects some edge transversely in one point, it is non-separating.

1.4. The above finite dimensional variational set up can be extended easily to find constant curvature cone metrics on 3-manifolds.

1.5. This paper is organized as follows. In §2, we establish a calculus Cosine Law and its derivative form. In §3, we establish a generalized Schlaefli formula. In §4, we define the edge lengths of 2-dimensional angle structures and state some of its consequences. In §5, we define the volume of angle structures on tetrahedra. In §6, we give a classification of angle structures on tetrahedra. In §7, we sketch the proof of theorem 1.1.

Acknowledgment. We thank X.S. Lin, Ben Chow, and Thomas Au for discussions on the related subjects. Bruno Martelli made a nice suggestion on improving the last statement in theorem 1.1. Part of the work was done while I was visiting the Chinese University of Hong Kong and Yale University. We thank Thomas Au and Yair Minsky for their hospitality.

2. A Calculus Cosine Law. Recall that if a spherical triangle has three inner angles x_1, x_2, x_3 and edge lengths y_1, y_2, y_3 where y_i is the length of the edge opposite to the angle x_i, then the Cosine Law says,

$$\cos(y_i) = \frac{\cos x_i + \cos x_j \cos x_k}{\sin x_j \sin x_k} \tag{2.1}$$

where $\{i, j, k\} = \{1, 2, 3\}$. It turns out (2.1) encodes all Cosine Laws for classical geometric triangles, i.e., hyperbolic, Euclidean triangles and even hyperbolic right-angled hexagons if we interprete the terms appropriately. To be more precise, if a classical geometric triangle in K^2 ($K^2 = S^2, E^2$ or H^2) has inner angles x_1, x_2, x_3, then the length y_i of the edge opposite to the angle x_i satisfies the Cosine Law,

$$\cos(\sqrt{\lambda} y_i) = \frac{\cos x_i + \cos x_j \cos x_k}{\sin(x_j)\sin(x_k)} \tag{2.2}$$

where $\{i, j, k\} = \{1, 2, 3\}$ and $\lambda = 1, 0, -1$ is the curvature of the space K^2. This prompts us to look at (2.1) from analysis point of view.

2.1. Suppose we have a single valued complex analytic function $y = y(x)$ where $y = (y_1, y_2, y_3) \in \mathbf{C}^3$ and $x = (x_1, x_2, x_3)$ is in some open connected set Ω in $\mathbf{C}^3$ so that y and x are related by (2.1).

Theorem 2.1. *Suppose $\Omega \subset \mathbf{C}^6$ contains a diagonal point (a, a, a) so that $y(a, a, a) = (b, b, b)$. Let the indices $\{i, j, k\}$ be $\{1, 2, 3\}$ and $A_{ijk} = \sin y_i \sin x_j \sin x_k$. Then*

(a) $$A_{ijk} = A_{jki}.$$

(b) $$A_{ijk}^2 = 1 - \cos^2 x_i - \cos^2 x_j - \cos^2 x_k - 2\cos x_i \cos x_j \cos x_k.$$

At the point x where $A_{ijk} \neq 0$, the following three hold,

(c) $$\partial y_i/\partial x_i = \sin x_i / A_{ijk},$$

(d) $$\partial y_i/\partial x_j = \partial y_i/\partial x_i \cos y_k,$$

(e) $$\cos(x_i) = \frac{\cos y_i - \cos y_j \cos y_k}{\sin y_j \sin y_k}.$$

2.2. REMARKS. 1. Formula (a) is sometimes called the *Sine Law.*
2. If we make a change of variable that $z_i = \pi - y_i$, then (2.1) and (e) show that the map $z = z(x)$ is an involution, i.e., $z(z(x)) = x$. This corresponds to the duality theorem for spherical triangles. Using this, we see that the partial derivatives $\partial x_i/\partial y_i$ and $\partial x_i/\partial y_j$ can be derived easily from theorem 2.1. To be more precise, we have $\partial x_i/\partial y_j = -\partial x_i/\partial y_i \cos x_k$.
3. Let $F_1(x) = (x_1, \pi - x_2, \pi - x_3)$, then $(\cos(y_1(F_1(x))), \cos(y_2(F_1(x))), \cos(y_3(F_1(x))))$ $= (\cos(y_1(x)), \cos(\pi - y_2(x)), \cos(\pi - y_3(x)))$. This symmetry of the Cosine Law (2.1) will be used extensively in the paper.

2.3. *Proof.* For simplicity, let $c_i = \cos x_i$ and $s_i = \sin x_i$ for $i = 1, 2, 3$. Then by definition,

$$\begin{aligned} A_{ijk}^2 &= \sin^2 y_i \sin^2 x_j \sin^2 x_k = (1 - \cos^2 y_i)\sin^2 x_j \sin^2 x_k \\ &= (1 - c_j^2)(1 - c_k^2) - (c_i + c_j c_k)^2 \\ &= 1 - c_j^2 - c_k^2 + c_j^2 c_k^2 - c_i^2 - 2c_i c_j c_k - c_j^2 c_k^2 \\ &= 1 - c_i^2 - c_j^2 - c_k^2 - 2c_i c_j c_k. \end{aligned}$$

This shows that (b) holds. Now consider the analytic function A_{ijk}/A_{jki}. By (b), it takes values ± 1. By the assumption that $y(a, a, a) = (b, b, b)$, we see that value 1 is achieved. Thus $A_{ijk} = A_{jki}$ in the connected set Ω. This shows that (a) holds.

By taking derivative of (2.1), we obtain,

$$-\sin y_i \partial y_i/\partial x_i = -\sin x_i/(\sin x_j \sin x_k).$$

This establishes part (c).

To see part (d), we take the derivative of (2.1) with respect to x_j. Thus

$$\begin{aligned} -\sin y_i \partial y_i/\partial x_j &= (1/s_k)[(-s_j \cos x_k)s_j - \cos x_j(\cos x_i + \cos x_j \cos x_k)]/s_j^2 \\ &= 1/(s_j^2 s_k)(-\cos x_k - \cos x_i \cos x_j) \\ &= [-s_i/(s_j s_k)][\cos x_k + \cos x_i \cos x_j]/(s_i s_j) \end{aligned}$$

$$= -(s_i/(s_j s_k)) \cos y_k$$

By dividing it by $-\sin y_i$ and using part (c), we obtain identity (d).

Finally, we derive (e). By (a) and (b), we have

$$(\cos y_i - \cos y_j \cos y_k)/(\sin y_j \sin y_k)$$
$$= [(c_i + c_j c_k)/(s_j s_k) - (c_j + c_i c_k)(c_k + c_i c_j)/(s_i s_k s_i s_j)]/(\sin y_j \sin y_k))$$
$$= [(1 - c_i^2)(c_i + c_j c_k) - (c_j c_k + c_i c_j^2 + c_i c_k^2 + c_i^2 c_j c_k)]/[(s_i s_k \sin y_j)(s_i s_j \sin y_k)]$$
$$= (c_i + c_j c_k - c_i^3 - c_j c_k c_i^2 - c_j c_k - c_i c_j^2 - c_i c_k^2 - c_i^2 c_j c_k)/(A_{ikj} A_{ijk})$$
$$= c_i(1 - c_i^2 - c_j^2 - c_k^2 - 2c_i c_j c_k)/A_{ijk}^2$$
$$= \cos x_i.$$

3. A Generalized Schlaefli Identity. In this section, we will fix a branch of the function $\arccos z$. Let U be the set $\{z \in \mathbf{C} | 0 < Re(z) < \pi\} \cup \{\sqrt{-1}x | x \in \mathbf{R}_{\geq 0}\} \cup \{\pi - \sqrt{-1}x | x \in \mathbf{R}_{\geq 0}\}$. Then the restriction map $\cos z : U \to \mathbf{C}$ is a bijection. We define $\arccos z : \mathbf{C} \to U$ to be the inverse and call it the principal branch of $\arccos z$. Note that $\arccos z$ is analytic in $\mathbf{C} - \{x \in \mathbf{R} | |x| \geq 1\}$.

3.1. Given a 3-simplex with vertices $\{v_1, ..., v_4\}$, a *complex weight* on the 3-simplex is an assignment of a complex number $x_{ij} = x_{ji} \in \mathbf{C} - \{\pi n | n \in \mathbf{Z}\}$ to the edge $v_i v_j$. We consider x_{ij} as a complex valued "dihedral angle". In this setting, the "face angle" y^i_{jk} of the weighted 3-simplex in the face triangle $\Delta v_i v_j v_k$ at the vertex v_i is defined to be the unique complex number y^i_{jk} in U so that

$$\cos y^i_{jk} = \frac{\cos x_{il} + \cos x_{ij} \cos x_{ik}}{\sin x_{ij} \sin x_{ik}} \tag{3.1}$$

where i, j, k, l are pairwise distinct. We will assume that indices i, j, k, l are always pairwise distinct in the sequel. Note that if $\{x_{12}, ..., x_{34}\}$ forms the dihedral angles of a classical geometric tetrahedron, then y^i_{jk} is the inner angle at v_i in the triangle $\Delta v_i v_j v_k$ by the Cosine Law (2.1).

PROPOSITION 3.1.(Compatibility) *Suppose the complex weight $x = (x_{12}, ..., x_{34})$ satisfies $\sin(y^i_{jk}) \neq 0$ for all indices. Then*

$$\frac{\cos(y^i_{jk}) + \cos(y^j_{ik}) \cos(y^k_{ij})}{\sin(y^j_{ik}) \sin(y^k_{ij})} = \frac{\cos(y^l_{jk}) + \cos(y^j_{lk}) \cos(y^k_{jl})}{\sin(y^j_{lk}) \sin(y^k_{jl})}. \tag{3.2}$$

The underlying geometric meaning of this proposition is that the length of the edge $v_j v_k$ in a classical geometric tetrahedron can be calculated from any of the two triangles $\Delta v_i v_j v_k$ or $\Delta v_l v_j v_k$. The proof is a direct computation using the Sine Law in theorem 2.1.

Note that the condition $\sin(y^i_{jk}) \neq 0$ for all indices i, j, k is equivalent to $x_{ij} \pm x_{ik} \pm x_{il} \neq (2n+1)\pi$ for some integer n for all $\{i, j, k, l\} = \{1, 2, 3, 4\}$. We call a complex weight $(x_{12}, ..., x_{34})$ *non-degenerate* if $\sin(y^i_{jk}) \neq 0$ for all indices. In this case, the common value in (3.2) is denoted by $\cos z_{jk}$ where $z_{jk} \in U$. We call z_{jk} the *complex length* of the edge v_jv_k. If the dihedral angles $(x_{12},...,x_{34})$ form the inner angles of a spherical or a hyperbolic 3-simplex, by the Cosine Laws, $\lambda\sqrt{\lambda}z_{ij}$ is equal to the length of edge v_iv_j in the 3-simplex in S^3 or H^3 where $\lambda=\pm 1$ is the curvature of S^3 or H^3.

3.2. For a non-degenerated complex weight $x=(x_{12},...,x_{34})$, the complex length $z_{ij}=z_{ij}(x)$ is a complex analytic function of the weight x.

THEOREM 3.2. *For a non-degenerate complex weight* $(x_{12},...,x_{34})$*, then*

$$\partial z_{ij}/\partial x_{ab}=\partial z_{ab}/\partial x_{ij},$$

where the indices satisfy $a\neq b$ *and* $i\neq j$. *In particular, the differential 1-form* $\sum_{i,j} z_{ij}dx_{ij}$ *is closed.*

The proof is computational using theorem 2.1.

4. Lengths of Edges in Moebius Triangles. Given a Moebius triangle of inner angles $(x_1,x_2,x_3)\in(0,\pi)^3$, we will define the Moebius length (or length for simplicity) of an edge in the triangle in this section. The definition of the length is guided by formula (2.2).

4.1. As a convention in this section, the function $\cos z$ is considered as a homeomorphism from the 1-dimensional subset $\mathbf{L}=\{\sqrt{-1}x|x\in\mathbf{R}_{\geq 0}\}\cup\{x\in\mathbf{R}|x\in[0,\pi]\}\cup\{\pi-\sqrt{-1}x|x\in\mathbf{R}_{\geq 0}\}$ to the real line $\mathbf{R}$. Let $\phi:\mathbf{R}\to\mathbf{L}$ be the homeomorphism given by $\phi(x)=x$ if $x\in[0,\pi]$; $\phi(x)=-\sqrt{-1}x$ if $x\leq 0$; and $\phi(x)=\pi+\sqrt{-1}(\pi-x)$ if $x\geq\pi$. In particular, the homeomorphism $f(x)=\cos(\phi(x)):\mathbf{R}\to\mathbf{R}$ is given by $f(x)=\cosh(x)$ when $x\leq 0$; $f(x)=\cos(x)$ when $x\in[0,\pi]$; and $f(x)=-\cosh(x-\pi)$ when $x\geq\pi$. The function f is C^1-smooth and f^{-1} is continuous.

DEFINITION 4.1. Given a Moebius triangle of inner angles (x_1,x_2,x_3), the *length* (or more precisely the *Moebius length)* z_i of the edge opposite to the angle x_i is the real number so that

$$\cos(\phi(z_i))=\frac{\cos x_i+\cos x_j\cos x_k}{\sin x_j\sin x_k}$$

where $\{i,j,k\}=\{1,2,3\}$.

By formula (2.2), if $(x_1,x_2,x_3)\in(0,\pi)^3$ is a classical geometric triangle in $K^2=S^2,H^2,E^2$ of curvature $\lambda=1,-1,0$ so that lengths of the edges are l_1,l_2,l_3 measured in K^2 geometry, then the Moebius lengths of it are $(\lambda l_1,\lambda l_2,\lambda l_3)$, i.e., the Moebius length is the signed length for classical geometric triangles.

4.2. Given a Moebius triangle $x=(x_1,x_2,x_3)\in(0,\pi)^3$, the i-th *flip* $F_i(x)$ of x is the Moebius triangle with inner angles $y=(y_a)$ where $y_i=x_i$ and $y_j=\pi-x_j$ for $j\neq i$.

Let $E(2),H(2),S(2)$ be the subspaces of $AS(2)$ corresponding the Euclidean, hyperbolic and spherical triangles respectively. Then evidently $F_i:S(2)\to S(2)$. It can be shown that for any $x\in AS(2)$, either $x\in E(2)\cup H(2)\cup S(2)$ or there is a flip $F_i(x)$ so that $F_i(x)\in E(2)\cup H(2)$. We classify the Moebius triangles into Euclidean type, hyperbolic type or spherical type according to x or $F_i(x)$ is the classical geometric triangle of the same type. This classification is invariant under the flip operation. The following result summerizes the basic properties of the Moebius lengths, the type of Moebius triangles and flip operations.

PROPOSITION 4.2. *The i-th edge length function $z_i : AS(2) = (0,\pi)^3 \to \mathbf{R}^1$ is continuous. If x is a classical geometric triangle in K^2 of curvature $\lambda = -1, 1, 0$, then the Moebius length $z_i(x)$ is λl_i where l_i is the length calculated in the classical geometry. Furthermore,*

(a) For a non-Euclidean type Moebius triangle, the three Moebius lengths determine the three inner angles.

(b) If $F_i(x)$ is the i-th flip of x, then the Moebius length and the flip are related by

$$z_j(F_i(x)) = \pi - z_j(x) \tag{4.1}$$

$$z_i(F_i(x)) = z_i(x) \tag{4.2}$$

(c) The type of a Moebius triangle is determined by one edge length. Namely, x is of Euclidean type if and only if $z_i(x) \in \{0, \pi\}$; x is of spherical type if and only if $z_i(x) \in (0,\pi)$; and x is of hyperbolic type if and only if $z_i(x) \in (-\infty, 0) \cup (\pi, \infty)$.

(d) A Moebius triangle is not a classical geometric triangle if and only if one edge length is at least π.

Part (a) follows from theorem 2.1 (e). Part (b) is a consequence of remark 2.2.3. Parts (c) and (d) follow from (b) and the definition. There are exactly two edges in a non-classical geometric triangle of length at least π. The flip operation about the vertex which is the intersection of the two edges of length $\geq \pi$ is a classical geometric triangle.

4.3. REMARK. One can take identities (4.1) and (4.2) as the definition of the Moebius length of edges in Moebius triangles.

5. Volume of an Angle Structure on a Tetrahedron.

5.1. In this section, we assume that the indices $\{i, j, k, l\} = \{1, 2, 3, 4\}$. An angle structure $x = (x_{12}, ..., x_{34})$ on a 3-simplex with vertices $v_1, ..., v_4$ has dihedral angle $x_{ij} = x_{ji}$ at the edge v_iv_j. The space of all angle structures on a tetrahedron $AS(3)$ is $\{x \in (0,\pi)^6 | x_{ij} + x_{ik} + x_{il} > \pi, x_{ij} + x_{ik} - x_{il} < \pi, \text{ for all i,j,k,l}\}$. Given $x \in AS(3)$ and a codimension-1 face $\Delta v_iv_jv_k$ of the tetrahedron, the *induced 2-dimensional angle structure* on $\Delta v_iv_jv_k$ is obtained by assigning the positive number $y^i_{jk} \in (0,\pi)$ to the vertex v_i where y^i_{jk} satisfies (3.1). Geometrically, if we construct a spherical triangle of inner angles x_{ij}, x_{ik}, x_{il} associated to the vertex v_i, then y^i_{jk} is the spherical length of the edge opposite to the angle x_{il} in the spherical triangle. There are four Moebius triangles $\Delta v_iv_jv_k$ appeared as codimension-1 faces of the tetrahedron. These Moebius triangles have Moebius lengths at the edges. By the definition of Moebius lengths and compatibility proposition 3.1, we have,

LEMMA 5.1. *The Moebius lengths of the edge v_iv_j in the Moebius triangles $\Delta v_iv_jv_k$ and $\Delta v_iv_jv_l$ are the same.*

We call the common value in lemma 5.1 the *length* (or Moebius length) of the edge v_iv_j in x, and denote it by l_{ij}. Note that if the angle structure is a classical geometric tetrahedron in the space $K^3 = S^3, H^3, E^3$ of curvature $\lambda = 1, -1, 0$, then the Moebius length is λl where l is the length measured in the classical geometry K^3.

5.2. Note that the regular Euclidean tetrahedron has the dihedral angle $\arccos(1/3)$. A continuous differential 1-form ω defined on a smooth manifold is said to be closed if its integration along each piecewise smooth loop is zero.

THEOREM 5.2. *The continuous differential 1-form $\omega = 1/2 \sum_{i>j} l_{ij} dx_{ij}$ on the open convex polytope $AS(3)$ is closed. Its integration*

$$V(x) = \int_a^x \omega$$

where $a = \arccos(1/3)(1, 1, .., 1)$ is a C^1-smooth function on $AS(3)$, called the volume (or Moebius volume). The function $V(x)$ has the following properties,

(a) (Schlaefli formula)

$$\partial V(x)/\partial x_{ij} = l_{ij}/2, \tag{5.1}$$

(b) if $x \in AS(3)$ is a classical geometric 3-simplex in the space of constant curvature λ ($\lambda = -1, 0, 1$), then the volume $V(x) = \lambda^2 vol(x)$ where vol is the volume measured in the classical geometry of constant curvature λ.

(c) The volume function V can be extended continuously to the compact closure $\bar{AS}(3) = \{x \in [0, \pi]^3 | x_{ij} + x_{ik} + x_{il} \geq \pi, x_{ij} + x_{ik} - x_{il} \leq \pi\}$.

6. The Classification of Angle Structures and Flip Operations. For an angle structure on a tetrahedron $x = (x_{rs}) \in AS(3)$, the i-th *flip* $F_i(x) = (y_{rs})$ is the angle structure so that $y_{ij} = x_{ij}$ and $y_{jk} = \pi - x_{jk}$ where $\{i, j, k, l\}$ are pairwise distinct. It follows from the definition and remark 2.2.3 that the codimension-1 faces $\Delta v_i v_j v_k$ of $F_i(x)$ and x are related by the i-th flip and the faces $\Delta v_j v_k v_l$ of x and $F_i(x)$ are the same. The corresponding vertex links of $F_i(x)$ and x, considered as spherical triangles, are either the same or related by a flip.

6.1. Let $E(3), H(3)$ and $S(3)$ be the subspaces of $AS(3)$ corresponding to the Euclidean, hyperbolic, and spherical tetrahedra respectively. It can be shown that,

PROPOSITION 6.1. *For any $x \in AS(3)$, exactly one of the following holds,*
(a) x is a classical geometric tetrahedron, i.e., $x \in E(3) \cup H(3) \cup S(3)$,
(b) there is a flip F_i so that $F_i(x) \in E(3) \cup H(3)$,
(c) there are two flips F_i, F_j, $i \neq j$, so that $F_iF_j(x) \in E(3) \cup H(3)$.

If an angle structure x is obtained from a single flip $F_i(y)$ where $y \in E(3) \cup H(3)$, then the edge length l_{ij} of x is at least π and all other edge lengths are at most 0 by (4.1) and (4.2). If x is obtained from a double-flip $F_iF_j(y)$ where $y \in E(3) \cup H(3)$ with $i \neq j$, then the edge lengths and dihedral angles of x satisfy: (1) $l_{ab} \geq \pi$, $x_{ab} = \pi - y_{ab}$ for $\{a, b\} = \{i, k\}, \{i, l\}, \{j, k\}, \{j, l\}$; and (2) $l_{ij}, l_{kl} \leq 0$ and $x_{ij} = y_{ij}, x_{kl} = y_{kl}$.

LEMMA 6.2. *Suppose $x = F_iF_j(y)$ for $y \in E(3) \cup H(3)$ is a double flip of a Euclidean or a hyperbolic tetrahedron y. Then the sum of the dihedral angles at the two pairs of opposite edges of length at least π is greater than 2π.*

Indeed, by the classification above, the lemma is equivalent to the following statement about Euclidean and hyperbolic tetrahedra. Namely, if a_1, a_2, a_3, a_4 are dihedral angles at two pairs of opposite edges in a Euclidean or hyperbolic tetrahedron, then $a_1 + a_2 + a_3 + a_4 < 2\pi$. The proof of it is a simple exercise in geometry.

6.2. Remark. The relationship between the flip operation and the volume is the following. For any angle structure $x \in AS(3)$, $V(F_i(x)) + V(x) = \pi/2(x_{ij} + x_{ik} + x_{il} - \pi)$. One can in fact use this as the definition of the volume of angle structure. Using this formula, we prove the continuous extension of the volume to the closure of $AS(3)$ using [Lu1].

7. The Sketch of the Proof of Theorem 1.1. The proof goes as follows.

7.1. Recall that the type of a Moebius triangle is determined by one edge length by proposition 4.2. Using proposition 3.1, it follows that the types of all Moebius triangles appeared as faces of an angle structure on a tetrahedron are the same. We define the type of an angle structure on a tetrahedron to be the type of a codimension-1 face of it. By definition, the type of an angle structure is determined by the length of one edge. To be more precise, if the length of the ij-th edge l_{ij} is in $\{0, \pi\}$, then it is of Euclidean type; if l_{ij} is in $(0, \pi)$, then it is of spherical type; and if l_{ij} is in $(-\infty, 0) \cup (\pi, \infty)$, then it is of hyperbolic type.

7.2. If $p \in AS(M, T)$ is a critical point of the volume, then Schlaefli formula shows that the following holds. Namely, if $e \in T^{(1)}$ is an edge in the triangulation so that e is adjacent to two 3-simplexes A, B, then the Moebius lengths of e in A and B (in p) are the same:

$$l_e(A, p) = l_e(B, p). \tag{7.1}$$

Indeed, suppose a_1 and a_2 are the dihedral angles of $p = (a_1, a_2, ..., a_n)$ at the edge e inside A and B. Now consider the deformation $r(t) = (t, -t, 0, ..., 0) + p$. Since $p = (a_1, a_2, ..., a_n)$ is the critical point of V, it follows that $d/dt|_{t=0} V(r(t)) = 0$. But by the Schlaefli formula (5.1), the derivative is $1/2(l_e(A, p) - l_e(B, p))$. Thus (7.1) follows.

7.3. By combining 7.1 and 7.2, the types of all angle structures on 3-simplexes in p are the same.

7.4. If one 3-simplex in p is spherical, then all 3-simplexes in p are spherical. They are all classical spherical tetrahedra so that their faces can be glued isometrically by (7.1). By the definition of angle structure that the sum of dihedral angles is 2π at each edge, we obtain a spherical metric on the 3-manifold M.

7.5. If all 3-simplexes in p are classical hyperbolic 3-simplexes, then the same argument as in 7.4 shows that the manifold M has a hyperbolic metric.

7.6. If all 3-simplexes in p are classical Euclidean tetrahedra, then we claim that p is not a local maximum point. In fact, the critical point p is a local minimum of the volume. This is due to the following two facts. First, if x is an angle structure on a tetrahedron sufficiently close to a Euclidean tetrahedron, then x is a classical geometric tetrahedron, i.e., $S(3) \cup E(3) \cup H(3)$ is open in $AS(3)$. Second, by theorem 5.2(b), the volume of spherical and hyperbolic tetrahedra are positive and the volume of a Euclidean tetrahedron is zero. Thus, the volume of points in $AS(M, T)$ sufficiently close to p are none negative. On the other hand, the volume of p is zero. This shows that the point p is a local minimum. It is easy to show that the volume function is not a constant on any open subset of $AS(M, T)$. Thus, the critical point p is not a local maximum.

7.7. We claim,

PROPOSITION 7.1. *If one 3-simplex in p is not a classical geometric tetrahedron, then the triangulation T contains a normal surface S of positive Euler characteristic which cuts each 3-simplex in at most one normal disk.*

To prove this, let X be the set of all edges in the triangulation so that its length is at least π (in p). This set is non-empty since there are non-classical geometric tetrahedra in p. By definition, the intersection of X with each 3-simplex A in T consists of the following three cases:

(1) $X \cap A = \emptyset$ if A is a classical geometric tetrahedron in p,

(2) $X \cap A$ consists of three edges from a vertex if A is $F_i(x)$ for a classical geometric tetrahedron $x \in E(3) \cup H(3)$,

(3) $X \cap A$ consists of two pairs of opposite edges if A is $F_iF_j(x)$ for a classical geometric tetrahedron $x \in E(3) \cup H(3)$.

For each 3-simplex A in cases (2) and (3), we construct a normal disk in A whose vertices are in X. These normal disks form a normal surface S in T. Note that the normal surface S intersects the each tetrahedron in T in at most one normal disk. We claim the normal surface S has positive Euler characteristic. To see this, let us consider the CW-decomposition of the surface S formed by the normal disks in S. We assign each vertex of each 2-cell in the CW-decomposition a number, called the *inner angle*, which is the corresponding dihedral angle in p. By the construction, these inner angles satisfy the following three conditions. First, the sum of all inner angles at each vertex in S is 2π. Second, the sum of the inner angles in each normal triangle greater than π. Finally, the sum of all inner angles in each normal quadrilateral is greater than 2π by lemma 6.2. By the Gauss-Bonnet theorem, it follows that the normal surface S has positive Euler characteristic.

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