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YAU'S PROBLEM ON A CHARACTERIZATION OF ROTATIONAL ELLIPSOIDS*

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Abstract. S.T. Yau stated the following problem:
Assume that the Euclidean principal curvatures k_1, k_2 of a closed surface in Euclidean 3-space satisfy the relation $k_1 = ck_2^3$ for some real constant c. Is the surface a rotational ellipsoid?
We give a proof for closed analytic surfaces and study related problems.

Key words. Characterization of quadrics, rotational ellipsoid, surface of revolution, principal curvatures, support functions

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1. Introduction. Let the principal curvatures k_1, k_2 of a surface in Euclidean 3-space E^3 satisfy a differentiable relation

$$W(k_1, k_2) = 0.$$

Such surfaces are called *Weingarten surfaces*. In [1] S.S. Chern generalized a theorem of D. Hilbert and proved:

THEOREM CHERN. *Consider a Weingarten ovaloid with the property that the principal curvature k_1 is a strictly decreasing function of k_2. Then the ovaloid is a sphere.*

On a rotational ellipsoid the principal curvatures satisfy the relation $k_1 = ck_2^3$ for some positive constant c. Chern used this as a counterexample in the sense that, for a characterization of spheres, one cannot modify the assumption "decreasing" to "increasing" in his Theorem. Chern's counterexample became of great importance in the study of relations between curvature functions (and also the support function) of compact hypersurfaces in Euclidean space.

In analogy to the characterization of spheres in terms of curvature functions there are also characterizations including the support function; such results are similar to the theorems about Weingarten surfaces with monotonicity properties; see e.g. [3], section 3.9. Following Chern's example, this led to a study of the support function on rotational surfaces and in particular on rotational ellipsoids; see see e.g. [3], p. 102. During the last decade, several authors proved additional results on the characterization of (hyper-)quadrics in terms of curvature and support functions. We recall the following result as an example; see section 8 in [6]:

THEOREM L-S-S-W. *Let $x : M \to E^{n+1}$ be a hyperovaloid. The following properties are equivalent:*
(i) x is a hyperellipsoid with center at the origin.
(ii) The Gauß-Kronecker curvature satisfies the following eigenvalue equation in terms of the third fundamental form metric $g^* = III$:

$$\Delta^*(H_n^{\frac{2}{n+2}} - c) + 2(n+1)(H_n^{\frac{2}{n+2}} - c) = 0,$$

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where Δ^* *denotes the Laplacian of* g^* *and* c *is an appropriate real positive constant.*

(iii) The support function $\rho :=< \mu, -x >$ *(with* μ *as unit normal) satisfies the following eigenvalue equation in terms of* g^* *:*

$$\Delta^*(\rho^2 - \gamma) + 2(n+1)(\rho^2 - \gamma) = 0,$$

where γ *is an appropriate real positive constant.*

Considering Chern's example of the rotational ellipsoid, S.T. Yau stated the following problem; see [10], Problem 58, p. 290:

Problem-Yau. *Does the relation* $k_1 = ck_2^3$*, where* c *is real, characterize rotational ellipsoids within the class of closed surfaces in Euclidean 3-space?*

A closely related problem was stated by Voss during the workshop [14]:

Problem-Voss. *Consider a local piece of surface admitting a principal curvature parametrization with Gauß basis* $\{\partial_1, \partial_2\}$*. Assume that the principal curvatures satisfy the relations*

$$\partial_1(\frac{k_1}{k_2^3}) = 0, \qquad \partial_2(\frac{k_2}{k_1^3}) = 0.$$

Is the surface an ellipsoid?

This second problem has been solved in [2] a few years ago:

Theorem I. *Assume that a non-degenerate surface in* E^3 *admits a parametrization in terms of principal curvature parameters. The principal curvatures satisfy the two relations stated in the Problem of Voss if and only if the surface is a non-degenerate quadric.*

The authors of [2] generalized the problem and its solution to hypersurfaces in space forms. Lemma 3.5.1 below shows the close relation of the problems of Yau and Voss, resp. There is another result that is related to the problems of Yau and Voss; see Corollary 8.6 in [6]:

Theorem II. *Let* $x : M \to E^3$ *be an analytic ovaloid. Assume that, in points with an appropriate parametrization in terms of principal curvature lines, the principal radii of curvature* R_1, R_2 *satisfy the following two relations:*

$$\partial_1(R_1 + R_2) = -(R_1 + 3R_2)\partial_1 \ln \rho,$$

$$\partial_2(R_1 + R_2) = -(3R_1 + R_2)\partial_2 \ln \rho.$$

Then x *is an ellipsoid.*

As far as we know Yau's problem has been considered to be open up to now. In [10] Yau already stated that an old result of Voss gives a partial answer:

Theorem Voss. [15] *An analytic Weingarten surface of genus zero is rotational.*

In our paper, we will use the Theorem of Voss and the following local characterization of rotational quadrics (Theorem III) as tools to tackle Yau's problem. A special global version of this result, namely a characterization of ellipsoids within the class of closed rotational surfaces, was stated by Kühnel in [4], p.90 (18). The proof he indicates there uses methods quite different from the methods we use for the proof of the following local Theorem.

THEOREM III. *Consider a non-degenerate rotational surface in E^3 with non-vanishing Gauß curvature K and without umbilics. The principal curvatures k_1, k_2 satisfy the relation $k_i = ck_j^3$ for $i, j \in \{1, 2\}, \quad i \neq j$, for some real non-zero constant c if and only if the surface is part of a non-degenerate quadric.*

Theorem III admits a proof of *Yau's problem* for analytic closed surfaces. Obviously one did not realize so far that another proof could be given combining the Theorem of Voss and the result of Kühnel cited above.

THEOREM IV. *Assume that the principal curvatures of a closed, analytic surface in Euclidean 3-space satisfy the relation $k_i = ck_j^3$ for $i, j \in \{1, 2\}, \quad i \neq j$, and some real constant c. Then the surface is a rotational ellipsoid.*

In section 5 below we will study relations between principal curvatures and the support function $\rho =< \mu, -x >$ on rotational quadrics; here we would like to point out that the support function depends on the fact how the surface is situated in relation to the origin; this fact influences the formulation of results. The examples in section 5 give rise to another local characterization:

THEOREM V. *Consider a non-degenerate rotational surface in E^3 with local representation as in (4.1) and its support function ρ. Then k_2 satisfies*

$$R^2 k_2^2 = \rho^2$$

with a non-zero real constant R if and only if the surface is a quadric with center at the origin.

As corollary we get:

THEOREM VI. *A closed rotational surface of genus zero is an ellipsoid with center at the origin if and only if*

$$R^2 k_2^2 = \rho^2$$

for some non-zero real constant R.

The foregoing result together with Theorem L-S-S-W from above gives:

COROLLARY VII. *On a rotational ellipsoid with center at the origin the differentiable function $k_2^2 - k_0$, where k_0 is an appropriate real constant, is a second eigenfunction of the Laplacian of the third fundamental form metric.*

There is an affine background of the topic and methods in our paper, and we would like to comment on this.
Recall that the proof of Theorem L-S-S-W in [6] uses transformation techniques for PDEs relating Euclidean and affine invariants. Such relation appear also as tool for our proof of Theorem III: In affine hypersurface theory it is well known that a non-degenerate hypersurface is a hyperquadric if and only if the unimodular cubic form vanishes. It is also known that the (traceless) symmetric (1.2)-tensor field associated to the cubic form is a so called *gauge invariant*, that means it is independent of the choice of the normalization of the hypersurface. We recall the basic facts for this in

section 3 and then calculate the cubic form in terms of a curvature line parametrization in section 3.5.
To motivate the results stated in Theorems V and VI we recall another fact well known in affine hypersurface theory; here we state it for non-degenerate surfaces. The Euclidean Gauß curvature, the Euclidean support function $\rho(E)$ and the equiaffine (unimodular) support function $\rho(e)$ satisfy

$$\rho(e)^4 = K^{-1}\rho(E)^4.$$

From this one easily verifies that, on rotational quadrics with center and with a representation of the form (4.1) below, the following two relations are equivalent for appropriate choice of the real constants $R > 0$ and c :

$$Rk_2 = \rho(E) \quad \text{and} \quad k_1 = ck_2^3.$$

The first relation appears in Theorems V and VI, the second in Theorem III. The method of proof for both Theorems V and VI differs from that of the proof of Theorem III.

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2. Curvature line parametrization. Consider a Euclidean space $E^3 = \mathbb{R}^3$ with its associated real vector space V and its inner product

$$< \quad , \quad >: V \times V \to \mathbb{R}.$$

We summarize some elementary well known facts about curvature line parameters.
Let M be a connected, oriented, differentiable manifold of dimension $n = 2$, and $x : M \to \mathbb{R}^3$ an immersion. Our considerations are local s.t. we can assume x to be an embedding. If the surface has no umbilics we can introduce curvature line parameters s.t. the first fundamental form $I =: g$, the second fundamental form II and the Weingarten operator S, resp., have the following matrix representations:

$$g : \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}, \tag{2.1}$$

$$II : \begin{pmatrix} k_1 g_{11} & 0 \\ 0 & k_2 g_{22} \end{pmatrix}, \tag{2.2}$$

$$S : \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}. \tag{2.3}$$

The Christoffel symbols of the Levi-Civita connection of the metric $g = I$ satisfy:

$$2\Gamma_{11}^1 = \partial_1 \ln g_{11}, \qquad 2\Gamma_{12}^1 = \partial_2 \ln g_{11}, \qquad 2\Gamma_{22}^1 = -\frac{1}{g_{11}}\partial_1 g_{22},$$

$$2\Gamma_{11}^2 = -\frac{1}{g_{22}}\partial_2 g_{11}, \quad 2\Gamma_{12}^2 = \partial_1 \ln g_{22}, \qquad 2\Gamma_{22}^2 = \partial_2 \ln g_{22}. \tag{2.4}$$

We calculate the covariant derivative ∇II of the second fundamental form II in terms of the Levi-Civita connection:

$$\nabla_k II_{ij} = \partial_k II_{ij} - \Gamma^r_{ik} II_{rj} - \Gamma^r_{jk} II_{ri}$$

and get:

$$\nabla_1 II_{11} = \partial_1 k_1 g_{11}, \quad \nabla_2 II_{11} = \partial_2 k_1 g_{11}, \quad \nabla_1 II_{22} = \partial_1 k_2 g_{22}, \quad \nabla_2 II_{22} = \partial_2 k_2 g_{22}.$$

The support function. Define the Euclidean support function (with respect to the origin) of a surface $x : M \to \mathbb{R}^3$ with Euclidean normal μ by $\rho := < \mu, -x >$; here x also denotes the position vector of the surface with respect to the origin.

3. The cubic form of a non-degenerate hypersurface. We are going to apply known results about the characterization of quadrics. For a better understanding we recall the affine context (a reader who is familiar with this theory can pass to Lemma 3.5). For details on relative hypersurface theory we refer to standard monographs like [13], [7], [9].

It is a classical result in the unimodular-affine hypersurface theory that quadrics can be characterized by the vanishing of the cubic form; in this theory the cubic form is traceless w.r.t. the Blaschke metric (apolarity condition). Within the class of all relative normalizations of a non-degenerate hypersurface the unimodular normalization can be characterized by this apolarity condition; thus, if one chooses a transverse field as "normal" field, different from the unimodular-affine normal, the associated cubic form does not satisfy an apolarity condition. But its traceless part w.r.t. the associated relative metric has two very interesting properties which we are going to recall.

3.1. Non-degenerate hypersurfaces in affine space. The duality of the real affine space $\mathbb{R}^{n+1}$ and its dual $\mathbb{R}^{(n+1)*}$ is described in terms of a non-degenerate scalar product

$$< \quad , \quad >: \mathbb{R}^{(n+1)*} \times \mathbb{R}^{n+1} \to \mathbb{R}.$$

By the same symbol $\overline{\nabla}$ we denote the canonical flat connections on $\mathbb{R}^{n+1}$ and $\mathbb{R}^{(n+1)*}$, resp.

Let M be a connected, oriented, differentiable manifold of dimension $n \geq 2$, and $x : M \to \mathbb{R}^{n+1}$ a hypersurface immersion. A *normalization* of x is a pair (Y, z) with $< Y, z >= 1$ where $z : M \to \mathbb{R}^{n+1}$ is a *transversal field* and $Y : M \to \mathbb{R}^{(n+1)*}$, satisfying $< Y, dz(v) >= 0$ for all tangent vectors v on M, is a *conormal field* of x. While a transversal field z extends a tangential basis to the ambient space, a conormal fixes the tangent plane. A *normalized hypersurface* is a triple (x, Y, z).

3.2. Structure equations. The geometry ot the triple (x, Y, z) can be described in terms of invariants defined via the *structure equations* of *Gauß* and *Weingarten*, resp.:

$$\begin{aligned} \overline{\nabla}_v dx(w) &= dx(\nabla_v w) \quad + h(v, w)z, \qquad (Gauß) \\ dz(v) &= dx(-S(v)) \ \ + \tau(v)z. \qquad (Weingarten) \end{aligned}$$

Here and in the following $u, v, w, ...$ denote tangent vectors and fields, resp. The *induced connection* ∇ is torsion free, h is bilinear and symmetric, S is the *shape* or

Weingarten operator and τ is a 1-form, the *connection form*; the sign in front of S in the Weingarten equation is a convention corresponding to an appropriate choice of the orientation of z. A normalization is called *relative* if $\tau = 0$ on M. All coefficients in the structure equations depend on the normalization, they are invariant under the affine group of transformations in $\mathbb{R}^{n+1}$. But it is well known that one can define affine invariants that are *independent of the choice of the normalization.* We call such affine invariants *gauge invariants*. We justified this terminology in [12]; there we present a study of gauge invariant structures.

A hypersurface is called *non-degenerate* if $rank\ \ h = dim M = n$. In this case h defines a *semi-Riemannian metric*. For a metric h we denote its Levi-Civita connection by $\nabla(h)$ and its associated volume form by $\omega(h)$.

It is well known that the class of all metrics, induced from all possible normalizations, is a conformal class, and the Euclidean second fundamental form II, induced from the Euclidean normalization, is sitting in this class. Thus a hypersurface is non-degenerate if and only if the Euclidean second fundamental form has maximal rank, that is the Euclidean Gauß-Kronecker curvature is nowhere zero. Obviously all conformal invariants like the conformal curvature tensor are gauge invariants.

The non-degeneracy of x is equivalent to the fact that any conormal field Y itself is an immersion $Y : M \to \mathbb{R}^{(n+1)*}$ with transversal position vector Y. The associated *Gauß structure equation* reads

$$\overline{\nabla}_v dY(w) = dY(\nabla^*_v w) + \frac{1}{n-1} Ric^*(v,w)(-Y)$$

where the *conormal connection* ∇^* is torsion free and *Ricci-symmetric*, i.e. its Ricci tensor Ric^* is symmetric. The Ricci symmetry is equivalent to the existence of a ∇^*-parallel volume form ω^* on M which is unique modulo a non-zero constant factor. It is well known that all conormal connections are projectively related, thus they define a projective class. This class is projectively flat. Obviously all projective invariants like the projective curvature tensor are gauge invariants.

3.3. The cubic form and the Tchebychev field. Consider the difference (1.2)-tensor field

$$C := \nabla(h) - \nabla^*.$$

As both connections are torsion free C is symmetric. Its associated *cubic form* $C^\flat$ is defined by $C^\flat(u,v,w) := h(u, C(v,w))$; it is totally symmetric and satisfies $\nabla^* h = 2C^\flat = -\nabla h$.

The trace of C is a closed one-form, defined by

$$nT^\flat(v) := trace(w \mapsto C(v,w));$$

it satisfies

$$nT^\flat = d \ln \frac{\omega(h)(v_1, ..., v_n)}{\omega^*(v_1, ..., v_n)}$$

on any local frame where the volume forms have the same orientation. The associated Tchebychev field T is implicitly defined via $h(v,T) := T^\flat(v)$. From the foregoing a geometric interpretation of $T^\flat$ and C is obvious: $T^\flat$ measures the deviation of volume forms, while C measures the deviation of the connections ∇^* and $\nabla(h)$.

3.4. The traceless cubic form. Define the symmetric (1.2) tensor $\widetilde{C}$ as traceless part of C (see [11], [8], [13], [12]):

$$\widetilde{C}(v,w) := C(v,w) - \frac{n}{n+2}(T^\flat(v)w + T^\flat(w)v + h(v,w)T);$$

then

(i) $\widetilde{C}$ is a gauge invariant, that means it is independent of the choice of the normalization;

(ii) the vanishing of $\widetilde{C}$ characterizes non-degenerate quadrics.

3.5. The traceless cubic form in terms of a Euclidean normalization. We use the notational mark (E) for Euclidean invariants. In section 6.1 of [13] we calculated the foregoing invariants in terms of a Euclidean normalization. In this case the induced connection coincides with the Levi-Civita connection of the Euclidean first fundamental form I: $\nabla = \nabla(I)$. We have:

(i) $2C^\flat(E)(u,v,w) = -(\nabla_u II)(v,w) = -(\nabla II)(u,v,w)$;

(ii)

$$nT^\flat(E) = d\ln\frac{\omega(II)(v_1,...,v_n)}{\omega(III)(v_1,...,v_n)}$$

where $\omega(II)$ and $\omega(III)$ denote the volume forms associated to the (semi-)Riemannian metrics II and III of a non-degenerate hypersurface; thus

$$nT^\flat(E) = -\frac{1}{2}d\ln|K|;$$

here $K := \det S(E)$ denotes the Euclidean Gauß-Kronecker curvature.

LEMMA 3.5.1 ([13], section 6.1). *Consider a non-degenerate hypersurface.*

(i) In terms of a Euclidean normalization, the traceless cubic form reads

$$2\widetilde{C}^\flat(E)(u,v,w) = -(\nabla II)(u,v,w) + $$
$$+\frac{1}{n+2}\{(d\ln|K|)(u)II(v,w) + (d\ln|K|)(v)II(w,u) + (d\ln|K|)(w)II(u,v)\}.$$

(ii) The vanishing of $\widetilde{C}^\flat(E)$ characterizes non-degenerate quadrics.

(iii) In dimension $n = 2$ and in terms of curvature line parameters, the coefficients of the traceless cubic form satisfy:

$$\widetilde{C^\flat}(E)_{111} = \frac{1}{8}k_1g_{11}[\partial_1\ln\frac{k_2^3}{k_1}]; \qquad \widetilde{C^\flat}(E)_{112} = \frac{1}{8}k_1g_{11}[\partial_2\ln\frac{k_2}{k_1^3}];$$

$$\widetilde{C^\flat}(E)_{122} = \frac{1}{8}k_2g_{22}[\partial_1\ln\frac{k_1}{k_2^3}]; \qquad \widetilde{C^\flat}(E)_{222} = \frac{1}{8}k_2g_{22}[\partial_2\ln\frac{k_1^3}{k_2}].$$

4. Surfaces of revolution in Euclidean 3-space. We summarize well known properties of surfaces of revolution which we will need for our discussion.

Consider a surface of revolution in Euclidean space E^3 given in terms of parameters (u^1, u^2) by

$$x(u^1,u^2) = (r(u^1)\cos u^2, r(u^1)\sin u^2, h(u^1)), \quad r \geq 0, \tag{4.1}$$

where u^1 parametrizes the meridians as arc length parameter and u^2 parametrizes the parallels of latitude with radius $r(u^1)$; r and h are differentiable functions.
For a function $f = f(u^1)$ we write $f' := df/du^1$; then we have

$$r'(u^1)^2 + h'(u^1)^2 = 1. \tag{4.2}$$

u^1, u^2 are curvature line parameters for all points with $r(u^1) > 0$. In case that the functions $r(u^1) > 0$, $h(u^1)$ are defined for $0 < u^1 < \Lambda$ and also for $u^1 = 0$ or $u^1 = \Lambda$ such that

$$r(0) = 0,\ r'(0) = 1 \quad \text{or} \quad r(\Lambda) = 0,\ r'(\Lambda) = -1, \tag{4.3}$$

we call such points "poles" P_S or P_N ($u^1 = 0$ or $u^1 = \Lambda$); for symmetry reasons they are umbilics. Near a pole, $x^1 = r(u^1)\cos u^2$ and $x^2 = r(u^1)\sin u^2$ are parameters for the surface. The parameter u^2, first of all, will be taken modulo 2π, i.e. $u^2 \in S^1$; if necessary one will have to pass to a covering surface by taking $u^2 \in \mathbb{R}$. In every case, according to the domain of the parameters, there is a manifold M of dimension two such that (4.1) defines an immersion $x : M \to E^3$ into Euclidean 3-space E^3.
It follows by straightforward computations that we have the following matrix representations for the first fundamental form $I = g$, the second fundamental form II, and the Weingarten (shape) operator S, resp., on $M\backslash\{P_N, P_S\}$; as already stated, k_1, k_2 denote the principal curvatures.

$$g : \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \tag{4.4}$$

$$II : \begin{pmatrix} k_1 & 0 \\ 0 & k_2 r^2 \end{pmatrix}, \tag{4.5}$$

$$S : \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}. \tag{4.6}$$

We calculate the principal curvatures in terms of the functions r and h :

$$S\partial_1 = k_1\partial_1 = (r'h'' - r''h')\partial_1 \tag{4.7}$$

and

$$S\partial_2 = k_2\partial_2 = \tfrac{h'}{r}\partial_2, \tag{4.8}$$

where $\{\partial_1, \partial_2\}$ denotes the Gauß basis associated to the local parameters. It is a particular consequence of the parametrization and of (4.7-8) that both principal curvatures k_1, k_2 only depend on the parameter u^1. The Codazzi equations reduce to the equation

$$k_2' = \frac{r'}{r}(k_1 - k_2), \tag{Cod}$$

while the Gauß integrability condition with $K = k_1 k_2$ as Gauß curvature reads

$$r'' + Kr = 0. \tag{$Gauß$}$$

The Christoffel symbols of the Levi-Civita connection of the metric satisfy:

$$\Gamma^1_{11} = 0, \qquad \Gamma^1_{12} = 0, \qquad \Gamma^1_{22} = -rr'$$

$$\Gamma^2_{11} = 0, \qquad \Gamma^1_{12} = (\ln r)', \quad \Gamma^1_{22} = 0. \tag{4.9}$$

We calculate the covariant derivative ∇II of the second fundamental form II in terms of the Levi-Civita connection $\nabla = \nabla(I)$

$$\nabla_k II_{ij} = \partial_k II_{ij} - \Gamma^r_{ik} II_{rj} - \Gamma^r_{jk} II_{ri}$$

and get:

$$\nabla_1 II_{11} = \partial_1 k_1, \quad \nabla_2 II_{11} = \partial_2 k_1 = 0, \quad \nabla_1 II_{22} = r^2 \partial_1 k_2, \quad \nabla_2 II_{22} = r^2 \partial_2 k_2 = 0. \tag{4.10}$$

The Tchebychev form reads

$$T_i = -\frac{1}{4} \partial_i \ln |K| \tag{4.11}$$

thus $T_1 = -(\ln \ |k_2|)'$ and $T_2 = 0$. Now the coefficients of $\widetilde{C^\flat}(E)$ read:

$$\widetilde{C^\flat}(E)_{111} = \frac{1}{8} k_1 g_{11} [\partial_1 \ln \frac{k_2^3}{k_1}]; \qquad \widetilde{C^\flat}(E)_{112} = 0;$$

$$\widetilde{C^\flat}(E)_{122} = \frac{1}{8} k_2 g_{22} [\partial_1 \ln \frac{k_1}{k_2^3}]; \qquad \widetilde{C^\flat}(E)_{222} = 0. \tag{4.12}$$

Here we can use (4.7-8) to express k_1, k_2 in terms of the functions r, h.

The support function. For a surface of revolution with representation (4.1) the support function satisfies the relation

$$\rho = rh' - r'h. \tag{4.13}$$

5. Examples: Rotational quadrics. We list relations between curvature functions and the support function $\rho =< \mu, -x >$ and we denote $r(u^1) = R \ \cos u^2$ and $h(u^1) = c \ \sin u^1$, where $R > 0$ and c are real constants. For central quadrics, we choose the origin as center, for the paraboloid we choose the origin as apex.

Ellipsoid.

$$k_1 = \frac{R^4}{c^2} k_2^3 \quad \text{and} \quad \rho = R^2 k_2.$$

Two-sheeted hyperboloid.

$$k_1 = \frac{R^4}{c^2} k_2^3 \quad \text{and} \quad \rho = -R^2 k_2.$$

One-sheeted hyperboloid.

$$k_1 = -\frac{R^4}{c^2} k_2^3 \quad \text{and} \quad \rho = R^2 k_2.$$

Elliptic paraboloid.

$$k_1 = \frac{R^4}{4} k_2^3 \quad \text{and} \quad \rho = \frac{1}{2} R^2 k_2.$$

6. Proofs.

Proof of Theorem I. According to the assumptions we consider a surface of revolution with the representation from section 4. The assertion immediately follows from Lemma 3.5.1 and (4.12).

Proof of Theorem III. We use the representation (4.1). As x is rotational we know that the principal curvatures only depend on the parameter u^1. This implies $\widetilde{C^\flat}(E)_{112} = 0$ and $\widetilde{C^\flat}(E)_{222} = 0$. Then x is a quadric if and only if the other two coefficients of $\widetilde{C^\flat}$ vanish, i.e. if and only if $k_1 = ck_2^3$ for some constant c.

Proof of Theorem IV. On a compact surface there is a point $p \in M$ where the Gauß curvature is positive. This together with the assumption $k_1 = ck_2^3$ implies that c is positive and thus $sign\ k_1 = sign\ k_2$. From this the Gauß curvature is non-negative on M and therefore the surface has genus zero.
For an umbilic $q \in M$ there are two possibilities:

$$k_1(q) = 0 = k_2(q) \quad \text{or} \quad k_1(q) = k_2(q) = c^{-\frac{1}{2}}.$$

The surface has genus zero thus from Voss' Theorem and the assumptions it is rotational and therefore the poles are umbilics. They are the only points that might be flat, and for $M \backslash \{P_N, P_S\}$ the surface is locally strictly convex and admits principal curvature parameters a.e. From Theorem III the surface is a quadric and thus an ellipsoid.

Proof of Theorem V. Again we use the representation (4.1) but drop the assumption that u^1 is an arc length parameter for the meridians. In analogy to section 4 one easily calculates the fundamental invariants. From the assumptions we have

$$Rk_2 = \rho \quad \text{or} \quad Rk_2 = -\rho.$$

These equations lead to the ODE

$$\pm R\frac{h}{r}(h'^2 + r'^2)^{-\frac{1}{2}} = \pm Rk_2 = \rho = (rh' - r'h)(h'^2 + r'^2)^{-\frac{1}{2}}.$$

(In fact, we arrive at the same ODE as in case that (4.2) yields). The ODE

$$-\frac{h'}{h} = \frac{1}{2}\frac{(r^2)'}{r^2 \pm R^2}$$

gives

$$\ln|h| = \ln\alpha^2 + \ln|r^2 \pm R^2|^{\frac{1}{2}}$$

for some non-zero real α. Depending on the sign of the arguments in the foregoing equation we have to discuss three different cases, given by

(i)

$$\frac{r^2}{R^2} + \frac{h^2}{\alpha^2 R^2} = 1$$

(ii-iii)

$$\frac{r^2}{R^2} - \frac{h^2}{\alpha^2 R^2} = \pm 1.$$

As solutions we get

(i) $r = R \ \cos u^1$ and $h = \alpha R \ \sin u^1$ (ellipsoid),
(ii) $r = R \ \cosh u^1$ and $h = \alpha R \ \sinh u^1$ (1-sheeted hyperboloid),
(iii) $r = R \ \sinh u^1$ and $h = \alpha R \ \cosh u^1$ (2-sheeted hyperboloid).

Note added in proof. Our Theorem IV was proved as Proposition 10 in the paper [5] cited below under the additional assumption that the analytic surface has genus zero.

REFERENCES

[1] S.S. Chern, *Some new characterizations of the Euclidean sphere*, Duke Math. J., 12 (1945), pp. 279–290.

[2] F. Dillen, T. Lusala, M. Scherfner, G. Verbouwe, *A characterization of quadrics by the principal curvature functions*, Arch. Math. (Basel), 81 (2003), pp. 342–347.

[3] H. Huck, R. Roitzsch, U. Simon, W. Vortisch, R. Walden, B. Wegner, W. Wendland, *Beweismethoden der Differentialgeometrie im Grossen*, Lecture Notes Math., 335, Springer, Berlin-New York, 1973.

[4] W. Kühnel, *Differentialgeometrie. Kurven, Flächen, Mannigfaltigkeiten*, 2nd. ed., Vieweg, Braunschweig, 2003.

[5] W. Kühnel, M. Steller, *On closed Weingarten surfaces*, Monatshefte Math., 146 (2005), pp. 113–126.

[6] J. Leder, A. Schwenk-Schellschmidt, U. Simon, M. Wiehe, *Generating higher order Codazzi tensors by functions,* Geometry and Topology of Submanifolds, IX, (Conf. Proc. 1997), pp. 174–191, World Sci. Publishing, River Edge, NJ, 1999.

[7] A. M. Li, U. Simon, G. Zhao, *Global affine differential geometry of hypersurfaces,* De Gruyter Expositions Mathematics, 11, Berlin 1993.

[8] K. Nomizu, U. Pinkall, *On the geometry of affine immersions*, Math. Z., 195 (1987), pp. 165–178.

[9] K. Nomizu, T. Sasaki, *Affine differential geometry,* Cambridge Tracts Math., 111, 1994.

[10] R. Schoen, S.T. Yau, *Lectures on differential geometry*, Conf. Proc. and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994. ISBN: 1-57146-012-8.

[11] U. Simon, *Zur Relativgeometrie: Symmetrische Zusammenhänge auf Hyperflächen*, Math. Z., 106 (1968), pp. 36–46.

[12] U. Simon, *Affine Hypersurface Theory Revisited: Gauge Invariant Structures* (in English), Russian Mathematics (Izv. vuz), 48 (2004), pp. 48–73.

[13] U. Simon, A. Schwenk-Schellschmidt, H. Viesel, *Introduction to the affine differential geometry of hypersurfaces*, Lecture Notes Science University Tokyo (SUT), 1991. ISBN: 3-7983-1529-9.

[14] U. Simon, L. Vrancken, C.P. Wang, M. Wiehe (organizers), *Workshop on PDEs and Submanifolds,* FB Math. TU Berlin, 2001. `http://www.math.tu-berlin.de/geometrie/adg/conf/workshop01.html`

[15] K. Voss, *Über geschlossene Weingartensche Flächen*, Math. Ann., 138 (1959), pp. 42–54.