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A CAVEAT ON THE CONVERGENCE OF THE RICCI FLOW FOR PINCHED NEGATIVELY CURVED MANIFOLDS*

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In his seminal paper [5], Hamilton initiated the Ricci flow method for finding Einstein metrics on a closed smooth n-dimensional manifold M^n starting with an arbitrary smooth Riemannian metric h on M^n. He considered the evolution equation

$$\frac{\partial}{\partial t} h = \frac{2}{n} r\, h \,-\, Ric$$

where $r = \int R\, d\mu / \int d\mu$ is the average scalar curvature (R is the scalar curvature) and Ric is the Ricci curvature tensor of h. Hamilton then spectacularly illustrated the success of this method by proving, when $n = 3$, that if the initial Riemannian metric has strictly positive Ricci curvature it evolves through time to a positively curved Einstein metric h_∞ on M^3. And, because $n = 3$, such a Riemannian metric automatically has constant sectional curvature; hence (M^3, h_∞) is a spherical space-form; i.e. its universal cover is the round sphere. Following Hamilton's approach G. Huisken [6], C. Margerin [7] and S. Nishikawa [9] proved that, for every n, sufficiently pinched to 1 n-manifolds (the pinching constant depending only on the dimension) can be deformed, through the Ricci flow, to a spherical-space form.

Ten years later R. Ye [10] studied the Ricci flow when the initial Riemannian metric h is negatively curved and proved that a negatively curved Einstein metric is strongly stable; that is, the Ricci flow starting near such a Riemannian metric h converges (in the C^∞ topology) to a Riemannian metric isometric to h, up to scaling. (We introduce the notation $h \equiv h'$ for two Riemannian metrics that are isometric up to scaling.) In [10] R. Ye also proved that sufficiently pinched to -1 manifolds can be deformed, through the Ricci flow, to hyperbolic manifolds, but the pinching constant in his theorem depends on other quantities (e.g the diameter or the volume). Ye's paper was motivated by the problem on whether the Ricci flow can be used to deform every sufficiently pinched to -1 Riemannian metric to an Einstein metric (the pinching constant depending only on the dimension). We would also like to mention the paper of Min-Oo about deforming almost Einstein metrics of negative scalar curvature to Einstein metrics [8].

In this short note we show that our previous results [4] imply the existence of pinched negatively curved metrics for which the Ricci flow does not converge in the C^2 topology (hence in the C^k topology, $2 \le k \le \infty$) to a negatively curved Einstein

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metric.

It is a consequence of Ye's paper [10] that the Ricci flow starting at some h converges (in the C^∞ topology) to a negatively curved Einstein metric if and only if the Ricci flow, starting at h, eventually gets into a stable neighborhood of some negatively curved Einstein metric h_∞. By a stable neighborhood of h_∞ we mean a neighborhood for which any Ricci flow starting there converges to a metric isometric (up to scaling) to h_∞.

Remark. The stable neighborhoods above can be taken as (sufficiently small) open sets just in the C^2 topology. It follows that the Ricci flow converges in the C^2 topology to a negatively curved Einstein metric if and only if it converges in the C^∞ topology to a negatively curved Einstein metric.

Theorem. *Given $n > 10$ and $\epsilon > 0$ there is a closed smooth n-dimensional manifold N such that*

(i) N admits a hyperbolic metric

(ii) N admits a Riemannian metric h with sectional curvatures in $[-1-\epsilon, -1+\epsilon]$ for which the Ricci flow does not converge in the C^2 topology (hence in the C^k topology, $2 \leq k \leq \infty$) to a negatively curved Einstein metric.

Proof. Let $n > 10$ and $\epsilon > 0$. From [4] we have the following.

There are closed smooth manifolds M_0, M_1, N, of dimension n, Riemannian metrics g_0, g_1 on M_0 and M_1, respectively, and smooth finite covers $p_0 : N \to M_0$, $p_1 : N \to M_1$ such that:

(1) M_0 and M_1 are homeomorphic but not PL-homeomorphic.

(2) g_0 is hyperbolic

(3) g_1 has sectional curvatures in $[-1-\epsilon, -1+\epsilon]$.

(4) There is a C^∞ family of C^∞ Riemannian metrics h_s on N, $0 \leq s \leq 1$, with $h_0 = p_0^* g_0$, and $h_1 = p_1^* g_1$, such that every h_s has sectional curvatures in $[-1-\epsilon, -1+\epsilon]$.

Note that h_0 is also hyperbolic. Now, since the Ricci flow preserves isometries (see [5]) we have that if the Ricci flow for g_1 does not converge in the C^2 topology to a negatively cuved Einstein metric, then the same happens to the Ricci flow for $h_1 = p_1^* g_1$, and we are done. Hence we assume that the Ricci flow for g_1 converges in the C^∞ topology (see remark above) to a negatively cuved Einstein metric. Let $g_{1,t}$ be the Ricci flow starting at $g_{1,0} = g_1$, $0 \leq t < \infty$, converging to the negatively curved Einstein metric $g_{1,\infty}$. Note that, by Mostow' Rigidity Theorem and (1) above, g_1 and $g_{1,\infty}$ are non-hyperbolic. It follows that $p_1^* g_1$ and $p_1^* g_{1,\infty}$ are also non-hyperbolic.

If the Ricci flow does not converge in the C^2 topology to a negatively cuved Einstein metric for some h_s, we are done. So, let us assume that the Ricci flow converges in the C^∞ topology to a negatively cuved Einstein metric, for all h_s. We will show a contradiction. Write $h_{s,t}$, for the Ricci flow starting at $h_{s,0} = h_s$, $0 \leq t < \infty$, converging to the negatively curved Einstein metric $h_{s,\infty}$. Then, from the form of the evolution equation we have that $h_{1,t} = p_1^* g_{1,\alpha t}$, for some constant $\alpha > 0$, and for all $0 \leq t \leq \infty$.

Claim. *$(s,t) \mapsto h_{s,t}$ is continuous for $0 \leq s \leq 1$, $0 \leq t < \infty$, where we consider the space of Riemannian metrics with the C^∞ topology.*

To prove the claim we have to prove that the Ricci flow depends continuously on the initial conditions. One way of doing this directly is by using Hamilton's proof of the local-in-time existence and uniqueness of the Ricci flow (see [5]). Let f_0, $\bar{f}$, $\bar{h}$ be as in the the proof of theorem 5.1 of [5], p.263. Let f_0' be another initial condition. If f_0' is close to f_0 (in the C^∞ topology) then we can find a $\bar{f}'$ close to $\bar{f}$ ($\bar{f}'$ with the same properties as $\bar{f}$, but with respect to f_0'). Then $\bar{h}'$ is close to $\bar{h}$, where $\bar{h}'$ is defined in a similar way as $\bar{h}$. Since the inverse function is continuous, it follows that f' and f are close, where f' and f are the inverses of some h' and h (which are chosen close to $\bar{h}'$ and $\bar{h}$ and vanishing on some small interval $[0, \epsilon)$. This proves the claim.

Since h_0 is hyperbolic we have that $h_{0,t} = h_0$ for all $0 \leq t \leq \infty$. Since every negatively curved Einstein metric is stable (see [1], p.357) we can assume that all negatively curved Einstein metrics $h_{s,\infty}$ have neighborhoods V_s for which any Ricci flow starting in V_s converges to a metric isometric (up to scaling) to $h_{s,\infty}$ (see [10], p.873) (in particular, $h_{s,\infty} \equiv h_0$, for sufficiently small s). It follows that every $s \in [0,1]$ has an open neighborhood I_s such that $h_{s',\infty} \equiv h_{s,\infty}$ for all $s' \in I_s$. Then the map $s \mapsto [h_{s,\infty}]$ from [0,1] to $\mathcal{M}_N/\equiv$ is locally constant and hence continuous. (Here $[h_{s,\infty}]$ denotes the equivalence class of $h_{s,\infty}$ in the quotient space $\mathcal{M}_N/\equiv$ of all isometry classes of Riemannian metrics on N.) This is a contradiction because $h_{0,\infty} = h_0$ is hyperbolic and $h_{1,\infty}$ is not hyperbolic. This proves the theorem.

Recall that $\mathcal{M}_P$ denotes the space of all Riemannian metrics on a smooth manifold P. For $\epsilon > 0$, let $\mathcal{M}_P^\epsilon$ denote the space of ϵ-pinched to -1 Riemannian metrics on P. Also, $\mathcal{E}_P \subset \mathcal{M}_P$ will denote the space of negatively curved Einstein metrics on P. Recall that $\mathcal{E}_P/\equiv$ is discrete, see [1], p.357.

Definition. Let $\epsilon > 0$ and n be a positive integer. A *negatively curved Einstein correspondence* $\Phi : \mathcal{M}^\epsilon \to \mathcal{E}$ *for n-dimensional manifolds* is a family of maps $\Phi_P : \mathcal{M}_P^\epsilon \to \mathcal{E}_P$, for every n-dimensional manifold P for which $\mathcal{M}_P^\epsilon$ is not empty. For the sake of brevity we will call Φ simply an *Einstein Correspondence*. We say that Φ is *cover-invariant* if $\Phi(p^*g) = p^*(\Phi(g))$ for every finite cover $p : P \to Q$ and $g \in \mathcal{M}_Q^\epsilon$, for which Φ_Q is defined.

We say that Φ is *continuous* if each $\Phi_P : \mathcal{M}_P^\epsilon \to \mathcal{E}_P$ is continuous. Here we consider $\mathcal{M}_P^\epsilon$ with the C^∞ topology and $\mathcal{E}_P$ with the C^2 topology.

Let h, $h' \in \mathcal{M}_P$. Write $h \equiv_0 h'$ provided (P,h) is isometric to (P,h'), up to scaling, via an isometry homotopic to id_P. Notice that the fibers of $\mathcal{E}_P/\equiv_0 \to \mathcal{E}_P/\equiv$ are discrete; and hence $\mathcal{E}_P/\equiv_0$ is also discrete.

The following corollary is a direct consequence of the Theorem above.

Corollary 1. *Suppose that there are $\epsilon > 0$ and $n > 10$ for which there exists* *a cover-invariant Einstein correspondence* Φ. *Then there is a closed* n*-dimensional Riemannian manifold* N, *with metric* $h \in \mathcal{M}_N^\epsilon$, *for which the Einstein metric* $\Phi(h)$ *is unreachable by the Ricci flow starting at* h.

COROLLARY 2. *Suppose that there are* $\epsilon > 0$ *and* $n \geq 6$ *for which there exists an Einstein correspondence* Φ. *Then there is a closed* n*-dimensional manifold* N *that admits, at least, two non-isometric negatively curved Einstein metrics. Moreover, one metric can be chosen to be hyperbolic.*

Proof. From [3] we have the following.

There are closed connected smooth manifolds M_0, M_1, N, of dimension n, Riemannian metrics g_0, g_1 on M_0 and M_1, respectively, and smooth two-sheeted covers $p_0 : N \to M_0$, $p_1 : N \to M_1$ such that:

(1) M_0 and M_1 are homeomorphic but not PL-homeomorphic.
(2) g_0 is hyperbolic
(3) g_1 has sectional curvatures in $[-1-\epsilon, -1+\epsilon]$.

Then the two non-isometric negatively curved Einstein metrics on N are $p_0^*(g_0)$ and $p_1^*(\Phi(g_1))$. This proves the corollary.

A comment of Rugang Ye motivated the following corollary.

COROLLARY 3. *A cover-invariant Einstein correspondence cannot be continuous.*

REMARK. Note that we are not assuming that Φ fixes hyperbolic metrics. If we assumed that Φ(*hyperbolic metric*) = (*hyperbolic metric*), the proof of the corollary would be much easier.

Proof. We use the notation from the proof of the Theorem. Let us suppose that there exists a continuous cover-invariant Einstein correspondence. We will show a contradiction.

Let $G_i \subset Diff(N)$, be (finite) subgroups of the group $Diff(N)$, of all self-diffeomorphisms of N, such that $N/G_i = M_i$, $i = 0, 1$. Note that $G_i \subset Iso(N, \Phi(h_i))$, since $\Phi(h_i) = p_i^*(\Phi(g_i))$, where $Iso(N, \Phi(h_i)) \subset Diff(N)$ is the subgroup consisting of all isometries of the negatively curved Einstein manifold $(N, \Phi(h_i))$. Let $Top(N)$ and $Out(\pi_1 N)$ denote the group of all self-homeomorphisms of N and the group of outer automorphisms of $\pi_1(N)$, respectively. Recall that $Out(\pi_1 N)$ can be identified with $\pi_0(E(N))$, where $E(N)$ is the H-space consisting of all self-homotopy equivalences of N. (This is because N is aspherical.) We have the following diagram of group homomorphisms

$$Diff(N) \xrightarrow{\alpha} Top(N) \xrightarrow{\beta} Out(\pi_1 N)$$

where α is the inclusion, and β is the composition of the inclusion $Top(N) \to E(N)$ and the quotient map $E(N) \to \pi_0(E(N))$. Write $\gamma = \beta\alpha$.

It was shown in [3], [4], that G_0 and G_1 are conjugate in $Top\, N$, via a homeomorphism homotopic to id_N; hence $\gamma G_0 = \gamma G_1$.

Now, since Φ continuous and $\mathcal{E}_N/\equiv_0$ is discrete, the composition

$$[0,1] \stackrel{h_t}{\to} \mathcal{M}_N^{\epsilon} \stackrel{\Phi}{\to} \mathcal{E}_N \to \mathcal{E}_N/\equiv_0$$

must be constant; hence $\Phi(h_0) \equiv_0 \Phi(h_1)$. It follows that G_1 is conjugate in $Diff\,(N)$ to a subgroup of $Iso\,(N, \Phi(h_0))$ via a diffeomorphism f homotopic to id_N; i.e. $f^{-1}G_1f \subset Iso\,(N, \Phi(h_0))$. Note that $\gamma(f^{-1}G_1f) = \gamma(G_1)$ since $f \sim id_N$; hence $\gamma(f^{-1}G_1f) = \gamma(G_0)$. This implies that $f^{-1}G_1f = G_0$ since both $f^{-1}G_1f$ and G_0 are subgroups of $Iso\,(N, \Phi(h_0))$ and Borel-Conner-Raymond showed (see [2], p.43) that γ restricted to compact subgroups of $Diff\,(N)$ is monic. (Recall that N is aspherical and the center of $\pi_1(N)$ is trivial.) It follows that f induces a diffeomorphism between $M_0 = N/G_0$ and $M_1 = N/G_1$, which is a contradiction. This proves the corollary.

The following is also a corollary of the proof of the Theorem above.

Corollary 4. *Given $n > 10$ and $\epsilon > 0$ there is a closed n-dimensional manifold N that admits a hyperbolic metric h_0 and a Riemannian metric h_a, with sectional curvatures in $[-1-\epsilon, -1+\epsilon]$, that satisfies the following. Either the Ricci flow for h_a does not converge (in the C^2 topology) or N supports a non-stable (hence not negatively curved) Einstein metric $\tilde{h}$ satisfying:*
(i) There is a C^∞ family of C^∞ Riemannian metrics h_s on N, $0 \le s \le a$, such that every h_s has sectional curvatures in $[-1-\epsilon, -1+\epsilon]$ and:
(a) the Ricci flow, starting at h_s, converges in the C^∞ topology to a metric isometric to the hyperbolic metric h_0 provided $0 \le s < a$.
(b) the Ricci flow, starting at h_a, converges in the C^∞ topology to the non-stable not negatively curved Einstein metric $\tilde{h}$.
(ii) There is a sequence of metrics h_n converging (in the C^∞topology) to $\tilde{h}$ such the the Ricci flow, starting at each h_n, converges to a metric isometric to the hyperbolic metric h_0.

Proof. We use all notation from the proof of the Theorem. As before, if the Ricci flow for g_1 does not converge in the C^2 topology, we are done. Let us assume then that the Ricci flow for g_1 and all h_s converges (in the C^∞ topology). Let

$$a \,=\, sup\,\{\, s \in [0,1] \,:\, h_{s',\infty} = h_0,\ s' \in [0,s]\,\}.$$

Then $0 < a \le 1$. It follows that $h_{a,\infty}$ is a non-stable (hence not negatively curved) Einstein metric. Take $\tilde{h} = h_{a,\infty}$. This proves part *(i)*.

To prove *(ii)*, note that we can choose a sequence s_n with $s_n \to a$, $s_n < a$, such that $h_n = h_{s_n,n} \to \tilde{h}$. Then the Ricci flow, starting at h_n, converges to $h_{s_n,\infty} \equiv h_0$. This proves the corollary.

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