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BERSTEIN TYPE THEOREMS WITHOUT GRAPHIC CONDITION*

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Abstract. We prove Bernstein type theorems for minimal n-submanifolds in $\mathbb{R}^{n+p}$ with flat normal bundle under certain growth conditions for $p \geq 2$ and $n \leq 5$, as well as for arbitrary n and $p = 1$. When M is a graphic minimal hypersurface we recover the known result.

Key words. minimal submanifold, flat normal bundle

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1. Introduction. There are various generalizations of the Bernstein theorem . For an entire minimal graph $M = (x, f(x))$ of dimension $n \leq 7$ in $\mathbb{R}^{n+1}$ the problem has been settled in [Si], [B-G-G]. If there is no dimension limitation, we have results in [M], [C-N-S], [Ni] and [E-H] under the growth condition on the function f.

For general minimal hypersurfaces, a natural condition is stable, or even ones with finite index. In this situation [C-S-Z] and [L-W] conclude that they have only one end or finitely many ends, respectively.

Higher codimensional Bernstein problem becomes more complicated. In [H-J-W] Moser's result has been generalized to minimal graphs of higher codimension. Recently, we obtained better results under a bound for the slope of the vector-valued functions f which is independent of the dimension and codimension [J-X1] [J-X2](also see [Wm]. On the other hand, the counter example of [L-O] prevents us from going further.

For general minimal surfaces in a Euclidean space, so-called parametric case, the Bernstein theorem was generalized to the beautiful theory of the value distribution of its Gauss image [Os] [Xa] [F].

In the present article we study minimal n-submanifolds of $\mathbb{R}^{n+p}$, $p \geq 2$, which are not necessary graphs. Normal bundle of the submanifold plays an important role in this situation. In the first step, it is natural to study the case when normal bundle is flat. One expects minimal submanifolds with flat normal bundle share similar properties with hypersurfaces in certain sense. As done in a series of papers [T1], [T2], [T3] and [H-P-T], Terng established a theory of isoparametric submanifolds in the frame work of flat normal bundle.

We define a w-function on M in §3. It is closely related to its generalized Gauss image of M in $\mathbb{R}^{n+p}$. It satisfies a nice formula in the case when M has parallel mean curvature with flat normal bundle. If w-function is always positive, its inverse $v = \frac{1}{w}$ is the volume element locally (globally when M is a graph). It is natural to consider its growth condition, instead of the growth condition on the function f for the minimal graph $(x, f(x))$ in $\mathbb{R}^{n+1}$, where $x \in \mathbb{R}^n$.

In section 2 we list known Bochner formula for the squared norm of the second fundamental form. The technique in [S-S-Y] can be applied to obtain a Kato type inequality in our case.

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In section 4 we derive volume growth of minimal submanifolds in a Cartan-Hadamard manifold by an elementary way. For the ambient Euclidean space the fact is well-known.

In the final section we prove our main results for a complete minimal submanifold M with flat normal bundle and positive w-function. First of all, it has only one end. Furthermore, under certain growth condition for v-function and polynomial volume growth we conclude that M is flat, if dimension $n \le 5$, and codimension $p \ge 2$. In the case when M is minimal graph, the conclusion follows only from the growth condition for v-function.

In the case when the codimension $p = 1$, the normal bundle of the minimal hypersurface is flat automatically. Our argument can also be carried out similarly.

By the Bochner formula for the squared norm of the second fundamental form, a Kato type inequality and the formula for the v-function, we obtain nonnegative subharmonic functions on minimal submanifolds. From the heat kernel estimate in [C-L-Y] we have mean-value inequality for subharmonic functions. Then some more integral estimates lead to our desired conclusion.

2. A Bochner-Type Formula. Let $M \to \bar{M}$ be an isometric immersion with the second fundamental form B, which can be viewed as a cross-section of the vector bundle $\mathrm{Hom}(\odot^2 TM, NM)$ over M, where TM and NM denote the tangent bundle and the normal bundle along M, respectively. A connection on $\mathrm{Hom}(\odot^2 TM, NM)$ can be induced from those of TM and NM naturally.

For $\nu \in \Gamma(NM)$ the shape operator $A^\nu : TM \to TM$ satisfies

$$\langle B_{XY}, \nu \rangle = \langle A^\nu(X), Y \rangle .$$

The second fundamental form, curvature tensors of the submanifold, curvature tensor of the normal bundle and that of the ambient manifold satisfy the Gauss equations, the Codazzi equations and the Ricci equations as follows.

$$\langle R_{XY} Z, W \rangle = \left\langle \bar{R}_{XY} Z, W \right\rangle - \langle B_{XW}, B_{YZ} \rangle + \langle B_{XZ}, B_{YW} \rangle , \tag{2.1}$$

$$(\nabla_X B)_{YZ} - (\nabla_Y B)_{XZ} = -(\bar{R}_{XY} Z)^N, \tag{2.2}$$

$$\langle R_{XY} \mu, \nu \rangle = \left\langle \bar{R}_{XY} \mu, \nu \right\rangle + \langle B_{X e_i}, \mu \rangle \langle B_{Y e_i}, \nu \rangle - \langle B_{X e_i}, \nu \rangle \langle B_{Y e_i}, \mu \rangle , \tag{2.3}$$

where $\{e_i\}$ is a local orthonormal frame field of M; X, Y and Z are tangent vector fields; μ, ν are normal vector fields in M. Here and in the sequel we use the summation convention.

There is the trace-Laplace operator ∇^2 acting on any cross-section of a Riemannian vector bundle E over M.

Now, we consider a minimal submanifold M of dimension n in Euclidean $(n+p)$-space $\mathbb{R}^{n+p}$. We have (see [Si])

$$\nabla^2 B = -\tilde{\mathcal{B}} - \underline{\mathcal{B}}. \tag{2.4}$$

We recall the following notations:

$$\tilde{\mathcal{B}} \overset{def.}{=} = B \circ B^t \circ B,$$

where B^t is the conjugate map of B,

$$\underline{\mathcal{B}}_{XY} \overset{def.}{=} \sum_{j=1}^{p} \left(B_{A^{\nu_j} A^{\nu_j}(X)\, Y} + B_{X\, A^{\nu_j} A^{\nu_j}(Y)} - 2\, B_{A^{\nu_j}(X)\, A^{\nu_j}(Y)} \right),$$

where ν_j are basis vectors of normal space. It is obvious that $\underline{\mathcal{B}}_{XY}$ is symmetric in X and Y, which is a cross-section of the bundle $\mathrm{Hom}(\odot^2 TM, NM)$. Simons also gave an estimate [Si]

$$\left\langle \tilde{\mathcal{B}} + \underline{\mathcal{B}}, B \right\rangle \le \left(2 - \frac{1}{p} \right) |B|^4.$$

It is optimal for the codimension 1.

In the case when $p \ge 2$, there is a refined estimate [L-L]

$$\left\langle \tilde{\mathcal{B}} + \underline{\mathcal{B}}, B \right\rangle \le \frac{3}{2} |B|^4.$$

Substituting it into (2.4) gives

$$\left\langle \nabla^2 B, B \right\rangle \ge -\frac{3}{2} |B|^4. \tag{2.5}$$

It follows that

$$\Delta |B|^2 \ge -3|B|^4 + 2|\nabla B|^2. \tag{2.6}$$

3. Submanifolds with Flat Normal Bundle. Let $M \to \mathbb{R}^{n+p}$ be an oriented submanifold. Around a point $x \in M$ choose the Darboux frame field $\{e_i,\, e_\alpha\}$, such that $e_i \in TM$, $e_\alpha \in NM$. We agree the following range of indices in the sequel

$$i,\ j,\ k, \cdots = 1, \cdots, n; \qquad \alpha,\ \beta,\ \gamma, \cdots = n+1, \cdots, n+p.$$

Set $B_{e_i e_j} = h_{\alpha ij} e_\alpha$ with $h_{\alpha ij} = h_{\alpha ji}$. We have

$$|B|^2 = \sum_{\alpha, i, j} h_{\alpha ij}^2,$$

$$(\nabla_{e_k} B)_{e_i e_j} = h_{\alpha ijk} e_\alpha.$$

By the Codazzi equations (2.2), $h_{\alpha ijk}$ is symmetric in all indices i, j, k.

If the curvature of the normal bundle vanishes, M is called a submanifold with flat normal bundle. By the Ricci equations (2.3), the coefficients of the second fundamental form $h_{\alpha ij}$ satisfy

$$h_{\alpha ij} h_{\beta ik} - h_{\beta ij} h_{\alpha ik} = 0, \tag{3.1}$$

which means that $p \quad (n \times n)$-matrices

$$h_{n+1\, ij}, \cdots, h_{n+p\, ij}$$

can be diagonalized simultaneously at a fixed point .

We need a Kato-type inequality in order to use the formula (2.6). Namely, we would estimate $|\nabla B|^2$ in terms of $|\nabla |B||^2$. Schoen-Simon-Yau [S-S-Y] did such an estimate for codimension $p = 1$. For any p with flat normal bundle their technique is also applicable.

$$
\begin{aligned}
|\nabla|B||^2 &= \left\langle \nabla_{e_k}\sqrt{\sum_{\alpha,i,j} h^2_{\alpha ij}}, \nabla_{e_k}\sqrt{\sum_{\beta,l,m} h^2_{\beta lm}} \right\rangle \\
&= \frac{1}{\sum_{\alpha,i,j} h^2_{\alpha ij}} \sum_k (\sum_{\beta l,m} h_{\beta lm} h_{\beta lmk})^2,
\end{aligned}
$$

$$
\begin{aligned}
|\nabla B|^2 - |\nabla|B||^2 &= \sum_{\alpha,i,j,k} h^2_{\alpha ijk} - \frac{1}{\sum_{\alpha,i,j} h^2_{\alpha ij}} \sum_k (\sum_{\beta,i,j} h_{\beta ij} h_{\beta ijk})^2 \\
&= \frac{1}{|B|^2}\left[\sum_{\beta,l,m} h^2_{\beta lm} \sum_{\alpha,i,j,k} h^2_{\alpha ijk} - \sum_k (\sum_{\alpha,i,j} h_{\alpha ij} h_{\alpha ijk})^2\right] \\
&= \frac{1}{2|B|^2} \sum_{\alpha,\beta,i,j,k,l,m} (h_{\alpha ij} h_{\beta lmk} - h_{\beta lm} h_{\alpha ijk})^2. \qquad (3.2)
\end{aligned}
$$

For any $x \in M$, we can choose a local frame field $\{e_1, \cdots, e_n\}$ around x so that $h_{\alpha ij} = \lambda_{\alpha i}\delta_{ij}$ at x, since normal bundle is flat. Hence, at the point x

$$
|B|^2 = \sum_{\alpha,i} h^2_{\alpha ii}.
$$

Then we have

$$
\begin{aligned}
\sum_{\alpha,\beta,i,j,k} (h_{\alpha ij} h_{\beta lmk} - h_{\beta lm} h_{\alpha ijk})^2 &= \sum_{\alpha,\beta,i,k,l,m} (h_{\alpha ii} h_{\beta lmk} - h_{\beta lm} h_{\alpha iik})^2 \\
&\quad + \sum_{\beta,l,m} h^2_{\beta lm} \sum_{i\neq j,k} h^2_{\alpha ijk} \\
&\geq \sum_{\alpha,\beta,i,k,l\neq m} h^2_{\alpha ii} h^2_{\beta lmk} + \sum_{\beta,l,m} h^2_{\beta lm} \sum_{\alpha,i\neq j,k} h^2_{\alpha ijk} \\
&= 2\,|B|^2 \sum_{i\neq j,k} h^2_{\alpha ijk}.
\end{aligned}
$$

Substituting it into (3.2) gives

$$
\begin{aligned}
|\nabla B|^2 - |\nabla|B||^2 &\geq \sum_{\alpha,i\neq j,k} h^2_{\alpha ijk} \\
&= \sum_{\alpha,i\neq j} h^2_{\alpha iji} + \sum_{\alpha,i\neq j} h^2_{\alpha ijj} + \sum_{\alpha,i\neq j,k\neq j,k\neq j} h^2_{\alpha ijk} \\
&\geq 2 \sum_{\alpha,i\neq j} h^2_{\alpha iij}. \qquad (3.3)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
|\nabla |B||^2 &= \frac{1}{|B|^2}\sum_k (\sum_{\alpha,i,j} h_{\alpha ij}h_{\alpha ijk})^2 \\
&= \frac{1}{|B|^2}\sum_k (\sum_{\alpha,i} h_{\alpha ii}h_{\alpha iik})^2 \\
&\le \frac{1}{|B|^2}\sum_k \left(\sum_{\alpha,i} h^2_{\alpha ii}\right)\left(\sum_{\beta,j} h^2_{\beta jjk}\right) \\
&= \sum_{\alpha,i,k} h^2_{\alpha iik} \\
&= \sum_{\alpha,i\ne k} h^2_{\alpha iik} + \sum_{\alpha,i} h^2_{\alpha iii} \\
&= \sum_{\alpha,i\ne k} h^2_{\alpha iik} + \sum_i (\sum_{\alpha,j\ne i} h_{\alpha jji})^2 \\
&\le \sum_{\alpha,i\ne k} h^2_{\alpha iik} + (n-1)\sum_{\alpha,j\ne i} h^2_{\alpha jji} = n\sum_{\alpha.i\ne j} h^2_{\alpha iij}.
\end{aligned}
$$

Substituting it into (3.3) gives

$$|\nabla B|^2 - |\nabla |B||^2 \ge \frac{2}{n}|\nabla |B||^2. \tag{3.4}$$

This is our desired Kato-type inequality. From (2.6) and (3.4) we have

Lemma 3.1. *Let $M \to \mathbb{R}^{n+p}$, $p \ge 2$, be a minimal n-submanifold with flat normal bundle. Its second fundamental form satisfies*

$$|B|\Delta|B| \ge -\frac{3}{2}|B|^4 + \frac{2}{n}|\nabla |B||^2. \tag{3.5}$$

For an n-dimensional oriented submanifold M in Euclidean space $\mathbb{R}^{n+p}$ we have the generalized Gauss map. By the parallel translation in the ambient Euclidean space, the tangent space T_xM at each point $x \in M$ is moved to the origin of $\mathbb{R}^{n+p}$ to obtain an n-subspace in $\mathbb{R}^{n+p}$, namely, a point of the Grassmannian manifold $\gamma(x) \in \mathbf{G}_{n,p}$. Thus, we define a generalized Gauss map $\gamma : M \to \mathbf{G}_{n,p}$.

For two simple n-vectors

$$A = a_1 \wedge \cdots \wedge a_n, \quad B = b_1 \wedge \cdots \wedge b_n,$$

their inner product is defined by

$$\langle A, B\rangle = \det\left(\langle a_i, b_j\rangle\right).$$

It is well-defined, if A and B are unit n-vectors. Choose an orthonormal frame field $\{e_i, e_\alpha\}$ along M such that $e_i \in TM$ and $e_\alpha \in NM$. Fix a unit simple n-vector $A = a_1 \wedge \cdots \wedge a_n$ define a function w on M by

$$w = \langle e_1 \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n\rangle = \det\left(\langle e_i, a_j\rangle\right).$$

Let $M = (x, f(x))$ be a graph in $\mathbb{R}^{n+p}$ defined by p functions $f^\alpha(x^1, \cdots, x^n)$. Choose A to be one representing $(x^1, \cdots, x^n)$ coordinate n-plane. The w is the inverse volume element of M [J-X1].

We derive now a basic formula of the function w. Since

$$\begin{aligned}
e_i(w) &= \sum_j \left\langle e_1 \wedge \cdots \wedge \bar\nabla_{e_i} e_j \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \right\rangle \\
&= \sum_j \left\langle e_1 \wedge \cdots \wedge (\bar\nabla_{e_i} e_j)^T \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \right\rangle \\
&\quad + \sum_j \left\langle e_1 \wedge \cdots \wedge (\bar\nabla_{e_i} e_j)^N \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \right\rangle \\
&= \sum_{\alpha, j} h_{\alpha ij} \left\langle e_1 \wedge \cdots \wedge e_{j-1} \wedge e_\alpha \wedge e_{j+1} \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \right\rangle,
\end{aligned}$$

$$\begin{aligned}
\Delta w =& -|B|^2 w + \sum_{\alpha,i,j} h_{\alpha iij} \left\langle e_1 \wedge \cdots \wedge e_{j-1} \wedge e_\alpha \wedge e_{j+1} \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \right\rangle \\
&+ \sum_{\alpha,\beta,i,j,k} \left\langle e_1 \wedge \cdots \wedge h_{\alpha ij} e_\alpha \wedge \cdots \wedge h_{\beta ik} e_\beta \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \right\rangle \\
=& -|B|^2 w + \sum_{\alpha,i,j} h_{\alpha iij} \left\langle e_1 \wedge \cdots \wedge e_{j-1} \wedge e_\alpha \wedge e_{j+1} \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \right\rangle \\
&+ \sum_{\alpha<\beta,i,j,k} (h_{\alpha ij} h_{\beta ik} - h_{\beta ij} h_{\alpha ik}) \\
&\qquad \left\langle e_1 \wedge \cdots \wedge e_\alpha \wedge \cdots \wedge e_\beta \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \right\rangle .
\end{aligned} \tag{3.6}$$

LEMMA 3.2. *Let M be an n-submanifold in $\mathbb{R}^{n+p}$ with parallel mean curvature and flat normal bundle. Then the above defined w-function satisfies*

$$\Delta w = -|B|^2 w \tag{3.7}$$

4. Volume Growth. Let N be a complete simply connected Riemannian manifold with non-positive sectional curvature, $M \to N$ be a minimal immersion. Fix a point $o \in M \subset N$, denote the distance function from o on N by ρ and that on M by r. It is obvious that $\rho \le r$. By using the triangle inequality we see that

$$\frac{\partial \rho}{\partial r} \le 1.$$

By using the Hessian comparison theorem on N

$$\overline{\text{Hess}}(\rho)(X, Y) \ge \frac{1}{\rho} \left(\langle X, Y\rangle - \langle X, \nabla\rho\rangle \langle Y, \nabla\rho\rangle \right), \tag{4.1}$$

where $X,\, Y \in TN$.

The restriction of ρ on M is a function on M. Then we have for $X,\, Y \in TM \subset TN$

$$\begin{aligned}
\text{Hess}(\rho)(X, Y) &= XY(\rho) - \left(\bar\nabla_X Y - \left(\bar\nabla_X Y\right)^N \right) \rho \\
&= \overline{\text{Hess}}(\rho)(X, Y) + \langle B_{XY}, \nabla\rho\rangle .
\end{aligned} \tag{4.2}$$

Taking trace in (4.2) and using (4.1), we obtain

$$\Delta\rho \geq \frac{1}{\rho}\left(n - |\nabla\rho|_M^2\right) + \langle n\,H, \nabla\rho\rangle\,,$$

$$\begin{aligned}\Delta\rho^2 &= 2\,|\nabla\rho|_M^2 + 2\,\rho\,\Delta\rho\\ &\geq 2\,n + 2\,\rho\,\langle n\,H, \nabla\rho\rangle\,,\end{aligned}$$

where H is the mean curvature vector of M in N. In particular, when M is minimal

$$\Delta\rho^2 \geq 2\,n. \tag{4.3}$$

Let $\bar{B}(\rho)$ be a geodesic ball of radius ρ and centered at $o \in M \subset N$ in N. Its restriction on M is denoted by

$$D(\rho) = \bar{B}(\rho) \cap M.$$

It is obvious that $B(s) \subset D(s)$, where $B(s)$ denotes the geodesic ball of radius s and centered at o in M. We know that $\frac{\partial}{\partial\rho}$ is the unit normal vector to $\partial\bar{B}(\rho)$. Its orthogonal projection to M is normal to $\partial D(\rho)$.

Let e_α be unit normal frame field of M in N at the concerned point. Then

$$\frac{\partial}{\partial\rho} - \left\langle \frac{\partial}{\partial\rho}, e_\alpha \right\rangle e_\alpha$$

is normal to $\partial D(\rho)$. Define

$$a = \frac{1}{\sqrt{1 - \sum_\alpha (e_\alpha(\rho))^2}}.$$

Hence,

$$\nu = a\left(\frac{\partial}{\partial\rho} - e_\alpha(\rho)\,e_\alpha\right)$$

is the unit normal vector to $\partial D(\rho)$.

Integrating (4.3) over $D(\rho)$ and using Stokes' theorem, we have

$$\begin{aligned}2\,n\,\mathrm{vol}(D(\rho)) &\leq \int_{D(\rho)} \Delta\,\rho^2\, * 1\\ &= \int_{\partial D(\rho)} \left\langle \nu, \nabla\rho^2 \right\rangle * 1.\end{aligned} \tag{4.4}$$

Noting

$$\left\langle \frac{\partial}{\partial\rho}, \nabla\rho^2 \right\rangle = 2\,\rho,$$

$$\left\langle \eta_\alpha, \nabla\rho^2 \right\rangle = 2\,\rho\,e_\alpha(\rho),$$

and then,

$$\begin{aligned}\left\langle \nu, \nabla\rho^2 \right\rangle &= a\,(2\,\rho - \sum_\alpha e_\alpha(\rho)\cdot 2\,\rho\,e_\alpha(\rho))\\ &= 2\,\rho\sqrt{1 - \sum_\alpha (\eta_\alpha(\rho))^2} \leq 2\,\rho,\end{aligned}$$

(4.4) becomes

$$2\,n\,\mathrm{vol}(D(\rho)) \le 2\int_{\partial D(\rho)} \rho * 1 = 2\,\rho\,\mathrm{vol}(\partial D(\rho)). \tag{4.5}$$

On the other hand, since $|\nabla\rho|_M \le 1$, the co-area formula gives us that

$$\frac{d\,(\mathrm{vol}(D(\rho)))}{d\,\rho} \ge \mathrm{vol}(\partial D(\rho)). \tag{4.6}$$

Combining (4.5) and (4.6), we obtain that

$$\frac{d\,(\mathrm{vol}(D(\rho)))}{\mathrm{vol} D(\rho)} \ge \frac{n\,d\,\rho}{\rho}.$$

It follows that

$$\frac{\mathrm{vol}(D(\rho))}{\rho^n} \tag{4.7}$$

is a nondecreasing function in ρ, which implies immediately that $\mathrm{vol}(M)$ is infinite and it has at least polynomial growth of order n. If

$$\lim_{\rho\to\infty} \frac{\mathrm{vol}(D(\rho))}{\rho^n} < \infty,$$

then M has the minimal volume growth and we say that M has Euclidean volume growth.

5. Proof of the Main Theorem.

THEOREM 5.1. *Let M be a complete minimal n-submanifold in $\mathbb{R}^{n+p}$ with flat normal bundle and positive w-function. Then any L^2-harmonic 1-form vanishes.*

Proof. If the function w, as defined in §3, is positive everywhere, then we have

$$\int_M |\nabla\phi|^2 \; * 1 \ge \int_M |B|^2\phi^2 * 1, \tag{5.1}$$

for any function with compact support $D \subset M$. In fact, let

$$L\phi = -\Delta\phi - |B|^2\phi.$$

Its first eigenvalue with the Dirichlet boundary condition in D is λ_1 and the corresponding eigenfunction is u. Without loss of generality, we assume that u achieves the positive maximum. Consider a smooth function

$$f = \frac{u}{w}.$$

Since $f|_{\partial D} = 0$, it achieves the positive maximum at $x \in D$. Therefore, at the point x,

$$\nabla f = 0,$$
$$\Delta f \le 0.$$

It follows that

$$w\Delta u \le u\,\Delta w.$$

Namely, at x,

$$\frac{\Delta u}{u} \le \frac{\Delta w}{w},$$

$$\frac{\Delta u+|B|^2u}{u}\le\frac{\Delta w+|B|^2w}{w}. \tag{5.2}$$

Since the normal bundle is flat, we have (3.7) and the right hand side of (5.2) is equal to zero. On the other hand,

$$\frac{\Delta u+|B|^2u}{u}=-\lambda_1. \tag{5.3}$$

(5.2) and (5.3) implies that $\lambda_1\ge 0$. We thus have

$$0\le\lambda_1=\inf\frac{\int_D\phi\,L\phi*1}{\int_D\phi^2*1}\le\frac{\int_D\phi\,L\phi*1}{\int_D\phi^2*1},$$

which shows (5.1) holds true.

Let ω be an L^2-harmonic 1-form on M. We have the Weitzenböck formula [X]

$$\Delta\omega=-\nabla^2\omega+S, \tag{5.4}$$

where for any $X\in TM$

$$S(X)=-(R(e_i,X)\omega)e_i=\omega(\mathrm{Ric}X).$$

We obtain

$$\Delta|\omega|^2=2|\nabla\omega|^2+2\left\langle\mathrm{Ric}\,\omega^\sharp,\omega^\sharp\right\rangle, \tag{5.5}$$

where $\omega^\sharp$ is the dual vector field of the 1-form ω. By using the Gauss equation (2.1)

$$\langle\mathrm{Ric}\,e_k,e_l\rangle=-\langle B_{e_ie_k},B_{e_ie_l}\rangle=-h_{\alpha ik}h_{\alpha il}\ge-|B|^2\delta_{kl},$$
$$\left\langle\mathrm{Ric}\,\omega^\sharp,\omega^\sharp\right\rangle\ge-|B|^2|\omega|^2.$$

So, (5.5) becomes

$$\Delta|\omega|^2\ge2\,|\nabla\omega|^2-2\,|B|^2|\omega|^2\ge\frac{2n}{n-1}|\nabla|\omega||^2-2\,|B|^2|\omega|^2, \tag{5.6}$$

here the second inequality holds because of the Kato inequality for harmonic 1-form [W]:

$$|\nabla\omega|^2\ge\frac{n}{n-1}|\nabla|\omega||^2.$$

Replacing ϕ by $|\omega|\phi$ in (5.1), integrating by parts and using (5.6), we arrive at the following inequality

$$\int_M|\omega|^2|\nabla\phi|^2*1\ge\frac{1}{n-1}\int_M|\nabla|\omega||^2\phi^2*1. \tag{5.7}$$

By taking the standard cut-off function ϕ in the above expression we have

$$\int_M|\nabla|\omega||^2*1\le\lim_{R\to\infty}\frac{C}{R^2}\int_{B(R)}|\omega|^2*1.$$

The finite assumption of the integral of $|\omega|^2$ forces that $|\omega|$ must be constant. On the other hand, by the previous argument that any minimal submanifold in Euclidean space has infinite volume. We conclude that any L^2-harmonic 1-form has to be zero. $\square$

More generally, we have

Theorem 5.2. *Let M be one as in Theorem 5.1, N be a manifold with non-positive sectional curvature. Then any harmonic map $f : M \to N$ with finite energy has to be constant.*

Proof. We also have the Bochner formula for the energy density of the harmonic map [X]

$$\begin{aligned}\Delta e(f) =& |\nabla df|^2 + \left\langle f_* \mathrm{Ric}^M e_i, f_* e_i \right\rangle - \left\langle R^N(f_*e_i, f_*e_j) f_*e_i, f_*e_j \right\rangle \\ \geq& |\nabla df|^2 - 2\,|B|^2\, e(f).\end{aligned} \tag{5.8}$$

There is also the Kato inequality [O]

$$|\nabla |df||^2 \leq \frac{n-1}{n} |\nabla df|^2.$$

Thus, (5.8) becomes

$$\Delta e(f) \geq \frac{n}{n-1} |\nabla |df||^2 - 2\,|B|^2 e(f). \tag{5.9}$$

Replacing ϕ by $\sqrt{e(f)}\phi$ in (5.1), integrating by parts and using (5.9), we obtain

$$\int_M e(f)\,|\nabla\phi|^2 * 1 \geq \frac{1}{n-1} \int_M |\nabla \sqrt{e(f)}|^2 \phi^2 * 1. \tag{5.10}$$

By the standard argument it follows that the energy density $e(f)$ is constant, which has to be zero by the finite energy and infinite volume of M. □

The argument in [C-S-Z] and Theorem 5.1 give

Theorem 5.3. *Any complete minimal submanifold in Euclidean space with flat normal bundle and positive w-function has only one end.*

In the case when w-function is positive, set $v = \frac{1}{w}$. We have a Bernstein type result.

Theorem 5.4. *Let M be a minimal n-submanifold in $\mathbb{R}^{n+p}$, $n \leq 5$, $p \geq 2$, , with flat normal bundle. If M has polynomial volume growth and v-function has growth*

$$v = O\left(R^{\frac{2}{3}\mu}\right),$$

where $0 \leq \mu < 1$ and R is the Euclidean distance from any point in M. Then M has to be an affine linear subspace.

Proof. From (3.7) we obtain

$$\Delta v = v\,|B|^2 + \frac{2}{v} |\nabla v|^2. \tag{5.11}$$

From (3.5) and (5.11) we obtain for any real q and s

$$\begin{aligned}\Delta\left(v^q |B|^s\right) \geq & q\,(q+1)\,v^{q-2} |B|^s\,|\nabla v|^2 + s\,\left(s - \frac{n-2}{n}\right)\,v^q |B|^{s-2} |\nabla |B||^2 \\ & + \left(q - \frac{3}{2}s\right)\,v^q |B|^{s+2} + 2\,q\,s\,v^{q-1} |B|^{s-1} \left\langle \nabla v, \nabla |B| \right\rangle.\end{aligned} \tag{5.12}$$

By using the Cauchy inequality with $\varepsilon > 0$

$$v^{q-1}|B|^{s-1}\langle \nabla v, \nabla |B|\rangle \le \frac{1}{2}\left(\varepsilon^{-1}v^{q-2}|B|^s|\nabla v|^2 + \varepsilon v^q |B|^{s-2}|\nabla |B||^2\right)$$

which is substituted in (5.12) we have

$$\begin{aligned}\Delta\left(v^q|B|^s\right) \ge\ & q\left(q+1-\varepsilon^{-1}s\right)v^{q-2}|B|^s|\nabla v|^2 \\ & + s\left(s-\frac{n-2}{n}-\varepsilon\, q\right)v^q|B|^{s-2}|\nabla|B||^2 + \left(q-\frac{3}{2}s\right)v^q|B|^{s+2}\end{aligned} \tag{5.13}$$

In the case of $n < 6$, we can choose an adequate

$$\frac{n-2}{n} \le \varepsilon < \frac{2}{3},$$

such that

$$\begin{cases} q+1-\varepsilon^{-1}s & \ge 0, \\ s-\frac{n-2}{n}-\varepsilon\, q & \ge 0, \\ q = \frac{3}{2}\, s + k, \end{cases} \tag{5.14}$$

where $k \ge 0$. From (5.13) we have for $k \ge 0$

$$\Delta(v^q|B|^s) \ge k\, v^q|B|^{s-2}. \tag{5.15}$$

For example, the following two cases satisfy (5.14)

1.

$$q = 3l,\ s = 2l,\ k = 0$$

and

$$\frac{2l}{3l+1} \le \varepsilon \le \frac{2l - \frac{n-2}{n}}{3l}$$

with

$$l \ge \frac{n-2}{6-n}.$$

2.

$$q = 3l,\ s = 2l-1,\ k = \frac{3}{2}$$

and

$$\frac{2l-1}{3l+1} \le \varepsilon \le \frac{2l - \frac{2(n-1)}{n}}{3l}$$

with

$$l \ge \frac{2(n-1)}{6-n}.$$

We notice that the ranges of ε and l in the second case include that of the first case.

Choosing the first case yields

$$\Delta(v^{3l}|B|^{2l}) \ge 0.$$

We have the mean value inequality for any subharmonic function on minimal submanifold M in $\mathbb{R}^{n+p}$ [C-L-Y], [N] which gives

$$v^{3l}|B|^{2l}(o) \le \frac{C}{R^n}\int_{D_R} v^{3l}|B|^{2l} * 1 \le \frac{C\,\mathrm{vol}(D_R)^{\frac{1}{2}}}{R^n}\left(\int_{D_R} v^{6l}|B|^{4l} * 1\right)^{\frac{1}{2}}, \tag{5.16}$$

where we assume $o \in M \subset \mathbb{R}^{n+p}$, C is a constant depending only on n. Choosing the second case, we obtain

$$\Delta(v^{3l}|B|^{2l-1}) \ge \frac{3}{2}\, v^{3l}|B|^{2l+1}.$$

Multiplying by $v^{3l}|B|^{2l-1}\phi^{4l}$, where ϕ is any smooth function with compact support, in the above inequality, then integrating by parts and using the Cauchy inequality, we have

$$\begin{aligned}\int_M v^{6l}|B|^{4l}\phi^{4l} * 1 &\le \frac{3}{2}\int_M v^{3l}|B|^{2l-1}\phi^{4l}\Delta(v^{3l}|B|^{2l-1}) * 1\\ &= -\frac{3}{2}\int_M \left\langle\nabla(v^{3l}|B|^{2l-1}\phi^{4l}), \nabla(v^{3l}|B|^{2l-1}\right\rangle * 1\\ &= -\frac{3}{2}\int_M |\nabla(v^{3l}|B|^{2l-1})|^2\phi^{4l}\\ &\quad - 6l\int_M \left\langle\phi^{2l-1}|B|^{2l-1}v^{3l}\nabla\phi, \phi^{2l}\nabla(v^{3l}|B|^{2l-1})\right\rangle\\ &\le C_1(l)\int_M v^{6l}|B|^{4l-2}\phi^{4l-2}|\nabla\phi|^2 * 1.\end{aligned} \tag{5.17}$$

By using Young's inequality

$$ab \le \frac{\alpha^p a^p}{p} + \frac{\alpha^{-q}b^q}{q}$$

for any real numbers p, q, α, a, b with $\frac{1}{p} + \frac{1}{q} = 1$, (5.17) becomes

$$\int_M v^{6l}|B|^{4l}\phi^{4l} * 1 \le C_2(l)\int_M v^{6l}|\nabla\phi|^{4l} * 1. \tag{5.18}$$

Choosing ϕ as the standard cut-off function, we obtain

$$\begin{aligned}\int_{D_R} v^{6l}|B|^{4l} * 1 &\le C_2(l)\, R^{-4l}\int_{D_{2R}} v^{6l} * 1\\ &\le C_2(l)\, R^{-4l}\mathrm{vol}(D_{2R})\sup_{D_{2R}} v^{6l},\end{aligned} \tag{5.19}$$

We know that M has polynomial volume growth of order $n+m,\ m \ge 0$. From (5.16) and (5.19) we obtain

$$v^{3l}|B|^{2l}(o) \le C_3(n)R^{-n-2l}R^{n+m}\sup_{D_{2R}} v^{3l},$$

then

$$v^{\frac{3}{2}}|B|(o) \le C(n)\, R^{-1+\frac{m}{2l}+\mu}. \tag{5.20}$$

For given $m \geq 0$ and $0 \leq \mu < 1$, we can choose l large enough such that

$$-1 + \frac{m}{2l} + \mu < 0.$$

Let R go to infinity, we have $|B|(o) = 0$. Since o is any point in M, we complete the proof. $\square$

In the case if M is an entire graph defined by p functions on $\mathbb{R}^{n+p}$, v-function is just the volume element. If

$$v = O\left(R^{\frac{2}{3}\mu}\right),$$

then

$$\mathrm{vol}(D_R) = O(R^{n+\frac{2}{3}\mu}).$$

In this case we need not assume that M has polynomial growth and have the following result.

Corollary 5.5. *Let $M = (x, f(x))$ be a minimal graph given by p functions $f^\alpha(x^1, \cdots, x^n)$, $p \geq 2$, $n < 6$, with flat normal bundle. If for* $0 \leq \mu < 1$

$$(det(\delta_{ij} + f_i^\alpha f_j^\alpha))^{\frac{1}{2}} = O(R^{\frac{2}{3}\mu}),$$

where $R^2 = |x|^2 + |f|^2$. Then f^α are affine linear functions.

Remark. For a minimal hypersurface M in $\mathbb{R}^{n+1}$ we have

$$\Delta|B|^2 \geq -2|B|^4 + 2\,|\nabla B|^2$$

and

$$|B|\Delta|B| \geq -|B|^4 + \frac{2}{n}|\nabla|B||^2,$$

instead of (2.6) and (3.5) for $p \geq 2$. We then have

$$\begin{aligned}\Delta\left(v^q|B|^s\right) \geq & q\left(q + 1 - \varepsilon^{-1}s\right) v^{q-2}|B|^s|\nabla v|^2 \\ & + s\left(s - \frac{n-2}{n} - \varepsilon\, q\right) v^q|B|^{s-2}|\nabla|B||^2 + (q-s)\ v^q|B|^{s+2}.\end{aligned}$$

Choose s sufficiently large, we have

$$\Delta(v^s|B|^s) \geq 0,$$

$$\Delta(v^{s+1}|B|^s) \geq v^{s+1}|B|^{s+2}.$$

Then by the similar argument as the higher codimension we obtain

$$v\,|B|(o) \leq C(n)R^{-1+\frac{m}{s}+\mu},$$

provided we assume that M has polynomial volume growth of order $n + m$, $m > 0$, and v-function has growth

$$v = O(R^\mu)$$

with $0 \leq \mu < 1$. We then can conclude that M is flat. In particular, when M is a graphic minimal hypersurface, we recover the result in [Ni] and [E-H].

Added in proof: In an author's joint work with K. Smoczyk and Guofang Wang, the dimension limitation in Theorem 5.4 has been removed. The paper will appear in Calculus of Variations and PDE.

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