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Existence of Solutions of IVPs for Differential Systems on Half Line with Sequential Fractional Derivative Operators*

Yuji Liu†
Guangdong University of Finance and Economics, Department of Mathematics,
Guangzhou, No. 21 Chisha Road,
Guangdong 510320 P R China

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Abstract. In this article, we establish some existence results for solutions of a initial value problem of a nonlinear fractional differential system on half line involving the sequential Riemann-Liouville fractional derivatives. Our analysis relies on the Schauder fixed point theorem. An efficiency example is presented to illustrate the main theorem. As far as the author knows, the present work is perhaps the first one that deals with such kind of initial value problems for fractional differential systems on half line.

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1 Introduction

Fractional differential equations have excited in recent years a considerable interest both in mathematics and in applications. They were used in modelling of many physical and chemical processes and in engineering, see the text books [21, 28, 19] and the references therein. For more details on the geometric and physical interpretation for derivatives see [22, 14, 30].

Furati and Tatar in [11], Zhang in [33], Agarwal, Benchohra and Hamani [2] established sufficient conditions for the existence of solutions for some boundary value problems of fractional differential equations with Caputo fractional derivative on the finite intervals.

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†e-mail address: liuyuji888@sohu.com

Arara, Benchohra, Hamidi and Nieto [3], Zhao and Ge [34], Liu and Jia [18], Su and Zhang [27] and Agarwal, Benchohra, Hamidi and Pinelas [1] studied the existence of solutions for some boundary value problems for fractional order differential equations on half line.

Applications of fractional order differential systems are in many fields, as for example, rheology, mechanics, chemistry, physics, bioengineering, robotics and many others, see [7]. Diethehm [8] proposed the model of the type (which is called a multi-order fractional differential system):

$$^{c}D_{0^+}^{n_i}y_i(t) = f_i(t, y_1(t), \cdots, y_n(t)), i = 1, 2, \cdots, n$$

subjected to the initial conditions

$$y_j(0) = y_{j,0}(j = 1, 2, \cdots, n).$$

This system contains many models as special cases, see Chen's fractional order system [29] with a double scroll attractor, Genesio-Tesi fractional-order system [13], Lu's fractional order system [10], Volta's fractional-order system [23, 20], Rossler's fractional-order system [17] and so on.

Boundary value problems of fractional order differential systems on finite intervals are a fascinating subject. See papers [26, 31, 32, 12, 24, 4, 9, 25]. One knows that

$$D_{0^+}^{p}D_{0^+}^{q}f(t) = D_{0^+}^{q}D_{0^+}^{p}f(t) = D_{0^+}^{p+q}f(t)$$

does not hold for $p > 0$ or $q > 0$, where D_{0^+} is Riemann-Liouville fractional derivative. $D_{0^+}^{p}D_{0^+}^{q}$ is called a sequential fractional derivative operator.

Sequential fractional derivative operators can appear in the formulation of various applied problems in physics and applied science. Indeed, differential equations modelling processes or objects arise usually as a result of a substitution of one relationship involving derivatives into another one. If the derivatives in both relationships are fractional derivatives, then the resulting expression (equation) will contain, in general case, sequential fractional derivative operators ([21], P.88). Therefore the consideration of sequential fractional derivative operators is of interest [21].

In [3, 34, 15, 29], the authors investigated the global existence of solutions of initial value problems of nonlinear fractional differential equations on a semi-axis. More precisely, they studied the folllowing initial value problem

$$\begin{cases} D_{0^+}^{\alpha}x(t) = f(t, x(t)) = 0, & t \in (0, +\infty), \\ \lim\limits_{t\to 0} t^{1-\alpha}x(t) = x_0, \end{cases}$$

where $0 < \alpha < 1$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order α, $f : (0, +\infty) \times R \to R$ is continuous.

There has been no paper concerned with the existence of solutions of initial value problems on half lines for fractional differential systems with sequential fractional derivative operators and the nonlinearities depending on the lower order derivatives [21].

In this paper, we fill this gap. We discuss the global existence of solutions of the following initial value problem of nonlinear fractional differential system on half line with sequential fractional derivative operators

$$\begin{cases} D^{\sigma_n}x(t) + \phi(t)f(t, y(t), D_{0^+}^{p}y(t)) = 0, & t \in (0, +\infty), \\ D^{\tau_m}y(t) + \psi(t)g(t, x(t), D_{0^+}^{q}x(t)) = 0, & t \in (0, +\infty), \\ \lim\limits_{t\to 0} t^{1-\alpha_i}D^{\sigma_{i-1}}x(t) = x_{i-1},\ i \in N_{1,n}, \\ \lim\limits_{t\to 0} t^{1-\beta_i}D^{\tau_{i-1}}y(t) = y_{i-1},\ i \in N_{1,n}, \end{cases} \tag{1.1}$$

where

- $D^{*}_{0^+}$ is the standard Riemann-Liouville fractional derivative of order $* > 0$,
- N_0 is the set of all nonnegative integers, $N_{a,b} = \{a, a+1, a+2, \cdots, b\}$ for $a, b \in N_0$ with $a \leq b$,
- $\alpha_i \in (0,1)(i \in N_{1,n})$, $\sigma_j = \alpha_1 + \cdots + \alpha_j (j \in N_{1,n})$, $q \in (0,1)$ with $q < \sigma_n$, $D^{\sigma_j}x = D^{\alpha_j}_{0^+} \cdots D^{\alpha_2}_{0^+} D^{\alpha_1}_{0^+} x (j \in N_{1,n})$ is a sequential fractional derivative operator, $D^{\sigma_0}x = x$,
- $\beta_i \in (0,1)(i \in N_{1,m})$, $\tau_j = \beta_1 + \cdots + \beta_j (j \in N_{1,m})$, $p \in (0,1)$ with $p < \tau_m$, $D^{\tau_j}y = D^{\beta_j}_{0^+} \cdots D^{\beta_2}_{0^+} D^{\beta_1}_{0^+} y (j \in N_{1,m})$ is a sequential fractional derivative operator, $D^{\beta_0}y = y$,
- $x_i \in R(i \in N_{0,n-1})$, $y_i \in R(i \in N_{0,m-1})$ are initial data,
- $\phi, \psi : (0, +\infty) \to R$ satisfy that there exist constants $k_i > -1 (i = 1, 2)$ such that

$$|\phi(t)| \leq t^{k_1}, \ \ |\psi(t)| \leq t^{k_2}, t \in (0, \infty),$$

- $f, g : (0, +\infty) \times R^2 \to R$ and f is a $\tau-$Caratheodory function and g a $\sigma-$Caratheodory function (see Definitions 2.3 and 2.4).

We establish sufficient conditions for the global existence of solutions of IVP(1.1). The methods used in this paper are based upon the Schauder fixed point theorem. The novelty of this paper is that IVP(1.1) is defined on a half line, and f, g involved with lower order fractional derivatives are allowed to be linear or supper linear functions. An example is presented to illustrate the main theorem.

The remainder of this paper is organized as follows: some preliminary results are given in Section 2. The main result and its proof are presented in Section 3. In Section 4, an example is given to show the efficiency of the main theorem.

2 Preliminary results

For the convenience of the reader, we present here the necessary definitions from fixed point theory and fractional calculus theory. These definitions and properties can be found in the literatures [21, 28, 19]. Denote the Gamma function and Beta function by

$$\Gamma(\alpha) = \int_0^{+\infty} s^{\alpha-1} e^{-s} ds, \ \ \mathbf{B}(\alpha, \beta) = \int_0^1 (1-x)^{\alpha-1} x^{\beta-1} dx.$$

Definition 2.1. . Let $c \in R$. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (c, +\infty) \to R$ is given by

$$I^{\alpha}_{c^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right-hand side exists.

Definition 2.2. Let $c \in R$. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : (c, +\infty) \to R$ is given by

$$D^{\alpha}_{c^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_c^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n - 1 \leq \alpha < n$, provided that the right-hand side exists.

Suppose that $\tau > \tau_m + k_2 + 1$ and $\sigma > \sigma_n + k_1 + 1$. Denote $\rho(t) = \frac{t^{1-\alpha_1}}{1+t^{\sigma}}$ and $\varrho(t) = \frac{t^{1-\beta_1}}{1+t^{\tau}}$.

Definition 2.3. $f : (0, +\infty) \times R^2 \to R$ is called a $\tau-$Caratheodory function if it satisfies the following assumptions:

(i) $t \to f\left(t, \frac{x}{\varrho(t)}, \frac{y}{t^p \varrho(t)}\right)$ is measurable on $(0, +\infty)$ for each $(x, y) \in R^2$;

(ii) $(x,y)\to f\left(t,\ \frac{x}{\varrho(t)},\ \frac{y}{t^p\varrho(t)}\right)$ is continuous on R^2 for each $t\in(0,+\infty)$;
(iii) for each $r>0$ there exists a constant $M_r\ge 0$ such that $|x|,|y|\le r$ imply

$$\left|f\left(t,\ \frac{x}{\varrho(t)},\ \frac{y}{t^p\varrho(t)}\right)\right|\le M_r, t\in(0,+\infty).$$

Definition 2.4. $g:(0,+\infty)\times R^2\to R$ is called a $\sigma-$Caratheodory function if it satisfies the following assumptions:
(i) $t\to g\left(t,\ \frac{x}{\rho(t)},\ \frac{y}{t^q\rho(t)}\right)$ is measurable on $(0,+\infty)$ for each $(x,y)\in R^2$;
(ii) $(x,y)\to g\left(t,\ \frac{x}{\rho(t)},\ \frac{y}{t^q\rho(t)}\right)$ is continuous on R^2 for each $t\in(0,+\infty)$;
(iii) for each $r>0$ there exists a constant $M_r>0$ such that $|x|,|y|\le r$ imply

$$\left|g\left(t,\ \frac{x}{\rho(t)},\ \frac{y}{t^q\rho(t)}\right)\right|\le M_r, t\in(0,+\infty).$$

Definition 2.5. Let Z_1 and Z_2 be Banach spaces and $T:Z_1\to Z_2$. T is called completely continuous if T is continuous and maps bounded sets into relatively compact sets.

For $\alpha>0$ and $\mu>-1$, it holds that

$$I_{0^+}^{\alpha}t^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}t^{\mu+\alpha},\ \ D_{0^+}^{\alpha}t^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}.$$

Let $A>B>0$. It is easy to show that

$$\sup_{t\in(0,+\infty)}\frac{t^B}{1+t^A}=\frac{A-B}{A}\left(\frac{B}{A-B}\right)^{\frac{B}{A}}=:M_{A,B}.$$

Let $\sigma>\sigma_n+k_1+1$ and $\tau>\tau_m+k_2+1$. $C(0,+\infty)$ denotes the set of all continuous functions on $(0,+\infty)$. Choose

$$X=\left\{x:\begin{array}{c}x,D_{0^+}^{q}x\in C(0,+\infty)\text{ and the following limits exist}\\ \lim\limits_{t\to0}\rho(t)x(t),\lim\limits_{t\to0}t^q\rho(t)D_{0^+}^{q}x(t),\\ \lim\limits_{t\to+\infty}\rho(t)x(t),\ \lim\limits_{t\to+\infty}t^q\rho(t)D_{0^+}^{q}x(t)\end{array}\right\}$$

and

$$Y=\left\{y:\begin{array}{c}y,D_{0^+}^{p}y\in C(0,+\infty)\text{ and the following limits exist}\\ \lim\limits_{t\to0}\varrho(t)y(t),\lim\limits_{t\to0}t^p\varrho(t)D_{0^+}^{p}y(t),\\ \lim\limits_{t\to+\infty}\varrho(t)y(t),\ \lim\limits_{t\to+\infty}t^p\varrho(t)D_{0^+}^{p}y(t)\end{array}\right\}.$$

For $x\in X$, define

$$||x||_X=\max\left\{\sup_{t\in(0,+\infty)}\rho(t)|x(t)|,\ \sup_{t\in(0,+\infty)}t^q\rho(t)|D_{0^+}^{q}x(t)|\right\}.$$

For $y\in Y$, define

$$||y||_Y=\max\left\{\sup_{t\in(0,+\infty)}\varrho(t)|y(t)|,\ \sup_{t\in(0,+\infty)}t^p\varrho(t)D_{0^+}^{p}y(t)|\right\}.$$

Lemma 2.6. *X is a Banach space with the norm* $||\cdot||_X$ *and Y a Banach space with the norm* $||\cdot||_Y$.

Proof. We prove that X is a Banach space. Similarly we can prove that Y is a Banach space.

It is easy to see that X is a normed linear space. Let $\{x_u\}$ be a Cauchy sequence in X. Then $\|x_u - x_v\| \to 0,\ u, v \to +\infty$. It follows that

$$\begin{aligned}
&\lim_{t\to 0}\rho(t)x_u(t),\ \lim_{t\to+\infty}\rho(t)x_u(t) \text{ exist},\\
&\lim_{t\to 0}t^q\rho(t)D^q_{0^+}x_u(t),\ \lim_{t\to+\infty}t^q\rho(t)D^q_{0^+}x_u(t) \text{ exist},\\
&\sup_{t\in(0,+\infty)}\rho(t)|x_u(t)-x_v(t)| \to 0, u, v \to +\infty,\\
&\sup_{t\in(0,+\infty)}t^q\rho(t)|D^q_{0^+}x_u(t)-D^q_{0^+}x_v(t)|, u, v \to +\infty.
\end{aligned} \tag{2.1}$$

Thus there exists two functions x_0, y_0 defined on $(0, +\infty)$ such that

$$\lim_{u\to+\infty}\rho(t)x_u(t) = x_0(t),\ \lim_{u\to+\infty}t^q\rho(t)D^q_{0^+}x_u(t) = y_0(t).$$

It follows that

$$\begin{aligned}
&\sup_{t\in(0,+\infty)}|\rho(t)x_u(t)-x_0(t)| \to 0, u \to +\infty,\\
&\sup_{t\in(0,+\infty)}\left|t^q\rho(t)D^q_{0^+}x_u(t)-y_0(t)\right|, u \to +\infty.
\end{aligned} \tag{2.2}$$

This means that functions $x_0, y_0 : (0, +\infty) \to R$ are well defined.

Step 1. Prove that $x_0, y_0 \in C(0, +\infty)$.

We have $t_0 \in (0, +\infty)$ that

$$\begin{aligned}
|x_0(t)-x_0(t_0)| &\le |x_0(t)-\rho(t)x_N(t)| + |\rho(t)x_N(t)-\rho(t)x_N(t_0)| + |\rho(t)x_N(t_0)-x_0(t_0)|\\
&\le 2\sup_{t\in(0,+\infty)}|\rho(t)x_N(t)-x_0(t)| + |\rho(t)x_N(t)-\rho(t)x_N(t_0)|.
\end{aligned}$$

Since $\sup_{t\in(0,+\infty)}|\rho(t)x_u(t)-x_0(t)| \to 0, u \to +\infty$ and $\rho(t)x_u(t)$ is continuous on $(0, +\infty)$, then for any $\epsilon > 0$ we can choose N and $\delta > 0$ such that $\sup_{t\in(0,+\infty)}|\rho(t)x_N(t)-x_0(t)| < \epsilon$ and $|\rho(t)x_N(t)-\rho(t)x_N(t_0)| < \epsilon$ for all $|t-t_0| < \delta$. Thus $|x_0(t)-x_0(t_0)| < 3\epsilon$ foe all $|t-t_0| < \delta$. So $x_0 \in C(0, +\infty)$. Similarly we can prove that $y_0 \in C(0, +\infty)$.

Step 2. Prove that the limits $\lim_{t\to 0}x_0(t),\ \lim_{t\to+\infty}x_0(t),\ \lim_{t\to 0}y_0(t),\ \lim_{t\to+\infty}y_0(t)$ exist.

Suppose that $\lim_{t\to 0}\rho(t)x_u(t) = A_u$. By $\sup_{t\in(0,+\infty)}\rho(t)|x_u(t)-x_v(t)| \to 0, u, v \to +\infty$, we know that A_u is a Cauchy sequence. Then $\lim_{u\to+\infty}A_u$ exists. By $\sup_{t\in(0,+\infty)}|\rho(t)x_u(t)-x_0(t)| \to 0, u \to +\infty$, we get that

$$\lim_{t\to 0}x_0(t) = \lim_{t\to 0}\lim_{u\to+\infty}\rho(t)x_u(t) = \lim_{u\to+\infty}\lim_{t\to 0}\rho(t)x_u(t) = \lim_{u\to+\infty}A_u.$$

Hence $\lim_{t\to 0}x_0(t)$ exists. Similarly we can prove that $\lim_{t\to+\infty}x_0(t), \lim_{t\to 0}y_0(t), \lim_{t\to+\infty}y_0(t)$ exist.

Step 3. Prove that $\frac{y_0(t)}{t^q\rho(t)} = D^q_{0^+}\left(\frac{x_0(t)}{\rho(t)}\right)$.

We have for some $c_u \in R$ that

$$\left|x_u(t)+c_ut^{q-1}-I_{0^+}^q\left(\frac{y_0(t)}{t^q\rho(t)}\right)\right|$$

$$=\left|I_{0^+}^qD_{0^+}^qx_u(t)-I_{0^+}^q\left(\frac{y_0(t)}{t^q\rho(t)}\right)\right|$$

$$=\left|\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\left(D_{0^+}^qx_u(s)-\frac{y_0(s)}{t^q\rho(s)}\right)ds\right|$$

$$\le\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}s^q\rho(s)ds\sup_{t\in(0,+\infty)}\left|t^q\rho(t)D_{0^+}^qx_u(t)-y_0(t)\right|$$

$$\le\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}s^qs^{1-\alpha_1}ds\sup_{t\in(0,+\infty)}\left|t^q\rho(t)D_{0^+}^qx_u(t)-y_0(t)\right|$$

$$=t^{1+2q-\alpha_1}\frac{\mathbf{B}(q,2+q-\alpha_1)}{\Gamma(q)}\sup_{t\in(0,+\infty)}\left|t^q\rho(t)D_{0^+}^qx_u(t)-y_0(t)\right|$$

$$\to 0 \text{ as } u\to+\infty.$$

So $\lim_{u\to+\infty}\left(x_u(t)+c_ut^{q-1}\right)=I_{0^+}^q\left(\frac{y_0(t)}{t^q\rho(t)}\right)$. Then $\frac{x_0(t)}{\rho(t)}+c_0t^{q-1}=I_{0^+}^q\left(\frac{y_0(t)}{t^q\rho(t)}\right)$. It follows that $\frac{y_0(t)}{t^q\rho(t)}=D_{0^+}^q\left(\frac{x_0(t)}{\rho(t)}\right)$.

So $t\to\frac{x_0(t)}{\rho(t)}$ is a element in X with $x_u\to\frac{x_0}{\rho(t)}$ as $u\to+\infty$. It follows that X is a Banach space. The proof of Lemma 2.1 is completed.

We define for $x\in X$ that $\rho(t)x(t)|_{t=0}=\lim_{t\to0}\rho(t)x(t)$ and $t^q\rho(t)D_{0^+}x(t)|_{t=0}=\lim_{t\to0}t^q\rho(t)D_{0^+}x(t)$. Then for $x\in X$, both $t\to\rho(t)x(t)$ and $t\to t^q\rho(t)D_{0^+}^qx(t)$ are continuous on $[0,+\infty)$.

□

Lemma 2.7. *Let M be a subset of X. Then M is relatively compact if and only if the following conditions are satisfied:*

(i) *both $\{t\to\rho(t)x(t):x\in M\}$ and $\{t\to t^q\rho(t)D_{0^+}^qx(t):x\in M\}$ are uniformly bounded,*

(ii) *both $\{t\to\rho(t)x(t):x\in M\}$ and $\{t\to t^q\rho(t)D_{0^+}^qx(t):x\in M\}$ are equicontinuous in any subinterval $[a,b]$ in $[0,+\infty)$,*

(iii) *both $\{t\to\rho(t)x(t):x\in M\}$ and $\{t\to t^q\rho(t)D_{0^+}^qx(t):x\in M\}$ are equiconverges as $t\to+\infty$.*

Proof. "$\Leftarrow$". From Lemma 2.1, we know X is a Banach space. In order to prove that the subset M is relatively compact in X, we only need to show M is totally bounded in X, that is for all $\epsilon>0$, M has a finite ϵ-net.

For any given $\epsilon>0$, by **(i)** and **(iii)**, there exist constants $A,B,T>0$, we have

$$|\rho(t_1)x(t_1)-\rho(t_2)x(t_2)|\le\frac{\epsilon}{3},t_1,t_2\ge T,x\in M,$$

$$|t_1^q\rho(t_1)D_{0^+}^qx(t_1)-t_2^q\rho(t_2)D_{0^+}^qx(t_2)|<\frac{\epsilon}{3},t_1,t_2\ge T,x\in M,$$

$$\rho(t)|x(t)|\le A,\ t^q\rho(t)|D_{0^+}^qx(t)|<A,t\in[0,+\infty),x\in M.$$

For $T>0$, define

$$X|_{(0,T]}=\left\{x:\begin{array}{c}x,D_{0^+}^qx\in C(0,T]\text{ and the following limits exist}\\ \lim_{t\to0}\rho(t)x(t),\lim_{t\to0}t^q\rho(t)D_{0^+}^qx(t)\end{array}\right\}.$$

For $x \in X|_{(0,T]}$, define

$$\|x\|_T = \max\left\{\sup_{t\in(0,T]} \rho(t)|x(t)|,\ \sup_{t\in(0,T]} t^q\rho(t)|D^q_{0^+}x(t)|\right\}.$$

Similarly to Lemma 2.1, we can prove that $X_{(0,T]}$ is a Banach space. Let $M|_{(0,T]} = \{t \to x(t), t \in (0,T] : x \in M\}$. Then $M|_{(0,T]}$ is a subset of $X|_{(0,T]}$. By **(i)** and **(ii)**, and Ascoli-Arzela theorem, we can know that $M|_{(0,T]}$ is relatively compact. Thus, there exist $x_1, x_2, \cdots, x_k \in M$ such that, for any $x \in M|_{(0,T]}$, we have that there exists some $i = 1, 2, \cdots, k$ such that

$$\|x - x_i\|_T = \max\left\{\sup_{t\in(0,T]} \rho(t)|x(t) - x_i(t)|,\ \sup_{t\in(0,T]} t^q\rho(t)|D^q_{0^+}x(t) - D^q_{0^+}x_i(t)|\right\} \le \frac{\epsilon}{3}.$$

Therefore,

$$\|x - x_i\|_X = \max\left\{\sup_{t\in(0,T]} \rho(t)|x(t) - x_i(t)|,\ \sup_{t\in(0,T]} t^q\rho(t)|D^q_{0^+}x(t) - D^q_{0^+}x_i(t)|,\right.$$

$$\left.\sup_{t\ge T}\rho(t)|x(t) - x_i(t)|,\ \sup_{t\ge T} t^q\rho(t)|D^q_{0^+}x(t) - D^q_{0^+}x_i(t)|\right\}.$$

For $t, t_1 \ge T$, we have

$$\rho(t)|x(t) - x_i(t)| \le |\rho(t)x(t) - \rho(t_1)x(t_1)| + |\rho(t_1)x(t_1) - \rho(t_1)x_i(t_1)| + |\rho(t_1)x_i(t_1) - \rho(t)x_i(t)|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Similarly we have $\sup_{t\ge T} t^q\rho(t)|D^q_{0^+}x(t) - D^q_{0^+}x_i(t)| < \epsilon$. Then $\|x - x_i\|_X < \epsilon$. So, for any $\epsilon > 0$, M has a finite ϵ-net $\{U_{x_1}, U_{x_2}, \cdots, U_{x_k}\}$, that is, M is totally bounded in X. Hence M is relatively compact in X.

$\Rightarrow$. Assume that M is relatively compact, then for any $\epsilon > 0$, there exists a finite ϵ-net of M. Let the finite ϵ-net be $\{U_{x_1}, U_{x_2}, \cdots, U_{x_k}\}$ with $U_{x_i} \subset M$. Then for any $x \in M$, there exists U_{x_i} such that $x \in U_{x_i}$ and

$$\rho(t)|x(t)| \le \rho(t)|x(t) - x_i(t)| + \rho(t)|x_i(t)| \le \epsilon + \max\left\{\sup_{t\in R}|x_i(t)| : i = 1, 2, \cdots, k\right\},$$

$$t^q\rho(t)|D_{0^+}qx(t)| \le \epsilon + \max\left\{\sup_{t\in R} t^q\rho(t)|D^q_{)^+}x_i(t)| : i = 1, 2, \cdots, k\right\}.$$

It follows that both M and $\{\rho(t)D^q_{0^+}x : x \in M\}$ are uniformly bounded. Then **(i)** holds.

Furthermore, there exists $T > 0$ such that $|\rho(t_1)x_i(t_1) - \rho(t_2)x_i(t_2)| < \epsilon$ for all $t_1, t_2 \ge T$ and $i = 1, 2, \cdots, k$. Then we have for $t_1, t_2 \ge T$ that

$$|\rho(t_1)x(t_1) - \rho(t_2)x(t_2)| \le |\rho(t_1)x(t_1) - \rho(t_1)x_i(t_1)| + |\rho(t_1)x_i(t_1) - \rho(t_2)x_i(t_2)|$$

$$+ |\rho(t_2)x_i(t_2) - \rho(t_2)x(t_2)| < 3\epsilon, x \in M.$$

Similarly we have for $t_1, t_2 \ge T$ that

$$|t_1^q\rho(t_1)D^q_{0^+}x(t_1) - t_2^q\rho(t_2)D^q_{0^+}x(t_2)| \le 3\epsilon, x \in M.$$

Thus **(iii)** is valid. Similarly we can prove that **(ii)** holds. Consequently, the Lemma is proved. □

Remark 2.8. Let $Z = X\times Y$ be normed with $\|(x,y)\| = \max\{\|x\|_X, \|y\|_Y\}$ for $(x,y)\in Z$. Then Z is a Banach space too. Let $\Omega = \{(x,y)\}\subset Z$. Then Ω is relatively compact if and only if both $\Omega|_X = \{x:$ there exists $y\in Y$ such that $(x,y)\in\Omega\}$ and $\Omega|_X = \{y:$ there exists $x\in X$ such that $(x,y)\in\Omega\}$ are relatively compact.

Lemma 2.9. *Suppose that $h:(0,+\infty)\to R$ satisfies that $|h(t)|\le t^{k_1}$ for all $t\in(0,+\infty)$. Then $u\in X$ is a solution of system*

$$\begin{cases} D^{\sigma_n}u(t)+h(t)=0, \quad t\in(0,\infty), \\ \lim_{t\to 0} t^{1-\alpha_i}D^{\sigma_{i-1}}u(t)=x_{i-1},\ i\in N_{1,n} \end{cases} \tag{2.3}$$

if and only if $u\in X$ satisfies

$$u(t) = -\int_0^t \frac{(t-s)^{\sigma_n-1}}{\Gamma(\sigma_n)}h(s)ds + \sum_{j=0}^{n-1}\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})}x_j t^{\sigma_{j+1}-1}. \tag{2.4}$$

Proof. Suppose that $x\in X$ is a solution of (2.3). From (2.3), we have that there exists a constant c_1 such that

$$D_{0^+}^{\alpha_{n-1}}\cdots D_{0^+}^{\alpha_1}u(t) = -\int_0^t \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)}h(s)ds + c_1t^{\alpha_n-1}. \tag{2.5}$$

By

$$\lim_{t\to 0} t^{1-\alpha_n}D_{0^+}^{\alpha_{n-1}}\cdots D_{0^+}^{\alpha_1}u(t) = x_{n-1},$$

we get that

$$D_{0^+}^{\alpha_{n-1}}\cdots D_{0^+}^{\alpha_1}u(t) = -\int_0^t \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)}h(s)ds + x_{n-1}t^{\alpha_n-1}.$$

Similarly we use

$$\lim_{t\to 0} t^{1-\alpha_{n-1}}D_{0^+}^{\alpha_{n-2}}\cdots D_{0^+}^{\alpha_1}u(t) = x_{n-2}$$

Then

$$\begin{aligned} D_{0^+}^{\alpha_{n-2}}\cdots D_{0^+}^{\alpha_1}u(t) &= -\int_0^t \frac{(t-s)^{\alpha_{n-1}+\alpha_n-1}}{\Gamma(\alpha_{n-1}+\alpha_n)}h(s)ds \\ &+x_{n-1}\frac{\Gamma(\alpha_n)}{\Gamma(\alpha_{n-1}+\alpha_n)}t^{\alpha_{n-1}+\alpha_n-1} + x_{n-2}t^{\alpha_{n-1}-1}. \end{aligned} \tag{2.6}$$

Using similar methods, by the other boundary conditions, we get

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha_1+\cdots+\alpha_n-1}}{\Gamma(\alpha_1+\cdots+\alpha_n)}h(s)ds + \sum_{j=0}^{n-1} x_j\frac{\Gamma(\alpha_{j+1})}{\Gamma(\alpha_1+\cdots+\alpha_{j+1})}t^{\alpha_1+\cdots+\alpha_{j+1}-1}.$$

It is easy to show that

$$\rho(t)u(t) = -\frac{1}{\Gamma(\sigma_n)}\frac{t^{1-\alpha_1}}{1+t^{\sigma}}\int_0^t (t-s)^{\sigma_n-1}h(s)ds + \sum_{j=0}^{n-1} x_j\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})}\frac{t^{\sigma_{j+1}-\alpha_1}}{1+t^{\sigma}}.$$

Since

$$\begin{aligned} \rho(t)\int_0^t (t-s)^{\sigma_n-1}|h(s)|ds &\le \frac{t^{1-\alpha_1}}{1+t^{\sigma}}\int_0^t (t-s)^{\sigma_n-1}s^{k_1}ds \\ &\le \frac{t^{1-\alpha_1}}{1+t^{\sigma}}t^{\sigma_n+k}\int_0^1 (t-w)^{\sigma_n-1}w^{k_1}dw \\ &= \frac{t^{\sigma_n+k_1+1-\alpha_1}}{1+t^{\sigma}}\mathbf{B}(\sigma_n, k_1+1), \end{aligned}$$

then $u \in C(0,\infty)$ and both $\lim_{t\to 0}\rho(t)u(t)$ and $\lim_{t\to+\infty}\rho(t)u(t)$ exist. One sees that

$$D_{0^+}^q u(t) = -\int_0^t \frac{(t-s)^{\sigma_n-q-1}}{\Gamma(\sigma_n-q)}h(s)ds + \sum_{j=0}^{n-1} x_j \frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1}-q)}t^{\sigma_{j+1}-q-1}.$$

So

$$t^q\rho(t)\int_0^t (t-s)^{\sigma_n-q-1}|h(s)|ds \le \frac{t^{1+q-\alpha_1}}{1+t^\sigma}\int_0^t (t-s)^{\sigma_n-q-1}s^{k_1}ds$$

$$\le \frac{t^{\sigma_n+k_1+1-\alpha_1}}{1+t^\sigma}\mathbf{B}(\sigma_n-q,k_1+1).$$

Similarly we can show that $D_{0^+}^q u \in C(0,\infty)$ and both $\lim_{t\to 0} t^q\rho(t)D_{0^+}^q u(t)$ and $\lim_{t\to+\infty} t^q\rho(t)D_{0^+}^q u(t)$ exist.

Then $u \in X$ satisfies (2.4). On the other hand, if $u \in X$ satisfies (2.4), then we can prove that $u \in X$ satisfies (2.3) easily. The proof is completed. □

Lemma 2.10. *Suppose that $h:(0,+\infty)\to R$ satisfies that $|h(t)| \le t^{k_2}$ for all $t \in (0,+\infty)$. Then $v \in Y$ is a solution of system*

$$\begin{cases} D^{\tau_m}v(t) + h(t) = 0, \quad t \in (0,\infty), \\ \lim_{t\to 0} t^{1-\beta_1}v(t) = y_0, \\ \lim_{t\to 0} t^{1-\beta_i}D^{\tau_{i-1}}v(t) = y_{i-1},\ i = 2,3,\cdots,m \end{cases} \tag{2.7}$$

if and only if $v \in Y$ satisfies

$$v(t) = -\int_0^t \frac{(t-s)^{\tau_m-1}}{\Gamma(\tau_m)}h(s)ds + \sum_{j=0}^{m-1} y_j \frac{\Gamma(\beta_{j+1})}{\Gamma(\tau_{j+1})}t^{\tau_{j+1}-1}. \tag{2.8}$$

Proof. The proof is similar to that of the proof of Lemma 2.3 and is omitted. □

For $(x,y) \in X \times Y$, let us define T by $T(x,y)(t) = ((T_1y)(t),(T_2x)(t))$ with

$$\begin{aligned} (T_1y)(t) &= -\frac{1}{\Gamma(\sigma_n)}\int_0^t (t-s)^{\sigma_n-1}\phi(s)f(s,y(s),D_{0^+}^p y(s))ds + \sum_{j=0}^{n-1} x_j \frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})}t^{\sigma_{j+1}-1}, \\ (T_2x)(t) &= -\frac{1}{\Gamma(\tau_m)}\int_0^t (t-s)^{\tau_m-1}\psi(s)g(s,x(s),D_{0^+}^q x(s))ds + \sum_{j=0}^{m-1} y_j \frac{\Gamma(\beta_{j+1})}{\Gamma(\tau_{j+1})}t^{\tau_{j+1}-1}. \end{aligned} \tag{2.9}$$

It is easy to show that

$$\begin{aligned} D_{0^+}^q(T_1y)(t) &= -\frac{1}{\Gamma(\sigma_n-q)}\int_0^t (t-s)^{\sigma_n-q-1}\phi(s)f(s,y(s),D_{0^+}^p y(s))ds \\ &+ \sum_{j=0}^{n-1} x_j \frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1}-q)}t^{\sigma_{j+1}-q-1}, \\ D_{0^+}^p(T_2x)(t) &= -\frac{1}{\Gamma(\tau_m-p)}\int_0^t (t-s)^{\tau_m-p-1}\psi(s)g(s,x(s),D_{0^+}^q x(s))ds \\ &+ \sum_{j=0}^{m-1} y_j \frac{\Gamma(\beta_{j+1})}{\Gamma(\tau_{j+1}-p)}t^{\tau_{j+1}-p-1}. \end{aligned} \tag{2.10}$$

Lemma 2.11. *Suppose that both f is a $\tau-$Caratheodory function and g a $\sigma-$Caratheodory function. Then*

(i) *both $T_1 : Y \to X$ and $T_2 : X \to Y$ are well defined and so $T : X \to X$ is well defined too;*

(ii) *the fixed point of the operator T coincides with the solution of IVP(1.1);*

(iii) *both $T_1 : Y \to X$ and $T_2 : X \to Y$ are completely continuous and so $T : X \to X$ is completely continuous.*

Proof. (i) For $y \in Y$, we get $\|y\| = r < +\infty$. Since f is a $\tau-$Caratheodory function, then there exist a positive number M_r such that

$$|f(t, y(t), D_{0^+}^p y(t))| = \left|f\left(t, \frac{\varrho(t)y(t)}{\varrho(t)}, \frac{t^p\varrho(t)D_{0^+}^p y(t)}{t^p\varrho(t)}\right)\right| \le M_r. \tag{2.11}$$

It is easy to show by similar methods used in the proof of Lemma 2.1 that $T_1y \in C(0,+\infty)$ and $D_{0^+}^q T_1 y \in C(0,\infty)$ and

$$\lim_{t\to 0}\rho(t)(T_1y)(t),\ \lim_{t\to+\infty}\rho(t)(T_1y)(t), \text{ and } \lim_{t\to 0}t^p\varrho(t)D_{0^+}^p y(t),\ \lim_{t\to+\infty}t^p\rho(t)D_{0^+}^q(T_1y)(t) \text{ exist.}$$

Hence $T_1y \in X$. Then $T_1 : Y \to X$ is well defined.

Similarly we can prove that $T_2 : X \to Y$ is well defined. So $T : Z \to Z$ is well defined.

(ii) It follows from Lemma 2.3 and Lemma 2.4 that the fixed point of the operator T coincides with the solution of IVP(1.1).

To prove that T is completely continuous, we must show that both T_1 and T_2 are completely continuous. We need to prove that

- both T_1 and T_2 are continuous,
- both T_1 and T_2 map bounded sets to relatively compact sets.

The remainder of the proof is completed by the following five steps.

Step 1. We prove that both T_1 and T_2 are continuous.

Let $y_n \in Y$ with $y_n \to y_0$ as $n \to \infty$. We will prove that $T_1y_n \to T_1y_0$ as $n \to \infty$. It is easy to see that there exists $r > 0$ such that $\|y_n\| \le r < \infty$ for all $n = 0, 1, 2, \cdots,$. Then there exists $M_r \ge 0$ such that (12) holds with y being replaced by y_n. One sees that

$$(T_1y_n)(t) = -\frac{1}{\Gamma(\sigma_n)}\int_0^t (t-s)^{\sigma_n-1}\phi(s)f(s, y_n(s), D_{0^+}^p y_n(s))ds + \sum_{j=0}^{n-1} x_j \frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})}t^{\sigma_{j+1}-1}, \tag{2.12}$$

and

$$\begin{aligned} D_{0^+}^q(T_1y_n)(t) = &-\frac{1}{\Gamma(\sigma_n-q)}\int_0^t (t-s)^{\sigma_n-q-1}\phi(s)f(s, y_n(s), D_{0^+}^p y_n(s))ds \\ &+ \sum_{j=0}^{n-1} x_j \frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1}-q)}t^{\sigma_{j+1}-q-1}. \end{aligned} \tag{2.13}$$

It follows from the Lebesgue dominated convergence theorem that

$$\lim_{n\to+\infty}\rho(t)|(T_1y_n)(t) - (T_1y_0)(t)| = 0,\ \lim_{n\to+\infty}q\rho(t)|D_{0^+}^q(T_1y_n)(t) - D_{0^+}^q(T_1y_0)(t)| = 0.$$

Hence we get

$$\lim_{n\to\infty} T_1y_n = T_1y_0.$$

Then T_1 is continuous. Similarly we can prove that T_2 is continuous.

Let $\Omega_1 \subseteq Y$ and $\Omega_2 \subseteq X$ be bounded subsets.

One sees that there exists $r > 0$ such that $\|y\| \le r$ for all $y \in \Omega_1$. Since f is a $\tau-$Caratheodory function, then there exist a positive number M_r such that (12) holds for all $y \in \Omega_1$.

Step 2. We prove that both $T_1(\Omega_1)$ and $T_2(\Omega_2)$ are uniformly bounded sets.

By the similar methods used in the proof of Lemma 2.1, it is easy to see that $T_1\Omega_1$ is uniformly bounded. We omit the details. Similarly we can prove that $T_2(\Omega_2)$ are uniformly bounded.

Step 3. We prove that both $T_1(\Omega_1)$ and $T_2(\Omega_2)$ are equi-continuous on finite closed interval on $(0,+\infty)$.

For $[a,b] \subset (0,+\infty)$ with $t_1, t_2 \in [a,b]$ with $t_1 > t_2$ and $y \in \Omega_1$, we have

$$\left|\frac{t_1^{1-\alpha_1}}{1+t_1^{\sigma}}|(T_1y)(t_1) - \frac{t_2^{1-\alpha_1}}{1+t_2^{\sigma}}(T_1y)(t_2)\right|$$

$$= \left| -\frac{1}{\Gamma(\sigma_n)}\frac{t_1^{1-\alpha_1}}{1+t_1^{\sigma}}\int_0^{t_1}(t_1-s)^{\sigma_n-1}\phi(s)f(s,y_n(s),D_{0^+}^p y_n(s))ds + \sum_{j=0}^{n-1} x_j \frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})}\frac{t_1^{\sigma_{j+1}-\alpha_1}}{1+t_1^{\sigma}}\right.$$

$$\left. -\left(-\frac{1}{\Gamma(\sigma_n)}\frac{t_2^{1-\alpha_1}}{1+t_2^{\sigma}}\int_0^{t_2}(t_1-s)^{\sigma_n-1}\phi(s)f(s,y_n(s),D_{0^+}^p y_n(s))ds + \sum_{j=0}^{n-1} x_j \frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})}\frac{t_2^{\sigma_{j+1}-\alpha_1}}{1+t_2^{\sigma}}\right)\right|$$

$$\le \frac{1}{\Gamma(\sigma_n)}\left|\frac{t_1^{1-\alpha_1}}{1+t_1^{\sigma}}\int_0^{t_1}(t_1-s)^{\sigma_n-1}\phi(s)f(s,y_n(s),D_{0^+}^p y_n(s))ds\right.$$

$$\left. -\frac{t_2^{1-\alpha_1}}{1+t_2^{\sigma}}\int_0^{t_2}(t_1-s)^{\sigma_n-1}\phi(s)f(s,y_n(s),D_{0^+}^p y_n(s))ds\right|$$

$$+\sum_{j=0}^{n-1}|x_j|\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})}\left|\frac{t_1^{\sigma_{j+1}-\alpha_1}}{1+t_1^{\sigma}} - \frac{t_2^{\sigma_{j+1}-\alpha_1}}{1+t_2^{\sigma}}\right|.$$

We know that

$$\left|\frac{t_1^{1-\alpha_1}}{1+t_1^{\sigma}}\int_0^{t_1}(t_1-s)^{\sigma_n-1}\phi(s)f(s,y_n(s),D_{0^+}^p y_n(s))ds\right.$$

$$\left. -\frac{t_2^{1-\alpha_1}}{1+t_2^{\sigma}}\int_0^{t_2}(t_1-s)^{\sigma_n-1}\phi(s)f(s,y_n(s),D_{0^+}^p y_n(s))ds\right|$$

$$\le \left|\frac{t_1^{1-\alpha_1}}{1+t_1^{\sigma}} - \frac{t_2^{1-\alpha_1}}{1+t_2^{\sigma}}\right|\int_0^{t_1}(t_1-s)^{\sigma_n-1}|\phi(s)f(s,y(s),D_{0^+}^p x(s)))|ds$$

$$+\frac{t_2^{1-\alpha_1}}{1+t_2^{\sigma}}\int_{t_2}^{t_2}(t_1-s)^{\sigma_n-1}|\phi(s)f(s,y(s),D_{0^+}^p y(s)))|ds$$

$$+\frac{t_2^{1-\alpha_1}}{1+t_2^{\sigma}}\int_0^{t_2}|(t_1-s)^{\sigma_n-1}-(t_2-s)^{\sigma_n-1}||\phi(s)f(s,y(s),D_{0^+}^p y(s)))|ds$$

$$\leq M_r\left|\frac{t_1^{1-\alpha_1}}{1+t_1^{\sigma}}-\frac{t_2^{1-\alpha_1}}{1+t_2^{\sigma}}\right|\int_0^{t_1}(t_1-s)^{\sigma_n-1}s^{k_1}ds$$

$$+M_r\frac{t_2^{1-\alpha_1}}{1+t_2^{\sigma}}\int_{t_2}^{t_1}(t_1-s)^{\sigma_n-1}s^{k_1}ds$$

$$+M_r\frac{t_2^{1-\alpha_1}}{1+t_2^{\sigma}}\int_0^{t_2}|(t_1-s)^{\sigma_n-1}-(t_2-s)^{\sigma_n-1}|s^{k_1}ds$$

$$\leq M_r\left|\frac{t_1^{1-\alpha_1}}{1+t_1^{\sigma}}-\frac{t_2^{1-\alpha_1}}{1+t_2^{\sigma}}\right|t_1^{\sigma_n+k_1}\int_0^{1}(1-w)^{\sigma_n-1}w^{k_1}dw$$

$$+M_r\frac{t_2^{1-\alpha_1}}{1+t_2^{\sigma}}t_1^{\sigma_n+k_1}\int_{\frac{t_2}{t_1}}^{1}(1-w)^{\sigma_n-1}w^{k_1}dw$$

$$+M_r\frac{t_2^{1-\alpha_1}}{1+t_2^{\sigma}}\int_0^{t_2}|(t_1-s)^{\sigma_n-1}-(t_2-s)^{\sigma_n-1}|s^{k_1}ds$$

$$\leq M_r\Bigg(\max\{a^{\sigma_n+k_1},b^{\sigma_n+k_1}\}\mathbf{B}(\sigma_n,k_1+1)\left|\frac{t_1^{1-\alpha_1}}{1+t_1^{\sigma}}-\frac{t_2^{1-\alpha_1}}{1+t_2^{\sigma}}\right|$$

$$+M_{\sigma,1-\alpha_1}\max\{a^{\sigma_n+k_1},b^{\sigma_n+k_1}\}\int_{\frac{t_2}{t_1}}^{1}(1-w)^{\sigma_n-1}w^{k_1}dw$$

$$+M_{\sigma,1-\alpha_1}\int_0^{t_2}|(t_1-s)^{\sigma_n-1}-(t_2-s)^{\sigma_n-1}|s^{k_1}ds\Bigg).$$

It is easy to show that $|u^\nu-v^\nu|\leq \nu b^{\nu-1}|u-v|$ for all $u,v\in[0,b]$, $\nu>1$ and $|u^\nu-v^\nu|\leq|u-v|^\nu$ for all $u,v\in[0,b]$, $\nu\in(0,1]$.

If $\sigma_n>2$, then

$$\int_0^{t_2}|(t_1-s)^{\sigma_n-1}-(t_2-s)^{\sigma_n-1}|s^{k_1}ds\leq\int_0^{b}[(t_1-s)^{\sigma_n-1}-(t_2-s)^{\sigma_n-1}]s^{k_1}ds$$

$$\leq\int_0^{b}[\sigma_n-1](t_1-t_2)s^{k_1}ds=(t_1-t_2)[\sigma_n-1]\frac{1}{k_1+1}b^{K_1+1}\to 0 \text{ as } t_2\to t_1.$$

If $1<\sigma_n\leq 2$, then

$$\int_0^{t_2}|(t_1-s)^{\sigma_n-1}-(t_2-s)^{\sigma_n-1}|s^{k_1}ds\leq\int_0^{b}[(t_1-s)^{\sigma_n-1}-(t_2-s)^{\sigma_n-1}]s^{k_1}ds$$

$$\leq\int_0^{b}(t_1-t_2)^{\sigma_n-1}s^{k_1}ds=(t_1-t_2)^{\sigma_n-1}\frac{1}{k_1+1}b^{k_1+1}\to 0 \text{ as } t_2\to t_1.$$

If $0 < \sigma_n \le 1$, then

$$\int_0^{t_2}|(t_1-s)^{\sigma_n-1}-(t_2-s)^{\sigma_n-1}|s^{k_1}ds=\int_0^{t_2}[(t_2-s)^{\sigma_n-1}-(t_1-s)^{\sigma_n-1}]s^{k_1}ds$$

$$=t_2^{\sigma_n+k_1}\int_0^1(1-w)^{\sigma_n-1}w^{k_1}dw-t_1^{\sigma_n+k_1}\int_0^{\frac{t_2}{t_1}}(1-w)^{\sigma_n-1}w^{k_1}dw$$

$$=[t_2^{\sigma_n+k_1}-t_1^{\sigma_n+k_1}]\mathbf{B}(\sigma_n,k_1+1)+t_1^{\sigma_n+k_1}\int_{\frac{t_2}{t_1}}^1(1-w)^{\sigma_n-1}w^{k_1}dw$$

$$\le[t_2^{\sigma_n+k_1}-t_1^{\sigma_n+k_1}]\mathbf{B}(\sigma_n,k_1+1)+\max\{a^{\sigma_n+k_1},b^{\sigma_n+k_1}\}\int_{\frac{t_2}{t_1}}^1(1-w)^{\sigma_n-1}w^{k_1}dw$$

$$\to 0 \text{ as } t_2\to t_1.$$

Hence

$$\left|\frac{t_1^{1-\alpha_1}}{1+t_1^{\sigma}}|(T_1y)(t_1)-\frac{t_2^{1-\alpha_1}}{1+t_2^{\sigma}}(T_1y)(t_2)\right|\to 0 \text{ uniformly in } \Omega_1 \text{ as } t_2\to t_1. \tag{2.14}$$

On the other hand, we have

$$\left|\frac{t_1^{1+q-\alpha_1}}{1+t_1^{\sigma}}D_{0^+}^q(T_1y)(t_1)-\frac{t_2^{1+q-\alpha_1}}{1+t_2^{\sigma}}D_{0^+}^q(T_1y)(t_2)\right|$$

$$=\left|-\frac{1}{\Gamma(\sigma_n-q)}\frac{t_1^{1+q-\alpha_1}}{1+t_1^{\sigma}}\int_0^{t_1}(t_1-s)^{\sigma_n-q-1}\phi(s)f(s,y_n(s),D_{0^+}^py_n(s))ds\right.$$

$$+\sum_{j=0}^{n-1}x_j\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1}-q)}\frac{t_1^{\sigma_{j+1}-\alpha_1}}{1+t_1^{\sigma}}$$

$$-\left(-\frac{1}{\Gamma(\sigma_n-q)}\frac{t_2^{1+q-\alpha_1}}{1+t_2^{\sigma}}\int_0^{t_2}(t_2-s)^{\sigma_n-q-1}\phi(s)f(s,y_n(s),D_{0^+}^py_n(s))ds\right.$$

$$\left.\left.+\sum_{j=0}^{n-1}x_j\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1}-q)}\frac{t_2^{\sigma_{j+1}-\alpha_1}}{1+t_2^{\sigma}}\right)\right|$$

$$\le\sum_{j=0}^{n-1}|x_j|\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1}-q)}\left|\frac{t_2^{\sigma_{j+1}-\alpha_1}}{1+t_2^{\sigma}}-\frac{t_1^{\sigma_{j+1}-\alpha_1}}{1+t_1^{\sigma}}\right|$$

$$+\frac{1}{\Gamma(\sigma_n-q)}\left|\frac{t_1^{1+q-\alpha_1}}{1+t_1^{\sigma}}\int_0^{t_1}(t_1-s)^{\sigma_n-q-1}\phi(s)f(s,y_n(s),D_{0^+}^py_n(s))ds\right.$$

$$\left.--\frac{t_2^{1+q-\alpha_1}}{1+t_2^{\sigma}}\int_0^{t_2}(t_2-s)^{\sigma_n-q-1}\phi(s)f(s,y_n(s),D_{0^+}^py_n(s))ds\right|.$$

Since

$$\left|\frac{t_1^{1+q-\alpha_1}}{1+t_1^{\sigma}}\int_0^{t_1}(t_1-s)^{\sigma_n-q-1}\phi(s)f(s,y_n(s),D_{0^+}^{p}y_n(s))ds\right.$$

$$\left.-\frac{t_2^{1+q-\alpha_1}}{1+t_2^{\sigma}}\int_0^{t_2}(t_2-s)^{\sigma_n-q-1}\phi(s)f(s,y_n(s),D_{0^+}^{p}y_n(s))ds\right|$$

$$\leq\left|\frac{t_1^{1+q-\alpha_1}}{1+t_1^{\sigma}}-\frac{t_2^{1+q-\alpha_1}}{1+t_2^{\sigma}}\right|\int_0^{t_1}(t_1-s)^{\sigma_n-q-1}L_1s^{k_1}M_rds$$

$$+\frac{t_2^{1+q-\alpha_1}}{1+t_2^{\sigma}}\int_{t_2}^{t_1}(t_1-s)^{\sigma_n-q-1}L_1s^{k_1}M_rds$$

$$+\frac{t_2^{1+q-\alpha_1}}{1+t_2^{\sigma}}\int_0^{t_2}|(t_1-s)^{\sigma_n-q-1}-(t_1-s)^{\sigma_n-q-1}|L_1s^{k_1}M_rds$$

$$=M_r\left(\left|\frac{t_1^{1+q-\alpha_1}}{1+t_1^{\sigma}}-\frac{t_2^{1+q-\alpha_1}}{1+t_2^{\sigma}}\right|t_1^{\sigma_n-q+k_1}\int_0^1(1-w)^{\sigma_n-q-1}w^{k_1}dw\right.$$

$$+\frac{t_2^{1+q-\alpha_1}}{1+t_2^{\sigma}}t_1^{\sigma_n-q+k_1}\int_{\frac{t_2}{t_1}}^{1}(1-w)^{\sigma_n-q-1}w^{k_1}dw$$

$$\left.+\frac{t_2^{1+q-\alpha_1}}{1+t_2^{\sigma}}\int_0^{t_2}|(t_1-s)^{\sigma_n-q-1}-(t_1-s)^{\sigma_n-q-1}|s^{k_1}ds\right)$$

$$\leq M_r\left(\left|\frac{t_1^{1+q-\alpha_1}}{1+t_1^{\sigma}}-\frac{t_2^{1+q-\alpha_1}}{1+t_2^{\sigma}}\right|\max\{a^{\sigma_n-q+k_1},b^{\sigma_n-q+k_1}\}\mathbf{B}(\sigma_n-q,k_1+1)\right.$$

$$+M_{\sigma,1+q-\alpha_1}\max\{a^{\sigma_n-q+k_1},b^{\sigma_n-q+k_1}\}\int_{\frac{t_2}{t_1}}^{1}(1-w)^{\sigma_n-q-1}w^{k_1}dw$$

$$\left.+\frac{t_2^{1+q-\alpha_1}}{1+t_2^{\sigma}}\int_0^{t_2}|(t_1-s)^{\sigma_n-q-1}-(t_1-s)^{\sigma_n-q-1}|s^{k_1}ds\right).$$

If $\sigma_n-q>2$, then

$$\int_0^{t_2}|(t_1-s)^{\sigma_n-q-1}-(t_2-s)^{\sigma_n-q-1}|s^{k_1}ds$$

$$\leq\int_0^{b}[(t_1-s)^{\sigma_n-q-1}-(t_2-s)^{\sigma_n-q-1}]s^{k_1}ds\leq\int_0^{b}[\sigma_n-q-1](t_1-t_2)s^{k_1}ds$$

$$=(t_1-t_2)[\sigma_n-q-1]\frac{1}{k_1+1}b^{k_1+1}\to 0\text{ as }t_2\to t_1.$$

If $1 < \sigma_n - q \le 2$, then

$$\int_0^{t_2} |(t_1 - s)^{\sigma_n - q - 1} - (t_2 - s)^{\sigma_n - q - 1}| s^{k_1} ds$$

$$\le \int_0^{b} [(t_1 - s)^{\sigma_n - q - 1} - (t_2 - s)^{\sigma_n - q - 1}] s^{k_1} ds$$

$$\le \int_0^{b} (t_1 - t_2)^{\sigma_n - q - 1} s^{k_1} ds = (t_1 - t_2)^{\sigma_n - q - 1} \frac{1}{k_1 + 1} b^{k_1 + 1} \to 0 \text{ as } t_2 \to t_1.$$

If $0 < \sigma_n - q \le 1$, then

$$\int_0^{t_2} |(t_1 - s)^{\sigma_n - q - 1} - (t_2 - s)^{\sigma_n - q - 1}| s^{k_1} ds$$

$$= \int_0^{t_2} [(t_2 - s)^{\sigma_n - q - 1} - (t_1 - s)^{\sigma_n - q - 1}] s^{k_1} ds$$

$$= t_2^{\sigma_n - q + k_1} \int_0^1 (1 - w)^{\sigma_n - q - 1} w^{k_1} dw - t_1^{\sigma_n - q + k_1} \int_0^{\frac{t_2}{t_1}} (1 - w)^{\sigma_n - q - 1} w^{k_1} dw$$

$$= [t_2^{\sigma_n - q + k_1} - t_1^{\sigma_n - q + k_1}] \mathbf{B}(\sigma_n - q, k_1 + 1) + t_1^{\sigma_n - q + k_1} \int_{\frac{t_2}{t_1}}^1 (1 - w)^{\sigma_n - q - 1} w^{k_1} dw$$

$$\le [t_2^{\sigma_n - q + k_1} - t_1^{\sigma_n - q + k_1}] \mathbf{B}(\sigma_n - q, k_1 + 1)$$

$$+ \max\{a^{\sigma_n - q + k_1}, b^{\sigma_n - q + k_1}\} \int_{\frac{t_2}{t_1}}^1 (1 - w)^{\sigma_n - q - 1} w^{k_1} dw$$

$$\to 0 \text{ as } t_2 \to t_1.$$

Hence

$$\left| \frac{t_1^{1+q-\alpha_1}}{1+t_1^{\sigma}} D_{0^+}^q (T_1 y)(t_1) - \frac{t_2^{1+q-\alpha_1}}{1+t_2^{\sigma}} D_{0^+}^q (T_1 y)(t_2) \right| \to 0 \text{ uniformly in } \Omega_1 \text{ as } t_2 \to t_1. \tag{2.15}$$

From (2.14) nd (2.15), we get that $\{t \to \rho(t)(T_1 y)(t) : y \in \Omega_1\}$ is equi-continuous on finite closed interval on $(0, \infty)$.

Similarly we can show that $\{t \to \rho(t)(T_2 x)(t) : x \in \Omega_2\}$ is equi-continuous on finite closed interval on $(0, \infty)$.

Step 4. We prove that both $\{t \to \rho(t)(T_1 y)(t) : y \in \Omega_1\}$ and $\{t \to \rho(t)(T_2 x)(t) : x \in \Omega_2\}$ are equi-convergent as $t \to 0$.

We see that

$$\left|\frac{t^{1-\alpha_1}}{1+t^\sigma}|(T_1y)(t)-x_0\right|$$

$$=\left|-\frac{1}{\Gamma(\sigma_n)}\frac{t^{1-\alpha_1}}{1+t^\sigma}\int_0^t(t-s)^{\sigma_n-1}\phi(s)f(s,y(s),D_{0^+}^py(s))ds+\sum_{j=1}^{n-1}x_j\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})}\frac{t^{\sigma_{j+1}-\alpha_1}}{1+t^\sigma}\right|$$

$$\leq\frac{1}{\Gamma(\sigma_n)}\frac{t^{1-\alpha_1}}{1+t^\sigma}\int_0^t(t-s)^{\sigma_n-1}L_1s^{k_1}M_rds+\sum_{j=1}^{n-1}|x_j|\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})}\frac{t^{\sigma_{j+1}-\alpha_1}}{1+t^\sigma}$$

$$=\frac{M_r}{\Gamma(\sigma_n)}\frac{t^{\sigma_n+k_1+1-\alpha_1}}{1+t^\sigma}\int_0^1(1-w)^{\sigma_n-1}w^{k_1}ds+\sum_{j=1}^{n-1}|x_j|\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})}\frac{t^{\sigma_{j+1}-\alpha_1}}{1+t^\sigma}$$

$$\to 0 \text{ uniformly in } \Omega_1 \text{ as } t\to 0.$$

Furthermore, we have

$$\left|\frac{t^{1+q-\alpha_1}}{1+t^\sigma}D_{0^+}^q(T_1y(t)-x_0\right|\leq\frac{M_r}{\Gamma(\sigma_n-q)}\frac{t^{1+q-\alpha_1}}{1+t^\sigma}\int_0^t(t-s)^{\sigma_n-q-1}s^{k_1}ds+\sum_{j=1}^{n-1}|x_j|\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1}-q)}\frac{t^{\sigma_{j+1}-\alpha_1}}{1+t^\sigma}$$

$$=\frac{M_r}{\Gamma(\sigma_n-q)}\frac{t^{\sigma_n+k_1+1-\alpha_1}}{1+t^\sigma}\int_0^1(1-w)^{\sigma_n-q-1}w^{k_1}dw+\sum_{j=1}^{n-1}|x_j|\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1}-q)}\frac{t^{\sigma_{j+1}-\alpha_1}}{1+t^\sigma}$$

$$\to 0 \text{ uniformly in } \Omega_1 \text{ as } t\to 0.$$

Hence $T_1(\Omega_1)$ id equi-convergent as $t\to 0$.

Similarly we can show that $T_2(\Omega_2)$ id equi-convergent as $t\to 0$.

Step 5. We prove that both $T_1(\Omega_1)$ and T_2 are equi-convergent as $t\to+\infty$. We get

$$\left|\frac{t^{1-\alpha_1}}{1+t^\sigma}|(T_1y)(t)\right|\leq\frac{M_r}{\Gamma(\sigma_n)}\frac{t^{1-\alpha_1}}{1+t^\sigma}\int_0^t(t-s)^{\sigma_n-1}s^{k_1}ds+\sum_{j=0}^{n-1}|x_j|\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})}\frac{t^{\sigma_{j+1}-\alpha_1}}{1+t^\sigma}$$

$$=\frac{M_r}{\Gamma(\sigma_n)}\frac{t^{\sigma_n+k_1+1-\alpha_1}}{1+t^\sigma}\int_0^1(1-w)^{\sigma_n-1}w^{k_1}dw+\sum_{j=0}^{n-1}|x_j|\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})}\frac{t^{\sigma_{j+1}-\alpha_1}}{1+t^\sigma}$$

$$\to 0 \text{ uniformly in } \Omega_1 \text{ as } t\to\infty.$$

Furthermore, we have

$$\left|\frac{t^{1+q-\alpha_1}}{1+t^\sigma}|(T_1y)(t)\right|\leq\frac{M_r}{\Gamma(\sigma_n-q)}\frac{t^{1+q-\alpha_1}}{1+t^\sigma}\int_0^t(t-s)^{\sigma_n-q-1}s^{k_1}ds+\sum_{j=0}^{n-1}|x_j|\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1}-q)}\frac{t^{\sigma_{j+1}-\alpha_1}}{1+t^\sigma}$$

$$=\frac{M_r}{\Gamma(\sigma_n-q)}\frac{t^{\sigma_n+k_1+1-\alpha_1}}{1+t^\sigma}\int_0^1(1-w)^{\sigma_n-q-1}w^{k_1}dw+\sum_{j=0}^{n-1}|x_j|\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1}-q)}\frac{t^{\sigma_{j+1}-\alpha_1}}{1+t^\sigma}$$

$$\to 0 \text{ uniformly in } \Omega_1 \text{ as } t\to\infty.$$

Hence $T_1(\Omega_1)$ is equi-convergent as $t\to\infty$.

Similarly we can show that $T_2(\Omega_2)$ id equi-convergent as $t \to \infty$.

From above discussion (Steps 1-5), we see that T is completely continuous. The proof is complete. □

3 Main results

We are in the position to prove the main results of the paper. We present the main assumptions:

(H1). f is a $\tau-$Caratheodory function and g a $\sigma-$Caratheodory function and satisfy the following assumptions: there exist non-zero functions $\Phi, \Psi : (0, +\infty) \to R$ measurable on each subinterval $(0, t]$ of $(0, +\infty)$ and non-decreasing functions

$$\left|f\left(t, \frac{u}{\rho(t)}, \frac{v}{t^p\rho(t)}\right) - \Phi(t)\right| \le F(u, v),$$

$$\left|g\left(t, \frac{u}{\varrho(t)}, \frac{v}{t^q\varrho(t)}\right) - \Psi(t)\right| \le G(u, v),$$

hold for all $t \in (0, +\infty), u, v \in R$.

Denote

$$\Phi_0(t) = -\frac{1}{\Gamma(\sigma_n)}\int_0^t (t-s)^{\sigma_n - 1}\phi(s)\Phi(s)ds + \sum_{j=0}^{n-1} x_j \frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})} t^{\sigma_{j+1}-1},$$
$$\Psi_0(t) = -\frac{1}{\Gamma(\tau_m)}\int_0^t (t-s)^{\tau_m - 1}\psi(s)\Psi(s)ds + \sum_{j=0}^{m-1} y_j \frac{\Gamma(\beta_{j+1})}{\Gamma(\tau_{j+1})} t^{\tau_{j+1}-1}. \quad (3.1)$$

It is easy to show that

$$D_{0^+}^q \Phi_0(t) = -\frac{1}{\Gamma(\sigma_n - q)}\int_0^t (t-s)^{\sigma_n - q - 1}\phi(s)\Phi(s)ds + \sum_{j=0}^{n-1} x_j \frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1}-q)} t^{\sigma_{j+1}-q-1},$$
$$D_{0^+}^p \Psi_0(t) = -\frac{1}{\Gamma(\tau_m - p)}\int_0^t (t-s)^{\tau_m - p - 1}\psi(s)\Psi(s)ds + \sum_{j=0}^{m-1} y_j \frac{\Gamma(\beta_{j+1})}{\Gamma(\tau_{j+1}-p)} t^{\tau_{j+1}-p-1}. \quad (3.2)$$

For $A > B > 0$, let $M_{A,B} = \frac{A-B}{A}\left(\frac{B}{A-B}\right)^{\frac{B}{A}}$ be defined in Section 2. Denote

$$M_0 = \max\left\{\frac{\mathbf{B}(\sigma_n - q, k_1+1)}{\Gamma(\sigma_n - q)} M_{\sigma, \sigma_n + k_1 + 1 - \alpha_1}, \frac{\mathbf{B}(\sigma_n, k_1+1)}{\Gamma(\sigma_n)} M_{\sigma, \sigma_n + k_1 + 1 - \alpha_1}\right\},$$

$$N_0 = \max\left\{\frac{\mathbf{B}(\tau_n - p, k_2+1)}{\Gamma(\tau_n - p)} M_{\tau, \tau_n + k_2 + 1 - \beta_1}, \frac{\mathbf{B}(\tau_n, k_2+1)}{\Gamma(\tau_n)} M_{\tau, \tau_n + k_2 + 1 - \beta_1}\right\},$$

$$a = M_0\left[\sum_{j=1}^{s-1}[A_j + B_j]||\Psi_0||^{\mu_j - \mu_s} + [A_s + B_s]\right],$$

$$b = N_0\left[\sum_{j=1}^{r-1}[C_j + D_j]||\Phi_0||]^{\delta_j - \delta_r} + [C_r + D_r]\right].$$

Theorem 3.1. *Suppose that (H1) holds. Then IVP(1.1) has at least one solution $(x, y) \in Z$ if the following inequality system*

$$M_0F(r_2 + ||\Psi_0||, r_2 + ||\Psi_0||) \le r_1, \; N_0G(r_1 + ||\Phi_0||, r_1 + ||\Phi_0||) \le r_2$$

has positive solution (r_1, r_2).

Proof. Let the Banach spaces X, Y and Z with their norms be defined in Section 2. Let $T : Z \to Z$ be defined by (2.9).

By Lemma 2.5, we seek solutions of IVP(1.1) by getting the fixed point of T in Z, and T is well defined and is completely continuous.

It is easy to show that $\Phi_0 \in X$, $\Psi_0 \in Y$. Let $r > 0$ and define

$$\overline{\Omega}_{r_1,r_2} = \{(x,y) \in Z : ||x - \Phi_0|| \le r_1,\ ||y - \Psi_0|| \le r_2\}.$$

For $(x,y) \in \overline{\Omega}_{r_1,r_2}$, we have $||x - \Phi_0|| \le r_1$ and $||y - \Psi_0|| \le r_2$. Then

$$\begin{aligned} ||x|| &\le ||x - \Phi_0|| + ||\Phi_0|| \le r_1 + ||\Phi_0||, \\ ||y|| &\le ||y - \Psi_0|| + ||\psi_0|| \le r_2 + ||\Psi_0||. \end{aligned}$$

Using (H1), using (3.1) and (3.2), we find

$$\begin{aligned} &\tfrac{t^{1-\alpha_1}}{1+t^\sigma}|(T_1 y)(t) - \Phi_0(t)| \\ &\le \tfrac{1}{\Gamma(\sigma_n)} \tfrac{t^{1-\alpha_1}}{1+t^\sigma} \int_0^t (t-s)^{\sigma_n - 1} \phi(s) |f(s, y(s), D_{0^+}^p y(s)) - \Phi(s)| ds \\ &\le \tfrac{1}{\Gamma(\sigma_n)} \tfrac{t^{1-\alpha_1}}{1+t^\sigma} \int_0^t (t-s)^{\sigma_n - 1} \phi(s) \left| f\left(s, \tfrac{1+s^\tau}{t^{1-\beta_1}} \tfrac{t^{1-\beta_1}}{1+s^\tau} y(s), \tfrac{1+s^\tau}{t^{1+p-\beta_1}} \tfrac{t^{1+p-\beta_1}}{1+s^\tau} D_{0^+}^p y(s)\right) - \Phi(s) \right| ds \\ &\le \tfrac{1}{\Gamma(\sigma_n)} \tfrac{t^{1-\alpha_1}}{1+t^\sigma} \int_0^t (t-s)^{\sigma_n - 1} \phi(s) F\left(\left| \tfrac{t^{1-\beta_1}}{1+s^\tau} y(s) \right|, \left| \tfrac{t^{1+p-\beta_1}}{1+s^\tau} D_{0^+}^p y(s) \right| \right) ds \\ &\le \tfrac{1}{\Gamma(\sigma_n)} \tfrac{t^{1-\alpha_1}}{1+t^\sigma} \int_0^t (t-s)^{\sigma_n - 1} s^{k_1} F(||y||, ||y||) ds \\ &\le \tfrac{\mathbf{B}(\sigma_n, k_1+1)}{\Gamma(\sigma_n)} \tfrac{t^{\sigma_n + k_1 + 1 - \alpha_1}}{1+t^\sigma} F(||y||, ||y||) \\ &\le \tfrac{\mathbf{B}(\sigma_n, k_1+1)}{\Gamma(\sigma_n)} M_{\sigma, \sigma_n + k_1 + 1 - \alpha_1} F(||y||, ||y||). \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\tfrac{t^{1+q-\alpha_1}}{1+t^\sigma} |D_{0^+}^q (T_1 y)(t) - D_{0^+}^q \Phi_0(t)| \\ &\le \tfrac{1}{\Gamma(\sigma_n - q)} \tfrac{t^{1+q-\alpha_1}}{1+t^\sigma} \int_0^t (t-s)^{\sigma_n - q - 1} |f(s, y(s), D_{0^+}^p y(s)) - \Phi(s)| ds \\ &\le \tfrac{\mathbf{B}(\sigma_n - q, k_1+1)}{\Gamma(\sigma_n - q)} M_{\sigma, \sigma_n + k_1 + 1 - \alpha_1} F(||y||, ||y||). \end{aligned}$$

It follows that

$$||T_1 y - \Phi_0|| \le M_0 F(||y||, ||y||) \le M_0 F(r_2 + ||\Psi_0||, r_2 + ||\Psi_0||). \tag{3.3}$$

Similarly we can show that

$$||T_2 x - \Psi_0|| \le N_0 G(||x||, ||x||) \le N_0 G(r_1 + ||\Phi_0||, r_1 + ||\Phi_0||).$$

From the assumption, we know that there exist $r_1 > 0, r_2 > 0$ such that $M_0F(r_2 + \|\Psi_0\|, r_2 + \|\Psi_0\|) \le r_1$ and $N_0G(r_1 + \|\Phi_0\|, r_1 + \|\Phi_0\|) \le r_2$.

Then, Schauder fixed point theorem implies that T has a fixed point $(x,y) \in \overline{\Omega}_{r_1,r_2}$, which is a solution of IVP (1.1). The proof is completed. □

(H2). f is a $\tau-$Caratheodory function and g a $\sigma-$Caratheodory function and satisfy the following assumptions: there exist non-zero functions $\Phi, \Psi : (0,+\infty) \to R$ measurable on each subinterval $(0,t]$ of $(0,+\infty)$ and numbers

$$A_i, B_i (i = 1,2,\cdots,s), C_i, D_i (i = 1,2,\cdots,r) \ge 0,$$

$$\mu_s > \mu_{s-1} > \cdots > \mu_1 > 0, \delta_r > \delta_{r-1} > \cdots > \delta_1 > 0,$$

such that

$$\left| f\left(t, \frac{u}{\rho(t)}, \frac{v}{t^p\rho(t)}\right) - \Phi(t)\right| \le \sum_{j=1}^{s} A_j|u|^{\mu_j} + \sum_{j=1}^{s} B_j|v|^{\mu_j},$$

$$\left| g\left(t, \frac{u}{\varrho(t)}, \frac{v}{t^q\varrho(t)}\right) - \Psi(t)\right| \le \sum_{j=1}^{r} C_j|u|^{\delta_j} + \sum_{j=1}^{r} D_j|v|^{\delta_j},$$

hold for all $t \in (0,+\infty), u, v \in R$.

Theorem 3.2. *Suppose that (H2) holds. Then IVP(1.1) has at least one solution $(x,y) \in Z$ if*

(i) $\mu_s\delta_r > 1$ *with*

$$\begin{aligned} &\frac{(\delta_r\mu_s)^{\delta_r\mu_s}}{(\delta_r\mu_s-1)^{\delta_r\mu_s-1}}\left[\|\Phi_0\| + \left(\frac{\|\Psi_0\|}{b}\right)^{\frac{1}{\delta_r}}\right]^{\delta_r\mu_s-1} \le \frac{1}{ab^{\mu_s}} \text{ for } \delta_r > 1, \\ &\frac{(\delta_r\mu_s)^{\delta_r\mu_s}}{(\delta_r\mu_s-1)^{\delta_r\mu_s-1}}\left[\|\Psi_0\| + \left(\frac{\|\Phi_0\|}{a}\right)^{\frac{1}{\mu_s}}\right]^{\delta_r\mu_s-1} \le \frac{1}{ba^{\delta_r}} \text{ for } \mu_s > 1 \end{aligned} \tag{3.4}$$

or

(ii) $\mu_s\delta_r = 1$ *with*

$$\text{either } a < \left(\frac{1}{b}\right)^{\frac{1}{\delta_r}} \text{ or } b < \left(\frac{1}{a}\right)^{\frac{1}{\mu_s}} \tag{3.5}$$

or

(iii) $\mu_s\delta_r < 1$.

Proof. Let the Banach spaces X, Y and Z with their norms be defined in Section 2. Let $T : Z \to Z$ be defined by (2.9).

By Lemma 2.5, we seek solutions of IVP(1.1) by getting the fixed point of T in Z, and T is well defined and is completely continuous.

Let Φ_0 and Ψ_0 be defined by (3.1). Then we get (3.2). It is easy to show that $\Phi_0 \in X$, $\Psi_0 \in Y$. Let $r > 0$ and define

$$\overline{\Omega}_{r_1,r_2} = \{(x,y) \in Z : \|x - \Phi_0\| \le r_1,\ \|y - \Psi_0\| \le r_2\}.$$

For $(x,y) \in \overline{\Omega}_{r_1,r_2}$, we have $\|x - \Phi_0\| \le r_1$ and $\|y - \Psi_0\| \le r_2$. Then

$$\begin{aligned} \|x\| &\le \|x - \Phi_0\| + \|\Phi_0\| \le r_1 + \|\Phi_0\|, \\ \|y\| &\le \|y - \Psi_0\| + \|\psi_0\| \le r_2 + \|\Psi_0\|. \end{aligned}$$

Using (H2), using (3.1) and (3.2), we find

$$\frac{t^{1-\alpha_1}}{1+t^\sigma}|(T_1y)(t)-\Phi_0(t)|$$

$$\leq \frac{1}{\Gamma(\sigma_n)}\frac{t^{1-\alpha_1}}{1+t^\sigma}\int_0^t (t-s)^{\sigma_n-1}\phi(s)|f(s,y(s),D_{0^+}^p y(s))-\Phi(s)|ds$$

$$\leq \frac{1}{\Gamma(\sigma_n)}\frac{t^{1-\alpha_1}}{1+t^\sigma}\int_0^t (t-s)^{\sigma_n-1}\phi(s)\left|f\left(s,\frac{1+s^\tau}{t^{1-\beta_1}}\frac{t^{1-\beta_1}}{1+s^\tau}y(s),\frac{1+s^\tau}{t^{1+p-\beta_1}}\frac{t^{1+p-\beta_1}}{1+s^\tau}D_{0^+}^p y(s)\right)-\Phi(s)\right|ds$$

$$\leq \frac{1}{\Gamma(\sigma_n)}\frac{t^{1-\alpha_1}}{1+t^\sigma}\int_0^t (t-s)^{\sigma_n-1}\phi(s)\left[\sum_{j=1}^s A_j\left|\frac{t^{1-\beta_1}}{1+s^\tau}y(s)\right|^{\mu_j}+\sum_{j=1}^s B_j\left|\frac{t^{1+p-\beta_1}}{1+s^\tau}D_{0^+}^p y(s)\right|^{\mu_j}\right]ds$$

$$\leq \frac{1}{\Gamma(\sigma_n)}\frac{t^{1-\alpha_1}}{1+t^\sigma}\int_0^t (t-s)^{\sigma_n-1}s^{k_1}\left[\sum_{j=1}^s [A_j+B_j]||y||^{\mu_j}\right]ds$$

$$\leq \frac{\mathbf{B}(\sigma_n,k_1+1)}{\Gamma(\sigma_n)}\frac{t^{\sigma_n+k_1+1-\alpha_1}}{1+t^\sigma}\left[\sum_{j=1}^s [A_j+B_j]||y||^{\mu_j}\right]$$

$$\leq \frac{\mathbf{B}(\sigma_n,k_1+1)}{\Gamma(\sigma_n)}M_{\sigma,\sigma_n+k_1+1-\alpha_1}\left[\sum_{j=1}^s [A_j+B_j]||y||^{\mu_j}\right].$$

Similarly, we have

$$\frac{t^{1+q-\alpha_1}}{1+t^\sigma}|D_{0^+}^q(T_1y)(t)-D_{0^+}^q\Phi_0(t)|$$

$$\leq \frac{1}{\Gamma(\sigma_n-q)}\frac{t^{1+q-\alpha_1}}{1+t^\sigma}\int_0^t (t-s)^{\sigma_n-q-1}|f(s,y(s),D_{0^+}^p y(s))-\Phi(s)|ds$$

$$\leq \frac{\mathbf{B}(\sigma_n-q,k_1+1)}{\Gamma(\sigma_n-q)}M_{\sigma,\sigma_n+k_1+1-\alpha_1}\left[\sum_{j=1}^s [A_j+B_j]||y||^{\mu_j}\right].$$

It follows that

$$||T_1y-\Phi_0||\leq M_0\left[\sum_{j=1}^s [A_j+B_j]||y||^{\mu_j}\right]\leq M_0\left[\sum_{j=1}^s [A_j+B_j][r+||\Psi_0||]^{\mu_j}\right]$$

$$\leq M_0[r_2+||\Psi_0||]^{\mu_s}\left[\sum_{j=1}^{s-1}[A_j+B_j][r_2+||\Psi_0||]^{\mu_j-\mu_s}+[A_s+B_s]\right]$$

$$\leq M_0[r_2+||\Psi_0||]^{\mu_s}\left[\sum_{j=1}^{s-1}[A_j+B_j]||\Psi_0||^{\mu_j-\mu_s}+[A_s+B_s]\right].$$

Hence

$$||T_1y-\Phi_0||\leq M_0\left[\sum_{j=1}^{s-1}[A_j+B_j]||\Psi_0||^{\mu_j-\mu_s}+[A_s+B_s]\right][r_2+||\Psi_0||]^{\mu_s}=a[r_2+||\Psi_0||]^{\mu_s}. \tag{3.6}$$

Similarly we can show that

$$\|T_2x-\Psi_0\| \le N_0\left[\sum_{j=1}^{r-1}[C_j+D_j]\|\Phi_0\|]^{\delta_j-\delta_r}+[C_r+D_r]\right][r_1+\|\Phi_0\|]^{\delta_r}=b[r_1+\|\Phi_0\|]^{\delta_r}. \tag{3.7}$$

Consider the following inequality system

$$\begin{cases} a[r_2+\|\Psi_0\|]^{\mu_s}\le r_1, \\ b[r_1+\|\Phi_0\|]^{\delta_r}\le r_2. \end{cases}$$

We will prove that it has a positive solution (r_1,r_2). This inequality system is changed to

$$b[r_1+\|\Phi_0\|]^{\delta_r}\le r_2\le\left(\frac{r_1}{a}\right)^{\frac{1}{\mu_s}}-\|\Psi_0\|, \tag{3.8}$$

or

$$a[r_2+\|\Psi_0\|]^{\mu_s}\le r_1\le\left(\frac{r_2}{b}\right)^{\frac{1}{\delta_r}}-\|\Phi_0\|. \tag{3.9}$$

Case (i). $\mu_s\delta_r>1$.

It is easy to show that $e^l+f^l\le(e+f)^l$ for all $e,f>0$ and $l>1$.

If $\delta_r>1$, choose

$$r_1=\frac{1}{\delta_r\mu_s-1}\left[\|\Phi_0\|+\left(\frac{\|\Psi_0\|}{b}\right)^{\frac{1}{\delta_r}}\right].$$

Then we get from

$$\frac{(\delta_r\mu_s)^{\delta_r\mu_s}}{(\delta_r\mu_s-1)^{\delta_r\mu_s-1}}\left[\|\Phi_0\|+\left(\frac{\|\Psi_0\|}{b}\right)^{\frac{1}{\delta_r}}\right]^{\delta_r\mu_s-1}\le\frac{1}{ab^{\mu_s}}$$

that

$$\frac{\left[r_1+\|\Phi_0\|+\left(\frac{\|\Psi_0\|}{b}\right)^{\frac{1}{\delta_r}}\right]^{\delta_r\mu_s}}{r_1}\le\frac{1}{ab^{\mu_s}}.$$

Since

$$b[r_1+\|\Phi_0\|]^{\delta_r}+\|\Psi_0\|\le b\left[r_1+\|\Phi_0\|+\left(\frac{\|\Psi_0\|}{b}\right)^{\frac{1}{\delta_r}}\right]^{\delta_r},$$

we get

$$b[r_1+\|\Phi_0\|]^{\delta_r}\le\left(\frac{r_1}{a}\right)^{\frac{1}{\mu_s}}-\|\Psi_0\|.$$

Choose r_2 such that

$$b[r_1+\|\Phi_0\|]^{\delta_r}\le r_2\le\left(\frac{r_1}{a}\right)^{\frac{1}{\mu_s}}-\|\Psi_0\|. \tag{3.10}$$

Then, for $(x,y)\in\overline{\Omega}_{r_1,r_2}$, using (3.10), we have

$$\|T_1y-\Phi_0\|\le a[r_2+\|\Psi_0\|]^{\mu_s}\le r_1,\ \|T_2x-\Psi_0\|\le b[r_1+\|\Phi_0\|]^{\delta_r}\le r_2.$$

Then $T(x,y)=(T_1y,T_2x)\in\overline{\Omega}_{r_1,r_2}$.

Then, Schauder fixed point theorem implies that T has a fixed point $(x,y)\in\overline{\Omega}_{r_1,r_2}$, which is a solution of IVP (1.1).

If $\mu_s > 1$, choose

$$r_2 = \frac{1}{\delta_r\mu_s - 1}\left[\|\Psi_0\| + \left(\frac{\|\Phi_0\|}{a}\right)^{\frac{1}{\mu_s}}\right].$$

Then we get from

$$\frac{(\delta_r\mu_s)^{\delta_r\mu_s}}{(\delta_r\mu_s - 1)^{\delta_r\mu_s - 1}}\left[\|\Psi_0\| + \left(\frac{\|\Phi_0\|}{a}\right)^{\frac{1}{\mu_s}}\right]^{\delta_r\mu_s - 1} \le \frac{1}{ba^{\delta_r}}$$

that

$$\frac{\left[r_2 + \|\Psi_0\| + \left(\frac{\|\Phi_0\|}{a}\right)^{\frac{1}{\mu_s}}\right]^{\delta_r\mu_s}}{r_2} \le \frac{1}{ba^{\delta_r}}.$$

Since

$$a[r_2 + \|\Psi_0\|]^{\mu_s} + \|\Phi_0\| \le a\left[r_2 + \|\Psi_0\| + \left(\frac{\|\Phi_0\|}{a}\right)^{\frac{1}{\mu_s}}\right]^{\mu_s},$$

we get

$$a[r_2 + \|\Psi_0\|]^{\mu_s} \le \left(\frac{r_2}{b}\right)^{\frac{1}{\delta_r}} - \|\Phi_0\|.$$

Choose r_1 such that

$$a[r_2 + \|\Psi_0\|]^{\mu_s} \le r_1 \le \left(\frac{r_2}{b}\right)^{\frac{1}{\delta_r}} - \|\Phi_0\|. \tag{3.11}$$

Then, for $(x,y) \in \overline{\Omega}_{r_1,r_2}$, using (3.11), we have

$$\|T_1y - \Phi_0\| \le a[r_2 + \|\Psi_0\|]^{\mu_s} \le r_1,\ \|T_2x - \Psi_0\| \le b[r_1 + \|\Phi_0\|]^{\delta_r} \le r_2.$$

Then $T(x,y) = (T_1y, T_2x) \in \overline{\Omega}_{r_1,r_2}$.

Then, Schauder fixed point theorem implies that T has a fixed point $(x,y) \in \overline{\Omega}_{r_1,r_2}$, which is a solution of IVP (1.1).

Case (ii). $\mu_s\delta_r = 1$.

For $a < \left(\frac{1}{b}\right)^{\frac{1}{\delta_r}}$, since

$$\lim_{r_1\to+\infty}\frac{a[r_2 + \|\Psi_0\|]^{\mu_s}}{\left(\frac{r_2}{b}\right)^{\frac{1}{\delta_r}} - \|\Phi_0\|} = \frac{a}{\left(\frac{1}{b}\right)^{\frac{1}{\delta_r}}} < 1,$$

we can choose $r_2 > 0$ sufficiently large such that

$$a[r_2 + \|\Psi_0\|]^{\mu_s} \le \left(\frac{r_2}{b}\right)^{\frac{1}{\delta_r}} - \|\Phi_0\|.$$

Then we can choose r_1 such that

$$a[r_2 + \|\Psi_0\|]^{\mu_s} \le r_1 \le \left(\frac{r_2}{b}\right)^{\frac{1}{\delta_r}} - \|\Phi_0\|. \tag{3.12}$$

Then, for $(x,y) \in \overline{\Omega}_{r_1,r_2}$, using (3.12), we have

$$\|T_1y - \Phi_0\| \le Ma[r_2 + \|\Psi_0\|]^{\mu_s} \le r_1,$$

$$\|T_2x - \Psi_0\| \le b[r_1 + \|\Phi_0\|]^{\delta_r} \le r_2.$$

Then $T(x,y) = (T_1 y, T_2 x) \in \overline{\Omega}_{r_1,r_2}$.

Then, Schauder fixed point theorem implies that T has a fixed point $(x,y) \in \overline{\Omega}_{r_1,r_2}$, which is a solution of IVP (1.1).

For $b < \left(\frac{1}{a}\right)^{\frac{1}{\mu_s}}$, since

$$\lim_{r_1 \to +\infty} \frac{b[r_1 + ||\Phi_0||]^{\delta_r}}{\left(\frac{r_1}{a}\right)^{\frac{1}{\mu_s}} - ||\Psi_0||} = \frac{b}{\left(\frac{1}{a}\right)^{\frac{1}{\mu_s}}} < 1,$$

we can choose $r_1 > 0$ sufficiently large such that

$$b[r_1 + ||\Phi_0||]^{\delta_r} \le \left(\frac{r_1}{a}\right)^{\frac{1}{\mu_s}} - ||\Psi_0||.$$

Then we can choose r_2 such that

$$b[r_1 + ||\Phi_0||]^{\delta_r} \le r_2 \le \left(\frac{r_1}{a}\right)^{\frac{1}{\mu_s}} - ||\Psi_0||. \tag{3.13}$$

Then, for $(x,y) \in \overline{\Omega}_{r_1,r_2}$, using (3.13), we have

$$||T_1 y - \Phi_0|| \le Ma[r_2 + ||\Psi_0||]^{\mu_s} \le r_1,$$

$$||T_2 x - \Psi_0|| \le b[r_1 + ||\Phi_0||]^{\delta_r} \le r_2.$$

Then $T(x,y) = (T_1 y, T_2 x) \in \overline{\Omega}_{r_1,r_2}$.

Then, Schauder fixed point theorem implies that T has a fixed point $(x,y) \in \overline{\Omega}_{r_1,r_2}$, which is a solution of IVP (1.1).

Case (iii). $\mu_s \delta_r < 1$.

It is easy to see that there exists $r_1 > 0$ sufficiently large such that

$$b[r_1 + ||\Phi_0||]^{\delta_r} \le \left(\frac{r_1}{a}\right)^{\frac{1}{\mu_s}} - ||\Psi_0||.$$

This allows us to choose r_2 such that

$$b[r_1 + ||\Phi_0||]^{\delta_r} \le r_2 \le \left(\frac{r_1}{a}\right)^{\frac{1}{\mu_s}} - ||\Psi_0||. \tag{3.14}$$

Then, for $(x,y) \in \overline{\Omega}_{r_1,r_2}$ using (3.14), we have

$$||T_1 y - \Phi_0|| \le a[r_2 + ||\Psi_0||]^{\mu_s} \le r_1, \; ||T_2 x - \Psi_0|| \le b[r_1 + ||\Phi_0||]^{\delta_r} \le r_2.$$

Then $T(x,y) = (T_1 y, T_2 x) \in \overline{\Omega}_{r_1,r_2}$.

Then, Schauder fixed point theorem implies that T has a fixed point $(x,y) \in \overline{\Omega}_{r_1,r_2}$, which is a solution of IVP (1.1).

The proof is complete. □

4 An example

In this section, we given an example to illustrate Theorem 3.1.

Example 4.1. consider the following problem

$$\begin{cases} D_0^{\frac{1}{2}} D_0^{\frac{1}{3}} x(t) + t^{-\frac{1}{2}} f(t, y(t), D_{0^+}^{\frac{1}{5}} y(t)) = 0, \quad t \in (0, +\infty), \\ D_0^{\frac{1}{4}} D_0^{\frac{1}{8}} y(t) + t^{-\frac{1}{2}} g(t, x(t), D_{0^+}^{\frac{1}{6}} x(t)) = 0, \quad t \in (0, +\infty), \\ \lim_{t\to 0} t^{\frac{2}{3}} x(t) = x_0, \ \lim_{t\to 0} t^{\frac{7}{8}} y(t) = x_0, \\ \lim_{t\to 0} t^{\frac{1}{2}} D^{\frac{1}{3}} x(t) = x_1, \ \lim_{t\to 0} t^{\frac{3}{4}} D^{\frac{1}{8}} x(t) = y_1, \end{cases} \tag{4.1}$$

where $x_0, x_1, y_0, y_1 \in R$, $\phi(t) = \psi(t) = t^{-\frac{1}{2}}$ and

$$f(t, u, v) = 1 + A \left(\frac{t^{\frac{7}{8}}}{1+t} u \right)^{\mu} + B \left(\frac{t^{\frac{43}{40}}}{1+t} v \right)^{\mu},$$

$$g(t, u, v) = 1 + C \left(\frac{t^{\frac{2}{5}}}{1+t^{\frac{3}{2}}} u \right)^{\delta} + D \left(\frac{t^{\frac{5}{6}}}{1+t^{\frac{3}{2}}} v \right)^{\delta}$$

with $A, B, C, D \geq 0, \delta, \mu > 0$. Then IVP(4.1) has at least one solution for all sufficiently small $A, B, C, D, |x_0|, |y_0|$.

Proof. Corresponding to (1.1), we have $m = n = 2$, $\alpha_1 = \frac{1}{3}, \alpha_2 = \frac{1}{2}$ and $\beta_1 = \frac{1}{8}$ and $\beta_2 = \frac{1}{4}$, $p = \frac{1}{5}$ and $q = \frac{1}{6}$.

It is easy to see that

- $x_i \in R(i = 0, 1)$, $y_i \in R(i = 0, 1)$, $p, q \in (0, 1)$ with $q < \sigma_2 = \alpha_1 + \alpha_2 = \frac{5}{6}$ and $p < \tau_2 = \beta_1 + \beta_2 = \frac{3}{8}$,
- $\alpha_i \in (0, 1)(i = 1, 2), \beta_i \in (0, 1)(i = 1, 2)$,
- $\phi, \psi : (0, \infty) \to [0, \infty)$ satisfy that

$$\phi(t) \leq t^{k_1}, \ \ \psi(t) \leq t^{k_2}, t \in (0, \infty),$$

with $k_1 = k_2 = -\frac{1}{2}$,.

Choose $\tau = 1, \sigma = \frac{3}{2}$. Then $\tau > \tau_2 + k_2 + 1$ and $\sigma > \sigma_2 + k_1 + 1$. By computation, one sees that

$$f\left(t, \frac{1+t^{\tau}}{t^{1-\beta_1}} u, \frac{1+t^{\tau}}{t^{1+p-\beta_1}} v\right) = 1 + Au^{\mu} + Bv^{\mu},$$
$$g\left(t, \frac{1+t^{\sigma}}{t^{1-\alpha_1}} u, \frac{1+t^{\sigma}}{t^{1+q-\alpha_1}} v\right) = 1 + Cu^{\delta} + Dv^{\delta}.$$

So

- $f, g : (0, \infty) \times R^2 \to R$ and f is a $\tau-$Caratheodory function and g a $\sigma-$Caratheodory function.

It is easy to see that

(H2). f is a a $\tau-$Caratheodory function and g a $\sigma-$Caratheodory function satisfying the following assumptions: there exist non-zero functions $\Phi(t) = \Psi(t) = 1$ and numbers

$$A, B \geq 0, \ C, D \geq 0, \ \mu > 0, \delta > 0,$$

such that

$$\left|f\left(t,\frac{1+t^\tau}{t^{1-\beta_1}}u,\frac{1+t^\tau}{t^{1+p-\beta_1}}v\right)-\Phi(t)\right|\le A|u|^\mu+B|v|^\mu,$$
$$\left|g\left(t,\frac{1+t^\sigma}{t^{1-\alpha_1}}u,\frac{1+t^\sigma}{t^{1+q-\alpha_1}}v\right)-\Psi(t)\right|\le C|u|^\delta+D|v|^\delta,$$

hold for all $t\in(0,+\infty), u,v\in R$ with $r=s=1$ and $\mu=\mu_1$, $\delta=\delta_1$.

We have

$$\Phi_0(t)=-\frac{1}{\Gamma(\sigma_n)}\int_0^t(t-s)^{\sigma_n-1}\phi(s)\Phi(s)ds+\sum_{j=0}^{n-1}x_j\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})}t^{\sigma_{j+1}-1}$$

$$=-\frac{1}{\Gamma(5/6)}\int_0^t(t-s)^{-\frac{1}{6}}s^{-\frac{1}{2}}ds+x_0t^{-\frac{2}{3}}=-\frac{\mathbf{B}(5/6,1/2)}{\Gamma(5/6)}t^{\frac{1}{3}}+x_0t^{-\frac{2}{3}}$$

$$\Psi_0(t)=-\frac{1}{\Gamma(\tau_m)}\int_0^t(t-s)^{\tau_m-1}\psi(s)\Psi(s)ds+\sum_{j=0}^{m-1}y_j\frac{\Gamma(\beta_{j+1})}{\Gamma(\tau_{j+1})}t^{\tau_{j+1}-1}$$

$$=-\frac{1}{\Gamma(3/8)}\int_0^t(t-s)^{-\frac{5}{8}}s^{-\frac{1}{2}}ds+y_0t^{-\frac{7}{8}}=-\frac{\mathbf{B}(3/8,1/2)}{\Gamma(3/8)}t^{-\frac{1}{8}}+y_0t^{-\frac{7}{8}},$$

$$D_{0^+}^q\Phi_0(t)=D_{0^+}^{\frac{1}{6}}\left(-\frac{\mathbf{B}(5/6,1/2)}{\Gamma(5/6)}t^{\frac{1}{3}}+x_0t^{-\frac{2}{3}}\right)=-\frac{\mathbf{B}(5/6,1/2)}{\Gamma(5/6)}\frac{\Gamma(4/3)}{\Gamma(7/6)}t^{\frac{1}{6}}+x_0\frac{\Gamma(1/3)}{\Gamma(1/6)}t^{-\frac{5}{6}}$$

$$D_{0^+}^p\Psi_0(t)=D_{0^+}^{\frac{1}{5}}\left(-\frac{\mathbf{B}(3/8,1/2)}{\Gamma(3/8)}t^{-\frac{1}{8}}+y_0t^{-\frac{7}{8}}\right)=-\frac{\mathbf{B}(3/8,1/2)}{\Gamma(3/8)}\frac{\Gamma(7/8)}{\Gamma(27/40)}t^{-\frac{13}{40}}+y_0\frac{\Gamma(1/8)}{\Gamma(3/40)}t^{-\frac{43}{40}}.$$

Then

$$||\Phi_0||=\max\left\{\sup_{t\in(0,\infty)}\frac{t^{\frac{2}{3}}}{1+t}|\Phi_0(t)|,\ \sup_{t\in(0,\infty)}\frac{t^{\frac{5}{6}}}{1+t}|D_{0^+}^{\frac{1}{6}}\Phi_0(t)|\right\}$$

$$\le\max\left\{\sup_{t\in(0,\infty)}\frac{\frac{\mathbf{B}(5/6,1/2)}{\Gamma(5/6)}t+|x_0|}{1+t},\ \sup_{t\in(0,\infty)}\frac{\frac{\mathbf{B}(5/6,1/2)}{\Gamma(5/6)}\frac{\Gamma(4/3)}{\Gamma(7/6)}t+|x_0|\frac{\Gamma(1/3)}{\Gamma(1/6)}}{1+t}\right\}$$

$$\le\max\left\{\frac{\mathbf{B}(5/6,1/2)}{\Gamma(5/6)}M_{1,1}+|x_0|,\ \frac{\mathbf{B}(5/6,1/2)}{\Gamma(5/6)}\frac{\Gamma(4/3)}{\Gamma(7/6)}M_{1,1}+|x_0|\frac{\Gamma(1/3)}{\Gamma(1/6)}\right\}$$

and

$$||\Psi_0||=\max\left\{\sup_{t\in(0,\infty)}\frac{t^{\frac{7}{8}}}{1+t^{\frac{3}{2}}}|\Phi_0(t)|,\ \sup_{t\in(0,\infty)}\frac{t^{\frac{43}{40}}}{1+^{\frac{3}{2}}}|D_{0^+}^{\frac{1}{5}}\Psi_0(t)|\right\}$$

$$\le\max\left\{\frac{\mathbf{B}(3/8,1/2)}{\Gamma(3/8)}M_{3/2,3/4}+|y_0|,\ \frac{\mathbf{B}(3/8,1/2)}{\Gamma(3/8)}\frac{\Gamma(7/8)}{\Gamma(27/40)}M_{3/2,3/4}+|y_0|\frac{\Gamma(1/8)}{\Gamma(3/40)}\right\}.$$

By direct computation, we get

$$M_0=\max\left\{\frac{\mathbf{B}(2/3,1/2)}{\Gamma(2/3)}M_{3/2,1},\ \frac{\mathbf{B}(5/6,1/2)}{\Gamma(5/6)}M_{3/2,1}\right\},$$

$$N_0=\max\left\{\frac{\mathbf{B}(7/40,1/2)}{\Gamma(7/40)}M_{1,3/4},\ \frac{\mathbf{B}(3/8,1/2)}{\Gamma(3/8)}M_{1,3/4}\right\},$$

$$a=M_0\left[\sum_{j=1}^{s-1}[A_j+B_j]||\Psi_0||^{\mu_j-\mu_s}+[A_s+B_s]\right]=M_0(A+B),$$

$$b=N_0\left[\sum_{j=1}^{r-1}[C_j+D_j]||\Phi_0||]^{\delta_j-\delta_r}+[C_r+D_r]\right]=N_0(C+D).$$

From Theorem 3.2, we have that IVP(4.1) has at least one solution $(x,y) \in Z$ if

(i) $\mu\delta > 1$ with

$$\begin{aligned} &ab^{\mu}\frac{(\delta\mu)^{\delta\mu}}{(\delta\mu-1)^{\delta\mu-1}}\left[||\Phi_0|| + \left(\frac{||\Psi_0||}{b}\right)^{\frac{1}{\delta}}\right]^{\delta\mu-1} \leq 1 \text{ for } \delta > 1,\\ &ba^{\delta}\frac{(\delta\mu)^{\delta\mu}}{(\delta\mu-1)^{\delta\mu-1}}\left[||\Psi_0|| + \left(\frac{||\Phi_0||}{a}\right)^{\frac{1}{\mu}}\right]^{\delta\mu-1} \leq 1 \text{ for } \mu > 1 \end{aligned} \tag{4.2}$$

or

(ii) $\mu\delta = 1$ with

$$\text{either } ab^{\frac{1}{\delta}} < 1 \text{ or } ba^{\frac{1}{\mu}} < 1 \tag{4.3}$$

or

(iii) $\mu\delta < 1$.

One sees that for sufficiently small A,B,C,D, we have $ab^{\frac{1}{\delta}} < 1$ and $ba^{\frac{1}{\mu}} < 1$, then IVP(4.1) has at least one solution when $\delta\mu = 1$ and sufficiently small A,B,C,D.

One sees that for sufficiently small $A,B,C,D,|x_0|,|y_0|$ that (4.2) holds. Then IVP(4.1) has at least one solution when $\delta\mu > 1$ and sufficiently small $A,B,C,D,|x_0|,|y_0|$.

IVP(4.1) has at least one solution when $\delta\mu < 1$. The proof is complete.

□

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