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Inverse Problems and Approximations in Quantum Calculus

Soumaya Chefai*
Faculté des Sciences de Tunis, 1060 Tunis, Tunisia

Lazhar Dhaouadi†
Institut Préparatoire aux Etudes d'Ingnieurs de Bizerte, 7021 Zarzouna, Tunisia

Ahmed Fitouhi‡
Faculté des Sciences de Tunis, 1060 Tunis, Tunisia

Abstract. In this paper we study in quantum calculus the theory of inverse problem and approximation in a large class of Hilbert spaces with reproducing kernels.

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1 Introduction

Inverse problem theory is driven by applied problems in sciences and engineering. Studies on inverse problems represent an exciting research area in recent decades. The special importance of inverse problems is that it is an interdisciplinary subject related with mathematics, physics, chemistry, geoscientific problems, biology, financial and business, life science, computing technology and engineering.

In this work we study in quantum calculus context the theory of inverse problem and approximation in a large class of Hilbert spaces with reproducing kernels using some result established by S. Saitoh [11]. Moreover, the approximate inverse method is adapted to bounded operators which are convolution products in the context of the quantum calculus. In fact for a given function e we can define a bounded linear operator $\mathbf{T} : \mathbf{H} \to \mathbf{H}$ as follows : put :

$$\mathbf{T}f = e *_q f,$$

where $\mathbf{H}$ is a Hilbert space with reproducing kernel. For a particular choose of the function e we can find particular operators as in the classic case (Weierstrass transform [10], Gabor transform [9], Laguerre-Type Weierstrass Transform [7]).

*e-mail address: klembi.soumaya56@hotmail.fr
†e-mail address: lazhardhaouadi@yahoo.fr
‡e-mail address: Ahmed.Fitouhi@fst.rnu.tn

The present dissertation consists of three sections which deal with a bounded linear operator and some of its properties in special Hilbert spaces with their reproducing kernels. In section 2, we recall the main results about the q-harmonic analysis. In section 3, we study the operator $\mathbf{T}$. If the space of departure of the operator $\mathbf{T}$ is a Sobolev space $\mathbf{H}_\omega$ included in $\mathbf{H}$, we lose the subjectivity, which leads us to study approximations of inverse problem and in this context, we use the theory of Saitoh [11, 12] to characterize the extremal functions. In section 4, we consider the continuous linear operator $\mathbf{T}_t : \mathbf{H} \to \mathbf{H}$ as follows

$$\mathbf{T}_t f = e_t *_q f,$$

where $e_t(\lambda) = e(\lambda t)$. We give the associated inversion and Plancherel formulas.

2 Preliminaries on q-Harmonic analysis

Throughout this paper, we will assume that $0 < q < 1$ and $\nu > -1$. We refer to [5] for the definitions, notations and properties of the q-shifted factorials, the Jackson's q-derivative and the Jackson's q-integrals.

Let $a \in \mathbb{C}$, the q-shifted factorial is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),$$

and

$$\mathbb{R}_q^+ = \{q^n \ : \ n \in \mathbb{Z}\}.$$

The q-Jackson integrals from 0 to a and from 0 to ∞ are defined by

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n,$$

$$\int_0^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n,$$

provided the sums converge absolutely.

The space $\mathcal{L}_{q,p,\nu}$ denotes the sets of real functions on $\mathbb{R}_q^+$ for which

$$\|f\|_{q,p,\nu} = \left[\int_0^\infty |f(x)|^2 x^{2\nu+1} d_qx\right]^{1/2}.$$

The third Jackson q-Bessel function $J_\nu(.;q)$ (also called Hahn-Exton q-Bessel function) is defined by the power series [13]

$$J_\nu(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} x^\nu \sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q^{\nu+1};q)_n(q;q)_n} x^{2n}.$$

The normalized form of the q-Bessel function is defined by

$$j_\nu(x;q) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_n(q^{\nu+1};q)_n} x^{2n}.$$

It satisfies the following estimate (see [1])

$$|j_\nu(q^n;q^2)| \leq \frac{(-q^2;q^2)_\infty(-q^{2\nu+2};q^2)_\infty}{(q^{2\nu+2};q^2)_\infty} \begin{cases} 1 & \text{if} \quad n \geq 0 \\ q^{n^2-(2\nu+1)n} & \text{if} \quad n < 0 \end{cases}.$$

The function $x \mapsto j_\nu(\lambda x;q^2)$ is a solution of the following q-difference equation

$$\Delta_{q,\nu} f(x) = -\lambda^2 f(x),$$

where $\Delta_{q,\nu}$ is the q-Bessel operator

$$\Delta_{q,\nu} f(x) = \frac{1}{x^2}\Big[f(q^{-1}x) - (1+q^{2\nu})f(x) + q^{2\nu} f(qx)\Big].$$

The q-Bessel Fourier transform $\mathcal{F}_{q,\nu}$ was introduced and studied in [1, 2, 4, 8]

$$\mathcal{F}_{q,\nu} f(x) = c_{q,\nu}\int_0^\infty f(t) j_\nu(xt;q^2)t^{2\nu+1} d_q t,$$

where

$$c_{q,\nu} = \frac{1}{(1-q)}\frac{\left(q^{2\nu+2};q^2\right)_\infty}{(q^2;q^2)_\infty}.$$

The q-Bessel translation operator is defined as follows [1, 2]

$$T^\nu_{q,x} f(y) = c_{q,\nu}\int_0^\infty \mathcal{F}_{q,\nu}(f)(t) j_\nu(xt;q^2) j_\nu(yt;q^2)t^{2\nu+1} d_q t,$$

and the q-convolution product of two functions is given by

$$f *_q g(x) = c_{q,\nu}\int_0^\infty T^\nu_{q,x} f(y) g(y) y^{2\nu+1} d_q y.$$

The following Theorem summarizes some results about q-Bessel Fourier transform [1, 2].

Theorem 2.1. *The q-Bessel Fourier transform satisfies*

1. *For all functions* $f \in \mathcal{L}_{q,p,\nu}$, $\quad \mathcal{F}^2_{q,\nu} f(x) = f(x), \quad \forall x \in \mathbb{R}^+_q$.
2. *For all functions* $f \in \mathcal{L}_{q,2,\nu}$, $\quad \|\mathcal{F}_{q,\nu} f\|_{q,2,\nu} = \|f\|_{q,2,\nu}$.
3. *Let* $f \in \mathcal{L}_{q,p,\nu}$ *and* $g \in \mathcal{L}_{q,r,\nu}$ *then* $f *_q g \in \mathcal{L}_{q,s,\nu}$ *and*

$$\mathcal{F}_{q,\nu}(f *_q g)(x) = \mathcal{F}_{q,\nu} f(x) \times \mathcal{F}_{q,\nu} g(x), \quad \forall x \in \mathbb{R}^+_q,$$

where $\frac{1}{p} + \frac{1}{r} - 1 = \frac{1}{s}$.

3 Hilbert spaces with reproducing kernels

We denote by **H** the space $\mathcal{L}_{q,2,\nu}$ which is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_0^\infty f(x) g(x) x^{2\nu+1} d_q x.$$

Given a positive function ω on $\mathbb{R}_q^+$ with support (not necessary compact) $\mathbf{I}_\omega$ an interval of $\mathbb{R}_q^+$ which satisfies

$$\int_0^{+\infty} \frac{\lambda^{2\nu+1}}{\omega(\lambda)} \mathbf{1}_\omega(\lambda) d_q\lambda$$

is finite, and

$$\left\| \frac{1}{\omega} \right\|_\infty < \infty.$$

Here $\mathbf{1}_\omega$ is the characteristic function of $\mathbf{I}_\omega$. We introduce also the following space

$$\mathbf{H}_\omega = \left\{ f \in \mathbf{H} \mid \mathrm{supp}\mathcal{F}_{q,\nu}(f) \subset \mathbf{I}_\omega \text{ and } \sqrt{\omega(\lambda)}\mathcal{F}_{q,\nu}(f)(\lambda) \in \mathbf{H} \right\}.$$

The space $\mathbf{H}_\omega$ is a sufficient large class of functions in the sense that we can find many classical spaces for the particular choose of the function ω.

When

$$\omega(\lambda) = \mathbf{1}_{[0,a]}(\lambda),$$

this functional space is the q-Paley-Wiener space $PW_{q,a}^\nu$ [3] and for

$$\omega(\lambda) = (1+\lambda^2)^\beta, \quad \beta > \nu + 1,$$

it is an analogue of the functional space introduced and studied in [10].

Proposition 3.1. *Equipped with the inner product*

$$\langle f, g \rangle_\omega = \int_0^{+\infty} \omega(\lambda)\mathcal{F}_{q,\nu}(f)(\lambda)\mathcal{F}_{q,\nu}(g)(\lambda)\mathbf{1}_\omega(\lambda)\lambda^{2\nu+1} d_q\lambda,$$

the space $\mathbf{H}_\omega$ *is a Hilbert space with the reproducing kernel*

$$\mathcal{K}_x^\omega(y) = c_{q,\nu}^2 \int_0^{+\infty} \frac{j_\nu(\lambda x; q^2) j_\nu(\lambda y; q^2)}{\omega(\lambda)} \mathbf{1}_\omega(\lambda)\lambda^{2\nu+1} d_q\lambda.$$

Proof. We have

$$\mathcal{F}_{q,\nu}(\mathcal{K}_x^\omega)(\lambda) = c_{q,\nu} \frac{j_\nu(\lambda x; q^2)}{\omega(\lambda)} \mathbf{1}_\omega(\lambda) \Rightarrow \mathcal{K}_x^\omega \in \mathbf{H}_\omega.$$

On the other hand if $f \in \mathbf{H}_\omega$, we obtain

$$\begin{aligned}
\langle f, \mathcal{K}_x^\omega \rangle_\omega &= \int_0^{+\infty} \omega(\lambda)\mathcal{F}_{q,\nu}(f)(\lambda)\mathcal{F}_{q,\nu}(\mathcal{K}_x^\omega)(\lambda)\mathbf{1}_\omega(\lambda)\lambda^{2\nu+1} d_q\lambda \\
&= c_{q,\nu} \int_0^{+\infty} \omega(\lambda)\mathcal{F}_{q,\nu}(f)(\lambda) \frac{j_\nu(\lambda x; q^2)}{\omega(\lambda)} \mathbf{1}_\omega(\lambda)\lambda^{2\nu+1} d_q\lambda \\
&= c_{q,\nu} \int_0^{+\infty} \mathcal{F}_{q,\nu}(f)(\lambda)\, j_\nu(\lambda x; q^2)\mathbf{1}_\omega(\lambda)\lambda^{2\nu+1} d_q\lambda \\
&= c_{q,\nu} \int_0^{+\infty} \mathcal{F}_{q,\nu}(f)(\lambda)\, j_\nu(\lambda x; q^2)\lambda^{2\nu+1} d_q\lambda \\
&= f(x).
\end{aligned}$$

In the following we put

$$\|f\|_\omega = \sqrt{\langle f, f \rangle_\omega} \text{ and } \|f\|_2 = \sqrt{\langle f, f \rangle}.$$

Proposition 3.2. *Let $e \in \mathcal{L}_{q,1,\nu}$, then*

$$\begin{array}{cccc} \mathbf{T}: & \mathbf{H}_\omega & \to & \mathbf{H} \\ & f & \mapsto & e *_q f \end{array}$$

is a bounded linear operator and we have

$$\|\mathbf{T}f\|_2 \leq \|e\|_1 \sqrt{\left\|\frac{1}{\omega}\right\|_\infty} \|f\|_\omega.$$

Proof. If $f \in \mathbf{H}_\omega$ then $\mathbf{T}f \in \mathbf{H}_\omega$. Using the Young inequality we show that, for $e \in \mathcal{L}_{q,1,\nu}$, we have $e *_q f \in \mathcal{L}_{q,2,\nu}$ and

$$\begin{array}{rcl} \|e *_q f\|_2 & \leq & \|e\|_1 \|f\|_2 \\ & \leq & \|e\|_1 \|\mathcal{F}_{q,\nu} f\|_2 \\ & \leq & \|e\|_1 \left[\int_0^\infty \frac{1}{\omega} \omega \mathbf{1}_\omega(t) |\mathcal{F}_{q,\nu} f(t)|^2 t^{2\nu+1} d_q t\right]^{\frac{1}{2}} \\ & \leq & \|e\|_1 \sqrt{\left\|\frac{1}{\omega}\right\|_\infty} \left[\int_0^\infty \omega \mathbf{1}_\omega(t) |\mathcal{F}_{q,\nu} f(t)|^2 t^{2\nu+1} d_q t\right]^{\frac{1}{2}} \\ & \leq & \|e\|_1 \sqrt{\left\|\frac{1}{\omega}\right\|_\infty} \|f\|_\omega. \end{array}$$

Proposition 3.3. *The space $\mathbf{H}_{\omega,\xi}$ (underline vector space of $\mathbf{H}_\omega$) equipped with the inner product*

$$\langle f, g\rangle_{\omega,\xi} = \xi \langle f, g\rangle_\omega + \langle \mathbf{T}(f), \mathbf{T}(g)\rangle$$

is a Hilbert space with the reproducing kernel

$$\mathcal{K}_x^{\omega,\xi}(y) = c_{q,\nu}^2 \int_0^{+\infty} \frac{j_\nu(\lambda x; q^2) j_\nu(\lambda y; q^2)}{\xi\omega(\lambda) + E(\lambda)^2} \mathbf{1}_\omega(\lambda) \lambda^{2\nu+1} d_q\lambda,$$

where

$$E(\lambda) = \mathcal{F}_{q,\nu} e(\lambda).$$

Proof. We have

$$\mathcal{F}_{q,\nu}\left(\mathcal{K}_x^{\omega,\xi}\right)(\lambda) = c_{q,\nu} \frac{j_\nu(\lambda x; q^2)}{\xi\omega(\lambda) + E(\lambda)^2} \mathbf{1}_\omega(\lambda).$$

Hence

$$\mathcal{K}_x^{\omega,\xi} \in \mathbf{H}_{\omega,\xi}.$$

On the other hand if $f \in \mathbf{H}_{\omega,\xi}$ then we have

$$\left\langle f, \mathcal{K}_x^{\omega,\xi}\right\rangle_{\omega,\xi} = \xi \left\langle f, \mathcal{K}_x^{\omega,\xi}\right\rangle_\omega + \left\langle \mathbf{T}(f), \mathbf{T}\left(\mathcal{K}_x^{\omega,\xi}\right)\right\rangle,$$

$$\left\langle f, \mathcal{K}_x^{\omega,\xi}\right\rangle_\omega = c_{q,\nu} \int_0^{+\infty} \frac{\omega(\lambda) j_\nu(\lambda x; q^2)}{\xi\omega(\lambda) + E(\lambda)^2} \mathcal{F}_{q,\nu}(f)(\lambda) \mathbf{1}_\omega(\lambda) \lambda^{2\nu+1} d_q\lambda.$$

$$\begin{array}{rcl} \mathbf{T}\left(\mathcal{K}_x^{\omega,\xi}\right)(\lambda) & = & e *_q \mathcal{K}_x^{\omega,\xi}(\lambda) \\ & = & \mathcal{F}_{q,\nu}\left(c_{q,\nu} \frac{j_\nu(xy; q^2) E(y)}{\xi\omega(y) + E(y)^2} \mathbf{1}_\omega(y)\right)(\lambda). \end{array}$$

Similarly,

$$\mathbf{T}(f)(x) = \mathcal{F}_{q,\nu}\left[E \times \mathcal{F}_{q,\nu}(f)\right](x).$$

Therefore according to the Parseval Theorem we get

$$\begin{aligned}
\left\langle \mathbf{T}(f), \mathbf{T}\left(\mathcal{K}_x^{\omega,\xi}\right)\right\rangle &= \left\langle \mathcal{F}_{q,\nu}\mathbf{T}(f), \mathcal{F}_{q,\nu}\mathbf{T}\left(\mathcal{K}_x^{\omega,\xi}\right)\right\rangle \\
&= \left\langle \mathcal{F}_{q,\nu}\left(e\right)\mathcal{F}_{q,\nu}(f)\right), \left(c_{q,\nu}\frac{j_\nu(\lambda x; q^2)E(\lambda)}{\xi\omega(\lambda) + E(\lambda)^2}\mathbf{1}_\omega(\lambda)\right)\right\rangle \\
&= c_{q,\nu}\int_0^{+\infty}\frac{E(\lambda)^2 j_\nu(\lambda x; q^2)}{\xi\omega(\lambda) + E(\lambda)^2}\mathcal{F}_{q,\nu}(f)(\lambda)\mathbf{1}_\omega(\lambda)\lambda^{2\nu+1}d_q\lambda,
\end{aligned}$$

and by the inversion formula for the q-Bessel transform we obtain

$$\begin{aligned}
\left\langle f, \mathcal{K}_x^{\omega,\xi}\right\rangle_{\omega,\xi} &= c_{q,\nu}\int_0^{+\infty} j_\nu(\lambda x; q^2)\mathcal{F}_{q,\nu}(f)(\lambda)\mathbf{1}_\omega(\lambda)\lambda^{2\nu+1}d_q\lambda \\
&= c_{q,\nu}\int_0^{+\infty} j_\nu(\lambda x; q^2)\mathcal{F}_{q,\nu}(f)(\lambda)\lambda^{2\nu+1}d_q\lambda \\
&= f(x).
\end{aligned}$$

4 Extremal functions

In this section we are interested by extremal functions to solve the problem of inverse approximation. We use reproducing kernel Hilbert spaces to give the best approximation for the bounded linear operator $\mathbf{T}$. Using the Saitoh's Theorem [11] we obtain the following result :

Theorem 4.1. *Let $\xi > 0$ and $g \in \mathbf{H}$. The approximation problem*

$$\inf_{f\in\mathbf{H}_\omega}\left(\xi\|f\|_\omega^2 + \|g - \mathbf{T}(f)\|_2^2\right) \tag{4.1}$$

is solvable and

$$f_{\xi,g}^* = \left\langle g, \mathbf{T}\left(\mathcal{K}_x^{\omega,\xi}\right)\right\rangle$$

is the element of $\mathbf{H}_\omega$ with the smallest $\mathbf{H}_\omega$-norm where the infimum (4.1) is attained.

Corollary 4.2. *For all $g_1, g_2 \in \mathbf{H}$ we have*

$$\left\|f_{\xi,g_1}^* - f_{\xi,g_2}^*\right\|_\omega^2 \le \frac{\|g_1 - g_2\|_2^2}{2\xi}.$$

Proof. The function $f_{\xi,g}^*$ can be written as follows :

$$\begin{aligned}
f_{\xi,g}^*(x) &= \left\langle g, \mathbf{T}\left(\mathcal{K}_x^{\omega,\xi}\right)\right\rangle \\
&= \int_0^{+\infty} g(\lambda)\mathbf{T}\left(\mathcal{K}_x^{\omega,\xi}\right)(\lambda)\lambda^{2\nu+1}d_q\lambda \\
&= \int_0^{+\infty} g(\lambda)\mathcal{F}_{q,\nu}\left(\frac{j_\nu(xy,q^2)E(y)}{\xi\omega(y) + E(y)^2}\mathbf{1}_\omega(y)\right)(\lambda)\lambda^{2\nu+1}d_q\lambda \\
&= \mathcal{F}_{q,\nu}\left(\mathcal{F}_{q,\nu}(g)(\lambda)\frac{E(\lambda)}{\xi\omega(\lambda) + E(\lambda)^2}\mathbf{1}_\omega(\lambda)\right)(x),
\end{aligned}$$

which implies that

$$\mathcal{F}_{\nu,q}\left(f^*_{\xi,g}\right)(\lambda) = \frac{E(\lambda)}{\xi\omega(\lambda)+E(\lambda)^2}\mathcal{F}_{\nu,q}(g)(\lambda)\mathbf{1}_\omega(\lambda).$$

We have

$$(f^*_{\xi,g_1} - f^*_{\xi,g_2})(x) = \int_0^{+\infty} \mathcal{F}_{q,\nu}(g_1-g_2)(\lambda)\frac{j_\nu(\lambda x;q^2)E(\lambda)}{\xi\omega(\lambda)+E(\lambda)^2}\mathbf{1}_\omega(\lambda)\lambda^{2\nu+1}d_q\lambda,$$

and

$$\mathcal{F}_{q,\nu}\left(f^*_{\xi,g_1} - f^*_{\xi,g_2}\right)(\lambda) = \frac{E(\lambda)}{\xi\omega(\lambda)+E(\lambda)^2}\mathcal{F}_{q,\nu}(g_1-g_2)(\lambda)\mathbf{1}_\omega(\lambda).$$

According to the fact that $a^2+b^2 \geq 2ab$ we obtain

$$\begin{aligned}
\left\|f^*_{\xi,g_1} - f^*_{\xi,g_2}\right\|^2_\omega &= \int_0^{+\infty}\omega(\lambda)\left|\mathcal{F}_{q,\nu}\left(f^*_{\xi,g_1} - f^*_{\xi,g_2}\right)(\lambda)\right|^2\mathbf{1}_\omega(\lambda)\lambda^{2\nu+1}d_q\lambda \\
&= \int_0^{+\infty}\frac{\omega(\lambda)E(\lambda)^2}{\left[\xi\omega(\lambda)+E(\lambda)^2\right]^2}\left|\mathcal{F}_{q,\nu}(g_1-g_2)(\lambda)\right|^2\mathbf{1}_\omega(\lambda)\lambda^{2\nu+1}d_q\lambda \\
&\leq \frac{1}{2\xi}\int_0^{+\infty}\left|\mathcal{F}_{q,\nu}(g_1-g_2)(\lambda)\right|^2\lambda^{2\nu+1}d_q\lambda = \frac{1}{2\xi}\|g_1-g_2\|_2^2.
\end{aligned}$$

Corollary 4.3. *For all $f \in \mathbf{H}_\omega$ and $g = \mathbf{T}(f)$ we have*

$$\lim_{\xi\to 0^+}\left\|f^*_{\xi,g} - f\right\|^2_\omega = 0.$$

Moreover, $\left(f^*_{\xi,g}\right)_{\xi>0}$ converges uniformly to f as $\xi \to 0^+$.

Proof. In fact we have

$$\begin{aligned}
\mathcal{F}_{q,\nu}\left(f^*_{\xi,g} - f\right)(\lambda) &= \frac{E(\lambda)\mathcal{F}_{\nu,q}(g)(\lambda)}{\xi\omega(\lambda)+E(\lambda)^2}\mathbf{1}_\omega(\lambda) - \mathcal{F}_{q,\nu}(f)(\lambda)\mathbf{1}_\omega(\lambda) \\
&= \frac{E(\lambda)\mathcal{F}_{\nu,q}(\mathbf{T}(f))(\lambda)}{\xi\omega(\lambda)+E(\lambda)^2}\mathbf{1}_\omega(\lambda) - \mathcal{F}_{q,\nu}(f)(\lambda)\mathbf{1}_\omega(\lambda) \\
&= \frac{-\xi\omega(\lambda)}{\xi\omega(\lambda)+E(\lambda)^2}\mathcal{F}_{q,\nu}(f)(\lambda)\mathbf{1}_\omega(\lambda).
\end{aligned}$$

Then

$$\begin{aligned}
\left\|f^*_{\xi,g} - f\right\|^2_\omega &= \int_0^{+\infty}\omega(\lambda)\left|\mathcal{F}_{q,\nu}\left(f^*_{\xi,g_1} - f\right)(\lambda)\right|^2\mathbf{1}_\omega(\lambda)\lambda^{2\nu+1}d_q\lambda \\
&= \int_0^{+\infty}\omega(\lambda)\left|\frac{-\xi\omega(\lambda)}{\xi\omega(\lambda)+E(\lambda)^2}\mathcal{F}_{q,\nu}(f)(\lambda)\right|^2\mathbf{1}_\omega(\lambda)\lambda^{2\nu+1}d_q\lambda \\
&= \int_0^{+\infty}\frac{\xi^2\omega(\lambda)^3}{\left[\xi\omega(\lambda)+E(\lambda)^2\right]^2}\left|\mathcal{F}_{q,\nu}(f)(\lambda)\right|^2\mathbf{1}_\omega(\lambda)\lambda^{2\nu+1}d_q\lambda.
\end{aligned}$$

For $\lambda \in \mathbb{R}_q^+$ we have,

$$\lim_{\xi\to 0^+}\frac{\xi^2\omega(\lambda)^3}{\left[\xi\omega(\lambda)+E(\lambda)^2\right]^2}\left|\mathcal{F}_{\nu,q}(f)(\lambda)\right|^2 = 0,$$

and

$$\frac{\xi^2\omega(\lambda)^3}{\left[\xi\omega(\lambda)+E(\lambda)^2\right]^2}\left|\mathcal{F}_{\nu,q}(f)(\lambda)\right|^2 \le \omega(\lambda)\left|\mathcal{F}_{\nu,q}(f)(\lambda)\right|^2.$$

Since $f \in \mathbf{H}_\omega$ and using the dominated convergence Theorem, we deduce that

$$\lim_{\xi\to 0^+}\left\|f^*_{\xi,g}-f\right\|^2_\omega = 0.$$

Therefore by the inversion formula in Theorem 2.1 we have for all $x \in \mathbb{R}^+_q$

$$\begin{aligned}\left|\left(f^*_{\xi,g}-f\right)(x)\right| &= \left|\int_0^{+\infty}\mathcal{F}_{q,\nu}\left(f^*_{\xi,g}-f\right)(\lambda)\, j_\nu\left(\lambda x;q^2\right)\lambda^{2\nu+1}d_q\lambda\right|\\ &= \left|\int_0^{+\infty}\frac{-\xi\omega(\lambda)\mathcal{F}_{q,\nu}(f)(\lambda)}{\xi\omega(\lambda)+E(\lambda)^2} j_\nu\left(\lambda x;q^2\right)\lambda^{2\nu+1}d_q\lambda\right|\\ &\le \int_0^{+\infty}\frac{\xi\omega(\lambda)}{\xi\omega(\lambda)+E(\lambda)^2}\left|\mathcal{F}_{q,\nu}(f)(\lambda)\right|\lambda^{2\nu+1}d_q\lambda.\end{aligned}$$

On the other hand

$$\frac{\xi\omega(\lambda)}{\xi\omega(\lambda)+E(\lambda)^2}\left|\mathcal{F}_{q,\nu}(f)(\lambda)\right| \le \left|\mathcal{F}_{q,\nu}(f)(\lambda)\right|.$$

Note that

$$\begin{aligned}\int_0^{+\infty}\left|\mathcal{F}_{q,\nu}(f)(\lambda)\right|\lambda^{2\nu+1}d_q\lambda &\le \left(\int_0^{+\infty}\omega(\lambda)\left|\mathcal{F}_{q,\nu}(f)(\lambda)\right|^2\mathbf{1}_\omega(\lambda)\lambda^{2\nu+1}d_q\lambda\right)^{1/2}\\ &\quad\times\left(\int_0^{+\infty}\frac{1}{\omega(\lambda)}\mathbf{1}_\omega(\lambda)\lambda^{2\nu+1}d_q\lambda\right)^{1/2}.\end{aligned}$$

The result follows from the dominated convergence Theorem.

5 Inverse problem

In this section, we will study the inverse formula of the continuous transform T_t from the Hilbert space $\mathbf{H}$ into itself :

$$\begin{array}{rccc} T_t: & \mathbf{H} & \to & \mathbf{H}\\ & f & \mapsto & e_t *_q f\end{array}$$

where

$$e_t(\lambda) = e(t\lambda) \Rightarrow E_t(x) = t^{-2(\nu+1)}E\left(\frac{x}{t}\right) > 0,$$

and

$$\overline{B}_\nu = \int_0^{+\infty}|E(z)|\,d_qz < \infty.$$

This transformation can be considered as a q-version of the wavelet transform. In our case we can recover the original signal via a unified inversion formula (modulo a constant).

Theorem 5.1. *Let $f \in \mathbf{H}$ then we have the following inversion formula :*

$$f(x) = \frac{c_{q,\nu}}{B_\nu}\int_0^{+\infty}\int_0^{+\infty}\mathcal{F}_{q,\nu}\circ\mathbf{T}_t f(\lambda)\, j_\nu(\lambda x;q^2)\lambda^{2\nu+2}t^{2\nu}d_q\lambda d_qt,$$

where

$$B_\nu = \int_0^{+\infty}E(z)\,d_qz \neq 0.$$

Proof. For $t > 0$, we have

$$
\begin{aligned}
&\frac{c_{q,\nu}}{B_\nu}\int_0^{+\infty}\int_0^{+\infty}\mathcal{F}_{q,\nu}\circ\mathbf{T}_t f(\lambda)\, j_\nu(\lambda x,q^2)\lambda^{2\nu+2}t^{2\nu}d_q\lambda d_q t\\
&= c_{q,\nu}\int_0^{+\infty}\mathcal{F}_{q,\nu}f(\lambda)j_\nu(\lambda x;q^2)\left[\frac{1}{B_\nu}\int_0^{+\infty}\frac{\lambda}{t}E_t(\lambda)t^{2\nu+1}d_q t\right]\lambda^{2\nu+1}d_q\lambda\\
&= c_{q,\nu}\int_0^{+\infty}\mathcal{F}_{q,\nu}f(\lambda)j_\nu(\lambda x;q^2)\left[\frac{1}{B_\nu}\int_0^{+\infty}\frac{\lambda}{t^2}E\left(\frac{\lambda}{t}\right)d_q t\right]\lambda^{2\nu+1}d_q\lambda\\
&= c_{q,\nu}\left[\int_0^{+\infty}\mathcal{F}_{q,\nu}f(\lambda)j_\nu(\lambda x;q^2)\lambda^{2\nu+1}d_q\lambda\right]\left[\frac{1}{B_\nu}\int_0^{+\infty}E(z)d_q z\right]\\
&= c_{q,\nu}\int_0^{+\infty}\mathcal{F}_{q,\nu}f(\lambda)j_\nu(\lambda x;q^2)\lambda^{2\nu+1}d_q\lambda = f(x).
\end{aligned}
$$

The computations are justified by the Fubini's theorem

$$
\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\left|\mathcal{F}_{q,\nu}\circ\mathbf{T}_t f(\lambda)\right|\left|j_\nu(\lambda x,q^2)\right|\lambda^{2\nu+2}t^{2\nu}d_q\lambda d_q t\\
&\leq \int_0^{+\infty}\left|\mathcal{F}_{q,\nu}f(\lambda)\right|\left|j_\nu(\lambda x,q^2)\right|\lambda^{2\nu+1}d_q\lambda\left[\frac{1}{B_\nu}\int_0^{+\infty}|E(z)|d_q z\right]\\
&\leq \frac{\overline{B}_\nu}{B_\nu}\|f\|_2\left\|j_\nu(.,q^2)\right\|_2 x^{-(\nu+1)},\quad \forall x\in\mathbb{R}_q^+.
\end{aligned}
$$

We can recover the energy of the original signal by the following Plancherel formula :

Theorem 5.2. *For f in $\mathbf{H}$, we have*

$$
\|f\|_2^2 = \frac{1}{B_\nu^*}\int_0^{+\infty}\int_0^{+\infty}|\mathcal{F}_{q,\nu}\circ\mathbf{T}_t f(\lambda)|^2\lambda^{2\nu+2}t^{4\nu+2}d_q\lambda d_q t
$$

where

$$
B_\nu^* = \int_0^{+\infty}|E(z)|^2 d_q z < \infty.
$$

Proof. Let $t > 0$. We have

$$
\begin{aligned}
&\frac{1}{B_\nu^*}\int_0^{+\infty}\int_0^{+\infty}|\mathcal{F}_{q,\nu}\circ\mathbf{T}_t f(\lambda)|^2\lambda^{2\nu+2}t^{4\nu+2}d_q\lambda d_q t\\
&= \int_0^{+\infty}|\mathcal{F}_{q,\nu}f(\lambda)|^2\left[\frac{1}{B_\nu^*}\int_0^{+\infty}E_t(\lambda)^2\, t^{4\nu+2}d_q t\right]\lambda^{2\nu+2}d_q\lambda\\
&= \int_0^{+\infty}|\mathcal{F}_{q,\nu}f(\lambda)|^2\left[\frac{1}{B_\nu^*}\int_0^{+\infty}\frac{\lambda}{t^2}E\left(\frac{\lambda}{t}\right)^2 d_q t\right]\lambda^{2\nu+1}d_q\lambda\\
&= \left[\int_0^{+\infty}|\mathcal{F}_{q,\nu}f(\lambda)|^2\lambda^{2\nu+1}d_q\lambda\right]\left[\frac{1}{B_\nu^*}\int_0^{+\infty}E(z)^2\, d_q z\right]\\
&= \int_0^{+\infty}|\mathcal{F}_{q,\nu}f(\lambda)|^2\lambda^{2\nu+1}d_q\lambda\\
&= \int_0^{+\infty}|f(\lambda)|^2\lambda^{2\nu+1}d_q\lambda.
\end{aligned}
$$

The computations are justified by the Fubini's theorem.

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