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Lightlike Hypersurfaces in Lorentzian Manifolds with Constant Screen Principal Curvatures

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Abstract. In this paper, we generalize the Cartan's fundamental formula on lightlike hypersurfaces, then we use it to show that a screen conformal lightlike hypersurface of a Lorentzian Euclidean space is locally a lightlike triple product manifold.

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1 Introduction

The theory of hypersurfaces, defined as submanifolds of codimension one, is one of the fundamental theories of submanifolds.

Let M be a submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. If the induced metric g on M is non-degenerate, then (M, g) becomes a Riemannian or a semi-Riemannian manifold. When g is degenerate, (M, g) is called a lightlike submanifold, and many different situations appear. The geometry of lightlike submanifolds is different and rather difficult since (contrary to Riemannian or semi-Riemannian submanifolds) its normal vector bundle intersects the tangent bundle.

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Thus, one cannot use, in the usual way, the theory of non-degenerate submanifolds (cf. Chen ([6])) to define the induced geometric objects (such as linear connection, second fundamental form, Gauss and Weingarten equations) on the lightlike submanifolds.

In 1996, Duggal and Bejancu ([8]) published a book on general theory of lightlike submanifolds of semi-Riemannian manifolds and their applications to general relativity. They introduced a non-degenerate screen distribution (or equivalently a null transversal vector bundle) so as to get three factors splitting the ambient tangent space. Then, they derived the main induced geometric objects (depending on the screen distribution, and hence is not unique in general) such as second fundamental forms, shape operators, induced connections, curvature, etc. Moreover, it is important to notice that the second fundamental form is independent from the choice of the screen distribution.

We know that the shape operator plays an important role in the study geometry of submanifolds. In the non-degenerate case, we have one shape operator which is diagonalizable and its eigenvalues are called *principal curvatures* of the hypersurface. In [4] and [5], E. Cartan studied and classified hypersurfaces in standard Riemannian space forms whose principal curvatures are all constant. The shape operator of a non-degenerate submanifold is related to the second fundamental form of the hypersurface. Contrary to this, we will see that in the case of lightlike hypersurfaces, there are two shape operators (A_N and $\overset{*}{A}_\xi$) and there are interrelations between these geometric objects and those of its screen distribution (see relations (2.19) and (2.20)). Moreover, the shape operator A_N of a lightlike hypersurface is not necessarily auto-adjoint, but the on $\overset{*}{A}_\xi$ of the screen distribution is diagonalizable. Since the null characteristic vector field is an eigenvector of $\overset{*}{A}_\xi$ with zero as eigenvalue, in the present paper, we consider the other eigenvalues.

The paper is organized as follow. Section 2 covers useful preliminaries for study the geometry of lightlike hypersurfaces. In Section 3, we prove the so-called Cartan's fundamental formula for lightlike hypersurfaces (Theorem 3.5). We give a proof along the same lines as Cartan's original proof, although Cartan used differential forms rather than vector fields. We apply Theorem 3.5 in Section 4 to show that a screen conformal lightlike hypersurface of a Lorentzian Euclidean space is locally a lightlike triple product manifold (Theorem 4.2). Using this theorem we prove a classification theorem for screen conformal lightlike hypersurfaces with constant screen principal curvatures (Theorem 4.4).

2 Preliminaries on Lightlike hypersurfaces

Let $(\overline{M}, \overline{g})$ be a $(m+2)$-dimensional semi-Riemannian manifold of index ν, $(0 < \nu < m+2)$. Consider a hypersurface M of $\overline{M}$ and denote by g the tensor field induced by $\overline{g}$ on M. We say that M is a lightlike (degenerate, null) hypersurface if $rank(g) = m$. Then the normal vector bundle $TM^\perp$ intersects the tangent bundle along a nonzero differentiable distribution called the radical distribution of M and denoted by $Rad(TM)$:

$$Rad(TM) : x \mapsto Rad(T_xM) = T_xM \cap T_xM^\perp. \tag{2.1}$$

A *screen distribution* $S(TM)$ on M is a non-degenerate vector bundle complementary to $TM^\perp$. A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As $TM^\perp$ lies in the tangent bundle, the following result has an important role in the study of the geometry of lightlike hypersurfaces.

Theorem 2.1. *([8]) Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \overline{g})$. Then there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that for any non zero section ξ of $TM^\perp$ on a* *coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of tr(TM) on $\mathcal{U}$ satisfying*

$$\overline{g}(N,\xi) = 1 \text{ and } \overline{g}(N,N) = \overline{g}(N,W) = 0, \tag{2.2}$$

for all $W \in \Gamma(S(TM)_{|\mathcal{U}})$.

With this theorem we may write the following decomposition

$$T\overline{M}_{|M} = S(TM)\perp(TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM), \tag{2.3}$$

where $\perp$ denotes an orthogonal direct sum and $\oplus$ a direct sum. Throughout the paper, we denoted by $\Gamma(E)$ the $C^{\infty}(M)$-module of smooth sections of a vector bundle E over M, while $C^{\infty}(M)$ represents the algebra of a smooth functions on M. Also, all manifolds are supposed to be smooth, paracompact and connected.

Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$, $\overline{\nabla}$ be the Levi-Civita connexion of $\overline{M}$, ∇ the induced connection on (M, g). Gauss and Weingarten formulas provide the following relations (see details in [8])

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.4}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{2.5}$$

for all $X, Y \in \Gamma(TM)$ and $V \in tr(TM)$, where $\nabla_X Y$ and $A_V X$ belong to $\Gamma(TM)$ while h is a $\Gamma(tr(TM))$-valued symmetric $C^{\infty}(M)$-bilinear form on $\Gamma(TM)$ and ∇^t is a linear connection on $tr(TM)$. It is easy to see that ∇ is a torsion-free connection. Define a symmetric $C^{\infty}(M)$-bilinear form B and a 1-form τ on the coordinate neighborhood $\mathcal{U} \subset M$ by

$$B(X,Y) = \overline{g}(h(X,Y),\xi), \tag{2.6}$$

$$\tau(X) = \overline{g}(\nabla^t_X N, \xi) \tag{2.7}$$

for all $X, Y \in \Gamma(TM_{j\mathcal{U}})$. Then, on $\mathcal{U}$, equations (2.4) and (2.5) become,

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{2.8}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.9}$$

respectively. It is important to stress the fact that the local second fundamental form B in Eq.(2.8) does not depend on the choice of the screen distribution and satisfies,

$$B(X,\xi) = 0, \tag{2.10}$$

for all $X \in \Gamma(TM|_{\mathcal{U}})$. Let P be the projection morphism of TM to $S(TM)$ with respect to the decomposition (2.2). We obtain: for all $X, Y \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$,

$$\begin{aligned} \nabla_X PY &= \overset{*}{\nabla}_X PY + \overset{*}{h}(X,PY), & (2.11)\\ \nabla_X U &= -\overset{*}{A}_U X + \overset{*}{\nabla}{}^t{}_X U, & (2.12) \end{aligned}$$

where $\overset{*}{\nabla}_X PY$ and $\overset{*}{A}_U X$ belong to $\Gamma(S(TM))$, $\overset{*}{\nabla}$ and $\overset{*}{\nabla}{}^t$ are linear connections on $\Gamma(S(TM))$ and $\Gamma(TM^{\perp})$ respectively, $\overset{*}{h}$ is a $\Gamma(TM^{\perp})$-valued $C^{\infty}(M)$-bilinear form on $\Gamma(TM) \times \Gamma(S(TM))$, $\overset{*}{A}_U$ is a $\Gamma(S(TM))$-valued $C^{\infty}(M)$-linear operator on $\Gamma(S(TM))$. $\overset{*}{h}$ and $\overset{*}{A}_U$ are the second fundamental form and the shape operator of the screen distribution $S(TM)$ respectively. Define on $\mathcal{U}$ the following relations

$$C(X,PY) = \overline{g}(\overset{*}{h}(X,PY),N), \tag{2.13}$$

$$\epsilon(X) = \overline{g}(\overset{*}{\nabla}{}^t{}_X\,\xi,N). \tag{2.14}$$

One shows that $\epsilon(X) = -\tau(X)$. Thus, locally (2.11) and (2.12) become

$$\nabla_X PY = \overset{*}{\nabla}_X PY + C(X,PY)\xi, \tag{2.15}$$

$$\nabla_X\xi = -\overset{*}{A}_\xi X - \tau(X)\xi, \tag{2.16}$$

respectively. The linear connection $\overset{*}{\nabla}$ is a metric connection on $\Gamma(S(TM))$. But, in general, the induced connection ∇ on M is not compatible with the induced metric g. Indeed, we have:

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \tag{2.17}$$

for all $X,Y \in \Gamma(TM|_{\mathcal{U}})$, where

$$\eta(X) = \overline{g}(X,N), \tag{2.18}$$

for all $Y \in \Gamma(TM|_{\mathcal{U}})$. Finally, it is straightforward to verify that

$$B(X,Y) = g(\overset{*}{A}_\xi X,Y), \quad g(A_N Y,N) = 0, \tag{2.19}$$

$$C(X,PY) = g(A_N X,Y), \quad \overset{*}{A}_\xi\,\xi = 0, \tag{2.20}$$

for $X,Y \in \Gamma(TM|_{\mathcal{U}})$.

We denote the curvature tensor associated with $\overline{\nabla}$ and ∇ by $\overline{R}$ and R, respectively. Then we have ([8]): for all $X,Y \in \Gamma(TM|_{\mathcal{U}})$

$$\overline{R}(X,Y)Z = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \tag{2.21}$$

$$\begin{aligned} g\Big(R(X,Y)PZ,PW\Big) &= g\Big(\overset{*}{R}(X,Y)PZ,PW\Big) + C(X,PZ)B(Y,PW) \\ &\quad - C(Y,PZ)B(X,PW), \end{aligned} \tag{2.22}$$

$$\overline{g}\Big(\overline{R}(X,Y)\xi,N\Big) = C(Y,\overset{*}{A}_\xi X) - C(X,\overset{*}{A}_\xi Y) - 2d\tau(X,Y). \tag{2.23}$$

3 Cartan's fundamental formula for lightlike hypersurfaces

In this section, we first consider a lightlike hypersurface M of a semi-Riemannian manifold $(\overline{M}(k),\overline{g})$ of constant curvature k. We start with the following proposition.

Proposition 3.1. *Let $(\overline{M}(k),\overline{g})$ be a semi-Riemannian manifold of constant curvature k and M be a lightlike hypersurface of $\overline{M}(k)$. Denote by R the curvature tensor of the induced connection ∇ on M by the Levi-civita connection $\overline{\nabla}$. For any $X,Y,Z \in \Gamma(TM)$, we have:*

(a) $R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\} - B(X,Z)A_N Y + B(Y,Z)A_N X$;

(b) $(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = B(X,Z)\tau(Y) - B(Y,Z)\tau(X)$;

(c) $B(A_N Y, X) - B(A_N X, Y) = 2d\tau(X, Y)$;

(d) $(\nabla_Y A_N)(X) - (\nabla_X A_N)(Y) + k\{\eta(X)Y - \eta(Y)X\} = \tau(Y)A_N X - \tau(X)A_N Y$;

(e) $(\nabla_X \overset{*}{A}_\xi)(Y) - (\nabla_Y \overset{*}{A}_\xi)(X) = \tau(Y)\overset{*}{A}_\xi X - \tau(X)\overset{*}{A}_\xi Y - 2d\tau(X, Y)\xi$;

(f) $\nabla_X PZ = \nabla_X Z - X \cdot \eta(Z)\xi + \eta(Z)\overset{*}{A}_\xi X + \eta(Z)\tau(X)\xi$.

Proof. For a semi-Riemannian manifold $(\overline{M}(k), \overline{g})$ of constant curvature k, the curvature tensor $\overline{R}$ of $\overline{M}$ has the following form:

$$\overline{R}(\overline{X}, \overline{Y})\overline{Z} = k\{\overline{g}(\overline{Y}, \overline{Z})\overline{X} - \overline{g}(\overline{X}, \overline{Z})\overline{Y}\}, \tag{3.1}$$

for $\overline{X}, \overline{Y}, \overline{Z} \in \Gamma(T\overline{M})$. Using Equations (2.6), (2.7), (2.21) and (3.1), we have

$$\begin{aligned} &R(X, Y)Z - k\{g(Y, Z)X - g(X, Z)Y\} + B(X, Z)A_N Y - B(Y, Z)A_N X \\ &+ [(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) - B(X, Z)\tau(Y) + B(Y, Z)\tau(X)]N = 0, \end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$. Then we obtain (a) and (b) by comparing the tangential and transversal parts. From (3.1), we have:

$$\begin{aligned} \overline{R}(X, Y)N &= \overline{\nabla}_X \overline{\nabla}_Y N - \overline{\nabla}_Y \overline{\nabla}_X N - \overline{\nabla}_{[X,Y]} N & (3.2) \\ &= k\{\eta(Y)X - \eta(X)Y\} \\ &= k\{\eta(Y)(PX + \eta(X)\xi) - \eta(X)(PY + \eta(Y)\xi\} \\ &= k\{\eta(Y)PX - \eta(X)PY\}. & (3.3) \end{aligned}$$

Now we compute (3.2). Using Equation (2.9), we have:

$$\begin{aligned} \overline{\nabla}_X \overline{\nabla}_Y N &= \overline{\nabla}_X\Big(-A_N Y + \tau(Y)N\Big) \\ &= -\overline{\nabla}_X A_N Y + X \cdot \tau(Y)N + \tau(Y)\overline{\nabla}_X N \\ &= -\nabla_X A_N Y - B(X, A_N Y)N + X \cdot \tau(Y)N \\ &\quad -\tau(Y)A_N X + \tau(X)\tau(Y)N \\ &= -(\nabla_X A_N)(Y) - A_N(\nabla_X Y) - \tau(Y)A_N X \\ &\quad +[-B(X, A_N Y) + X \cdot \tau(Y) + \tau(X)\tau(Y)]N. & (3.4) \end{aligned}$$

Interchanging X and Y, we get

$$\begin{aligned} \overline{\nabla}_Y \overline{\nabla}_X N &= -(\nabla_Y A_N)(X) - A_N(\nabla_Y X) - \tau(X)A_N Y \\ &\quad +[-B(Y, A_N X) + Y \cdot \tau(X) + \tau(Y)\tau(X)]N. & (3.5) \end{aligned}$$

We have also the equation

$$\overline{\nabla}_{[X,Y]} N = -A_N([X, Y]) + \tau([X, Y])N. \tag{3.6}$$

Since ∇ is a torsion-free connection, by using (3.3), (3.4), (3.5) and (3.6), we get

$$\begin{aligned} &-(\nabla_X A_N)(Y) + (\nabla_Y A_N)(X) - \tau(Y)A_N X + \tau(X)A_N Y \\ &+ \{-B(X, A_N Y) + B(Y, A_N X) + X \cdot \tau(Y) - Y \cdot \tau(X) \\ &-\tau([X, Y])\}N = k\{\eta(Y)X - \eta(X)Y\}, \end{aligned}$$

Again we have (c) and (d) by comparing the tangential and transversal parts in view of

$$2d\tau(X,Y) = X\cdot\tau(X) - Y\cdot\tau(Y) - \tau([X,Y]).$$

From (a), we have $R(X,Y)\xi = 0 = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi$, by the same computation as above, we have (e). Since $PX = X - \eta(X)$, assertion (f) follows by direct calculation.

Now, we recall the definition of a screen conformal lightlike hypersurface of a semi-Riemannian manifold $\overline{M}$.

Definition 3.2. ([1]). A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $\overline{M}$ is said to be locally screen (resp. globally) conformal if on any coordinate neighborhood $\mathcal{U}$ (resp. $\mathcal{U} = M$), the shape operators A_N and $\overset{*}{A}_\xi$ of M and its screen distribution $S(TM)$ are related by

$$A_N = \varphi\,\overset{*}{A}_\xi, \tag{3.7}$$

where φ is a non-vanishing smooth function on $\mathcal{U}$ (resp. $\mathcal{U} = M$).

$\mathcal{U}$ will be connected and maximal in the sense that there is no larger domain $\mathcal{U}' \supset \mathcal{U}$ on which Eq. (3.7) holds. It is easy to see that (3.7) is equivalent to

$$C(Y, PZ) = \varphi B(Y,Z), \tag{3.8}$$

for all $X, Y \in \Gamma(TM|_{\mathcal{U}})$.

In the sequel, we consider a lightlike hypersurface M of an $(m+2)$-dimensional Lorentzian manifold $(\overline{M}(k), \overline{g})$ of constant curvature k. For this class of screen conformal lightlike hypersurface M, the screen distribution $S(TM)$ is Riemannian, integrable and the induced Ricci tensor on M is symmetric ([1]). Then, according to Proposition 3.4 in [8], there exists a canonical null pair $\{\xi, N\}$ satisfying (2.2) such that the corresponding 1-form τ from (2.9) vanishes. Since ξ is an eigenvector field of $\overset{*}{A}_\xi$ corresponding to the eigenvalue 0 and $\overset{*}{A}_\xi$ is $\Gamma(S(TM))$-valued real symmetric, $\overset{*}{A}_\xi$ has m orthonormal eigenvector fields in $S(TM)$ and is diagonalizable. Consider a frame field of eigenvectors $\{\xi, E_1, \ldots, E_m\}$ of $\overset{*}{A}_\xi$ such that $\{E_1, \ldots, E_m\}$ is an orthonormal frame field of $S(TM)$. Then, $\overset{*}{A}_\xi E_i = \lambda_i E_i, \quad 1 \le i \le m$. We call the eigenvalues λ_i the *screen principal curvatures* for all i.

In the following, we assume that all screen principal curvatures are constant along $S(TM)$ and $\tau = 0$. Consider the following distribution on M:

$$T_\lambda = \{X \in \Gamma(S(TM)) : \overset{*}{A}_\xi X = \lambda X\}.$$

Lemma 3.3. *For any $X \in \Gamma(TM)$, it follows that*

(i) $(\nabla_X \overset{*}{A}_\xi)Y = (\nabla_Y \overset{*}{A}_\xi)X$, *for all* $Y \in \Gamma(TM)$;

(ii) $(\overset{*}{\nabla}_X \overset{*}{A}_\xi)Y = (\overset{*}{\nabla}_Y \overset{*}{A}_\xi)X$, *for all* $Y \in \Gamma(TM)$;

(iii) $\nabla_X \overset{*}{A}_\xi$ *is symmetric with respect to g i.e. for all* $Y, Z \in \Gamma(S(TM))$

$$g\big((\nabla_X \overset{*}{A}_\xi)Y, Z) = g(Y, (\nabla_X \overset{*}{A}_\xi)Z\big);$$

(iv) for any Y, Z *in* $\Gamma(S(TM))$,

$$g\big((\nabla_X \overset{*}{A}_\xi)Y, Z\big) = g\big(Y, (\nabla_Z \overset{*}{A}_\xi)X\big) = g\big((\nabla_Z \overset{*}{A}_\xi)Y, X\big);$$

(v) for $Y \in \Gamma(T_\lambda)$, $Z \in \Gamma(T_\mu)$, we have

$$g\Big((\nabla_X \overset{*}{A}_\xi)Y,Z\Big) = g\Big((\nabla_Z \overset{*}{A}_\xi)Y,X\Big) = (\lambda-\mu)g(\nabla_X Y,Z).$$

Proof. Let $X,Y \in \Gamma(TM)$. Then (i) is a consequence of (e) in Proposition 3.1 by using $\tau = 0$.

$$\begin{aligned}
(\overset{*}{\nabla}_X\overset{*}{A}_\xi)Y &= \overset{*}{\nabla}_X\overset{*}{A}_\xi\, Y - \overset{*}{A}_\xi\overset{*}{\nabla}_X\, Y\\
&\overset{(2.15)}{=} \nabla_X \overset{*}{A}_\xi\, Y - C(X,\overset{*}{A}_\xi\, Y)\xi - \overset{*}{A}_\xi\, \nabla_X Y\\
&= (\nabla_X \overset{*}{A}_\xi)Y - C(X,\overset{*}{A}_\xi\, Y)\\
&= (\nabla_Y \overset{*}{A}_\xi)X - C(X,\overset{*}{A}_\xi\, Y)\\
&= (\overset{*}{\nabla}_Y\overset{*}{A}_\xi)X + C(Y,\overset{*}{A}_\xi\, X) - C(X,\overset{*}{A}_\xi\, Y).
\end{aligned}$$

Since $\overline{R}(X,Y)\xi = 0$ (Eq. (3.1), by (2.23), we have $C(Y,\overset{*}{A}_\xi\, X) - C(X,\overset{*}{A}_\xi\, Y) = 0$. Then we infer (ii). For (iii), let $X \in \Gamma(TM)$ and $Y,Z \in \Gamma(S(TM))$. We use the symmetry of $\overset{*}{A}_\xi$ with respect to g and equation (2.17),

$$\begin{aligned}
g\Big((\nabla_X \overset{*}{A}_\xi)Y,Z\Big) &= g\Big(\nabla_X \overset{*}{A}_\xi\, Y,Z\Big) - g\Big(\overset{*}{A}_\xi\,(\nabla_X Y),Z\Big)\\
&= g\Big(\nabla_X \overset{*}{A}_\xi\, Y,Z\Big) - g\Big((\nabla_X Y),\overset{*}{A}_\xi\, Z\Big)\\
&= g\Big(\nabla_X \overset{*}{A}_\xi\, Y,Z\Big) - g\Big(\nabla_X Y,\overset{*}{A}_\xi\, Z\Big)\\
&= -(\nabla_X g)(\overset{*}{A}_\xi\, Y,Z) + X\cdot g(\overset{*}{A}_\xi\, Y,Z) - g(\overset{*}{A}_\xi\, Y,\nabla_X Z)\\
&\quad +(\nabla_X g)(Y,\overset{*}{A}_\xi\, Z) - X\cdot g(Y,\overset{*}{A}_\xi\, Z) + g(Y,\nabla_X \overset{*}{A}_\xi\, Z)\\
&= -g(Y,\overset{*}{A}_\xi\, \nabla_X Z) + g(Y,\nabla_X \overset{*}{A}_\xi\, Z)\\
&= g(Y,(\nabla_X \overset{*}{A}_\xi)Z).
\end{aligned}$$

Now (iv) comes from (i) and (iii). To prove (v), let $X \in \Gamma(TM)$, $Y \in \Gamma(T_\lambda)$ and $Z \in \Gamma(T_\mu)$. By the symmetry of $\overset{*}{A}_\xi$ with respect to g, we have

$$\begin{aligned}
g((\nabla_X \overset{*}{A}_\xi)Y,Z) &= g(\nabla_X \overset{*}{A}_\xi\, Y,Z) - g(\overset{*}{A}_\xi\,(\nabla_X Y),Z)\\
&= \lambda g(\nabla_X Y,Z) - g(\nabla_X Y,\overset{*}{A}_\xi\, Z)\\
&= (\lambda-\mu)g(\nabla_X Y,Z).
\end{aligned}$$

Thus we have (v) by using (iv)

Lemma 3.4. *Let λ and μ be screen principal curvatures of M. Then we have*

(1) $\overset{*}{\nabla}_X\, Y \in \Gamma(T_\lambda)$ if $X,Y \in \Gamma(T_\lambda)$,

(2) $\nabla_X Y \perp T_\lambda$; $\nabla_Y X \perp T_\mu$ if $X \in \Gamma(T_\lambda)$, $Y \in \Gamma(T_\mu)$, $\lambda \neq \mu$.

Proof. Let $Z \in \Gamma(TM)$ and $X, Y \in \Gamma(T_\lambda)$. By (ii) and (iv) in Lemma 3.3, it follows that

$$\begin{aligned} g(\overset{*}{A}_\xi \overset{*}{\nabla}_X Y, Z) &= g(\overset{*}{\nabla}_X \overset{*}{A}_\xi Y, Z) - g((\overset{*}{\nabla}_X \overset{*}{A}_\xi)Y, Z) \\ &= \lambda g(\overset{*}{\nabla}_X Y, Z) - g(Y, (\overset{*}{\nabla}_Z \overset{*}{A}_\xi)X) \\ &= \lambda g(\overset{*}{\nabla}_X Y, Z) - g(Y, \overset{*}{\nabla}_Z \overset{*}{A}_\xi X) + g(\overset{*}{\nabla}_Z X, \overset{*}{A}_\xi Y) \\ &= \lambda g(\overset{*}{\nabla}_X Y, Z) - \lambda g(Y, \overset{*}{\nabla}_Z X) + \lambda g(\overset{*}{\nabla}_Z X, Y) \\ &= \lambda g(\overset{*}{\nabla}_X Y, Z), \end{aligned}$$

and we conclude that $\overset{*}{A}_\xi \overset{*}{\nabla}_X Y - \lambda \overset{*}{\nabla}_X Y = \alpha\xi$, where α is a smooth function. Since $\eta(\overset{*}{A}_\xi \overset{*}{\nabla}_X Y - \lambda \overset{*}{\nabla}_X Y) = 0 = \alpha$, then $\overset{*}{A}_\xi \overset{*}{\nabla}_X Y - \lambda \overset{*}{\nabla}_X Y = 0$. That is $\overset{*}{A}_\xi \overset{*}{\nabla}_X Y = \lambda \overset{*}{\nabla}_X Y$ This proves (1). For $Z \in \Gamma(T_\lambda)$ and $X \in \Gamma(T_\lambda), Y \in \Gamma(T_\mu)$, using (v) of Lemma 3.3, it follows that

$$g\Big((\nabla_X \overset{*}{A}_\xi)Y, Z\Big) = (\mu - \lambda) g(\nabla_X Y, Z). \tag{3.9}$$

On the other hand, by (iv) in Lemma 3.3, we compute

$$g\Big((\nabla_X \overset{*}{A}_\xi)Y, Z\Big) = g\Big((\nabla_Z \overset{*}{A}_\xi)X, Y\Big) = -(\mu - \lambda) g(\overset{*}{\nabla}_Z X, Y). \tag{3.10}$$

By (1), it comes that $\overset{*}{\nabla}_Z X \in \Gamma(T_\lambda)$ for $X, Z \in \Gamma(T_\lambda)$ and therefore $g(\nabla_Z X, Y) = 0$. Combining relations (3.9) and (3.10), we obtain

$$-(\mu - \lambda) g(\overset{*}{\nabla}_Z X, Y) = (\mu - \lambda) g(\nabla_X Y, Z) = 0.$$

Hence, if $\lambda \neq \mu$, then $\nabla_X Y \perp T_\lambda$. Similarly, we have $\nabla_Y X \perp T_\mu$ if $\lambda \neq \mu$.

Now we prove the following theorem which extends Cartan's fundamental formula on lightlike hypersurfaces of Lorentzian manifolds with constant curvature.

Theorem 3.5. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an $(m+2)$-dimensional Lorentzian manifold $(\overline{M}(k), \overline{g})$ of constant curvature k. Assume that $E_0 = \xi$, $E_1, \ldots, E_m$ are eigenvectors of $\overset{*}{A}_\xi$ satisfying $\overset{*}{A}_\xi E_0 = 0$ and $\overset{*}{A}_\xi E_i = \lambda_i E_i$, such that λ_i is constant along $S(TM)$ for all i and $\tau = 0$ ($\{E_i\}_{i=1,\ldots,m}$ represents an orthonormal basis of $S(TM)$). Then for every $i \in \{1, \ldots, m\}$, we have*

$$\sum_{\substack{j=1 \\ \lambda_j \neq \lambda_i}}^{m} \frac{k + \lambda_j g(A_N E_i, E_i) + \lambda_i g(A_N E_j, E_j)}{\lambda_i - \lambda_j} = 0. \tag{3.11}$$

Moreover, if the screen is conformal with conformal factor φ, then for all $i \in \{1, \ldots, m\}$

$$\sum_{\substack{j=1 \\ \lambda_j \neq \lambda_i}}^{m} \frac{k + 2\varphi \lambda_i \lambda_j}{\lambda_i - \lambda_j} = 0. \tag{3.12}$$

Proof. From (a) in Proposition 3.1 and (2.19), we have

$$R(E_i, E_j)E_j = kE_i + \lambda_j A_N E_i. \tag{3.13}$$

On the other hand, using the definition of a curvature tensor R, for $\lambda_i \neq \lambda_j$, Lemma 3.4, (2.17) and (2.20), we compute

$$\begin{aligned} g\Big(R(E_i,E_j)E_j,E_i\Big) &= g(\nabla_{E_i}\nabla_{E_j}E_j,E_i) - g(\nabla_{E_j}\nabla_{E_i}E_j,E_i) - g(\nabla_{[E_i,E_j]}E_j,E_i) \\ &= g(\nabla_{E_i}E_j,\nabla_{E_j}E_i) - g(\nabla_{[E_i,E_j]}E_j,E_i) - \lambda_i.g(A_N E_j,E_j). \end{aligned} \tag{3.14}$$

From relation (3.13), we get

$$g\Big(R(E_i,E_j)E_j,E_i\Big) = k + \lambda_j g(A_N E_i,E_i). \tag{3.15}$$

Now the equality between (3.14) and (3.15) gives

$$k + \lambda_j g(A_N E_i,E_i) + \lambda_i g(A_N E_j,E_j) = g(\nabla_{E_i}E_j,\nabla_{E_j}E_i) - g(\nabla_{[E_i,E_j]}E_j,E_i). \tag{3.16}$$

By (v) of Lemma 3.3, we get

$$\begin{aligned} g\Big((\nabla_{[E_i,E_j]}\overset{*}{A}_\xi)E_i,E_j\Big) &= (\lambda_i-\lambda_j)g(\nabla_{[E_i,E_j]}E_i,E_j) \\ &\overset{(2.17)}{=} (\lambda_j-\lambda_i)g(\nabla_{[E_i,E_j]}E_j,E_i), \end{aligned}$$

from which it follows that

$$g(\nabla_{[E_i,E_j]}E_j,E_i) = \frac{g\Big((\nabla_{[E_i,E_j]}\overset{*}{A}_\xi)E_i,E_j\Big)}{\lambda_j-\lambda_i}. \tag{3.17}$$

Using (i), (iv) and (v) of Lemma 3.3, we compute

$$\begin{aligned} g\Big((\nabla_{[E_i,E_j]}\overset{*}{A}_\xi)E_i,E_j\Big) &= g\Big((\nabla_{E_i}\overset{*}{A}_\xi)E_j,[E_i,E_j]\Big) \\ &= g\Big((\nabla_{E_i}\overset{*}{A}_\xi)E_j,\nabla_{E_i}E_j\Big) - g\Big((\nabla_{E_i}\overset{*}{A}_\xi)E_j,\nabla_{E_j}E_i\Big) \\ &= g\Big((\nabla_{E_j}\overset{*}{A}_\xi)E_i,\nabla_{E_i}E_j\Big) - g\Big((\nabla_{E_i}\overset{*}{A}_\xi)E_j,\nabla_{E_j}E_i\Big) \\ &= 2(\lambda_i-\lambda_j)g(\nabla_{E_i}E_j,\nabla_{E_j}E_i), \end{aligned}$$

that is,

$$g\Big((\nabla_{[E_i,E_j]}\overset{*}{A}_\xi)E_i,E_j\Big) = 2(\lambda_i-\lambda_j)g(\nabla_{E_i}E_j,\nabla_{E_j}E_i). \tag{3.18}$$

Combining (3.16), (3.17) and (3.18), we have

$$k + \lambda_j g(A_N E_i,E_i) + \lambda_i g(A_N E_j,E_jE_i) = 2g(\nabla_{E_i}E_j,\nabla_{E_j}E_i). \tag{3.19}$$

Since $\nabla_{E_i}E_j = \sum_{s=1}^m g(\nabla_{E_i}E_j,E_s)E_s + \eta(\nabla_{E_i}E_j)\xi$, relation (3.19) becomes

$$k + \lambda_j g(A_N E_i,E_i) + \lambda_i g(A_N E_j,E_j) = 2\sum_{s=1}^m g(\nabla_{E_i}E_j,E_s)g(\nabla_{E_j}E_i,E_s). \tag{3.20}$$

Again, by using (i) and (v) of Lemma 3.3, we get

$$k + \lambda_j g(A_N E_i,E_i) + \lambda_i g(A_N E_j,E_j) = 2\sum_{\substack{s=1\\ s\neq i,j}}^m \frac{g((\nabla_{E_i}\overset{*}{A}_\xi)E_j,E_s)^2}{(\lambda_i-\lambda_s)(\lambda_j-\lambda_s)}, \tag{3.21}$$

and therefore we have

$$
\begin{aligned}
&\sum_{\substack{j=1\\ \lambda_j\neq\lambda_i}}^{m} \frac{k+\lambda_j g(A_N E_i, E_i)+\lambda_i g(A_N E_j, E_j)}{\lambda_i-\lambda_j}\\
= &\sum_{\substack{j=1\\ \lambda_j\neq\lambda_i,\lambda_s}}^{m} 2 \sum_{\substack{s=1\\ \lambda_j\neq\lambda_i,\lambda_s}}^{m} \frac{g((\nabla_{E_i}\overset{*}{A}_{\xi})E_j, E_s)^2}{(\lambda_i-\lambda_s)(\lambda_j-\lambda_s)(\lambda_i-\lambda_j)}\\
= &\sum_{\substack{s=1\\ \lambda_j\neq\lambda_i,\lambda_s}}^{m} -\frac{1}{\lambda_i-\lambda_s} 2 \sum_{\substack{j=1\\ \lambda_j\neq\lambda_i,\lambda_s}}^{m} \frac{g((\nabla_{E_i}\overset{*}{A}_{\xi})E_j, E_s)^2}{(\lambda_s-\lambda_j)(\lambda_i-\lambda_j)}\\
\overset{(3.21)}{=} &-\sum_{\substack{s=1\\ \lambda_s\neq\lambda_i}}^{m} \frac{k+\lambda_j g(A_N E_i, E_i)+\lambda_i g(A_N E_j, E_j)}{\lambda_i-\lambda_s},
\end{aligned}
$$

thus (3.11) follows. Using (3.7), we get (3.12).

4 Application

In this section, we consider a lightlike hypersurface M of $\mathbb{R}_1^{m+2}$ whose screen principal curvatures are constant along $S(TM)$. We assume that M has at most two distinct screen principal curvatures. We prove the following

Theorem 4.1. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of $\mathbb{R}_1^{m+2}$ whose screen principal curvatures are constant along the screen distribution $S(TM)$ and at most two are distinct. If M has two distinct screen principal curvatures, then one of them must be zero.*

Proof Since we assume that M has at most two distinct screen principal curvatures, then there exists $p \in \{1, \ldots, m\}$ such that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_p = \alpha \ \text{ and } \ \lambda_{p+1} = \lambda_{p+2} = \cdots = \lambda_m = \beta.$$

By using this together with equation (3.7) and $k = 0$, the equation (3.19) becomes

$$\varphi\alpha\beta = g(\nabla_{E_\alpha} E_\beta, \nabla_{E_\beta} E_\alpha),$$

where φ is the conformal factor. Using (2) in Lemma 3.4, we have $g(\nabla_{E_\alpha} E_\beta, \nabla_{E_\beta} E_\alpha) = 0$. Thus $\varphi\alpha\beta = 0$. Since φ is a nowhere-vanishing and smooth function defined on a connected and maximal neighborhood, $\alpha \neq \beta$, then $\alpha = 0$ and $\beta \neq 0$ or $\alpha \neq 0$ and $\beta = 0$.

Now, suppose that M has exactly two distinct screen principal curvatures. Then, by Theorem 4.1 one of them must be 0. We denote by λ the non-zero screen principal curvatures and r the multiplicity of λ. The sets,

$$
\begin{aligned}
T_\lambda &= \{X \in \Gamma(S(TM)) |\, \overset{*}{A}_{\xi} X = \lambda X\}\\
T_0 &= \{X \in \Gamma(S(TM)) |\, \overset{*}{A}_{\xi} X = 0\}
\end{aligned}
$$

define the distributions of dimension r and dimension $m-r$, respectively. By Lemma 3.4, T_λ and T_0 are both involutive and if $X \in \Gamma(T_\lambda)$, $Y \in \Gamma(T_0)$, then $\nabla_X Y \in \Gamma(T_0)$, $\nabla_Y X \in \Gamma(T_\lambda)$, which shows that T_λ and T_0 are parallel along their normals in $S(TM)$. Moreover, it is known that (see [1]) if M is a screen conformal lightlike hypersurface of a Lorentzian manifold, the screen distribution $S(TM)$ is Riemannian, integrable and the induced Ricci tensor on M is symmetric. More precisely, a screen conformal lightlike hypersurface is locally a product $C \times M'$ where C is a null curve, M' is an integral manifold of $S(TM)$ ([1]). We have the following local decomposition.

Theorem 4.2. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of a Lorentzian Euclidean space $\mathbb{R}_1^{m+2}$ with exactly two distinct screen curvatures which are constant along $S(TM)$. Then, M is locally a lightlike triple product manifold $C \times (M' = M_\lambda \times M_0)$, where C is a null curve, M' is an integral manifold of $S(TM)$, M_λ and M_0 are leaves of some distributions M such that M_λ is a totally geodesic Riemannian manifold of constant curvature $2\varphi\lambda^2$ and M_0 is an $(m-r)$-dimensional totally geodesic Euclidean space.*

Proof Since M has exactly two distinct screen curvatures, by Theorem 4.1 one must be zero and we denote by λ the non-zero one. Then we take T_λ and T_0 as above. On the other hand, the leaf M' of $S(TM)$ is Riemannian and $S(TM) = T_\lambda \perp T_0$, where T_λ and T_0 are parallel distributions with respect to the induced connection on M'. By the decomposition theorem of de Rham ([7]), we have $M' = M_\lambda \times M_0$, where M_λ and M_0 are some leaves of T_λ and T_0 respectively. Thus M is locally a product $C \times M' = C \times M_\lambda \times M_0$. Now, let $X \in \Gamma(T_\lambda)$ and $Y \in \Gamma(T_0)$, we have $g(X, Y) = 0$, and then $g(\overset{*}{\nabla}_Z X, Y) + g(X, \overset{*}{\nabla}_Z Y) = 0$. If $Z \in \Gamma(T_\lambda)$, by Lemma 3.4, $\overset{*}{\nabla}_Z X \in \Gamma(T_\lambda)$ and $g(\overset{*}{\nabla}_Z X, Y) = 0$. This shows that M_λ is totally geodesic in $S(TM)$. In entirely the same way, we can see that M_0 is totally geodesic in $S(TM)$. Consider the frame field of eigenvectors $\{E_1, \ldots, E_r\}$ of $\overset{*}{A}_\xi$ such that $\{E_i\}_{i=1,\ldots,r}$ is an orthonormal frame field of T_λ, then using (3.13) and (2.22) we have: $g\Big(R(E_i, E_j)E_j, E_i\Big) = \varphi\lambda^2 = g\Big(\overset{*}{R}(E_i, E_j)E_j, E_i\Big) - \lambda^2\varphi$. Then $g\Big(\overset{*}{R}(E_i, E_j)E_j, E_i\Big) = 2\lambda^2\varphi$. Thus the sectional curvature K of the leaf M_λ of T_λ is given by

$$K(E_i, E_j) = \frac{g\Big(\overset{*}{R}(E_i, E_j)E_j, E_i\Big)}{g(E_i, E_i)g(E_j, E_j) - g^2(E_i, E_j)} = 2\varphi\lambda^2.$$

By the same way, we can see that the sectional curvature K' of the leaf M_0 of T_0 is 0. This completes the proof.

Next, we say that M is totally umbilical if for any coordinate neighborhood $\mathcal{U} \subset M$, there exists a smooth function ρ such that

$$B(X, Y) = \rho g(X, Y), \tag{4.1}$$

for all $X, Y \in \Gamma(TM|_{\mathcal{U}})$, or equivalently,

$$\overset{*}{A}_\xi X = \rho PX, \tag{4.2}$$

for all $X \in \Gamma(TM|_{\mathcal{U}})$. It is easy to see that if the screen principal curvatures are all identical and non-zero then M is totally umbilical.

M is said to be a totally geodesic lightlike hypersurface if the second fundamental form $B = 0$ or equivalently $\overset{*}{A}_\xi = 0$. It is easy to see that if the screen principal curvatures are all zero then M is totally geodesic.

Remark 4.3. Since we assume that M has at most two distinct screen principal curvatures, thus, if the screen principal curvatures are all identical, M is either totally geodesic or totally umbilical and if the two screen principal curvatures are distinct, then $M = C \times M_\lambda \times M_0$.

Thus we have the following classifcation theorem.

Theorem 4.4. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of a Lorentzian Euclidean space $\mathbb{R}_1^{m+2}$ whose screen principal curvatures are all contant along the screen distribution $S(TM)$ such that at most two of them are distinct. Then we have one of the following:*

(1) M is either totally geodesic or totally umbilical;

(2) M is locally a lightlike triple product manifold $C \times M_\lambda \times M_0$, where C is a null curve, M_λ and M_0 are leaves of some distributions of M such that $M_\lambda$is a totally geodesic Riemannian manifold of constant curvature $2\varphi\lambda^2$ and M_0 is an $(m-r)$-dimensional totally geodesic Euclidean space.

Theorem 4.5. *Let M_λ and M_0 be as in theorem 4.2. Then M_λ is a totally umbilical submanifold of $\mathbb{R}_1^{m+2}$ and M_0 is a totally geodesic submanifold of $\mathbb{R}_1^{m+2}$.*

Proof As M' is a Riemannian submanifold of codimension 2 of $\mathbb{R}_1^{m+2}$, consider in the normal bundle $TM'^{\perp}$, the vector fields

$$\zeta_1 = \frac{\varphi}{\sqrt{2|\varphi|}}\xi + \frac{1}{\sqrt{2|\varphi|}}N \quad \text{and} \quad \zeta_2 = \frac{\varphi}{\sqrt{2|\varphi|}}\xi - \frac{1}{\sqrt{2|\varphi|}}N.$$

Clearly, $\{\zeta_1, \zeta_2\}$ is an orthonormal basis , where ζ_1 and ζ_2 are spacelike and timelike respectively. Then for any $X,\ Y \in \Gamma(TM_\lambda)$, we have

$$\overline{\nabla}_X Y = \nabla^\lambda_X Y + \sum_{a=r+1}^{m+2} g_\lambda(A_{\xi^\lambda_a} X, Y)\xi^\lambda_a, \tag{4.3}$$

where g_λ, ∇^λ are the induced metric and the induced connection on M_λ respectively, ξ^λ_a are orthonormal normals to TM_λ in $\mathbb{R}_1^{m+2}$ such that $\xi^\lambda_{m+1} = \zeta_1$ and $\xi^\lambda_{m+2} = \zeta_2$, $A_{\xi^\lambda_a}$ are corresponding shape operators of ξ^λ_a. In addition,

$$\begin{aligned}
\overline{\nabla}_X Y &= \nabla_X Y + B(X,Y)N \\
&= \nabla_X Y + g(\overset{*}{A}_\xi X, Y)N \\
&= \overset{*}{\nabla}_X Y + C(X,Y)\xi + g(\overset{*}{A}_\xi X, Y)N \\
&= \overset{*}{\nabla}_X Y + g(A_N X, Y)\xi + g(\overset{*}{A}_\xi X, Y)N \\
&= \overset{*}{\nabla}_X Y + \varphi g(\overset{*}{A}_\xi X, Y)\xi + g(\overset{*}{A}_\xi X, Y)N \\
&= \nabla^\lambda_X Y + \sum_{a=r+1}^{m} g_\lambda(A'_{\xi^\lambda_a} X, Y)\xi^\lambda_a + g(\overset{*}{A}_\xi X, Y)(\varphi\xi + N) \\
&= \nabla^\lambda_X Y + \sum_{a=r+1}^{m} g_\lambda(A'_{\xi^\lambda_a} X, Y)\xi^\lambda_a + \lambda g(X,Y)(\varphi\xi + N),
\end{aligned} \tag{4.4}$$

where $A'_{\xi^\lambda_a}$ denotes the shape operator of M_λ with respect to ξ^λ_a in $S(TM)$. By Theorem 4.2, M_λ is totally geodesic in $S(TM)$, and consequently the equation (4.4) can be written as follows:

$$\overline{\nabla}_X Y = \nabla^\lambda_X Y + \lambda g_\lambda(X,Y)(\varphi\xi + N) = \nabla^\lambda_X Y + \sqrt{2|\varphi|}\lambda g_\lambda(X,Y)\zeta_1. \tag{4.5}$$

Comparing (4.3) and (4.5), we have $A_{\xi^\lambda_a} X = 0$, for all $a \neq m+1$ and $A_{\xi^\lambda_{m+1}} X = A_{\zeta_1} X = \sqrt{2|\varphi|}\lambda X$. Thus, M_λ is a totally umbilical submanifold of $\mathbb{R}_1^{m+2}$. Similarly, we can prove that M_0 is a totally geodesic submanifold in $\mathbb{R}_1^{m+2}$.

5 Examples

Example 5.1. Let $(\mathbb{R}^4_1, \bar{g})$ be a 4-dimensional semi-Euclidean space with Lorentzian signature. Consider a Monge hypersurface M of $\mathbb{R}^4_1$ given by

$$t = \frac{1}{\sqrt{2}}\left(x + \sqrt{y^2+z^2}\right).$$

It is easy to check that M is a lightlike hypersurface whose radical distribution $\mathrm{Rad}TM$ is spanned by

$$\xi = \partial_t + \frac{y}{\sqrt{2}\sqrt{y^2+z^2}}\partial_y + \frac{z}{\sqrt{2}\sqrt{y^2+z^2}}\partial_z + \frac{1}{\sqrt{2}}\partial_x.$$

It is readily checked that, one gets an orthonormal basis $\{E_1, E_2\}$ of $S(TM)$ given by

$$\begin{aligned} E_1 &= \frac{1}{\sqrt{y^2+z^2}}\left(-z\partial_y + y\partial_z\right); \\ E_2 &= \epsilon\frac{1}{\sqrt{2}\sqrt{y^2+z^2}}\left(\sqrt{y^2+z^2}\partial_x - y\partial_y - z\partial_z\right) \quad \epsilon = \pm. \end{aligned}$$

Then the lightlike transversal vector bundle is given by

$$tr(TM) = \mathrm{Span}\left\{N = -\frac{1}{2}\partial_t + \frac{y}{\sqrt{8}\sqrt{y^2+z^2}}\partial_y + \frac{z}{\sqrt{8}\sqrt{y^2+z^2}}\partial_z + \frac{1}{\sqrt{8}}\partial_x\right\}.$$

By direct computation, we obtain

$$\overline{\nabla}_{E_1}\xi = \nabla_{E_1}\xi = \frac{1}{\sqrt{2}\sqrt{y^2+z^2}}E_1 \text{ and } \overline{\nabla}_{E_2}\xi = \nabla_{E_2}\xi = 0. \quad (5.1)$$

Thus, from the Weingarten formula (2.16), we have

$$\overset{*}{A}_\xi E_1 = -\frac{1}{\sqrt{2}\sqrt{y^2+z^2}}E_1, \ \overset{*}{A}_\xi E_2 = 0 \text{ and } \tau = 0.$$

Then, M has two distinct screen principal curvatures $\lambda_1 = -\frac{1}{\sqrt{2}\sqrt{y^2+z^2}}$ and $\lambda_2 = 0$. On the other hand, we have

$$\overline{\nabla}_{E_1}N = \frac{1}{\sqrt{8}\sqrt{y^2+z^2}}E_1, \ \overline{\nabla}_{E_2}N = 0 \text{ and } \overline{\nabla}_\xi N = 0. \quad (5.2)$$

Then, from the Weingarten formula (2.9), we have

$$A_N E_1 = -\frac{1}{\sqrt{8}\sqrt{y^2+z^2}}E_1 = \frac{1}{2}\overset{*}{A}_\xi E_1, \ A_N E_2 = 0 \text{ and } A_N\xi = 0.$$

Next, any $X \in \Gamma(TM)$, is expressed by $X = \alpha E_1 + \beta E_2 + \gamma\xi$, where α, β, γ are smooth functions, and then $A_N X = \alpha A_N E_1 + \beta A_N E_2 + \gamma A_N \xi = \frac{1}{2}\overset{*}{A}_\xi X$, that is M is a screen conformal lightlike hypersurface of $\mathbb{R}^4_1$ with conformal factor $\varphi = \frac{1}{2}$. Thus, M is a screen conformal lightlike hypersurface of $\mathbb{R}^4_1$ with two distinct screen principal curvatures.

Example 5.2. (The lightlike cone Λ_0^3 of $\mathbb{R}_1^4$)
Let $\mathbb{R}_1^4$ be the space $\mathbb{R}^4$ endowed with the semi-Euclidean metric

$$\overline{g}(u,v) = -xx' + yy' + zz' + tt',$$

where $u = (x,y,z,t)$ and $v = (x',y',z',t')$. The light cone Λ_0^3 is given by the equation $-x^2+y^2+z^2+t^2 = 0$ with $(x,y,z,t) \neq (0,0,0,0)$. It is known that Λ_0^3 is a lightlike hypersurface of $\mathbb{R}_1^4$ and the radical distribution is spanned by a global vector field

$$\xi = x\partial_x + y\partial_y + z\partial_z + t\partial_t \tag{5.3}$$

on Λ_0^3. It is easy to see that, one gets an orthonormal basis $\{E_1, E_2\}$ of $S(T\Lambda_0^3)$ given by

$$\begin{aligned} E_1 &= \left(\frac{t^2+y^2}{t^2}\right)^{\frac{1}{2}} \left(\partial_y - \frac{y}{t}\partial_t\right), \\ E_2 &= \left(\frac{t^2+y^2}{x^2}\right)^{\frac{1}{2}} \left(-\frac{yz}{t^2+y^2}\partial_y + \partial_z + -\frac{zt}{t^2+y^2}\partial_t\right). \end{aligned}$$

As ξ is a position vector field, we get for all $i = 1,2$

$$\overline{\nabla}_{E_i}\xi = \nabla_{E_i}\xi = E_i.$$

Using (2.16), we have $\overset{*}{A}_\xi E_i + \tau(E_i)\xi + E_i = 0$. As $\overset{*}{A}_\xi$ is $\Gamma(S(TM))$-valued we obtain

$$\overset{*}{A}_\xi E_i = -E_i, \tag{5.4}$$

for all $i = 1,2$ This proves that $\lambda_1 = \lambda_2 = -1$ and $\tau = 0$.

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