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The Weyl Algebra and Noetherian Operators

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Abstract

We give an explicit and purely algebraic proof for the existence of *noetherian differential operators* for primary ideals of polynomial algebras. The proof of this important result in [1] uses complicated algebraic and analytic techniques. Later U.Oberst gave an elementary and constructive proof in [6]. In this paper we propose a different proof from the one in [6].

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1 Introduction

Let K be a field of characteristic zero. Denote by $K[X] = K[x_1,\dots,x_n]$ the polynomial ring in n variables and by $A_n(K)$ the *n-th Weyl algebra*.
Let P be a prime ideal of $K[X]$ and $Q \subset P$ be a primary ideal so that $\sqrt{Q} = P$. Let

$$F(X,\partial) = \sum p_\alpha(X)\partial^\alpha$$

be in $A_n(K)$ such that $F(Q) \subset P$. Then we say that F is a Noetherian operator with respect to Q. Denote by $\mathcal{N}(Q)$ the set of all Noetherian operators $F \in A_n(K)$.

Main Theorem There exist $F_1,\dots,F_l \in \mathcal{N}(Q)$ such that if $\varphi \in K[X]$ satisfies

$$F_\nu(\varphi) \in P, \quad 1 \le \nu \le l$$

then $\varphi \in Q$.

The proof requires several steps. First we use Noether's Normalization Theorem for prime ideals and reduce the proof to the case when P is a maximal ideal. In this situation we employ Kashiwara's Decomposition Theorem to reduce the proof to the case $n = 1$. The 1-dimensional case is treated in a separate section where certain facts about the 1-dimensional Weyl algebra appear.

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1.1 A Special basic case

Let J be an ideal in $K[X]$ and assume there exists an integer $w \geq 1$ such that $(x_1,\dots,x_n)^{w+1} = J$ (J is the $w+1$-th power of the maximal $\mathfrak{M} = (x_1,\dots,x_n)$). Consider differential operators with constant coefficients

$$F(\partial) = \sum c_\alpha \partial^\alpha, c_\alpha \in K, \alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n.$$

Put $\mathcal{N}_c(J) = \{F(\partial) : F(J) \subset \mathfrak{M}\}$, such $F \in \mathcal{N}(J)$.
Now we have the finite dimensional K-vector space $V = K[X]/J$. Each $F \in \mathcal{N}_c(J)$ gives a K-linear map $\varphi_F : V \to K$ as follows. If a vector $v \in V$ is image of p, $p \in K[X]$, let $\varphi_F(v) = F(p)(0)$. Since for $p \in J, F(p)$ has no constant term, φ_F is well-defined. Thus every Noetherian operator $F \in \mathcal{N}_c(J)$ produces a $\varphi_F \in V^* = \mathrm{Hom}_K(V,K)$. So we have constructed a K-linear map $\mathcal{N}_c(J) \to V^*$. With these notations we can announce the following duality theorem.

Theorem 1.1. *$\mathcal{N}_c(J) \cong V^*$. In particular $\dim_K \mathcal{N}_c(J) = \dim_K V^* = dim_K K[X]/J$.*

Proof. For α,β multi-indices we have

$$\begin{cases} \partial^\alpha(x^\beta)(0) = 0 \text{ if } \alpha \neq \beta \\ \partial^\alpha(x^\alpha)(0) = \alpha! \end{cases}$$

where $\alpha! = \alpha_1!\cdots\alpha_n!$.

We know that $V = \oplus_{|\alpha|\leq w} Kx^\alpha$, also for $|\beta| \leq w$, $\partial^\beta(J) \subset \mathfrak{M}$ and $\varphi_{\partial^\beta}(\overline{c_\alpha x^\alpha}) = c_\beta \beta!$. Then $\ker\varphi_{\partial^\beta} \oplus Kx^\beta = V$ and $\cap_{|\beta|\leq w} \ker\varphi_{\partial^\beta} = \{0\}$, it follows that $\mathrm{Hom}(V,K) \cong \oplus_{|\beta|\leq w} K\partial^\beta \cong \mathcal{N}_c(J)$. □

In this case we thus see that $\mathcal{N}_c(J) = \mathcal{N}(J) \cap K[\partial]$ is a finite dimensional vector space. And by taking a basis $F_1,\dots,F_k$ we have found Noetherian operators as in the main theorem.

2 The 1-dimensional case

Notation: $A_1 = A_1(K) = K\langle t, \partial_t\rangle$

Theorem 2.1. *Let $q(t) \in K[t]$ be an irreducible polynomial. Then $A_1 q$ is a maximal left ideal in A_1, so that $\dfrac{A_1}{A_1 q}$ is simple as A_1-module.*

Proof. Without loss of generality, let us change the variable t to ∂_t. For simplicity we write ∂ for ∂_t so $q(\partial) = \partial^e + k_1\partial^{e-1} + \cdots + k_e,\ \ k_j \in K, e = \deg(q) \geq 2$ (the case $\deg(q) = 1$ is immediate). We have

$$A_1 = A_1 q \oplus K[t] \oplus K[t]\partial \oplus \cdots \oplus K[t]\partial^{e-1}$$

since $K[\partial] = (q) + K + K\partial + \cdots + K\partial^{e-1}$. Let $0 \neq \xi \in K[t] \oplus K[t]\partial \oplus \cdots \oplus K[t]\partial^{e-1}$ and let us show that $A_1\xi + A_1 q = A_1$. There exists $m \in \mathbb{N}$ and $\gamma_j(\partial) \in K[\partial], j = 0,1,\dots,m$ such that $\xi = \sum_{j=0}^m t^j \gamma_j(\partial)$ with $\deg\gamma_j \leq e-1$. We proceed by induction on the degree of t in ξ ; if

$m = 0$ then $(\xi, q) = 1$ (since q is irreducible) and $A_1\xi + A_1q = A_1$. If $m \geq 1$, suppose that the statement is true when the degree of t is less than m. We have that $(\gamma_m, q) = 1$ and there exists a_m and b_m in A_1 such that

$$a_m\gamma_m + b_mq = 1$$

and $a_m\xi = t^m - t^m b_m q + \sum_{j=0}^{m-1} a_m t^j \gamma_j(\partial) \equiv t^m + \sum_{j=0}^{m-1} a_m t^j \gamma_j(\partial) (\mod A_1q)$ so $qa_m\xi = qt^m + \sum_{j=0}^{m-1} qa_m t^j \gamma_j(\partial)$. Let $\eta = qa_m\xi$, it is sufficient to show that $A_1\eta + A_1q = A_1$. But by [5, Chapter 1] we know that

$$[\partial^j, t^k] = \sum_{i\geq 1} \frac{k(k-1)\cdots(k-i+1)j(j-1)\cdots(j-i+1)}{i!} t^{k-i}\partial^{j-i}$$

and this yields that $qt^m = t^m q + [q, t^m]$ where $[q, t^m] \in \sum_{j\leq m-1} t^j K[\partial]$. Hence $\eta \in \sum_{j\leq m-1} t^j K[\partial] + A_1q$ and by the hypothesis of induction $A_1\eta + A_1q = A_1$. Thus $A_1\xi + A_1q = A_1$. This means that A_1q is a maximal left ideal, and finishes the proof. □

Theorem 2.2. *If $q(t)$ is an irreducible polynomial in $K[t]$. Then*

(i) $\mathrm{Ext}^1_{A_1}(\frac{A_1}{A_1q}, \frac{A_1}{A_1q}) = 0$

(ii) $\mathrm{Hom}_{A_1}(\frac{A_1}{A_1q}, \frac{A_1}{A_1q}) = \mathcal{K} = \frac{K[t]}{(q)}$.

Proof. First we prove $\mathrm{Ext}^1_{A_1}(\frac{A_1}{A_1q}, \frac{A_1}{A_1q}) = 0$.
Let us consider the following short exact sequence

$$0 \to A_1q \xrightarrow{i} A_1 \to \frac{A_1}{A_1q} \to 0.$$

We get the induced long exact sequence

$$\begin{aligned} 0 \quad \to \quad & \mathrm{Hom}_{A_1}(\frac{A_1}{A_1q}, \frac{A_1}{A_1q}) \to \mathrm{Hom}_{A_1}(A_1, \frac{A_1}{A_1q}) \xrightarrow{i_*} \mathrm{Hom}_{A_1}(A_1q, \frac{A_1}{A_1q}) \\ \to \quad & \mathrm{Ext}^1_{A_1}(\frac{A_1}{A_1q}, \frac{A_1}{A_1q}) \to \mathrm{Ext}^1_{A_1}(A_1, \frac{A_1}{A_1q}) \to \mathrm{Ext}^1_{A_1}(A_1q, \frac{A_1}{A_1q}) \to \cdots \end{aligned}$$

Since $\mathrm{Ext}^1_{A_1}(A_1, \frac{A_1}{A_1q}) = 0$, it is sufficient to show that the map $i_* : \mathrm{Hom}_{A_1}(A_1, \frac{A_1}{A_1q}) \longrightarrow \mathrm{Hom}_{A_1}(A_1q, \frac{A_1}{A_1q})$ is surjective.
Let $\varphi : A_1q \longrightarrow \frac{A_1}{A_1q}$ be a left A_1-linear map. Then φ belongs to the i_*-image if and only if there exists $\psi : A_1 \longrightarrow \frac{A_1}{A_1q}$ a left A_1-linear map such that $\psi|_{A_1q} = \varphi$, hence $\varphi(q) = \psi(q) = q \cdot \psi(1)$.
Since $\mathrm{Hom}_{A_1}(A_1, \frac{A_1}{A_1q}) \cong \frac{A_1}{A_1q}$, we conclude that the i_*-image is equal to $\left\{\varphi \in \mathrm{Hom}_{A_1}(A_1q, \frac{A_1}{A_1q}) : \varphi(q) \in q \cdot \frac{A_1}{A_1q}\right\}$. We claim that $qA_1 + A_1q = A_1$. From $qA_1 + A_1q = A_1$ we get $q \cdot \frac{A_1}{A_1q} \cong \frac{A_1}{A_1q}$.

Then the i_*-image is $\mathrm{Hom}_{A_1}(A_1q, \frac{A_1}{A_1q})$ and i_* is surjective.

Let us prove our claim. If $q(t) = t^e + k_1t^{e-1} + \cdots + k_e,\ \ k_j \in K, e = deg(q) \geq 2$ then $A_1 = A_1q \oplus K[\partial] \oplus K[\partial]t \oplus \cdots \oplus K[\partial]t^{e-1}$ since $K[t] = (q) + K + Kt + \cdots + Kt^{e-1}$. Put $M = qA_1 + A_1q$ and it is sufficient to show that $K[\partial]t^j \subset M, j = 0, \ldots, e-1$. We have $t^jq' = t^j\partial q - qt^j\partial \in M$. Since q is irreducible there exist $a, b \in K[t]$ such that $aq + bq' = 1$, so $1 \in M$ and $t^j = t^jaq + t^jbq' \in M$ for all $j \in \mathbb{N}$, hence $K[t] \subset M$. Moreover $2\partial q't^j = \partial^2 qt^j - q\partial^2 t^j + q''t^j \in M$ and $\partial^m q't^j \in M$ for all $m, j \in \mathbb{N}$. Then $\partial^m t^j = \partial^m t^j aq + \partial^m q'bt^j \in M$ for all $m, j \in \mathbb{N}$. Therefore $K[\partial]t^j \subset M, j = 0, \ldots, e-1$.

Secondly, we prove that $\mathrm{Hom}_{A_1}(\frac{A_1}{A_1q}, \frac{A_1}{A_1q}) = \frac{K[t]}{(q)}$.

Let $q(t) = t^e + k_1t^{e-1} + \cdots + k_e,\ \ k_j \in K, e = deg(q) \geq 2$ then $A_1 = A_1q \oplus K[\partial] \oplus K[\partial]t \oplus \cdots \oplus K[\partial]t^{e-1}$.

Let $\psi : \frac{A_1}{A_1q} \longrightarrow \frac{A_1}{A_1q}$ be a left A_1-linear map. There exists $p \in A_1$ such that $\psi(\bar{1}) = \bar{p}$ in $\frac{A_1}{A_1q}$. So $p = \sum_{\nu=0}^{e-1} \rho_\nu(\partial)t^\nu(\mod A_1q), \rho_\nu \in K[\partial]$. We claim that one may choose $p \in K[t]$. Note that p has a very special property : $q \cdot p \in A_1q$. Let us rewrite

$$p = \sum_{j=0}^{m} \partial^j \cdot r_j(t),\ \ r_j(t) \in K + Kt + \cdots + Kt^{e-1}.$$

Suppose $m \geq 1$, then

$$q \cdot p = \sum_{j=0}^{m} (q \cdot \partial^j) \cdot r_j(t).$$

We know that

$$q \cdot \partial^j = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \partial^{j-k} q^{(k)}.$$

We get

$$q \cdot p = \sum_{j=0}^{m} \left(\sum_{k=0}^{j} (-1)^k \binom{j}{k} \partial^{j-k} q^{(k)} \right) \cdot r_j(t).$$

By the fact that $q \cdot p \in A_1q$ we have found that there exist polynomials $\varphi_j \in K[t], j = 0, \ldots, n-2$ such that

$$-\binom{m}{1} \partial^{m-1} q'(t) r_m(t) + \sum_{j=0}^{m-2} \partial^j \varphi_j(t) \in A_1q.$$

Since q is irreducible, q is relatively prime with both q' and r_m; q is relatively prime with $q'r_m$ and by Euclidean division in $K[t]$ we get

$$-mq'(t)r_m = \rho_{m-1}(t) + \gamma_{m-1}(t)q(t),\ \rho_{m-1} \neq 0\ \ \deg(\rho_{m-1}) \leq e-1.$$

In the same way for $j = 0, 1, \ldots, n-2$

$$\varphi_j(t) = \rho_j(t) + \gamma_j(t)q(t),\ \ \deg(\rho_j) \leq e-1.$$

So we get

$$\partial^{m-1}\rho_{m-1}(t)+\sum_{j=0}^{m-2}\partial^{j}\rho_{j}(t)\in A_1q,\ \ deg(\rho_j)\le e-1.$$

Because of the direct sum $\bigoplus_0^{e-1} K[\partial]t^j \oplus A_1q = A_1$, this is absurd if $m \geq 1$. We have $\rho_{m-1}(t) = 0$, this is in contradiction to $\rho_{m-1}(t) \neq 0$ shown above, hence $p = r_0(t) \in K[t]$.
We have proved that $\psi(\bar{1}) = p(t) \in K[t]$, and $\psi \equiv 0$ if an only if $p(t) \in (q)$. Therefore $\mathrm{Hom}_{A_1}(\frac{A_1}{A_1q}, \frac{A_1}{A_1q}) \cong \mathcal{K}$. □

2.1 Conclusion

Lemma 2.3. *Let $q(t) \in K[t]$ be an irreducible polynomial then*

$$\frac{A_1}{A_1q^m} \cong \bigoplus_m \frac{A_1}{A_1q}.$$

Proof. We proceed by induction on m. Let $M = \frac{A_1}{A_1q^2}$. By right multiplication by q we get the following exact sequence;

$$0\to \frac{A_1q}{A_1q^2} \to M\to \frac{A_1}{A_1q} \to 0.$$

Since q is a non-zerodivisor in A_1 we have $\frac{A_1q}{A_1q^2} \cong \frac{A_1}{A_1q}$. We then also have

$$0\to \frac{A_1}{A_1q} \to M\to \frac{A_1}{A_1q} \to 0.$$

By [4, Chapter 3] we know that $\mathrm{Ext}^1_{A_1}(\frac{A_1}{A_1q}, \frac{A_1}{A_1q})$ corresponds to extensions of this form. Since by Theorem 2.2 $\mathrm{Ext}^1_{A_1}(\frac{A_1}{A_1q}, \frac{A_1}{A_1q}) = 0$, and we get

$$M \cong \frac{A_1}{A_1q} \bigoplus \frac{A_1}{A_1q}.$$

Now suppose by induction that $\frac{A_1}{A_1q^{m-1}} \cong \bigoplus_{m-1} \frac{A_1}{A_1q}$ for $m \geq 3$, and let $M = \frac{A_1}{A_1q^m}$. By right multiplication by q we get the following exact sequence

$$0\to \frac{A_1}{A_1q} \to M\to \frac{A_1}{A_1q^{m-1}} \to 0.$$

And

$$\mathrm{Ext}^1_{A_1}(\frac{A_1}{A_1q}, \frac{A_1}{A_1q^{m-1}}) = \mathrm{Ext}^1_{A_1}(\frac{A_1}{A_1q}, \bigoplus_{m-1} \frac{A_1}{A_1q}) = \bigoplus_{m-1} \mathrm{Ext}^1_{A_1}(\frac{A_1}{A_1q}, \frac{A_1}{A_1q}) = 0.$$

Then it follows that

$$M = \frac{A_1}{A_1q} \oplus \frac{A_1}{A_1q^{m-1}} = \oplus_m \frac{A_1}{A_1q}.$$

□

3 Facts from commutative algebra

Noether's description of prime ideals Let us recall the description of prime ideals in $K[x_1,\dots,x_n]$ where K is a field of characteristic zero [1, Appendix 1]. Take $n \geq 3$ and $1 \leq k \leq n-3$. Up to K-linear transformation a prime ideal P for which $K[x_1,\dots,x_n]/P$ has dimension k is determined as follows. Put $X' = (x_1,\dots x_k)$ and let
(a) $q(X',x_n) = x_n^e + \sum_0^{e-1} \varrho_\nu(X')x_n^\nu$ be an irreducible polynomial in x_n.
(b) $q_j = \delta_q(X')x_{k+j} - h_j(X',x_n)$; $1 \leq j \leq n-k-1$ where h_j are polynomials in $K[X',x_n]$ and δ_q the discriminant of q.
To $(q,h_1,\dots,h_{n-k-1})$ we associate the prime ideal
$P = \{\varphi \in K[X] : \exists\gamma(X') \in K[X'], \gamma(X') \neq 0 \text{ and } \gamma(X')\varphi(X) \in (q,q_1,\dots,q_{n-k-1})\}$.
Noether's Theorem [8, Theorem 25] asserts that all prime ideals arise in this way .

3.1 Passage to maximal Ideals

Let P as above and put

$$\tilde{P} = K(X') \otimes_{K[X']} P.$$

Then $\tilde{P}$ is a maximal ideal in $K(X')[x_{k+1},\dots,x_n]$ (where $K(X')$ is the fraction fields of $K[X']$) and $\tilde{P}$ is generated by $q,q_1,\dots,q_{n-k-1}$.

In general consider a maximal ideal $\mathfrak{M}$ in $\mathcal{K}[t_1,\dots,t_p]$ (in our case $p = n-k$ and $\mathcal{K} = K(X')$). Up to change of variables $\mathfrak{M}$ is generated by $q(t_p) = t_p^e + \sum_0^{e-1} c_\nu t_p^\nu \in \mathcal{K}[t_p]$ and $q_j = t_j - h_j(t_p)$ (since we may invert the discriminant in the last equations above).
Let us make a change of variables

$$\begin{cases} u_j = t_j - h_j(t_p) \\ u_p = t_p. \end{cases}$$

Now $K[t] \cong K[u]$ and using the variables $u_1,\dots,u_p$ it follows that

$$\mathfrak{M} = (u_1,\dots,u_{p-1},q(u_p))$$

holds in the polynomial ring in $\mathcal{K}[u_1,\dots,u_p] = \mathcal{K}[t_1,\dots,t_p]$.

4 Kashiwara's Decomposition Theorem

We need the following version of Kashiwara's embedding theorem.

Theorem 4.1. *Let $A_p(K) = K\langle u_1,\dots,u_p,\partial_{u_1},\dots,\partial_{u_p}\rangle$ be the p-th Weyl algebra, and M a left $A_p(K)$-module such that every $m \in M$ is annihilated by some power of u_j for each $1 \leq j \leq p-1$, i.e. there exists $w_j \in \mathbb{N}$ (depending on m) such that $u_j^{w_j}m = 0$ in M. Then*

$$M = \bigoplus \partial_{u_1}^{\alpha_1} \dots \partial_{u_{p-1}}^{\alpha_{p-1}} M_0$$

where $M_0 = \{m \in M : u_1 m = \dots = u_{p-1}m = 0\}$ is a finitely generated left $A_1 = K\langle u_p,\partial_p\rangle$-module.

Proof. By [1, Theorem 6.2] and [3, Chapter 17] □

As an application we can prove the following result.

Proposition 4.2. *Let* $\mathfrak{M}$ *be a maximal ideal in* $K[u_1,\dots,u_p]$ *and* $A_p(K) = K\langle u_1,\dots,u_p,\partial_{u_1},\dots,\partial_{u_p}\rangle$, $s \in \mathbb{N}$. *Then*

$$\frac{A_p(K)}{A_p(K)\mathfrak{M}^s} \cong \bigoplus \frac{A_p(K)}{A_p(K)\mathfrak{M}}$$

and $\dfrac{A_p(K)}{A_p(K)\mathfrak{M}}$ *is a simple* $A_p(K)$*-module .*

Proof. The case $\mathfrak{M} = (u_1,\dots,u_p)$ is well known and we exclude it. The field $K[u_1,\dots,u_p]/\mathfrak{M}$ is a finite dimensional vector space over K. There exists a primitive element which we can assume to be u_p and let $q(u_p)$ be the minimal polynomial of u_p. There exist polynomials h_j such that $u_j = h_j(u_p) : 1 \le j \le p$. Now by change of variables

$$\begin{cases} t_j = u_j - h_j(u_p) \\ t_p = u_p, \end{cases}$$

we get an algebra isomorphism $K[u] \longrightarrow K[t]; u \longmapsto t$ and under that isomorphism $\mathfrak{M} = (t_1,\dots,t_{p-1},q(t_p))$. So all is reduced to the case $\mathfrak{M} = (t_1,\dots,t_{p-1},q(t_p))$.
We may thus assume that $\mathfrak{M} = (u_1,\dots,u_{p-1},q(u_p))$, with $q(u_p)$ an irreducible polynomial. Now put

$$M = \frac{A_p(K)}{A_p(K)\mathfrak{M}^s}.$$

That $m \in M$ means $m = Q(u,\partial) + A_p(K)\mathfrak{M}^s$ where $Q(u,\partial) = \sum_{|\alpha|\le w} q_\alpha(u)\partial^\alpha$ for some $w \in \mathbb{N}$ and $q_\alpha \in K[u_1,\dots,u_p]$. For $j = 1,\dots,p-1$ we have that $u_j^{w+s+1} \cdot Q \in A_p(K)u_j^s \subset A_p(K)\mathfrak{M}^s$, so $u_j^{w+s+1}m = 0$. It follows from Kashiwara's Decomposition Theorem above that

$$M = \bigoplus \partial_{u_1}^{\alpha_1} \dots \partial_{u_{p-1}}^{\alpha_{p-1}} M_0,$$

where $M_0 = \{m \in M : u_1 m = \dots = u_{p-1}m = 0\}$ is a finitely generated left $A_1 = K\langle u_p,\partial_p\rangle$-module. Let $M_0 = \sum_v A_1\zeta_v$, it is clear that some power of $q(u_p)$ annihilates every element $\zeta_v \in M_0$ i.e, there exist $w \in \mathbb{N}$ such that $q(u_p)^w\zeta_v = 0$. It follows that there exists a surjective map $\dfrac{A_1}{A_1q^w} \twoheadrightarrow A_1\zeta_v$. By Lemma 2.3 and Theorem 2.1 we know that $\dfrac{A_1}{A_1q^w} \cong \bigoplus_w \dfrac{A_1}{A_1q}$ is semisimple and each $\dfrac{A_1}{A_1q}$ is simple, so $A_1\zeta_v \cong \oplus\dfrac{A_1}{A_1q}$. We then get $M_0 \cong \oplus\dfrac{A_1}{A_1q}$ and M_0 is semisimple as A_1-module. This gives by Theorem 4.1 that

$$M = \bigoplus\Big(\bigoplus_{\alpha\in\mathbb{N}^{p-1}} \partial^\alpha \frac{A_1}{A_1q}\Big).$$

Now it is easily seen that

$$\bigoplus_{\alpha\in\mathbb{N}^{p-1}} \partial^\alpha \frac{A_1}{A_1q} \cong \frac{A_p(K)}{A_p(K)\mathfrak{M}},$$

and this implies that $M' = \dfrac{A_p(K)}{A_p(K)\mathfrak{M}}$ is simple, since otherwise, we may write $M' = M_1 + M_2$ with $M_1 \neq 0, M_2 \neq 0$. Then

$$M'_0 = \frac{A_1}{A_1 q} = (M_1)_0 \oplus (M_2)_0$$

with $(M_1)_0 \neq 0, (M_2)_0 \neq 0$. This is however in contradiction to the simplicity of $\dfrac{A_1}{A_1 q}$ proved in Theorem 2.1. □

Corollary 4.3. *Suppose that J is an ideal in $K[u_1,\dots,u_p]$ such that $\mathfrak{M}^s \subset J$ for some $s \geq 2$. Then*

$$\frac{A_p(K)}{A_p(K)J} \cong \bigoplus \frac{A_p(K)}{A_p(K)\mathfrak{M}}(*).$$

Moreover the number N of the copies $\dfrac{A_p(K)}{A_p(K)\mathfrak{M}}$ in (*) is equal to the length of $K[u_1,\dots,u_p]/J$.

Proof. If J is an ideal in $K[u_1,\dots,u_p]$ such that $\mathfrak{M}^s \subset J$ for some $s \geq 2$, then there exists a surjective map $\dfrac{A_p(K)}{A_p(K)\mathfrak{M}^s} \twoheadrightarrow \dfrac{A_p(K)}{A_p(K)J}$. Since $\dfrac{A_p(K)}{A_p(K)\mathfrak{M}^s}$ is semisimple as $A_p(K)$-module from Proposition 4.2, $\dfrac{A_p(K)}{A_p(K)J}$ is also semisimple and $\dfrac{A_p(K)}{A_p(K)J} \cong \bigoplus_N \dfrac{A_p(K)}{A_p(K)\mathfrak{M}}$. Let us prove that the number N of the copies $\dfrac{A_p(K)}{A_p(K)\mathfrak{M}}$ in (*) is equal to the length of $K[u_1,\dots,u_p]/J$. We proceed by induction on l, the length of $K[u_1,\dots,u_p]/J$. The statement is trivial when $l = 1$, let us consider a maximal chain of ideals $0 \subsetneq J = J_0 \subsetneq J_1 \subsetneq \dots \subsetneq J_n = \mathfrak{M}$ ($\sqrt{J_0} = \mathfrak{M}$) and the following exact sequence

$$0 \to \frac{A_p(K)J_1}{A_p(K)J_0} \to \frac{A_p(K)}{A_p(K)J_0} \to \frac{A_p(K)}{A_p(K)J_1} \to 0. \tag{4.1}$$

Since $\dfrac{A_p(K)}{A_p(K)J_0}$ is semi-semiple the exact sequence (4.1) splits. We need to prove that $\dfrac{A_p(K)J_1}{A_p(K)J_0} \cong \dfrac{A_p(K)}{A_p(K)\mathfrak{M}}$, but $J_1/J_0 \cong K[u_1,\dots,u_p]/\mathfrak{M}$ and $J_1 = K[u_1,\dots,u_p]\eta + J_0$ for $\eta \in J_1 \backslash J_0$. It follows that $A_p(K)J_1 = A_p(K)\eta + A_p(K)J_0$ and

$$\frac{A_p(K)J_1}{A_p(K)J_0} = \frac{A_p(K)\eta + A_p(K)J_0}{A_p(K)J_0} \cong \frac{A_p(K)\eta}{A_p(K)\eta \cap A_p(K)J_0}.$$

Since $\mathfrak{M}\eta \subset J_0$, $\mathfrak{M}\bar{\eta} = 0$, $\dfrac{A_p(K)\eta}{A_p(K)\eta \cap A_p(K)J_0} \cong \dfrac{A_p(K)}{A_p(K)\mathfrak{M}}$. Therefore

$$\frac{A_p(K)}{A_p(K)J_0} \cong \frac{A_p(K)}{A_p(K)J_1} \oplus \frac{A_p(K)}{A_p(K)\mathfrak{M}}.$$

Since by induction the statement is true for $\dfrac{A_p(K)}{A_p(K)J_1}$, it is also true for $\dfrac{A_p(K)}{A_p(K)J_0}$. □

5 Final part of the proof

Let $P \subset K[x_1,\ldots,x_n]$ be a prime ideal, $X' = (x_1,\ldots,x_k)$, $X'' = (x_{k+1},\ldots,x_n)$ and $\partial'' = (\partial_{k+1},\ldots,\partial_n)$. Suppose that $K[X'] \cap P = \{0\}$ and $K(X') \otimes_{K[X']} P = \tilde{P}$ is a maximal ideal of $\mathcal{K}[X'']$ where $\mathcal{K} = K(X')$. If Q is a primary ideal with $\sqrt{Q} = P$, then $\tilde{Q} = K(X') \otimes_{K[X']} Q$ is a primary ideal in $\mathcal{K}[X'']$.
Put $p = n-k$ and $A_p(\mathcal{K}) = \mathcal{K}\langle x_{k+1},\ldots,x_n,\partial_{x_{k+1}},\ldots,\partial_{x_n}\rangle$. From Corollary 4.3 we have

$$\frac{A_p(\mathcal{K})}{A_p(\mathcal{K})\tilde{Q}} \cong \bigoplus_N \frac{A_p(\mathcal{K})}{A_p(\mathcal{K})\tilde{P}}.$$

Without loss of generality we from [3, Proposition 16.2.1] get

$$\frac{A_p(\mathcal{K})}{\tilde{Q}A_p(\mathcal{K})} \cong \bigoplus_N \frac{A_p(\mathcal{K})}{\tilde{P}A_p(\mathcal{K})}. \tag{5.1}$$

Then there exist right $A_p(\mathcal{K})$-linear surjections
$\varphi_j : \dfrac{A_p(\mathcal{K})}{\tilde{Q}A_p(\mathcal{K})} \to \dfrac{A_p(\mathcal{K})}{\tilde{P}A_p(\mathcal{K})}, j = 1,\ldots,N$. There also exist $F_j \in A_p(\mathcal{K})$ such that $\varphi_j(\bar{1}) = \overline{F_j}$, so for all $p \in K[X], \varphi_j(\bar{p}) = \overline{F_j \cdot p}$ and $F_j \cdot p \in \tilde{P}A_p(\mathcal{K})$ if $p \in \tilde{Q}$. Let $F_j = \sum_\alpha q_\alpha(X)(\partial'')^\alpha$. The product $F_j \cdot p$ is taken in the Weyl algebra $A_p(\mathcal{K})$, and we may write

$$F_j \cdot p = \rho_0(X) + \sum_{|\alpha|\geq 1} \rho_\alpha(X)(\partial'')^\alpha.$$

Then

$$\rho_0(X) = \sum_\alpha q_\alpha(X)(\partial'')^\alpha(p) = F_j(p).$$

Here $F_j(p)$ is the result of the F_j-action on $p(X) \in K[X]$. Furthermore we have

$$A_p(\mathcal{K}) = \bigoplus_{\alpha\in\mathbb{N}^p} \mathcal{K}[X''](\partial'')^\alpha \text{ so } \tilde{P}A_p(\mathcal{K}) = \tilde{P} \bigoplus_{|\alpha|\geq 1} \tilde{P}\cdot(\partial'')^\alpha.$$

Now let $p \in \tilde{Q}$ since $F_j \cdot p \in \tilde{P} \bigoplus_{|\alpha|\geq 1} \tilde{P}\cdot(\partial'')^\alpha$ and $F_j \cdot p = F_j(p) + \sum_{|\alpha|\geq 1} \rho_\alpha(X)(\partial'')^\alpha$, we get $F_j(p) \in \tilde{P}$. Conversely if $p \in K[X]$ such that $F_j(p) \in \tilde{P}; 1 \leq j \leq N$. Since $\tilde{Q}$ is an ideal $F \in \mathcal{N}(\tilde{Q}) \Rightarrow F(X,(\partial)'')x^\alpha \in \mathcal{N}(\tilde{Q})$, by this we get that $F \in \mathcal{N}(\tilde{Q}) \Rightarrow F^{(\beta)} \in \mathcal{N}(\tilde{Q})$ where $F = \sum q_\alpha(X)\partial^\alpha$ and $F^{(\beta)} = q_\alpha(X)\binom{\alpha}{\beta}\partial^{\alpha-\beta}$. The family $\mathcal{N}(\tilde{Q})$ is closed under derivations with repect to ∂-monomials. By [2, Proposition 1.1.11] we have that: $F^{(\beta)}(p) \in \tilde{P}$ for all β implies that $F \cdot p \subset \tilde{P}A_p(\mathcal{K})$. Then $F_j \cdot p \in \tilde{P}A_p(\mathcal{K}); 1 \leq j \leq N$ and $\varphi_j(p) = 0; 1 \leq j \leq N$. From the isomorphism in (5.1) we get $p \in \tilde{Q}A_p(\mathcal{K})$, and it follows that $p \in \tilde{Q}$. Therefore we have found differential operators $F_1(X,\partial''),\ldots,F_N(X,\partial'')$ such that

$$F_j(\tilde{Q}) \subset \tilde{P} : 1 \leq j \leq N. \tag{5.2}$$

And if $p \in K[X]$ such that

$$F_j(p) \in \tilde{P}; 1 \leq j \leq N \text{ then } p \in \tilde{Q}. \tag{5.3}$$

Denote by $\gamma(X') \in K[X']$ the common denominator of the F_j. By multiplying the F_j by $\gamma(X')$ we get differential operators
$F'_1(X,\partial),\dots,F'_N(X,\partial) \in A_n(K) = K\langle x_1,\dots,x_n,\partial_{x_1},\dots,\partial_{x_n}\rangle$ such that

$$F'_j(Q) \subset P : 1 \le j \le N. \tag{5.4}$$

Suppose that $p \in K[X]$ such that $F'_j(p) \in P$ for $1 \le j \le N$ then $p \in \tilde{Q}$. We can find $\delta(X') \in K[X']$ such that $\delta(X')p \in Q$. Since Q is primary and $K[X'] \cap P = \{0\}$, From [7, Lemma 4.14] we have $(Q : \delta(X')) = Q$, so $p \in Q$. We conclude that $(F'_1,\dots,F'_N)$ gives the requested family of Noetherian operators in our Main Theorem.

6 Some examples

(1) Let $n = 2$ and let us consider the primary ideal $Q = (x_1^k, x_2^k)$ in $K[x_1,x_2]$ and $P = (x_1,x_2) = \sqrt{Q}$. We know that

$$K[x_1,x_2] = Q \oplus_{\alpha_i<k} Kx^\alpha$$

then

$$K[x_1,x_2]/Q = \oplus_{\alpha_i<k} K\bar{x}^\alpha.$$

So $\dim K[x_1,x_2]/Q = k^2$,

$$\mathcal{N}(Q) = K[X]\{\partial^\alpha : \alpha_i < k\}.$$

(2) Let $n = 3$ and let Q be the ideal generated by the x_2^2, x_3^2 and $x_2 - x_1x_3$ in $\mathbb{C}[x_1,x_2,x_3]$. It is easily seen that Q is a primary ideal and the affine variety $V(Q)$ defined by $Q^{-1}(0)$ is the subspace $V(Q) = \{(a,b,c) | b = c = 0\}$ then the ideal $I(V(Q))$ of $V(Q)$ is generated by x_2 and x_3 ; $\sqrt{Q} = (x_2,x_3)$. Moreover

$$\mathbb{C}[x_1,x_2,x_3] = \mathbb{C}[x_1] \oplus \mathbb{C}[x_1]x_3 \oplus Q$$

then

$$\mathbb{C}[x_1,x_2,x_3]/Q = \mathbb{C}[x_1] \oplus \mathbb{C}[x_1]\bar{x}_3$$

and the rank of the $\mathbb{C}[x_1]$-module $\mathbb{C}[x_1,x_2,x_3]/Q$ is 2. We have that

$$\mathcal{N}(Q) = \mathbb{C}[X]\{1, x_1\partial_2 + \partial_3\}.$$

(3) Let $Q = (x_1^k,\dots,x_n^k)$ a primary ideal in $K[x_1,\dots,x_n]$, $\sqrt{Q} = (x_1,\dots,x_n)$. As above we get $\dim K[x_1,x_2]/Q = k^n$, and

$$\mathcal{N}(Q) = K[X]\{\partial^\alpha : \alpha_i < k\}.$$

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