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Admissible Monomials and Generating Sets for the Polynomial Algebra as a Module Over the Steenrod Algebra

Mbakiso F. Mothebe*
Department of Mathematics, University of Botswana,
Pvt Bag 00704, Gaborone, Botswana

Abstract

For $n \geq 1$, let $\mathbf{P}(n) = \mathbb{F}_2[x_1, \ldots, x_n]$ be the polynomial algebra in n variables x_i, of degree one, over the field $\mathbb{F}_2$ of two elements. The mod-2 Steenrod algebra $\mathcal{A}$ acts on $\mathbf{P}(n)$ according to well known rules. Let $\mathcal{A}^+\mathbf{P}(n)$ denote the image of the action of the positively graded part of $\mathcal{A}$. A major problem is that of determining a basis for the quotient vector space $\mathbf{Q}(n) = \mathbf{P}(n)/\mathcal{A}^+\mathbf{P}(n)$. Both $\mathbf{P}(n) = \oplus_{d\geq 0}\mathbf{P}^d(n)$ and $\mathbf{Q}(n)$ are graded where $\mathbf{P}^d(n)$ denotes the set of homogeneous polynomials of degree d.

In this paper we show that if $n \geq 2$, and $d \geq 1$ can be expressed in the form $d = \sum_{i=1}^{n-1}(2^{\lambda_i} - 1)$ with $\lambda_1 > \lambda_2 > \ldots > \lambda_{n-2} \geq \lambda_{n-1} \geq 1$, then

$$\dim(\mathbf{Q}^d(n)) \geq \left(\sum_{q=1}^{\min\{\lambda_{n-1}, n\}} \binom{n}{q}\right)(\dim(\mathbf{Q}^{d'}(n-1)))$$

where $d' = \sum_{i=1}^{n-1}(2^{\lambda_i - \lambda_{n-1}} - 1)$.

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1 Introduction

For $n \geq 1$, let $\mathbf{P}(n)$ be the polynomial algebra

$$\mathbf{P}(n) = \mathbb{F}_2[x_1, \ldots, x_n]$$

in n variables x_i, of degree one, over the field $\mathbb{F}_2$ of two elements. We identify $\mathbf{P}(n)$ with the mod-2 cohomology group of the n-fold product of $\mathbb{R}P^\infty$ with itself.

The mod-2 Steenrod algebra $\mathcal{A}$ is the algebra of stable operations of the mod-2 cohomology of topological spaces. It is generated over $\mathbb{F}_2$ by certain linear transformations Sq^i for $i \geq 0$, called Steenrod squares, subject to the Adem relations [1] and $Sq^0 = 1$. Let

*E-mail address: mothebem@mopipi.ub.bw

$\mathbf{P}^d(n)$ denote the homogeneous polynomials of degree d. The action of the Steenrod squares $Sq^i : \mathbf{P}^d(n) \to \mathbf{P}^{d+i}(n)$ is determined by the formula:

$$Sq^i(u) = \begin{cases} u & \text{if} \quad i = 0 \\ u^2 & \text{if} \quad \deg(u) = i \\ 0 & \text{if} \quad \deg(u) < i \end{cases}$$

and the Cartan formula:

$$Sq^i(uv) = \sum_{r=0}^{i} Sq^r(u) Sq^{i-r}(v).$$

A polynomial $u \in \mathbf{P}^d(n)$ is said to be hit if it is in the image of the action of $\mathcal{A}$ on $\mathbf{P}(n)$, that is, if

$$u = \sum_{i>0} Sq^i(u_i)$$

for some $u_i \in \mathbf{P}(n)$ of degree $d - i$. Let $\mathcal{A}^+\mathbf{P}(n)$ denote the subspace of all hit polynomials. The problem of determining $\mathcal{A}^+\mathbf{P}(n)$ is called the hit problem and has been studied by several authors [9], [11], [15]. A closely related problem is that of determining a basis for the quotient vector space

$$\mathbf{Q}(n) = \mathbf{P}(n)/\mathcal{A}^+\mathbf{P}(n)$$

which has also been studied by several authors [2], [4], [6], [7], [12], [13]. Some of the motivation for studying these problems is mentioned in [6]. It stems from the Peterson conjecture proved in [15] and various other sources [8], [10].

The following result is useful for determining $\mathcal{A}$-generators for $\mathbf{P}(n)$. Let $\alpha(m)$ denote the number of digits 1 in the binary expansion of m.

Theorem 1.1. (Wood [15]) *Let $u \in \mathbf{P}(n)$ be a monomial of degree d. If $\alpha(n+d) > n$ then u is hit.*

Thus $\mathbf{Q}^d(n)$ is zero unless $\alpha(n+d) \leq n$ or, equivalently, unless d can be written in the form, $d = \sum_{i=1}^{n}(2^{\lambda_i} - 1)$ where $\lambda_i \geq 0$. Thus $\mathbf{Q}^d(n) \neq 0$ only if $\mathbf{P}^d(n)$ contains monomials

$$v = x_1^{2^{\lambda_1}-1} \cdots x_n^{2^{\lambda_n}-1}$$

called spikes.

We note that a spike can never appear as a term in a hit polynomial.

$\mathbf{Q}(n)$ has been explicitly calculated by Peterson [7] for $n = 1, 2$, by Kameko [3] for $n = 3$ and independently by Kameko [4] and N. Sum [12] for $n = 4$.

2 Preliminaries

In this section we recall some results in Kameko [3] and Singer [11] on admissible monomials and hit monomials in $\mathbf{P}(n)$.

If $b = x_1^{e_1} \cdots x_n^{e_n}$ is a monomial in $\mathbf{P}(n)$, write $e_i = \sum_{j \geq 0} \alpha_j(e_i) 2^j$ for the binary expansion of each exponent e_i. The expansions are then assembled into a matrix

$$\beta(b) = (\alpha_j(e_i))$$

of digits 0 or 1 with $\alpha_j(e_i)$ in the (i, j)th position of the matrix.

We shall associate with a monomial b two sequences

$$w(b) = (w_0(b), w_1(b), \dots, w_j(b), \dots),$$

$$e(b) = (e_1, e_2, \dots, e_n),$$

where $w_j(b) = \sum_{i=1}^{n} \alpha_j(e_i)$ for each $j \geq 0$. $w(b)$ is called the **weight vector** of the monomial b and $e(b)$ is called the **exponent vector** of the monomial b. Note that $w_j(b) \leq n$ for all j. The monomial b is said to have length l if $w_l(b) \neq 0$ and $w_j(b) = 0$ for all $j > l$.

Given two sequences

$$p = (u_0, u_1, \dots, u_l, 0, 0 \dots), \;\; q = (v_0, v_1, \dots, v_l, 0, 0, \dots),$$

we say $p < q$ if there is a positive integer k such that $u_i = v_i$ for all $i < k$ and $u_k < v_k$. We are now in a position to define an order relation on monomials.

Definition 2.1. Let a, b be monomials in $\mathbf{P}(n)$. We say that $a < b$ if one of the following holds:
(i) $w(a) < w(b)$
(ii) $w(a) = w(b)$ and $e(a) < e(b)$.

Note that the order relation on the set of sequences is the lexicographical one.

Let $a = x_1^{e_1} \cdots x_n^{e_n}$ and $b = x_1^{t_1} \cdots x_n^{t_n}$ be monomials in $\mathbf{P}(n)$, for which $e_i \neq 0$ and $t_i \neq 0$ for any i, $1 \leq i \leq n$. If a and b have length less that or equal to l and $a < b$, then for any non-zero monomial $c \in \mathbf{P}(n)$, $ac^{2^{l+1}} < bc^{2^{l+1}}$.

Following Kameko [2] we define

Definition 2.2. A monomial $b \in \mathbf{P}(n)$ is said to be **inadmissible** if there exist monomials $b_1, b_2, \dots, b_r \in \mathbf{P}(n)$ with $b_j < b$ for each j, $1 \leq j \leq r$, such that

$$b \equiv (\sum_{j=1}^{r} b_j) \bmod \mathcal{A}^+\mathbf{P}(n).$$

b is said to be **admissible** if it is not inadmissible.

Clearly the set of all admissible monomials in $\mathbf{P}(n)$ is a minimal set of $\mathcal{A}$-generators of $\mathbf{P}(n)$.

Let b be a monomial of length l and let k be an integer such that $0 \leq k < l$. Then k determines uniquely a k-factorization of b of the form

$$b = b_1 b_2^{2^{k+1}}$$

obtained by splitting $\beta(b)$ such that $w_j(b) = w_j(b_1)$, $0 \leq j \leq k$ and $w_j(b) = w_{j-(k+1)}(b_2)$ for $j > k$. For example if $b = x_1^{47} x_2^{28} x_3^{13} \in \mathbf{P}(3)$, then $k = 2$ determines the factorization

$$b = b_1 b_2^8 = (x_1^7 x_2^4 x_3^5)(x_1^5 x_2^3 x_3)^8.$$

We require the following result due to Kameko:

Definition 2.3. Let d be a positive integer. Define a linear mapping $h : \mathbf{P}^{2d+n}(n) \to \mathbf{P}^{d}(n)$ by

$$h(b) = \begin{cases} c & \text{if} \quad b = x_1 x_2 \dots x_n c^2 \\ 0 & \quad\ \text{otherwise} \end{cases}$$

for any monomial $b \in \mathbf{P}^{2d+n}(n)$.

Then h induces a homomorphism $h_* : \mathbf{Q}^{2d+n}(n) \to \mathbf{Q}^{d}(n)$.

Let $\beta(d) = \min\{m \in \mathbf{Z} \mid \alpha(m+d) \leq m\}$. In [3, Theorem 4.2] Masaki Kameko proved that:

Theorem 2.4. (Kameko) *Let d be a positive integer. If $\beta(2d+n) = n$, then $h_* : \mathbf{Q}^{2d+n}(n) \to \mathbf{Q}^{d}(n)$ is an isomorphism.*

From Wood's theorem and the above result of Kameko the problem of determining $\mathcal{A}$-generators for $\mathbf{P}(n)$ is reduced to the cases $\beta(d) < n$.

We recall the following result of Singer on hit polynomials in $\mathbf{P}(n)$.

Definition 2.5. A spike $v = x_1^{2^{\lambda_1}-1} \cdots x_n^{2^{\lambda_n}-1}$ is called a **minimal spike** if its weight order is minimal with respect to other spikes of degree d or, equivalently, if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s \geq 0$ and $\lambda_{j-1} = \lambda_j$ only if $j = s$ or $\lambda_{j+1} = 0$.

In [11, Theorem 1.2] W. M. Singer proved that:

Theorem 2.6. (Singer) *Let $b \in \mathbf{P}(n)$ be a monomial of degree d, where $\alpha(n+d) \leq n$. Let v be the minimal spike of degree d. If $w(b) < w(v)$, then b is hit.*

Finally we note that for any element $Sq^k \in \mathcal{A}^+$ and any polynomial $u \in \mathbf{P}(n)$ we have

$$Sq^{k2^\lambda}(u^{2^\lambda}) = (Sq^k(u))^{2^\lambda} \tag{2.1}$$

for a given $\lambda \geq 0$.

3 Preliminary result

In this section we outline a proof of the result below, obtained in [5], which serves as the foundation of our main result, which we shall state and prove in the next section. Some of the notation introduced in this section is adopted in the rest of the paper.

Let $\lambda \geq 1$ be an integer and let $d(\lambda) = (n-1)(2^\lambda - 1)$ if $n \geq 2$ and $d(\lambda) = 1$ if $n = 1$. In [5] it is shown that:

Theorem 3.1. *If $n \geq 1$, then*

$$\dim\left(\mathbf{Q}^{d(\lambda)}(n)\right) \geq \begin{cases} \sum_{q=1}^{\lambda} \binom{n}{q} & \text{if} \quad \lambda < n \\ 2^n - 1 & \text{if} \quad \lambda \geq n. \end{cases}$$

The result above is a generalization of special cases when $n = 1, 2, 3$ for which equality holds. The result was achieved by showing the equivalent to that, if $\lambda \geq n \geq 2$, then

$$b = b(n) = x_1^{2^{n-1}-1} x_2^{2^\lambda - 2^{n-2}-1} \cdots x_i^{2^\lambda - 2^{n-i}-1} \cdots x_n^{2^\lambda - 2}$$

is the only admissible monomial among the class of monomials of degree $d(\lambda)$ and of weight vector

$$(\overbrace{n-1, n-1, \dots, n-1}^{\lambda})$$

and for which no factor x_i has exponent $2^\lambda - 1$.

To see this let $\mathbf{D}^{d(\lambda)}(n)$ denote the subspace of $\mathbf{P}^{d(\lambda)}(n)$ spanned by monomials $a = x_1^{t_1} \cdots x_n^{t_n}$ such that $w(a) = w(b)$ and $\mathbf{D}^{d(\lambda)}(n,b)$ denote the subspace of $\mathbf{D}^{d(\lambda)}(n)$ for which $t_i \neq 2^\lambda - 1$ for any i. Let

$$\pi \;:\; \mathbf{P}^{d(\lambda)}(n) \;\to\; \mathbf{D}^{d(\lambda)}(n)$$

denote the projection of $\mathbf{P}^{d(\lambda)}(n)$ onto its summand $\mathbf{D}^{d(\lambda)}(n)$ and

$$\pi_b \;:\; \mathbf{D}^{d(\lambda)}(n) \;\to\; \mathbf{D}^{d(\lambda)}(n,b)$$

denote the projection of $\mathbf{D}^{d(\lambda)}(n)$ onto $\mathbf{D}^{d(\lambda)}(n,b)$. Let $\mathbf{H}^{d(\lambda)}(n,b)$ be the subspace of $\mathbf{D}^{d(\lambda)}(n,b)$ spanned by

$$\left\{b_1 + b_2 \mid b_1,\, b_2 \text{ monomials in } \mathbf{D}^{d(\lambda)}(n,b)\right\}$$

Then

$$\pi_b \pi\left(\mathcal{A}^+ \mathbf{P}(n) \cap \mathbf{P}^{d(\lambda)}(n)\right) = \mathbf{H}^{d(\lambda)}(n,b) \tag{3.1}$$

Clearly then $\mathbf{Q}^{d(\lambda)}(n)$ must contain at least one element of $\mathbf{D}^{d(\lambda)}(n,b)$ and consequently b, being the monomial of least order in $\mathbf{D}^{d(\lambda)}(n,b)$, is admissible.

The proof of (3.1) is by induction on n and λ and is a mirror image of the following inductive procedure. We show that in the special case when $n = 4$ and $\lambda = 4$ we have

$$\pi_{b(4)} \pi\left(\mathcal{A}^+ \mathbf{P}(4) \cap \mathbf{P}^{45}(4)\right) = \left\{b_1 + b_2 \mid b_1,\, b_2 \text{ monomials in } \mathbf{D}^{45}(4, b(4))\right\}$$

Proceeding by induction on n we assume that (3.1) is true when $n = 3$ and $d = 14$ (known case).

Let $\sigma = (1\ 2\ 3\ 4) \in S_4$ and for each j, $1 \le j \le 4$, let $\mathbf{P}^{45}_j(4)$ be the vector subspace of $\mathbf{P}^{45}(4)$ generated by monomials

$$\{(x^{e_1}_{\sigma^j(1)} x^{e_2}_{\sigma^j(2)} x^{e_3}_{\sigma^j(3)} x^{7}_{\sigma^j(4)})^2 x_j^0 v \;\mid\; x_1^{e_1} x_2^{e_2} x_3^{e_3} \in \mathbf{D}^{14}(3, b(3)) \text{ and } w(v) = (3)\}.$$

Then $\mathbf{D}^{14}(3, b(3))$ is isomorphic to $\mathbf{P}^{45}_j(4)$ for each j. It is sufficient to show that for each j, we can find monomials $u_j \in \mathbf{P}^{45}_j(4)$ such that

$$u_1 + u_2,\; u_2 + u_3,\; u_3 + u_4$$

all belong to $\pi_{b(4)} \pi\left(\mathcal{A}^+ \mathbf{P}(4) \cap \mathbf{P}^{45}(4)\right)$.

Let $b = u_2 = (x_1^3 x_2^7 x_3^6 x_4^5)^2 (x_1 x_3 x_4) \in \mathbf{P}^{45}_2(4)$. Put

$$u_3 = (x_1^3 x_2^6 x_3^7 x_4^5)^2 (x_1 x_2 x_4) \in \mathbf{P}^{45}_3(4).$$

Then $u_2 + u_3 \in \pi_{b(4)} \pi\left(Sq^1(\mathbf{P}(4)) \cap \mathbf{P}^{45}(4)\right)$. Put

$$r_3 = (x_1^3 x_2^5 x_3^7 x_4^6)^2 (x_1 x_2 x_4) \in \mathbf{P}^{45}_3(4)$$

and $u_4 = (x_1^3x_2^5x_3^6x_4^7)^2(x_1x_2x_3) \in \mathbf{P}_4^{45}(4)$. By the induction hypothesis

$$u_3 + r_3 \in \pi_{b(4)}\pi\left(\mathcal{A}^+\mathbf{P}(4) \cap \mathbf{P}^{45}(4)\right).$$

But $r_3 + u_4 \in \pi_{b(4)}\pi\left(Sq^1(\mathbf{P}(4)) \cap \mathbf{P}^{45}(4)\right)$. Finally put

$$r_4 = (x_1^6x_2^5x_3^3x_4^7)^2(x_1x_2x_3) \in \mathbf{P}_4^{45}(4)$$

and $u_1 = (x_1^7x_2^5x_3^3x_4^6)^2(x_2x_3x_4) \in \mathbf{P}_1^{45}(4)$. By the induction hypothesis

$$u_4 + r_4 \in \pi_{b(4)}\pi\left(\mathcal{A}^+\mathbf{P}(4) \cap \mathbf{P}^{45}(4)\right).$$

But $r_4 + u_1 \in \pi_{b(4)}\pi\left(Sq^1(\mathbf{P}(4)) \cap \mathbf{P}^{45}(4)\right)$ and this establishes (3.1) in the case $n = 4$ and $\lambda = 4$. The case $\lambda \geq 4$ follows similarly by induction on both n and λ and the general case follows similarly.

Now to derive the result of Theorem 3.1 let, for $q, 1 \leq q \leq n$, c_q be the following monomial in $\mathbf{D}^{d(\lambda)}(n)$;

$$c_q = \begin{cases} x_1^{2^{q-1}-1} \cdots x_i^{(2^\lambda - 2^{q-i})-1} \cdots x_q^{2^\lambda-2} x_{q+1}^{2^\lambda-1} \cdots x_n^{2^\lambda-1} & \text{if} \quad \lambda \geq q \geq 2 \\ x_1^0 x_2^{2^\lambda-1} \cdots x_j^{2^\lambda-1} \cdots x_n^{2^\lambda-1} & \text{if} \quad q = 1 \\ 0 & \text{if} \quad q < \lambda \end{cases}$$

and let

$$W(n,q) = \{\sigma \in S_n \mid \sigma(j) < \sigma(k) \text{ if } j < k \leq q \text{ and } \sigma(s) < \sigma(r) \text{ if } q < s < r\}.$$

Then

$$|W(n,q)| = \binom{n}{q}$$

and for each $\sigma \in W(n,q)$

$$c_q(\sigma) = \begin{cases} x_{\sigma(1)}^{2^{q-1}-1} \cdots x_{\sigma(i)}^{(2^\lambda - 2^{q-i})-1} \cdots x_{\sigma(q)}^{2^\lambda-2} x_{\sigma(q+1)}^{2^\lambda-1} \cdots x_{\sigma(n)}^{2^\lambda-1} & \text{for} \quad \lambda \geq q \geq 2 \\ x_{\sigma(1)}^0 x_{\sigma(2)}^{2^\lambda-1} \cdots x_{\sigma(j)}^{2^\lambda-1} \cdots x_{\sigma(n)}^{2^\lambda-1} & \text{for} \quad q = 1 \\ 0 & \text{for} \quad q < \lambda \end{cases}$$

is admissible. $c_q(\sigma)$, therefore, denotes the monomial resulting from the natural right action of the permutation σ over the element $c_q(\sigma) \in \mathbf{P}(n)$.

Example 3.2. If $n = 3$, $\lambda = 3$ and $q = 2$, then $W(3,2) = \{(1), (2\ 3), (1\ 2\ 3)\}$ and in matrix notation

$$c_2(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad c_2(2\ 3) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad c_2(1\ 2\ 3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Clearly each $c_q(\sigma)$ is admissible if $q = 1$. To see that $c_q(\sigma)$ is admissible if $\lambda \geq q \geq 2$, let $d(\lambda)$ denote $d(\lambda, q) = (q-1)(2^\lambda - 1)$. Then

$$u_q = x_1^{2^{q-1}-1} x_2^{2^\lambda - 2^{q-2}-1} \cdots x_i^{2^\lambda - 2^{q-i}-1} \cdots x_q^{2^\lambda - 2}$$

is an admissible element in $\mathbf{D}^{d(\lambda)}(q, u_q)$. For each $\sigma \in W(n,q)$, q fixed, let

$$g_q^\sigma \;:\; \mathbf{D}^{d(\lambda)}(q, u_q) \to \mathbf{D}^{d(\lambda)}(n)$$

be the mapping given on monomials by

$$g_q^\sigma(x_1^{e_1} \cdots x_q^{e_q}) = x_{\sigma(1)}^{e_1} \cdots x_{\sigma(q)}^{e_q} x_{\sigma(q+1)}^{2^\lambda - 1} \cdots x_{\sigma(j)}^{2^\lambda - 1} \cdots x_{\sigma(n)}^{2^\lambda - 1}.$$

For instance if we assume $n = 3, \lambda = 3$ and $q = 2$, as in Example (3.2), then $\mathbf{D}^{d(3)}(2, u_2)$ is generated by

$$\left\{x_1 x_2^6,\ x_1^2 x_2^5,\ x_1^4 x_2^3,\ x_1^3 x_2^4,\ x_1^5 x_2^2,\ x_1^6 x_2\right\}$$

and $g_3^{(1)}(\mathbf{D}^{d(3)}(2, u_2))$ is generated by

$$\left\{x_1 x_2^6 x_3^7,\ x_1^2 x_2^5 x_3^7,\ x_1^4 x_2^3 x_3^7,\ x_1^3 x_2^4 x_3^7,\ x_1^5 x_2^2 x_3^7,\ x_1^6 x_2 x_3^7\right\}$$

We then have

$$\pi\left(\mathcal{A}^+ \mathbf{P}(n) \cap \mathbf{P}^{d(\lambda)}(n)\right) = \bigoplus_{\sigma, q} g_q^\sigma \left(\mathbf{H}^{d(\lambda)}(q, u_q)\right).$$

Since

$$\mathbf{D}^{d(\lambda)}(n) = \bigoplus_{\sigma, q} g_q^\sigma \left(\mathbf{D}^{d(\lambda)}(q, u_q)\right)$$

and each g_q^σ preserves the order of monomials we see that $c_q(\sigma)$ is admissible for each pair (σ, q) for which $\lambda \geq q \geq 2$. We note that if $\lambda \geq n$, and we let $\mathbf{C}^{d(\lambda)}(n)$ denote the subspace of $\mathbf{Q}^{d(\lambda)}(n)$ with basis

$$\mathcal{B}(d(\lambda)) = \{c_q(\sigma) \mid 1 \leq q \leq n,\ \sigma \in W(n,q)\},$$

then the mapping

$$f : \mathbf{C}^{d(\lambda)}(n) \to \mathbf{C}^{d(\lambda+1)}(n) \tag{3.2}$$

defined on monomials by $f(c_q(\sigma)) = c_q(\sigma)(x_{\sigma(2)} \cdots x_{\sigma(n)})^{2^\lambda}$ is an isomorphism between the vector subspaces $\mathbf{C}(n)$. Since $\mathbf{C}^{d(\lambda)}(n)$ is a subspace of $\mathbf{Q}^{d(\lambda)}(n)$, we have

$$\begin{aligned}
\dim\left(\mathbf{Q}^{d(\lambda)}(n)\right) &\geq \dim\left(\mathbf{C}^{d(\lambda)}(n)\right) \\
&= |\mathcal{B}(d(\lambda))| \\
&= \textstyle\sum_{q=1}^{\min\{\lambda, n\}} |W(n,q)| \\
&= \begin{cases} \sum_{q=1}^{\lambda} \binom{n}{q} & \text{if} \quad \lambda < n \\ 2^n - 1 & \text{if} \quad \lambda \geq n. \end{cases}
\end{aligned}$$

This establishes the result of Theorem 3.1.

4 Main result

In this section we adopt some of the notation introduced in Section 3 and prove our main result, namely, Theorem 4.1 below. To give some insight into our argument, suppose $n \geq 2$ and $d > 0$ are integers such that $d = \sum_{i=1}^{n-1}(2^{\lambda_i} - 1)$ with $\lambda_1 > \lambda_2 > \ldots > \lambda_{n-2} \geq \lambda_{n-1} \geq 1$. Let $\lambda = \lambda_{n-1}$. By Theorem 2.6 if $u \in \mathbf{P}^d(n)$ is a monomial in a basis for $\mathbf{Q}^d(n)$ then the weight vector of u has the form

$$w(u) = (\overbrace{n-1, n-1, \ldots, n-1}^{\lambda}, w_\lambda(a), w_{\lambda+1}(a), \ldots)$$

for some monomial a of degree $d' = \sum_{i=1}^{n-1}(2^{\lambda_i - \lambda_{n-1}} - 1)$. Adopting the notation of Section 3 we see, as a result, that $\mathbf{P}^d(n)$ has a subspace isomorphic to the direct product $\mathbf{D}^{d(\lambda)}(n) \times \mathbf{P}^{d'}(n-1)$ and this splitting is natural to the image of the action of the Steenrod algebra. The results of Section 3 carry over and we may suppose that $u = c_q(\sigma)a^{2^\lambda}$ for some $c_q(\sigma) \in \mathbf{C}^{d(\lambda)}(n)$.

Theorem 4.1. *Suppose that $n \geq 2$ and $d > 0$ may be expressed in the form $d = \sum_{i=1}^{n-1}(2^{\lambda_i} - 1)$ with $\lambda_1 > \lambda_2 > \ldots > \lambda_{n-2} \geq \lambda_{n-1} \geq 1$. Then*

$$\dim(\mathbf{Q}^d(n)) \geq \left(\sum_{q=1}^{\min\{\lambda_{n-1}, n\}} \binom{n}{q}\right)(\dim(\mathbf{Q}^{d'}(n-1)))$$

where $d' = \sum_{i=1}^{n-1}(2^{\lambda_i - \lambda_{n-1}} - 1)$.

Proof. Under the hypothesis of Theorem 4.1 let $\lambda = \lambda_{n-1}$. By Theorem 2.6 a monomial in a basis for $\mathbf{Q}^d(n)$ has the form ba^{2^λ} for some $b \in \mathbf{D}^{d(\lambda)}(n)$ and $a \in \mathbf{P}^{d'}(n)$ of weight order greater than or equal to that of the minimal spike of degree d'. For each i, $1 \leq i \leq n$, let $\mathbf{C}^{d(\lambda)}(n, i)$ be the subspace of $\mathbf{C}^{d(\lambda)}(n)$ generated by

$$\{c_q(\sigma) \in \mathbf{C}^{d(\lambda)}(n) \mid \sigma(1) = i\}$$

and $\mathbf{P}^{d'}(n, i)$ be the subspace of $\mathbf{P}^{d'}(n)$ generated by

$$\{x_1^{e_1} \cdots x_n^{e_n} \in \mathbf{P}^{d'}(n) \mid e_i = 0\}$$

and consider the subspace $\mathbf{C}^{d(\lambda)}(n, i) \times \mathbf{P}^{d'}(n, i)$ of $\mathbf{P}^d(n)$ generated by

$$\mathcal{S}(n, d, i) = \{c_q(\sigma)a^{2^\lambda} \in \mathbf{P}^d(n) \mid c_q(\sigma) \in \mathbf{C}^{d(\lambda)}(n, i) \text{ and } a \in \mathbf{P}^{d'}(n, i)\}.$$

Then $\cup_{i=1}^{n}\mathcal{S}(n, d, i)$ generates a subspace of $\mathbf{P}^d(n)$ isomorphic to $\mathbf{C}^{d(\lambda)}(n) \times \mathbf{P}^{d'}(n-1)$.

Now for each $c_q(\sigma) \in \mathbf{C}^{d(\lambda)}(n, i)$ consider the injective linear mapping

$$h_q^\sigma : \mathbf{P}^{d'}(n, i) \rightarrow \mathbf{P}^d(n)$$

given on monomials by $h_q^\sigma(a) = c_q(\sigma)a^{2^\lambda}$. Then h_q^σ is onto the subspace $\{c_q(\sigma)\} \times \mathbf{P}^{d'}(n, i)$ of $\mathbf{C}^{d(\lambda)}(n, i) \times \mathbf{P}^{d'}(n, i)$.

Let $Sq^k \in \mathcal{A}^+$ and suppose that $Sq^k(u) \in \mathbf{P}^{d'}(n,i)$ for some $u \in \mathbf{P}(n)$. Then, by formula (2.1),

$$\begin{aligned} h_q^{\sigma}(Sq^k(u)) &= c_q(\sigma)(Sq^k(u))^{2^{\lambda}} \\ &= c_q(\sigma)(Sq^{k2^{\lambda}}(u^{2^{\lambda}})) \end{aligned}$$

Further, modulo hit monomials, every hit element in $\{c_q(\sigma)\} \times \mathbf{P}^{d'}(n,i)$ is obtained uniquely in this way. Thus $h_q^{\sigma}(\mathbf{Q}^{d'}(n,i))$ is a direct summand of $\mathbf{Q}^d(n)$ isomorphic to $\mathbf{Q}^{d'}(n-1)$. Hence if we let $\mathcal{B}(n,d,i)$ be the set of monomials

$$\{c_q(\sigma)a^{2^{\lambda}} \in \mathcal{S}(n,d,i) \mid c_q(\sigma) \in \mathbf{C}^{d(\lambda)}(n,i) \text{ and } a \in \mathbf{Q}^{d'}(n,i)\}$$

then $\mathcal{B}(n,d,\lambda) = \cup_{i=1}^{n} \mathcal{B}(n,d,i)$ generates a subspace of $\mathbf{P}^d(n)$ isomorphic to $\mathbf{C}^{d(\lambda)}(n) \times \mathbf{Q}^{d'}(n-1)$. But then $\mathbf{C}^{d(\lambda)}(n) \times \mathbf{Q}^{d'}(n-1)$ is a direct summand of $\mathbf{Q}^d(n)$ and $\mathcal{B}(n,d,\lambda)$ is a basis for this subspace.

Finally we note that if we let f be the mapping defined in equation (3.2), then the mapping

$$f_* \;:\; \mathbf{C}^{d(\lambda)}(n) \times \mathbf{Q}^{d'}(n-1) \to \mathbf{C}^{d(\lambda+1)}(n) \times \mathbf{Q}^{d'}(n-1)$$

defined on monomials by $f_*(c_q(\sigma)a^{2^{\lambda}}) = f(c_q(\sigma))a^{2^{\lambda+1}}$ is an isomorphism. Since $\mathbf{C}^{d(\lambda)}(n) \times \mathbf{Q}^{d'}(n-1)$ is a subspace of $\mathbf{Q}^d(n)$, we have

$$\begin{aligned} \dim\left(\mathbf{Q}^d(n)\right) &\geq \dim\left(\mathbf{C}^{d(\lambda)}(n) \times \mathbf{Q}^{d'}(n-1)\right) \\ &= \dim\left(\mathbf{C}^{d(\lambda)}(n)\right)\dim\left(\mathbf{Q}^{d'}(n-1)\right) \\ &= \textstyle\sum_{q=1}^{\min\{\lambda,n\}} \binom{n}{q}(\dim(\mathbf{Q}^{d'}(n-1))). \end{aligned}$$

This proves the theorem.

Remark 4.2. In general strict inequality holds in the statement of Theorem 4.1. In [[14], Theorem 1.3] N. Sum proved that equality holds when $\lambda_{n-1} \geq n-1$. It has, in particular, been proved, by T. N. Nam [6], that if $\alpha(n+d) = n$ and $\lambda_{n-1} \geq n$ then equality holds in the statement of the theorem .

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References

[1] J. Adem, The relations on Steenrod powers of cohomology classes, Algebraic Geometry and Topology, a symposium in honour of S. Lefschetz, 191 – 238, Princeton Univ. Press, Princeton NJ (1957).

[2] Kameko, M. Products of projective spaces as Steenrod modules. Thesis, John Hopkins University (1990).

[3] M. Kameko, Generators of the cohomology of BV_3. *J. Math. Kyoto Univ.* **38** (1998), 587-593.

[4] M. Kameko, Generators of the cohomology of BV_4. preprint, Toyama Univ. (2003).

[5] M. F. Mothebe, Generators of the polynomial algebra $\mathbb{F}_2[x_1,\ldots,x_n]$ as a module over the Steenrod algebra. Thesis, University of Manchester (1997).

[6] T. N. Nam, $\mathcal{A}$-générateurs génériques pour l' algèbre polynomiale, *Adv. Math.* **186** (2004), 334-362.

[7] F. P. Peterson, Generators of $\mathbf{H}^*(\mathbb{R}P^\infty \bigwedge \mathbb{R}P^\infty)$ as a module over the Steenrod algebra. *Abstracts Amer. Math. Soc.* **833** (1987).

[8] F. P. Peterson, A-generators for certain polynomial algebras. *Math. Proc. Camb. Phil. Soc.* **105** (1989), 311–312.

[9] J. H. Silverman, Hit polynomials and conjugation in the dual Steenrod algebra. *Proc. Math. Camb. Phil. Soc.* **123** (1998), 531–547.

[10] W. M. Singer, The transfer in homological algebra. *Math. Zeitschrift.* **202** (1989), 493–523.

[11] W. M. Singer, On the action of Steenrod squares on polynomials. *Proc. Amer. Math. Soc.* **111** (1991), 577–583.

[12] N. Sum, The hit problem for the polynomial algebra of four variables. University of Quy Nhon, Viet Nam, preprint (2007).

[13] N. Sum, The negative answer to Kameko's conjecture on the hit problem. *Adv. Math.* **225** (2010), 2365-2390.

[14] N. Sum, On the Peterson hit problem. Preprint (2011).

[15] R.M.W. Wood, Steenrod squares of polynomials and the Peterson conjecture. *Math. Proc. Camb. Phil. Soc.* **105** (1989), 307–309.