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My Life with Augustin*

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Abstract

This is a very brief and partial account of Augustin's mathematics and our common interests. In the end is presented some recent work of mine whose birth he encouraged.

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In August 2004, after visiting the ethnographic museum in Porto Novo (Benin), I wanted to write a few words in the visitors' book... and found out that Augustin Banyaga had been there five hours before. This might be a symbol of our relationship, even though the intertwining of our destinies owes little to coincidence: indeed, we are mathematical cousins, our masters André Haefliger and René Thom being two sons of Charles Ehresmann.

Symplectic geometry

I heard of Augustin for the first time in 1982, during the weekly meetings on symplectic geometry organized by Daniel Bennequin at the École Normale Supérieure: among the results discussed was Banyaga's theorem[1] that a symplectic transformation of a compact symplectic manifold (M,ω) is Hamiltonian isotopic to the identity[2] if an only if it belongs to the subgroup $[G_\omega,G_\omega]$ of commutators of the component G_ω of the identity in the group of symplectic transformations of M.

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[1] In his thesis [3] under the supervision of André Haefliger.

[2] That is, obtained by integrating a time-dependent Hamiltonian vector field; in other words, its Calabi invariant is zero.

These meetings were very fruitful[3], as they coincided with a breakthrough in what is now called symplectic topology, the proof by Conley and Zehnder [20] of an Arnold conjecture [2] that had been defying topologists for almost twenty years: when ω is a standard symplectic structure on the $2n$-torus, every $h \in [G_\omega, G_\omega]$ has at least $2n+1$ fixed points, and at least 2^{2n} if all of them are non-degenerate[4].

I was lucky enough to see at once that a more general conjecture of Arnold [1] could be proven along the same lines [9]: every symplectic transformation of the cotangent bundle $T^*\mathbb{T}^n$, Hamiltonian isotopic to the identity, sends the zero section $d0$ to a submanifold L intersecting $d0$ at least at $n+1$ points, and at least 2^n if all these intersections are transversal[5].

One year later, in order to extend this result from $\mathbb{T}^n$ to an arbitrary closed manifold M, I gave a much simpler proof of the same theorem [10], constructing by purely elementary means a generating phase for L. The extension of the theorem to an arbitrary M was obtained by Hofer [21] via "hard analysis" following the idea of [9] but, very little later, Laudenbach and Sikorav [23] found a simpler proof using my new method.

The method was immediately taught all around the world, in particular by Augustin.

Smooth local conjugacies

Even though I may be essentially known for those two little contributions to symplectic geometry, they constitute a parenthesis in my work, whose first years had been devoted to various aspects of the smooth local classification of dynamical systems [11, 12]. With Rafael de la Llave and Gene Wayne, Augustin contributed to the subject [4, 5], using the so-called *path method* introduced by Samuel and Thom in singularity theory and applied by Moser to the conjugacy problem for volume or symplectic forms. The device having been the key of Roussarie's work [25] on the local classification of vector fields (and differential forms), Banyaga, de la Llave and Wayne deal chiefly with the case of diffeomorphisms.

You wish to prove that two local diffeomorphisms $h_0, h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ are conjugate near 0. Setting $R := h - h_0$ the idea is to consider the path $h_t := h_0 + tR$ from h_0 to $h_1 = h$ and solve the seemingly more difficult problem: find a smooth path g_t with $g_0 = \mathrm{Id}$ in the space of local diffeomorphisms $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that, in a neighbourhood of the origin, $g_t^* h_t = h_0$ for all $t \in [0,1]$, where $g_t^* h_t := g_t^{-1} \circ h_t \circ g_t$; as $g_0^* h_0 = h_0$ by definition, this is equivalent to solving the equation $\frac{d}{dt} g_t^* h_t = 0$ near $0 \in \mathbb{R}^n$ for all $t \in [0,1]$. Now, the local isotopy g_t will determine (and be determined by) the time-dependent local vector field $X_t := (\frac{d}{dt} g_t) \circ g_t^{-1}$, called its *infinitesimal generator*[6], which must satisfy $X_t(0) = 0$ for $0 \le t \le 1$; near $0 \in \mathbb{R}^n$, setting $Y_t := (\frac{d}{dt} h_t) \circ h_t^{-1} = R \circ h_t^{-1}$, the equation $\frac{d}{dt} g_t^* h_t = 0$ writes[7] as the "cohomological equation"

$$X_t = h_{t*} X_t + Y_t$$

[3] Other participants were Michèle Audin, Alain Chenciner, Ivar Ekeland, Albert Fathi, Misha Gromov, Michael R. Herman, François Laudenbach, Jean-Claude Sikorav, Claude Viterbo...

[4] This implies for example [9] the smooth case of the Poincaré-Birkhoff theorem ("Poincaré's last geometric theorem") on the number of fixed points of area-preserving twist transformations of the annulus.

[5] If L projects diffeomorphically onto $\mathbb{T}^n$, it is (the image of) the differential $d\varphi$ of a function φ on $\mathbb{T}^n$; thus, the result generalizes the classical Lyusternik-Schnirelmann/Morse estimates for the number of fixed points of a C^2 function on $\mathbb{T}^n$.

[6] For each x near $0 \in \mathbb{R}^n$, the path $t \mapsto g_t(x)$ is the solution of the equation $\frac{dy}{dt} = X_t(y)$ equal to x for $t = 0$.

[7] If $g_t^* Y(x) = Dg_t(x)^{-1} Y(g_t(x))$ for each vector field Y defined near $0 \in \mathbb{R}^n$ and $h_{t*} := (h_t^{-1})^*$.

in the unknown X_t. This affine fixed point problem is solved as usual when h_t is a strict dilation, yielding

$$X_t = \sum_{k\geq 0} h_{t*}^k Y_t,$$

which converges near $0 \in \mathbb{R}^n$ provided Y_t vanishes to a high enough order at the origin[8]. Similar arguments work in the general hyperbolic case.

In [8, 12] such problems had been solved more directly. Since then, I have treated the conjugacy problem (and related questions like the solution of cohomological equations) as a part of invariant manifold theory [13, 17, 14, 15]; the starting point is the remark that $h : X \to Y$ is a semi-conjugacy between $f : X \to X$ and $g : Y \to Y$ (i.e. $g \circ h = h \circ f$) if and only if its graph is invariant by $f \times g : (x,y) \mapsto (f(x), g(y))$.

Contact geometry and Weinstein's conjecture

I first heard this story from Albert Fathi, then a professor at the University of Florida, Gainesville. The result [6] is the following: let M be a $(2n+1)$-dimensional oriented closed manifold with a 1-form α such that the 2-form $\omega = d\alpha$ has rank $2n$ everywhere. If there exists a locally free circle action ρ on M which preserves ω, then the characteristic foliation of ω has at least two closed leaves for $n \geq 1$.

The wonderfully simple proof should be contrasted with the very hard prize-winning work of Cliff Taubes on the Weinstein conjecture in dimension three[9]; it goes as follows: if we view the circle as $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, the 1-form $\alpha_0 := \int_{\mathbb{T}} \rho^{\theta *}\alpha\, d\theta$ is ρ-invariant and satisfies

$$\begin{aligned} d\alpha_0 &= d\int_{\mathbb{T}} \rho^{\theta *}\alpha\, d\theta = \int_{\mathbb{T}} d(\rho^{\theta *}\alpha)\, d\theta = \int_{\mathbb{T}} (\rho^{\theta *}d\alpha)\, d\theta = \int_{\mathbb{T}} \rho^{\theta *}\omega\, d\theta = \int_{\mathbb{T}} \omega\, d\theta \\ &= \omega \end{aligned}$$

because ω is ρ-invariant; denoting by $Z := \frac{d}{d\theta}\rho^\theta|_{\theta=0}$ the infinitesimal generator of ρ, the invariance of α_0 under ρ reads

$$\mathcal{L}_Z\alpha_0 := \tfrac{d}{d\theta}\rho^{\theta *}\alpha_0|_{\theta=0} = 0,$$

which, by the Cartans' formula[10] $\mathcal{L}_Z\alpha_0 = (d\alpha_0)Z + d(\alpha_0 Z) = \omega Z + d(\alpha_0 Z)$, can be rewritten

$$dS = \omega Z \quad \text{if} \quad S := -\alpha_0 Z.$$

It follows that the critical points of S are those $x \in M$ such that $Z(x)$ belongs to the characteristic direction $\ker\omega(x)$ of ω at x; now, since Z and α_0 are ρ-invariant, so is S, hence the whole orbit of the critical point x under ρ consists of critical points y of S, at which $Z(y)$ belongs to the characteristic direction of ω; thus, the orbit of x under ρ is a closed characteristic of ω. As M is compact, the function S has a maximum and a minimum, which are critical values because M has no boundary; if they are distinct, the corresponding critical points yield distinct closed characteristics of ω; if they coincide, every point of M is a critical point of S and therefore every characteristic of ω is closed.

[8] Which means that R does. Convergence occurs in the same open neighbourhood of 0 for all $t \in [0,1]$; hence, as $X_t(0) = 0$, the time-dependent vector field X_t does generate a local isotopy.

[9] As far as I know, the general case, on which I spent quite a lot of infructuous time in the early nineties, is still open.

[10] Where βX denotes the interior product of a differential form β and a vector field X.

Generalized Hopf bifurcations

In January 2010, for my sixtieth birthday, a conference took place at the Institut Henri Poincaré in Paris. Among the speakers were Augustin[11], whose talk began with a minute of silence for the victims of the earthquake in Haiti, and Mark Levi, who invited me to spend some time at Penn State in October and November the same year.

This visit, during which I enjoyed the hospitality of Augustin, Mark and Aissa Wade, played a key role in my work on generalized Hopf bifurcations, as the (polite?) interest it arose[12] gave me the courage to complete a substantial part of its redaction [16].

Overview of the results. They are about the birth of dynamics out of statics (or the nonlinear coupling of oscillators). In generic smooth one-parameter families of vector fields, simple examples are *the Hopf bifurcation*, in which an attracting equilibrium point becomes unstable while giving rise to an attracting periodic orbit, and the *Sacker-Naimark bifurcation*, in which an attracting periodic orbit becomes unstable while giving rise to an attracting invariant 2-torus.

These $\mathbb{T}^0 \to \mathbb{T}^1$ and $\mathbb{T}^1 \to \mathbb{T}^2$ bifurcations are *not* paralleled by a $\mathbb{T}^2 \to \mathbb{T}^3$ bifurcation which, far from being generic, requires infinitely many conditions [19]: in generic one-parameter families, the invariant 2-torus will break down when it loses attractivity and chaos ("turbulence" [26]) will develop. Thus, to study the birth of n-tori with $n > 2$, it is best to follow René Thom's advice: "Look for the organising center or phenomena" and consider families depending on more parameters.

Loosely speaking, the result is that $\mathbb{T}^0 \to \mathbb{T}^n$ and $\mathbb{T}^1 \to \mathbb{T}^{n+1}$ bifurcations (among others) occur smoothly in generic families depending on at least n parameters.

More precisely, attracting[13] invariant n-tori—and more suprising invariant submanifolds—are born smoothly at partially elliptic stationary points in generic families of vector fields (resp., transformations) depending on at least n parameters. Here, "partially elliptic" means that the eigenvalues of the linearised dynamics which lie on the imaginary axis (resp., unit circle) are simple and consist of n pairs of conjugate complex numbers[14].

The corresponding values u_0 of the parameter u form a submanifold of codimension n, but we shall see soon that the set $\mathcal{V}$ of those u for which the attracting invariant submanifold exists (and depends differentiably on u) contains an open subset with nonempty *open* tangent cone at u_0, implying that the phenomenon is not negligible[15].

The birth lemma. Under a mild nonresonance condition, taking a suitable chart and restricting the dynamics to a central manifold, one may assume that, near u_0 and the partially elliptic stationary point considered in phase space, the dynamics under study form a local

[11] In the nineties, he and his wife Judith had become our friends when they had spent six months in Paris.

[12] At the colloquium of the mathematics department and the Fall meeting of the Penn State/Maryland Workshop in Dynamical Systems and Related Topics.

[13] More generally, normally hyperbolic.

[14] For maps, [16] treats similarly the case where there are $n-1$ pairs, plus -1 but not $+1$.

[15] Here, Thom's advice is spectacularly good: indeed, for $n > 1$, the codimension n submanifold consisting of "birth values" u_0 occupies almost no room in the boundary of $\mathcal{V}$; near the other boundary points, turbulence will appear, making it practically impossible to understand the phenomenon via families with fewer parameters.

family Z_u (resp., h_u) of vector fields on (resp., transformations of) $\mathbb{C}^n$ having third order contact at $0 \in \mathbb{C}^n$ with a normal form

$$N_u(z) = \Big(z_j\Big(\lambda_j(u) + i\mu_j(u) - \sum_{\ell=1}^{n}(a_{j\ell}(u) + ib_{j\ell}(u))|z_\ell|^2\Big)\Big)_{1\le j\le n},$$

(resp., the time one of its flow), where $\lambda_j, \mu_j, a_{j\ell}, b_{j\ell}$ are differentiable real functions with $\lambda_j(u_0) = 0$ (ellipticity).

Under those hypotheses[16], the main result of [16] is the following *birth lemma*: assume that, for some tangent vector v_0 at u_0, the vector field

$$\widetilde{\xi}_{u_0,v_0}(\zeta) = \Big(\zeta_j\Big(D(\lambda_j + i\mu_j)(u_0)v_0 - \sum_{\ell=1}^{n}(a_{j\ell}(u_0) + ib_{j\ell}(u_0))|\zeta_\ell|^2\Big)\Big)_{1\le j\le n}$$

on $\mathbb{C}^n$ admits a normally hyperbolic compact invariant manifold $\widetilde{\Sigma} \subset \mathbb{C}^n$. Then, there is an open subset $\mathcal{U}_{u_0,v_0}$ of parameter space $\mathcal{U}$ with the following properties:

i) Its closure contains u_0.

ii) Its tangent cone at u_0 is an open cone with vertex 0 containing $\mathbb{R}_+^* v_0$.

iii) Every Z_u (resp. h_u) with $u \in \mathcal{U}_{u_0,v_0}$ has a compact normally hyperbolic invariant manifold S_u diffeomorphic to $\widetilde{\Sigma}$, whose index[17] is that of $\widetilde{\Sigma}$ for $\widetilde{\xi}_{u_0,v_0}$, depending nicely on u and tending to $\{0\}$ when $u \to u_0$.

iv) Precisely, there is an open cone $V \ni v_0$ of $T_{u_0}\mathcal{U}$ with vertex 0 such that each $\widetilde{\xi}_{u_0,v}$ with $v \in V$ has a unique normally hyperbolic compact invariant manifold $\widetilde{\Sigma}_v$ diffeomorphic to $\widetilde{\Sigma}$ and C^1-close to it up to homothety[18]; every smooth $\gamma : (\mathbb{R}_+, 0) \to (\mathcal{U}, u_0)$ with $\dot\gamma(0) = v \in V$ satisfies $\gamma(\varepsilon) \in \mathcal{U}_{v_0,v_0}$ for $\varepsilon > 0$ small enough, and $\lim\limits_{\varepsilon\to 0} \varepsilon^{-\frac12} S_{\gamma(\varepsilon)} = \widetilde{\Sigma}_v$ in the at least C^1 sense[19].

A key argument in the proof is that $\widetilde{\xi}_{u_0,v}$ is the limit when $\varepsilon \to 0$ of the vector field obtained from $N_{\gamma(\varepsilon)}$ by substracting to it the infinitesimal rotation $z \mapsto (i\mu_j(u_0)z_j)_{1\le j\le n}$ (which commutes with it) and making the variable change $z = \varepsilon^{\frac12}\zeta$, and the time change $t = \varepsilon^{-1}\tau$

Consequences. The vector field $\widetilde{\xi}_{u_0,v_0}$ being $\mathbb{U}(1)^n$-invariant, so is $\widetilde{\Sigma}$ by local uniqueness; passing to the quotient, we see that the $\mathbb{O}(1)^n$-invariant submanifold $\Sigma = \widetilde{\Sigma} \cap \mathbb{R}^n$ of $\mathbb{R}^n$ is a normally hyperbolic invariant manifold of the $\mathbb{O}(1)^n$-invariant vector field

$$\xi_{u_0,v_0}(r) = \sum_j r_j\Big(D\lambda_j(u_0)v_0 - \sum_\ell a_{j\ell}(u_0)\, r_\ell^2\Big)\frac{\partial}{\partial r_j}$$

on $\mathbb{R}^n$. Here is a weak converse: if ξ_{u_0,v_0} has a normally hyperbolic $\mathbb{O}(1)^n$-invariant submanifold Σ *on which it vanishes identically*, then the hypothesis of the birth lemma is satisfied by $\widetilde{\Sigma} = \{z \in \mathbb{C}^n : (|z_1|, \ldots, |z_n|) \in \Sigma\}$. Two cases of interest:

[16]Even when there are less than n parameters or the family is not generic.

[17]Dimension of the leaves of the stable foliation.

[18]Because normal hyperbolicity is open and $\eta^{\frac12}\widetilde{\Sigma}_v$ is invariant by the flow of $\widetilde{\xi}_{u_0,\eta v}$ for all positive η when $\widetilde{\Sigma}_v$ is invariant by the flow of $\widetilde{\xi}_{u_0,v}$

[19]Hence the invariant manifold S_u arises rather suddenly, as in classical Hopf bifurcations.

– *Tori.* When $\Sigma = \{r : \forall j\, D\lambda_j(u_0)v_0 = \sum_\ell a_{j\ell}(u_0)\, r_\ell^2\}$ with $(a_{j\ell}(u_0))$ invertible, it consists of equilibrium points; if they are hyperbolic, we get the n-tori mentioned before.

– *Moment-angle manifolds.* If $\Sigma = \{r : F(r) = b\}$ with $F(r) = \sum_j \Lambda_j r_j^2$, $b, \Lambda_1, \ldots, \Lambda_n \in \mathbb{R}^c$, $\mathrm{conv}(\Lambda_1, \ldots, \Lambda_n) \not\ni 0$ and b a regular value of F, then $\widetilde{\Sigma}$ is called a *moment-angle manifold* and can have various topologies [7, 24]. When $\xi_{u_0,v_0}(r) = \sum_j r_j \Lambda_j \cdot (b - F(r)) \frac{\partial}{\partial r_j}$ (the dot stands for the scalar product), it equals $-\frac{1}{2}\nabla|F(r) - b|^2$ and therefore admits Σ as a pointwise invariant normally hyperbolic attractor[20]; hence, the birth lemma applies.

In this example, $\widetilde{\Sigma}$ is a $(2n-1)$-sphere if $c = 1$, an n-torus if $c = n$. This ξ_{u_0,v_0} is too particular to arise in *generic* n-parameter families[21] but, as normal hyperbolicity is open, the birth of normally hyperbolic attractors diffeomorphic to $\widetilde{\Sigma}$ will be observed in generic n-parameter families nearby. This is the idea of the following corollary of the birth lemma:

Assume that the vector field $\xi(r) = \sum_j r_j\Big(\nu_j v - \sum_\ell \gamma_{j\ell} r_\ell^2\Big) \frac{\partial}{\partial r_j}$ on $\mathbb{R}^n$, $\nu_j, \gamma_{j\ell} \in \mathbb{R}$, has an $\mathbb{O}(1)^n$-invariant normally hyperbolic invariant manifold Σ whose intersection with $\mathbb{R}^n_+$ is connected[22]. Then, if M and $\mathcal{U}$ are separable manifolds with $\dim M \geq 2n$ and $\dim \mathcal{U} \geq n$, there exists a *nonempty*, C^3-open set of smooth families $X : \mathcal{U} \times M \to TM$ of vector fields (resp., $f : \mathcal{U} \times M \to M$) for which the birth lemma ensures at some point (u_0, x_0) the birth of normally hyperbolic invariant submanifolds of X_u (resp., f_u) diffeomorphic to $\widetilde{\Sigma}$. The same holds true if "normally hyperbolic" is replaced by "normally hyperbolic and attracting".

For example, if $n = 3$, Σ can be a periodic orbit, yielding a 4-torus $\widetilde{\Sigma}$ in $\mathbb{C}^3$.

Spheres. Of special interest is the case of the birth lemma where $\widetilde{\Sigma}$ is an attracting embedded sphere of codimension 1 around the origin, a *bona fide* generalisation of the Hopf bifurcation in which every nonzero forward orbit of Z_u (resp., h_u) in a fixed neighbourhood of the origin in $\mathbb{C}^n$ tends to S_u for $u \in \mathcal{U}_{u_0,v_0}$ close enough to u_0.

The approach *via* the case $c = 1$ of moment-angle manifolds [22] is interesting because the dynamics on S_u can vary a lot, ξ_{u_0,v_0} having no dynamics on Σ, but this provides quite a narrow set in parameter space.

A very wide set is furnished by the rough birth lemma [16] stating that, for positive $D\lambda_j(u_0)v_0$ and $a_{j\ell}(u_0)$, a family of attracting Čech homology $(2n-1)$-spheres ("Birkhoff attractors") bifurcates as in the birth lemma. Conditions for these "spheres" to be normally hyperbolic differentiable hypersurfaces S_u will be studied in a forthcoming paper with Santiago López de Medrano, together with the bifurcations that can occur inside S_u—the birth lemma can indeed apply for the same (u_0, v_0) (with different $\mathcal{U}_{u_0,v_0}$) to many manifolds $\widetilde{\Sigma}$, among which a big embedded $(2n-1)$-sphere containing all the others.

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[20] This remark is due to S. López de Medrano [18].

[21] Normally hyperbolic invariant manifolds Σ of large dimension are not always easy to find.

[22] Technically, normal hyperbolicity is assumed absolute when Σ meets some coordinate hyperplane $\{r_j = 0\}$ in which it is not contained.

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