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Entropy Solution for Some $p(x)-$Quasilinear Problem with Right-Hand Side Measure

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Abstract

In this paper we study the existence of entropy solution for the following $p(x)$-quasilinear elliptic problem

$$-\mathrm{div}(a(x,u,\nabla u)) + g(x,u,\nabla u) = \mu$$

where the right-hand side μ is a measure, which admits a decomposition in $L^1(\Omega) + W^{-1,p'(x)}(\Omega)$ and $g(x,s,\xi)$ is a nonlinear term which has a growth condition with respect to ξ and has no growth with respect to s while satisfying a sign condition on s.

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1 Introduction

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$, $p \in C(\bar{\Omega})$, $p(x) > 1$. Let A be the nonlinear operator defined from $W_0^{1,p(x)}(\Omega)$ into its dual $W^{-1,p'(x)}(\Omega)$ by the formula

$$Au = -\mathrm{div}(a(x,u,\nabla u)). \tag{1.1}$$

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In this paper we investigate the problem of existence solutions of the following Dirichlet problem

$$Au + g(x,u,\nabla u) = \mu \text{ in } \Omega, \tag{1.2}$$

where μ is some measure which can be decomposed as, $\mu = f - \operatorname{div} F$.

In this context of nonlinear operators in the degenerated case for the Sobolev spaces with constant exponent $p(x) = p = cte$, if μ belongs to $W^{-1,p'}(\Omega, w)$ the existence results have been proved in [3], where the authors have used the approach based on the strong convergence of the positive part u_ε^+ (resp. negative part u_ε^-), and the case where μ in $L^1(\Omega)$ is investigated in [4] under the following coercivity condition,

$$|g(x,s,\xi)| \geq \beta \sum_{i=1}^{N} w_i |\xi_i|^p \text{ for } |s| \geq \gamma. \tag{1.3}$$

Let us recall that the result given in [3, 4] have been proved under some additional conditions on the weight function σ and the parameter q introduced in Hardy inequality.

It will turn out that in the L^p case, Boccardo, Gallouët and Orsina, have studied in [13] the following particular case

$$Au = \mu \text{ in } \Omega, \tag{1.4}$$

where $Au = -\operatorname{div}(a(x,\nabla u))$.

However Porreta has proved in [23] the existence of a solution u of (1.2) which belongs to the Sobolev space $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$ where the datum μ is assumed to be in $L^1(\Omega) + H^{-1}(\Omega)$.

Recently, when $g \equiv 0$, $\mu = f(x,u,\nabla u)$, Benboubker, Azroul and Barbara have proved the existence result on Sobolev spaces with variable exponent by using a classical theorem of J.L. Lions on operators of the calculus of variations (see [8]), besides, when $a(x,s,\xi) = |\xi|^{p(x)-2}\xi$, $g \equiv 0$ Bendahmane and Wittbold in [9] proved the existence and uniqueness of renormalized solutions to problem (1.2) with $\mu \in L^1$. Then, Zhang and Zhou (see [26]) have obtained the above results for measure data $\mu \in L^1(\Omega) + W^{-1,p'(x)}(\Omega)$.

Concerning the notion of entropy solution (introduced by Bénilan et al in [11]), Sanchón and Urbano in [25] studied a Dirichlet problem of p(x)-Laplace equation and obtained the existence and uniqueness of entropy solutions for L^1 data, as well as integrability results for the solution and its gradient. The proofs rely crucially on a priori estimates in Marcinkiewicz spaces with variable exponents. Furthermore the notion of measure data which can be decomposed is verified when $p(x) = p = cte$, has been introduced by Boccardo, Gallouët and Orsina (see [13]), in the context that they considered a signed measure $\mu \in L^1(\Omega) + W^{-1,p'}(\Omega)$ if and only if $\mu \in \mathcal{M}_b^p(\Omega)$: every signed measure that is zero on the sets of zero p-capacity can be splitted in the sum of a function in $L^1(\Omega)$, and an element in $W^{-1,p'}(\Omega)$ (the dual space of $W_0^{1,p}(\Omega)$), and conversely, every signed measure in $L^1(\Omega) + W^{-1,p'}(\Omega)$ is zero measure for the sets of zero p-capacity. For the variable exponent case, using the same arguments as in [13], we feel that the similar decomposition result should be true by the properties of $L^{p(x)}(\Omega)$ and the relative $p(x)$-capacity (see [20]).

The natural framework to solve problem (1.2) is that of Sobolev spaces with variable exponent. Recent applications in elasticity [27], non-Newtonian fluid mechanics [28, 24, 7], or image processing [15], gave rise to a revival of the interest in these spaces, the origins of which can be traced back to the work of Orlicz in the 1930's. An account of recent advances, some open problems, and an extensive list of references can be found in the interesting surveys by Diening [16] and Antontsev [6] (cf. also the work of Kováčik and Rákosník [21], where many of the basic properties of these spaces are established).

The interest of the study of Lebesgue and Sobolev spaces with variable exponent lies on the fact that most materials can be modelled with sufficient accuracy using classical Lebesgue and Sobolev spaces L^p and $W^{1,p}$ where p is a fixed constant, but for some materials with inhomogeneities, for instance electrorheological fluids (sometimes referred to as "smart fluids", this is not adequate, but rather the exponent p should be able to vary (cf. [24]). These fluids are smart materials which are concentrated suspensions of polarizable particles in a non-conducting dielectric liquid. By applying an electric field, the viscosity can be changed by a factor up to 10^5, and the fluid can be transformed from liquid state into semi-solid state within milliseconds. The process is reversible. An example of electrorheological fluids are alumina Al_2O_3 particles.

It would be interesting at this work to refer the reader to the previous work in degenerated case [1]. For different approach used in the setting of Orlicz Sobolev space the reader can refer to [12] , and for same results in L^p case to [23].

The present paper is organized as follows: In section 2, we introduce a framework for function spaces. In section 3, we give our basic assumptions and we prove some fundamental lemmas concerning convergence in Sobolev spaces with variable exponent. In section 4, we prove our results and we study the positivity of solution.

2 A framework for function spaces

In this section, we define Lebesgue and Sobolev spaces with variable exponent and give some of their properties.

Let Ω be an open bounded set in $I\!R^N$ $(N \geq 2)$, we denote

$$C_+(\overline{\Omega}) = \{p | p \in C(\overline{\Omega}),\ p(x) > 1 \text{ for any } \ x \in \bar{\Omega}\},$$

For every $p \in C_+(\overline{\Omega})$ we define,

$$p_+ = \sup_{x\in\Omega} p(x) \qquad \text{and} \qquad p_- = \inf_{x\in\Omega} p(x).$$

and we define the variable exponent Lebesque space by:

$$L^{p(x)}(\Omega) = \{u | u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}.$$

We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$||u||_{p(x)} = \inf\left\{\lambda > 0, \quad \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \leq 1\right\}.$$

The variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces (Kováčik and Rákosník [21]; theorem 2.5), the Hölder inequality holds (Kováčik and Rákosník [21] ; theorem 2.1), they are reflexive if and only if $1 < p_- \leq p_+ < \infty$, (Kováčik and Rákosník [21] ; corollary 2.7) and continuous functions are dense in $L^{p(x)}$, if $p_+ < \infty$ (Kováčik and Rákosník [21] ; theorem 2.11).

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$ (see [18], [30]). For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, the Generalized Hölder inequality

$$\left|\int_\Omega u\,v\,dx\right| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|u\|_{p(x)}\,\|v\|_{p'(x)}\,,$$

holds true.

Proposition 2.1. ***(see [18],[29])***
If $f : \Omega \times I\!R \to I\!R$ is a Carathéodory function and satisfies

$$|f(x,s)| \leq a(x) + b|s|^{p_1(x)/p_2(x)} \qquad \text{for any } x \in \Omega, s \in I\!R,$$

where $p_1, p_2 \in C_+(\bar{\Omega})$, $a(x) \in L^{p_2(x)}(\Omega)$, $a(x) \geq 0$ and $b \geq 0$ is a constant, then the Nemytskii operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $(N_f(u))(x) = f(x,u(x))$ is a continuous and bounded operator.

Proposition 2.2. ***(see [18], [30])***
If we denote

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx, \quad \forall u \in L^{p(x)}(\Omega),$$

then the following assertions holds:
(i) $\|u\|_{p(x)} < 1 \quad (resp, = 1, > 1) \Leftrightarrow \rho(u) < 1 \quad (resp, = 1, > 1)$,
(ii) $\|u\|_{p(x)} > 1 \Rightarrow \quad \|u\|^{p_-}_{p(x)} \leq \rho(u) \leq \|u\|^{p_+}_{p(x)}$; $\quad \|u\|_{p(x)} < 1 \Rightarrow \quad \|u\|^{p_+}_{p(x)} \leq \rho(u) \leq \|u\|^{p_-}_{p(x)}$,
(iii) $\|u\|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho(u) \to 0$; $\|u\|_{p(x)} \to \infty \quad \Leftrightarrow \quad \rho(u) \to \infty$.

We define the variable Sobolev space by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega)\}.$$

normed by,

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)} \qquad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{for } p(x) < N, \\ \infty & \text{for } p(x) \geq N. \end{cases}$$

Proposition 2.3. ***(see [18])***
(i) *Assuming $p_- > 1$, the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*
(ii) *If $q \in C_+(\bar{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous.*

In particular, we have $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$ *is compact and continuous (for more details we refer to Theorem 8.4.2 [17]).*
(iii) *Let* $p \in C_+(\bar{\Omega})$*. Then, for* $u \in W_0^{1,p(x)}(\Omega)$*, the* $p(x)$*-Poincaré inequality*

$$||u||_{p(x)} \leq C\,||\nabla u||_{p(x)}$$

holds, where the positive constant C *depends on* $p(x)$ *and* Ω*.*

Remark 2.4. By (iii) of Proposition 2.3, we know that $||\nabla u||_{p(x)}$ and $||u||_{1,p(x)}$ are equivalent norms on $W_0^{1,p(x)}$.

3 Basic assumptions and some fundamental Lemmas

Let $p \in C_+(\bar{\Omega})$ such that $1 < p_- \leq p(x) \leq p_+ < \infty$, and denote

$$Au = -\mathrm{div}(a(x,u,\nabla u)),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following assumptions :

$$|a(x,s,\xi)| \leq \beta[k(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}], \tag{3.1}$$

$$[a(x,s,\xi) - a(x,s,\eta)](\xi - \eta) > 0 \text{ for all } \ \xi \neq \eta \in \mathbb{R}^N, \tag{3.2}$$

$$a(x,s,\xi)\xi \geq \alpha|\xi|^{p(x)}, \tag{3.3}$$

for a.e. $x \in \Omega$, all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$,
where $k(x)$ is a positive function lying in $L^{p'(x)}(\Omega)$ and $\beta, \alpha > 0$.
Assume that $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \longmapsto \mathbb{R}$ is a Carathéodory function satisfying :

$$g(x,s,\xi).s \geq 0, \tag{3.4}$$

$$|g(x,s,\xi)| \leq b(|s|)(c(x) + |\xi|^{p(x)}), \tag{3.5}$$

where $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive increasing function and $c(x)$ is a positive function which belong to $L^1(\Omega)$. Furthermore, we suppose that

$$\mu = f - \mathrm{div}F, \ f \in L^1(\Omega) \text{ and } F \in (L^{p'(x)}(\Omega))^N, \tag{3.6}$$

We introduce the functional spaces, we will need later.
For $p \in C_+(\bar{\Omega})$ such that $1 < p_- \leq p(x) \leq p_+ < \infty$, $\mathcal{T}_0^{1,p(x)}(\Omega)$ is defined as the set of measurable functions $u : \Omega \to \mathbb{R}$ such that for $k > 0$ the truncated functions $T_k(u) \in W_0^{1,p(x)}(\Omega)$.

We give the following lemma which is a generalization of Lemma 2.1 [11] in Sobolev spaces with variable exponent. Note that its proof is a slight modification of the previous lemma.

Lemma 3.1. *For every* $u \in \mathcal{T}_0^{1,p(x)}(\Omega)$*, there exists a unique measurable function* $v : \Omega \to \mathbb{R}^N$ *such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}}, \ \textit{a.e. in } \Omega, \textit{ for every } k > 0.$$

where χ_E *denotes the characteristic function of a measurable set E. Moreover, if* u *belongs to* $W_0^{1,1}$*, then* v *coincides with the standard distributional gradient of* u*, and we will denote it by* $v = \nabla u$

Proof The result follows from ([5], Theorem 1.5), since

$$T_k(u) \in W_0^{1,p(x)}(\Omega) \subset W_0^{1,p_-}(\Omega), \text{ for all } k > 0.$$

Lemma 3.2. *Let $\lambda \in I\!R$ and let u and v be two functions which are finite almost everywhere, and which belong to $\mathcal{T}_0^{1,p(x)}(\Omega)$. Then,*

$$\nabla(u+\lambda v) = \nabla u + \lambda \nabla v \text{ a.e. in } \Omega,$$

where ∇u, ∇v and $\nabla(u+\lambda v)$ are the gradients of u, v and $u+\lambda v$ introduced in Lemma 3.1.

Proof Let $E_n = \{|u| < n\} \cap \{|v| < n\}$. On E_n, we have $T_n(u) = u$ and $T_n(v) = v$, so that for every $k > 0$,

$$T_k(T_n(u) + \lambda T_n(v)) = T_k(u+\lambda v) \text{ a.e. in } E_n,$$

and therefore, since both functions belong to $W_0^{1,p(x)}(\Omega)$,

$$\nabla T_k(T_n(u) + \lambda T_n(v)) = \nabla T_k(u+\lambda v) \text{ a.e. in } E_n. \tag{3.7}$$

Since $T_n(u)$ and $T_n(v)$ belong to $W_0^{1,p(x)}(\Omega)$, we have, using a classical property of the truncated functions in $W_0^{1,p(x)}$, and the definition of ∇u and ∇v,

$$\begin{aligned} \nabla T_k(T_n(u) + \lambda T_n(v)) &= \chi_{\{|T_n(u)+\lambda T_n(v)| \le k\}} (\nabla T_n(u) + \lambda \nabla T_n(v)) \\ &= \chi_{\{|T_n(u)+\lambda T_n(v)| \le k\}} (\chi_{\{|u| \le n\}} \nabla u + \lambda \chi_{\{|v| \le n\}} \nabla v) \quad \text{a.e. in } \Omega. \end{aligned}$$

Therefore

$$\nabla T_k(T_n(u) + \lambda T_n(v)) = \chi_{\{|u+\lambda v| \le k\}} (\nabla u + \lambda \nabla v) \text{ a.e. in } E_n. \tag{3.8}$$

On the other hand, by definition of $\nabla(u+\lambda v)$,

$$\nabla T_k(u+\lambda v) = \chi_{\{|u+\lambda v| \le k\}} \nabla(u+\lambda v) \text{ a.e. in } E_n. \tag{3.9}$$

Putting together (3.7), (3.8) and (3.9), we obtain

$$\chi_{\{|u+\lambda v| \le k\}} \nabla(u+\lambda v) = \chi_{\{|u+\lambda v| \le k\}} (\nabla u + \lambda \nabla v) \text{ a.e. in } E_n. \tag{3.10}$$

Since $\bigcup_{n \in I\!N} E_n$ at most differs from Ω by a set of zero Lebesgue measure (since u and v are almost everywhere finite), (3.10) also holds almost everywhere in Ω. Since $\bigcup_{k \in I\!N} \{|u+\lambda v| \le k\}$ at most differs from Ω by a set of zero Lebesgue measure, we have proved Lemma 3.2.

The symbole $\rightharpoonup$ denote the weak convergence.

Lemma 3.3. *[8] Let $g \in L^{r(x)}(\Omega)$ and $g_n \in L^{r(x)}(\Omega)$ with $\|g_n\|_{L^{r(x)}(\Omega)} \le C$ for $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ in $L^{r(x)}(\Omega)$.*

Lemma 3.4. *[8] Assume that (3.1), (3.2) and (3.3) hold, and let $(u_n)_n$ be a sequence in $W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and*

$$\int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)] \nabla(u_n - u)\,dx \to 0. \tag{3.11}$$

Then, $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$.

Lemma 3.5. *Let $F : I\!R \longrightarrow I\!R$ be uniformly Lipschitzian with $F(0) = 0$ and $p \in C_+(\overline{\Omega})$. Let $u \in W_0^{1,p(x)}(\Omega)$. Then $F(u) \in W_0^{1,p(x)}(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial(F \circ u)}{\partial x_i} = \begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} & \text{a.e. in} \quad \{x \in \Omega : \ u(x) \notin D\}, \\ \\ 0 & \text{a.e. in} \quad \{x \in \Omega : \ u(x) \in D\}. \end{cases}$$

Remark 3.6. The previous lemma is a generalization of the corresponding in ([19], pp. 151-152), where $p(x) = p = cte$, $F \in C^1(I\!R)$ and $F' \in L^\infty(I\!R)$, and of the corresponding one in [10], where $p(x) = p = cte$, $w \equiv w_1 \equiv w_2 \equiv \cdots \equiv w_N \equiv 1$ is some weight function, $F \in C^1(I\!R)$ and $F' \in L^\infty(I\!R)$. Also note that the previous lemma implies that functions in $W_0^{1,p(x)}(\Omega)$ can be truncated.

Proof Consider firstly the case $F \in C^1(\Omega)$ and $F' \in L^\infty(\Omega)$.

Let u in $W_0^{1,p(x)}(\Omega)$. Since $\overline{C_0^\infty(\Omega)}^{W^{1,p(x)}(\Omega)} = W_0^{1,p(x)}(\Omega)$, then there exists a sequence u_n of elements of $C_0^\infty(\Omega)$ such that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. Passing to a subsequence, we can assume that $u_n \to u$ a.e. in Ω and $\nabla u_n \to \nabla u$ a.e. in Ω.
Then,

$$F(u_n) \to F(u) \ \text{ a.e. in } \ \Omega. \tag{3.12}$$

On the other hand, from the relation

$$|F(u_n)| = |F(u_n) - F(0)| \leq \|F'\|_\infty |u_n| \tag{3.13}$$

we obtain

$$|F(u_n)|^{p(x)} \leq (1 + \|F'\|_\infty)^{p_+} |u_n|^{p(x)} \quad \text{and} \quad \left|\frac{\partial F}{\partial x_i}(u_n)\right|^{p(x)} = \left|F'(u_n)\frac{\partial u_n}{\partial x_i}\right|^{p(x)} \leq M\left|\frac{\partial u_n}{\partial xi}\right|^{p(x)}, \tag{3.14}$$

for some constant M which does not depend on $p(x)$.
Then, we deduce that $F(u_n)$ remains bounded in $W_0^{1,p(x)}(\Omega)$. Thus, going to a further subsequence, we obtain

$$F(u_n) \rightharpoonup v \ \text{ in } W_0^{1,p(x)}(\Omega) \tag{3.15}$$

According to the proposition 2.3, $F(u_n) \to v$ in $L^{p(x)}(\Omega)$

$$F(u_n) \to v \ \text{ a.e. in } \ \Omega \tag{3.16}$$

Thanks to (3.12), (3.15) and (3.16) we conclude that

$$v = F(u) \in W_0^{1,p(x)}(\Omega).$$

We now turn our attention to the general case. Taking convolutions with a regularizing sequence ρ_n in $I\!R$, we have $F_n = F * \rho_n$, $F_n \in C^1(I\!R)$ and $F'_n \in L^\infty(I\!R)$.
Then, by the first case we have $F_n(u) \in W_0^{1,p(x)}(\Omega)$. Since $F_n \to F$ uniformly in every compact, we have $F_n(u) \to F(u)$ a.e. in Ω. On the other hand , $F_n(u)$ is bounded in $W_0^{1,p(x)}(\Omega)$, then $F_n(u) \rightharpoonup \overline{v}$ in $W_0^{1,p(x)}(\Omega)$ and a.e. in Ω (due to the proposition 2.3), hence

$$\overline{v} = F(u) \in W_0^{1,p(x)}(\Omega).$$

The following lemma follow from the previous lemma.

Lemma 3.7. *Let $u \in W_0^{1,p(x)}(\Omega)$. Then $T_k(u) \in W_0^{1,p(x)}(\Omega)$, with $k > 0$. Moreover, we have $T_k(u) \to u$ in $W_0^{1,p(x)}(\Omega)$ as $k \to \infty$.*

Proof Let $k > 0$,

$$\begin{array}{rccl} T_k : & \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ & s & \longmapsto & T_k(s) = \begin{cases} s & \text{if} \quad |s| \le k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k. \end{cases} \end{array}$$

Since T_k is a uniformly Lipschitzian function and $T_k(0) = 0$, then by Lemma 3.5 we have $T_k(u) \in W_0^{1,p(x)}(\Omega)$, and

$$\begin{aligned} &\int_\Omega |T_k(u) - u|^{p(x)} dx + \int_\Omega |\nabla T_k(u) - \nabla u|^{p(x)} dx \\ &= \int_{\{|u| \le k\}} |T_k(u) - u|^{p(x)} dx + \int_{\{|u| > k\}} |T_k(u) - u|^{p(x)} dx \\ &\quad + \int_{\{|u| \le k\}} |\nabla T_k(u) - \nabla u|^{p(x)} + \int_{\{|u| > k\}} |\nabla T_k(u) - \nabla u|^{p(x)} dx \\ &= \int_{\{|u| > k\}} |T_k(u) - u|^{p(x)} dx + \int_{\{|u| > k\}} |\nabla u|^{p(x)} dx. \end{aligned}$$

Since $T_k(u) \to u$ as $k \to \infty$, using the dominated convergence theorem, we have

$$\int_{\{|u| > k\}} |T_k(u) - u|^{p(x)} dx + \int_{\{|u| > k\}} |\nabla u|^{p(x)} dx \to 0 \text{ as } k \to \infty.$$

Finally $\|T_k(u) - u\|_{W_0^{1,p(x)}(\Omega)} \to 0$ as $k \to \infty$.

Definition 3.8. Let Y be a reflexive Banach space, a bounded operator B from Y to its dual Y^* is called pseudo-monotone if

$$\left. \begin{array}{l} u_n \rightharpoonup u \text{ in } Y \\ Bu_n \rightharpoonup \chi \text{ in } Y^* \\ \limsup\limits_{n \to \infty} \langle Bu_n, u_n \rangle \le \langle \chi, u \rangle \end{array} \right\} \Longrightarrow \chi = Bu \text{ and } \langle Bu_n, u_n \rangle \to \langle \chi, u \rangle.$$

4 Statement of the result

Consider the nonlinear problem with Dirichlet boundary condition

$$(\mathcal{P}) \begin{cases} Au + g(x, u, \nabla u) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We are now in a position to define the notion of entropy solution.

Definition 4.1. A function $u \in \mathcal{T}_0^{1,p(x)}(\Omega)$ is called an entropy solution of the Dirichlet problem $(\mathcal{P})$ if,

$$\int_\Omega a(x, u, \nabla u) \nabla T_k(u - v)\, dx + \int_\Omega g(x, u, \nabla u) T_k(u - v)\, dx \le \int_\Omega T_k(u - v)\, d\mu. \tag{4.1}$$

for every $v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ and for every $k > 0$ and $g(x, u, \nabla u) \in L^1(\Omega)$.

We shall prove the following existence theorem

4.1 Quasilinear $p(x)$−problem with right-hand side measure

First of all we write $\mu = f - \mathrm{div}F$, with $f \in L^1(\Omega)$ and $F \in (L^{p'(x)}(\Omega))^N$.
We obtain the following problem

$$(\mathcal{P}')\begin{cases} u \in \mathcal{T}_0^{1,p(x)}(\Omega), \qquad g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \\ \qquad\qquad \displaystyle\leq \int_\Omega fT_k(u-v)\,dx + \int_\Omega F\nabla T_k(u-v)\,dx \\ \forall v \in W_0^{1,p(x)}(\Omega)\cap L^\infty(\Omega) \quad \forall k>0. \end{cases}$$

Theorem 4.2. *Let assumptions* (3.1) − (3.3) *hold true and let* $g(x,s,\xi)$ *satisfy* (3.4) − (3.5). *Then for every* $\mu \in L^1(\Omega) + (L^{p'(x)}(\Omega))^N$ *there exists at least one entropy solution of the problem* $(\mathcal{P}')$.

Remark 4.3. (1) If $p(x) = p = cte$, the result of the above theorem coincides with the analogous one in [23].
(2) Theorem 4.2, generalizes to Sobolev spaces with variable exponent the analogous statement in [2] (in the non degenerated case).

4.2 Proof of Theorem 4.2

In order to prove the existence result of theorem 4.2, we need the following:

STEP 1. Quasilinear variational problem

Let $(f_n)_n$ be a sequence of smooth functions such that $f_n \to f$ in $L^1(\Omega)$ and $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$.
We consider the sequence of the approximate problems :

$$(\mathcal{P}_n)\begin{cases} u_n \in W_0^{1,p(x)}(\Omega) \\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\nabla v dx + \int_\Omega g_n(x,u_n,\nabla u_n)v dx = \int_\Omega f_n v dx + \int_\Omega F\nabla v dx \\ \forall v \in W_0^{1,p(x)}(\Omega). \end{cases} \tag{4.2}$$

where $g_n(x,s,\xi) = \dfrac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$.
Note that $g_n(x,s,\xi)$ satisfies the following conditions

$$g_n(x,s,\xi).s \geq 0, \qquad |g_n(x,s,\xi)| \leq |g(x,s,\xi)| \text{ and } |g_n(x,s,\xi)| \leq n.$$

We define the operator $G_n : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ by,

$$\langle G_n u, v\rangle = \int_\Omega g_n(x,u,\nabla u)v\,dx,$$

and

$$\langle Au, v\rangle = \int_\Omega a(x,u,\nabla u)\nabla v\,dx,$$

Thanks to Hölder's inequality, we have for all $u,v \in W_0^{1,p(x)}(\Omega)$,

$$\begin{aligned} \left| \int_\Omega g_n(x,u,\nabla u)v\,dx \right| &\leq (\frac{1}{p_-} + \frac{1}{p'_-})\|g_n(x,u,\nabla u)\|_{p'(x)}\|v\|_{p(x)}, \\ &\leq (\frac{1}{p_-} + \frac{1}{p'_-})(\int_\Omega g_n(x,u,\nabla u)^{p'(x)}\,dx + 1)^{\frac{1}{p'_-}}\|v\|_{p(x)}, \\ &\leq (\frac{1}{p_-} + \frac{1}{p'_-})n^{\frac{p_+}{p_-}}(meas(\Omega)+1)^{\frac{1}{p'_-}}\|v\|_{p(x)}, \\ &\leq C_n\|v\|_{1,p(x)}, \end{aligned} \tag{4.3}$$

for every fixed n.

Lemma 4.4. *The operator $B_n = A + G_n$ from $W_0^{1,p(x)}(\Omega)$ into $W^{-1,p'(x)}(\Omega)$ is pseudo-monotone, Moreover, B_n is coercive, in the following sence:*

$$\frac{\langle B_n v, v\rangle}{\|v\|_{1,p(x)}} \to +\infty \qquad if \quad \|v\|_{1,p(x)} \to +\infty, \qquad \forall v \in W_0^{1,p(x)}(\Omega).$$

Proof of Lemma 4.4

Using Hölder's inequality and the growth condition (3.1) we can show that A is bounded, and by (4.3), we have B_n bounded in $W_0^{1,p(x)}(\Omega)$. The coercivity follows from (3.3) and (3.4). It remain to show that B_n is pseudo-monotone.
Let $(u_k)_k$ be a sequence in $W_0^{1,p(x)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u \quad \text{in } W_0^{1,p(x)}(\Omega), \\ B_n u_k \rightharpoonup \chi \text{ in } W^{-1,p'(x)}(\Omega), \\ \limsup_{k\to\infty}\langle B_n u_k, u_k\rangle \leq \langle \chi, u\rangle. \end{cases} \tag{4.4}$$

We will prove that

$$\chi = B_n u \ \text{ and } \ \langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle \ \text{ as } \ k \to +\infty.$$

Firstly, since $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$, then

$$u_k \to u \text{ in } L^{p(x)}(\Omega) \tag{4.5}$$

for a subsequence denoted again by $(u_k)_k$.
Since $(u_k)_k$ is a bounded sequence in $W_0^{1,p(x)}(\Omega)$, then by (3.1) $(a(x,u_k,\nabla u_k))_k$ is bounded in $(L^{p'(x)}(\Omega))^N$, therefore there exists a function $\varphi \in (L^{p'(x)}(\Omega))^N$ such that

$$a(x,u_k,\nabla u_k) \rightharpoonup \varphi \ \text{ in } \ (L^{p'(x)}(\Omega))^N \ \text{ as } \ k \to \infty. \tag{4.6}$$

Similarly, it is easy to see that $(g_n(x,u_k,\nabla u_k))_k$ is bounded in $L^{p'(x)}(\Omega)$ with respect to k, then there exist a function $\psi_n \in L^{p'(x)}(\Omega)$ such that

$$g_n(x,u_k,\nabla u_k) \rightharpoonup \psi_n \ \text{ in } \ L^{p'(x)}(\Omega) \ \text{ as } \ k \to \infty. \tag{4.7}$$

It is clear that, for all $v \in W_0^{1,p(x)}(\Omega)$, we get

$$\begin{aligned}\langle \chi, v\rangle &= \lim_{k\to\infty}\langle B_n u_k, v\rangle,\\ &= \lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla v\,dx + \lim_{k\to\infty}\int_\Omega g_n(x,u_k,\nabla u_k)v\,dx.\\ &= \int_\Omega \varphi\nabla v\,dx + \int_\Omega \psi_n v\,dx.\end{aligned} \tag{4.8}$$

On the one hand, by (4.5) we have

$$\int_\Omega g_n(x,u_k,\nabla u_k)u_k\,dx \to \int_\Omega \psi_n u\,dx \;\text{ as }\; k\to\infty. \tag{4.9}$$

and by (4.4) and (4.8), we have

$$\begin{aligned}\limsup_{k\to\infty}\langle B_n(u_k),u_k\rangle &= \limsup_{k\to\infty}\Big\{\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx + \int_\Omega g_n(x,u_k,\nabla u_k)u_k\,dx\Big\},\\ &\qquad\le \int_\Omega \varphi\nabla u\,dx + \int_\Omega \psi_n u\,dx.\end{aligned} \tag{4.10}$$

Therefore

$$\limsup_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx \le \int_\Omega \varphi\nabla u\,dx. \tag{4.11}$$

Thanks to (3.2), we have

$$\int_\Omega (a(x,u_k,\nabla u_k) - a(x,u_k,\nabla u))(\nabla u_k - \nabla u)\,dx > 0. \tag{4.12}$$

Then

$$\begin{aligned}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx \ge -\int_\Omega a(x,u_k,\nabla u)\nabla u\,dx&\\ +\int_\Omega a(x,u_k,\nabla u_k)\nabla u\,dx + \int_\Omega a(x,u_k,\nabla u)\nabla u_k\,dx,&\end{aligned}$$

By (4.6), we get

$$\liminf_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx \ge \int_\Omega \varphi\nabla u\,dx.$$

This implies by using (4.11)

$$\lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx = \int_\Omega \varphi\nabla u\,dx. \tag{4.13}$$

By means of (4.8), (4.9) and (4.13), we obtain

$$\langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle \;\text{ as }\; k\to+\infty.$$

On the other hand, by (4.13), we can deduce that

$$\lim_{k\to+\infty}\int_\Omega (a(x,u_k,\nabla u_k) - a(x,u_k,\nabla u))(\nabla u_k - \nabla u)\,dx = 0,$$

and so, by virtue of Lemma 3.4

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega.$$

We conclude that

$$a(x,u_k,\nabla u_k) \rightharpoonup a(x,u,\nabla u) \text{ in } (L^{p'(x)}(\Omega))^N,$$

and

$$g_n(x,u_k,\nabla u_k) \rightharpoonup g_n(x,u,\nabla u) \text{ in } L^{p'(x)}(\Omega).$$

Which implies that $\chi = B_n u$.
Finally, by using the classical theorem in [22] and as a conclusion of this step, there exists at least one solution $u_n \in W_0^{1,p(x)}(\Omega)$ of the problem $(\mathcal{P}_n)$.

STEP 2. Estimates on the sequences $\{\nabla T_k(u_n)\}$, $\{u_n\}$.

Assertion 1. We will show that $\nabla T_k(u_n)$ is bounded in $L^{p(x)}(\Omega)$.
If we take $T_k(u_n)$ as test function in (4.2), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)\,dx = \int_\Omega f_n T_k(u_n)\,dx + \int_\Omega F\nabla T_k(u_n)\,dx.$$

Using the fact that $g_n(x,u_n,\nabla u_n)T_k(u_n) \geq 0$ and by (3.3) and Young's inequality, we have

$$\begin{aligned}
\alpha\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx &\leq \int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)\,dx,\\
&\leq \int_\Omega f_n T_k(u_n)\,dx + \int_\Omega F\nabla T_k(u_n)\,dx,\\
&\leq k\int_\Omega |f_n|\,dx + \int_\Omega \frac{F}{(\frac{\alpha}{2}p(x))^{\frac{1}{p(x)}}}\Big(\Big(\frac{\alpha}{2}p(x)\Big)^{\frac{1}{p(x)}}\nabla T_k(u_n)\Big)\,dx,\\
&\leq k\|f_n\|_{L^1(\Omega)} + \int_\Omega \frac{|F|^{p'(x)}}{p'(x)(\frac{\alpha}{2}p(x))^{\frac{p'(x)}{p(x)}}}\,dx + \int_\Omega \frac{\frac{\alpha}{2}p(x)|\nabla T_k(u_n)|^{p(x)}}{p(x)}\,dx,\\
&\leq k\|f\|_{L^1(\Omega)} + C_0\int_\Omega |F|^{p'(x)}\,dx + \frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx,
\end{aligned}$$

where $C_0 = p'_+(\frac{\alpha}{2}p_+)^{\frac{p'_+}{p_-}} = p'_+\exp\Big(\frac{p'_+}{p_-}\ln(\frac{\alpha}{2}p_+)\Big)$, then

$$\frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx \leq k\|f\|_{L^1(\Omega)} + C_1.$$

This implies that by Proposition 2.2 we get

$$\begin{aligned}
\frac{\alpha}{2}\|\nabla T_k(u_n)\|^{\gamma}_{p(x)} &\leq k\|f\|_{L^1(\Omega)} + C_1\\
&\leq C_2 k \text{ for all } \quad k > 1
\end{aligned} \tag{4.14}$$

with

$$\gamma = \begin{cases} p_+ & \text{if} \quad \|\nabla T_k(u_n)\|_{p(x)} \leq 1,\\ p_- & \text{if} \quad \|\nabla T_k(u_n)\|_{p(x)} > 1,\end{cases}$$

Assertion 2. We prove that u_n converges to some function u in measure.
To prove this, we show that u_n is a Cauchy sequence in measure.
Let k be large enough. Combining Poincaré's inequality and (4.14), one has

$$\begin{aligned} k\ \text{meas}(\{|u_n|>k\}) &= \int_{\{|u_n|>k\}} |T_k(u_n)|\,dx \leq \int_\Omega |T_k(u_n)|\,dx, \\ &\leq C_2' \|\nabla T_k(u_n)\|_{p(x)} \\ &\leq C_3 k^{\frac{1}{\gamma}} \end{aligned} \tag{4.15}$$

Which yields,

$$\text{meas}(\{|u_n|>k\}) \leq \frac{C_3}{k^{1-\frac{1}{\gamma}}} \quad \forall k>1. \tag{4.16}$$

then

$$\text{meas}(\{|u_n|>k\}) \to 0 \text{ as } k \to +\infty \ \text{ since } 1-\frac{1}{\gamma} > 1.$$

Moreover, for every fixed $\delta > 0$ and every positive k ,we know that

$$\{|u_n-u_m|>\delta\} \subset \{|u_n|>k\} \cup \{|u_m|>k\} \cup \{|T_k(u_n)-T_k(u_m)|>\delta\},$$

and hence

$$\begin{aligned} \text{meas}\ (\{|u_n-u_m|>\delta\}) \leq \text{meas}\ (\{|u_n|>k\}) + \text{meas}\ (\{|u_m|>k\}) \\ +\text{meas}\ (\{|T_k(u_n)-T_k(u_m)|>\delta\}). \end{aligned} \tag{4.17}$$

Since $(T_k(u_n))_n$ is bounded in $W_0^{1,p(x)}(\Omega)$, then there exists some $v_k \in W_0^{1,p(x)}(\Omega)$ such that

$$T_k(u_n) \rightharpoonup v_k \qquad \text{in} \quad W_0^{1,p(x)}(\Omega)$$

and by the compact imbedding, we have

$$T_k(u_n) \to v_k \qquad \text{in} \quad L^{p(x)}(\Omega) \quad \text{ and a.e. in } \Omega.$$

Consequently, we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Ω.
Let $\varepsilon > 0$. Then by (4.16) and (4.17), there exists some $k(\varepsilon) > 0$ such that $\text{meas}(\{|u_n - u_m| > \delta\}) < \varepsilon$ for all $n, m \geq n_0(k(\varepsilon), \delta)$. This proves that $(u_n)_n$ is a Cauchy sequence in measure, thus converges almost everywhere to some measurable function u. Then

$$\begin{aligned} T_k(u_n) &\rightharpoonup T_k(u) \quad \text{in } W_0^{1,p(x)}(\Omega), \\ T_k(u_n) &\to T_k(u) \quad \text{in} \quad L^{p(x)}(\Omega) \ \text{ and a.e. in } \Omega. \end{aligned} \tag{4.18}$$

STEP 3. Strong convergence of truncations.

We fix $k > 0$, and let $h > k$.
We shall use in (4.2) the test function

$$\begin{cases} v_n &= \varphi(\omega_n) \\ \omega_n &= T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \end{cases} \tag{4.19}$$

with $\varphi(s) = se^{\lambda s^2}$, $\lambda = (\frac{b(k)}{\alpha})^2$.
It is well known that ([14], lemma 1),

$$\varphi'(s) - \frac{b(k)}{\alpha}|\varphi(s)| \geq \frac{1}{2}, \quad \forall s \in \mathbb{R}. \tag{4.20}$$

It follows that,

$$\begin{aligned} \int_\Omega a(x,u_n,\nabla u_n)\nabla\omega_n\varphi'(\omega_n)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)\varphi(\omega_n)\,dx \\ = \int_\Omega f_n\varphi(\omega_n)\,dx + \int_\Omega F\nabla\varphi(\omega_n)\,dx. \end{aligned} \tag{4.21}$$

Since $g_n(x,u_n,\nabla u_n)\varphi(\omega_n) > 0$ on the subset $\{x \in \Omega,\ |u_n(x)| > k\}$ (because they have the same sign on this subset), then by (4.21), we deduce that,

$$\begin{aligned} \int_\Omega a(x,u_n,\nabla u_n)\nabla\omega_n\varphi'(\omega_n)\,dx + \int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\varphi(\omega_n)\,dx \\ \leq \int_\Omega f_n\varphi(\omega_n)\,dx + \int_\Omega F\nabla\varphi(\omega_n)\,dx. \end{aligned} \tag{4.22}$$

Denote by $\varepsilon_h^1(n), \varepsilon_h^2(n), ...$ various sequences of real numbers which converge to zero as n tends to infinity for any fixed value of h.
We will deal with each term of (4.22). First of all, observe that,

$$\int_\Omega f_n\varphi(\omega_n)\,dx = \int_\Omega f\varphi(T_{2k}(u - T_h(u)))\,dx + \varepsilon_h^1(n), \tag{4.23}$$

and

$$\int_\Omega F\nabla\varphi(\omega_n)\,dx = \int_\Omega F\nabla T_{2k}(u - T_h(u))\varphi'(T_{2k}(u - T_h(u)))\,dx + \varepsilon_h^2(n). \tag{4.24}$$

Splitting the first integral on the left hand side of (4.22), where $|u_n| \leq k$ and $|u_n| > k$ we can write,

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\nabla\omega_n\varphi'(\omega_n)\,dx \\ &= \int_{\{|u_n|\leq k\}} a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(u)]\varphi'(\omega_n)\,dx \\ &+ \int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla\omega_n\varphi'(\omega_n)\,dx. \end{aligned} \tag{4.25}$$

Choosing $M = 4k + h$, using $a(x,s,\xi)\xi \geq 0$ and the fact that $\nabla\omega_n = 0$ on the set $\{|u_n| > M\}$, we have

$$\begin{aligned} \int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla\omega_n\varphi'(\omega_n)\,dx \\ \geq -\varphi'(2k)\int_{\{|u_n|>k\}} |a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|\,dx, \end{aligned} \tag{4.26}$$

and since $a(x,s,0)=0\ \ \forall s\in I\!R$, we have

$$\int_{\{|u_n|\le k\}} a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(u)]\varphi'(\omega_n)\,dx \\ = \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(u)]\varphi'(\omega_n)\,dx. \tag{4.27}$$

Combining (4.26) and (4.27), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla\omega_n\varphi'(\omega_n)\,dx \ \ \ge \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(u)]\varphi'(\omega_n)\,dx \\ -\varphi'(2k)\int_{\{|u_n|>k\}} |a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|\,dx. \tag{4.28}$$

The second term of the right hand side of the last inequality tends to 0 as n tends to infinity. Indeed, since the sequence $(a(x,T_M(u_n),\nabla T_M(u_n)))_n$ is bounded in $(L^{p'(x)}(\Omega))^N$ while $\nabla T_k(u)\chi_{|u_n|>k}$ tends to 0 in $(L^{p(x)}(\Omega))^N$ strongly, which yields

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla\omega_n\varphi'(\omega_n)\,dx \\ \ge \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(u)]\varphi'(\omega_n)\,dx+\varepsilon_h^3(n). \tag{4.29}$$

On the other hand, the term of the right hand side of (4.29) reads as,

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(u)]\varphi'(\omega_n)\,dx \\ = \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))] \\ \times[\nabla T_k(u_n)-\nabla T_k(u)]\varphi'(\omega_n)\,dx \\ +\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u_n)\varphi'(T_k(u_n)-T_k(u))\,dx \\ -\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u)\varphi'(w_n)\,dx. \tag{4.30}$$

since $a(x,T_k(u_n),\nabla T_k(u))\varphi'(T_k(u_n)-T_k(u))\to a(x,T_k(u),\nabla T_k(u))\varphi'(0)$ in $(L^{p'(x)}(\Omega))^N$ by using the continuity of Nemytskii's operator, while $\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)$ in $(L^{p(x)}(\Omega))^N$, we have

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u_n)\varphi'(T_k(u_n)-T_k(u))\,dx \\ = \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\varphi'(0)\ dx+\varepsilon_h^4(n). \tag{4.31}$$

In the same way, we have

$$-\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u)\varphi'(w_n)\,dx \\ = -\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\varphi'(0)\ dx+\varepsilon_h^5(n). \tag{4.32}$$

Combining (4.29)-(4.32), we get

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla\omega_n\varphi'(\omega_n)\,dx \\ \geq \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))] \\ \times[\nabla T_k(u_n)-\nabla T_k(u)]\varphi'(\omega_n)\,dx+\varepsilon_h^6(n).\end{aligned}\tag{4.33}$$

The second term of the left hand side of (4.22),
can be estimated as by using (3.5) and (3.3),

$$\begin{aligned}\Big|\int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\varphi(\omega_n)\,dx\Big| &\leq \int_{\{|u_n|\leq k\}} b(k)(c(x)+|\nabla T_k(u_n)|^{p(x)})|\varphi(\omega_n)|\,dx,\\ &\leq b(k)\int_\Omega c(x)|\varphi(\omega_n)|\,dx \\ &\quad+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi(\omega_n)|\,dx,\end{aligned}\tag{4.34}$$

since $c(x)$ belongs to $L^1(\Omega)$, it is easy to see that,

$$b(k)\int_\Omega c(x)|\varphi(\omega_n)|\,dx = b(k)\int_\Omega c(x)|\varphi(T_{2k}(u-T_h(u)))|\,dx+\varepsilon_h^7(n).\tag{4.35}$$

On the other side, we have

$$\begin{aligned}\int_\Omega & a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi(\omega_n)|\,dx \\ &= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))] \\ &\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)]|\varphi(\omega_n)|\,dx \\ &\quad+\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)|\varphi(\omega_n)|\,dx \\ &\quad+\int_\Omega a(x,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n)-\nabla T_k(u)]|\varphi(\omega_n)|\,dx.\end{aligned}\tag{4.36}$$

As above, by letting n tends to infinity, we can easily see that each one of the last two integrals in the right hand side of the last equality is of the form $\varepsilon_h^8(n)$ and then

$$\begin{aligned}\Big|\int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\varphi(\omega_n)\,dx\Big| &\leq \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))] \\ &\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)]|\varphi(\omega_n)|\,dx \\ &\quad+b(k)\int_\Omega c(x)|\varphi(T_{2k}(u-T_h(u)))|\,dx+\varepsilon_h^9(n)\end{aligned}\tag{4.37}$$

Combining(4.22) – (4.24), (4.33) and (4.37), we obtain

$$\begin{aligned}\int_\Omega & [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))] \\ &\times[\nabla T_k(u_n)-\nabla T_k(u)](\varphi'(\omega_n)-\tfrac{b(k)}{\alpha}|\varphi(\omega_n)|)\,dx \\ &\leq b(k)\int_\Omega c(x)|\varphi(T_{2k}(u-T_h(u)))|\,dx+\int_\Omega f\varphi(T_{2k}(u-T_h(u)))\,dx, \\ &\quad+\int_\Omega F\nabla T_{2k}(u-T_h(u))\varphi'(T_{2k}(u-T_h(u)))\,dx+\varepsilon_h^{10}(n),\end{aligned}\tag{4.38}$$

which together with (4.20) imply that,

$$\begin{aligned}&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\,dx\\&\qquad\le 2b(k)\int_\Omega c(x)|\varphi(T_{2k}(u-T_h(u)))|\,dx+2\int_\Omega f\varphi(T_{2k}(u-T_k(u)))\,dx,\\&\qquad\quad+2\int_\Omega F\nabla T_{2k}(u-T_h(u))\varphi'(T_{2k}(u-T_h(u)))\,dx+\varepsilon_h^{11}(n),\end{aligned}\tag{4.39}$$

We can pass to the limit as $n\to+\infty$ in the last inequality and obtain,

$$\begin{aligned}&\limsup_{n\to\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\,dx\\&\qquad\le 2b(k)\int_\Omega c(x)|\varphi(T_{2k}(u-T_h(u)))|\,dx+2\int_\Omega f\varphi(T_{2k}(u-T_k(u)))\,dx,\\&\qquad\quad+2\int_\Omega F\nabla T_{2k}(u-T_h(u))\varphi'(T_{2k}(u-T_h(u)))\,dx.\end{aligned}\tag{4.40}$$

We are going to prove that all terms on the right-hand side of (4.40) converges to 0 as h goes to infinity. The only difficulty that exists is in the last term. For the two first terms it suffices to apply Lebesque's theorem.
We deal with this term. Let us observe that, if we take $\varphi(T_{2k}(u_n-T_k(u_n)))$ as test function in (4.2), we obtain

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla\varphi(T_{2k}(u_n-T_h(u_n)))\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)\varphi(T_{2k}(u_n-T_h(u_n)))\,dx\\&\qquad\le\int_\Omega f_n\varphi(T_{2k}(u_n-T_h(u_n)))\,dx+\int_\Omega F\nabla u_n\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx,\end{aligned}\tag{4.41}$$

and using (3.3) and the sign condition (3.4), we obtain

$$\begin{aligned}&\alpha\int_{\{h\le|u_n|\le 2k+h\}}|\nabla u_n|^{p(x)}\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx\\&\qquad\le\int_\Omega f_n\varphi(T_{2k}(u_n-T_h(u_n)))\,dx\\&\qquad\quad+\int_{\{h\le|u_n|\le 2k+h\}}F\nabla u_n\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx.\end{aligned}\tag{4.42}$$

Using the Young inequality, we have

$$\begin{aligned}&\int_{\{h\le|u_n|\le 2k+h\}}F\nabla u_n\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx,\\&=\int_{\{h\le|u_n|\le 2k+h\}}\left(\frac{F}{(\frac{\alpha}{2}p(x))^{\frac{1}{p(x)}}}\right)\left(\nabla u_n(\frac{\alpha}{2}p(x))^{\frac{1}{p(x)}}\right)\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx,\\&\le\int_{\{h\le|u_n|\le 2k+h\}}\left(\frac{|F|^{p'(x)}}{p'(x)(\frac{\alpha}{2}p(x))^{\frac{p'(x)}{p(x)}}}+\frac{\alpha}{2}|\nabla u_n|^{p(x)}\right)\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx,\\&\le C_4\int_{\{h\le|u_n|\}}|F|^{p'(x)}\,dx\\&\quad+\frac{\alpha}{2}\int_{\{h\le|u_n|\le 2k+h\}}|\nabla u_n|^{p(x)}\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx,\end{aligned}\tag{4.43}$$

Then from (4.42), we obtain,

$$\begin{aligned}
&\frac{\alpha}{2}\int_{\{h\le|u_n|\le 2k+h\}}|\nabla u_n|^{p(x)}\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx \\
&\qquad\le \int_\Omega f_n\varphi(T_{2k}(u_n-T_h(u_n)))\,dx + C_4\int_{\{h\le|u_n|\}}|F|^{p'(x)}\,dx.
\end{aligned}\tag{4.44}$$

Moreover, as ρ is weakly lower semi-continuous (see Theorem 3.2.9 [17]) and $\varphi'\ge 1$, we get

$$\begin{aligned}
&\int_\Omega |\nabla T_{2k}(u-T_h(u))|^{p(x)}\varphi'(T_{2k}(u-T_h(u)))\,dx,\\
&\le C_5\int_\Omega |\nabla T_{2k}(u-T_h(u))|^{p(x)}\,dx,\\
&\le C_5\liminf_{n\to\infty}\int_\Omega |\nabla T_{2k}(u_n-T_h(u_n))|^{p(x)}\,dx,\\
&\le C_5\liminf_{n\to\infty}\int_\Omega |\nabla T_{2k}(u_n-T_h(u_n))|^{p(x)}\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx,\\
&\le \frac{2}{\alpha}C_5\liminf_{n\to\infty}\int_\Omega f_n\varphi(T_{2k}(u_n-T_h(u_n)))\,dx\\
&\quad +C_6\liminf_{n\to\infty}\int_{\{h\le|u_n|\}}|F|^{p'(x)}\,dx.
\end{aligned}\tag{4.45}$$

Finally, by the strong convergence in $L^1(\Omega)$ of f_n, we have, as first n and then h tend to infinity,

$$\limsup_{h\to\infty}\int_{\{h\le|u|\le 2k+h\}}|\nabla u|^{p(x)}\varphi'(T_{2k}(u-T_h(u)))\,dx = 0,$$

hence

$$\lim_{h\to\infty}\int_\Omega F\nabla T_{2k}(u-T_h(u))\varphi'(T_{2k}(u-T_h(u)))\,dx = 0.$$

Therefore by(4.40), letting h tend to infinity, we deduce,

$$\lim_{n\to\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\,dx = 0.$$

Using Lemma 3.4 we conclude that

$$T_k(u_n)\to T_k(u) \text{ in } W_0^{1,p(x)}(\Omega)\ \ \forall k>0. \tag{4.46}$$

STEP 4. Behavior as $n\to\infty$.

By using $T_k(u_n-\psi)$ as test function in (4.2), with $\psi\in W_0^{1,p(x)}(\Omega)\cap L^\infty(\Omega)$, and putting $M = k+\|\psi\|_\infty$, we get,

$$\begin{aligned}
&\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\nabla T_k(u_n-\psi)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-\psi)\,dx\\
&\qquad = \int_\Omega f_nT_k(u_n-\psi)\,dx + \int_\Omega F\nabla T_k(u_n-\psi)\,dx.
\end{aligned}\tag{4.47}$$

since

$$a(x,T_M(u_n),\nabla T_M(u_n)) \rightharpoonup a(x,T_M(u),\nabla T_M(u)) \ \text{ in } (L^{p'(x)}(\Omega))^N.$$

and by Fatou's lemma, we obtain

$$\begin{aligned}\int_\Omega a(x,T_M(u),\nabla T_M(u))\nabla T_k(u-\psi)\,dx \\ \leq \liminf_{n\to\infty}\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\nabla T_k(u_n-\psi)\,dx.\end{aligned} \tag{4.48}$$

For the second term of the right hand side of (4.47), we have

$$\int_\Omega F\nabla T_k(u_n-\psi)\,dx \to \int_\Omega F\nabla T_k(u-\psi)\,dx \quad \text{as } n\to\infty, \tag{4.49}$$

since $\nabla T_k(u_n-\psi) \rightharpoonup \nabla T_k(u-\psi)$ in $(L^{p(x)}(\Omega))^N$, while $F \in (L^{p'(x)}(\Omega))^N$.
On the other hand, we have

$$\int_\Omega f_n T_k(u_n-\psi)\,dx \to \int_\Omega f T_k(u-\psi)\,dx \ \text{ as } n\to\infty. \tag{4.50}$$

In order to pass to the limit in the approximate equation, we now show that

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \ \text{ in } \ L^1(\Omega). \tag{4.51}$$

In particulary, it is enough to prove the equi-integrability of the sequence $\{|g_n(x,u_n,\nabla u_n)|\}$. To this purpose, we take $T_{l+1}(u_n)-T_l(u_n)$ as test function in (4.2), we obtain

$$\int_{\{|u_n|>l+1\}}|g_n(x,u_n,\nabla u_n)|\,dx \leq \int_{\{|u_n|>l\}}|f_n|\,dx.$$

Let $\varepsilon>0$ be fixed. Then there exists $l(\varepsilon)\geq 1$ such that

$$\int_{\{|u_n|>l(\varepsilon)\}}|g_n(x,u_n,\nabla u_n)|\,dx < \frac{\varepsilon}{2}. \tag{4.52}$$

For any measurable subset $E\subset\Omega$, we have

$$\begin{aligned}\int_E|g_n(x,u_n,\nabla u_n)|\,dx \leq \int_E b(l(\varepsilon))(c(x)+|\nabla T_{l(\varepsilon)}(u_n)|^{p(x)})\,dx \\ +\int_{\{|u_n|>l(\varepsilon)\}}|g_n(x,u_n,\nabla u_n)|\,dx.\end{aligned} \tag{4.53}$$

In view of (4.46), there exists $\eta(\varepsilon)>0$ such that

$$\int_E b(l(\varepsilon))(c(x)+|\nabla T_{l(\varepsilon)}(u_n)|^{p(x)})\,dx < \frac{\varepsilon}{2} \quad \text{for all } E \text{ such that } \mathrm{meas}(E)<\eta(\varepsilon). \tag{4.54}$$

Finally, by combining (4.52) and (4.54), one easily has

$$\int_E|g_n(x,u_n,\nabla u_n)|\,dx < \varepsilon \quad \text{for all } E \text{ such that } \mathrm{meas}(E)<\eta(\varepsilon),$$

we then deduce that $(g_n(x,u_n,\nabla u_n))_n$ are uniformly equi-integrable in Ω.
Thanks to (4.48) − (4.51) we can pass to the limit in (4.47) and we obtain that u is a solution of the problem $(\mathcal{P})$, which completes the proof of Theorem 4.2.

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