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# Bumps of Potentials and Almost Periodic Oscillations

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**Abstract**

We establish the existence of a Besicovitch almost periodic solution of a second-order differential equation, $u''(t) + D_1V(u(t),t) = 0$, in a Hilbert space, when the potential $V(.,t)$ possesses a bump surrounded with a hollow. We use a variational method on a Hilbert space of Besicovitch almost periodic functions.

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## 1 Introduction

Let $H$ be a real Hilbert space, $V : H \times \mathbb{R} \to \mathbb{R}$ be a function which is differentiable with respect to its first vector variable and which is almost periodic with respect to its second real variable. We consider the problem of the existence of almost periodic solutions of the following second-order differential equation.

$$u''(t) + D_1V(u(t),t) = 0 \tag{1.1}$$

where $D_1V$ denotes the partial differential of $V$ with respect to its first vector variable. We assimilate $H$ and its dual space $H^* = \mathcal{L}(H,\mathbb{R})$.

We assume that there exists a subset $S$ in $H$ which satisfies the following condition.

$$S \text{ is nonempty convex, closed, and bounded.} \tag{1.2}$$

On the function $V$ we assume that the following conditions are fulfilled.

$$V \in APU(H \times \mathbb{R}, \mathbb{R}) \tag{1.3}$$

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i.e. $V$ is almost periodic in $t$ uniformly with respect to $x$ in the sense of Yoshizawa, [11] p. 45. $D_1 V$ denotes (when it exists) the partial differential of $V$ with respect its first variable.

$$\text{For all } (x,t) \in H \times \mathbb{R}, D_1 V(x,t) \text{ exists, and } D_1 V \in APU(H \times \mathbb{R}, H^*). \tag{1.4}$$

$$\text{For all } t \in \mathbb{R}, V(.,t) \text{ is concave on } S. \tag{1.5}$$

We set $R_0 := \sup_{x \in S} \|x\|$.

$$\text{There exists } R \in (R_0, \infty) \text{ s.t. } \forall t \in \mathbb{R}, \forall x \in S, \forall y \in B(0,R) \setminus S, V(x,t) \geq V(y,t) \tag{1.6}$$

And so we can see the graph of $V(.,t)$ on $S$ as a mount, and the graph of $V(.,t)$ on $B(0,R) \setminus S$ as a moat around this mount.

We use a variational formalism on a space of Besicovitch almost periodic functions by using the functional

$$J(u) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (\frac{1}{2} \|\nabla u(t)\|^2 - V(u(t),t)) dt \tag{1.7}$$

where $\nabla u$ is a generalized derivative of $u$. We obtain the existence of a minimizer on a subset of functions with values into $S$, and by using the "moat around $S$" we prove that this minimizer is a critical point of $J$ on a space of functions with values into $H$, from which we deduce that this minimizer is an almost periodic solution (in a Besicovitch sense) to (1.1).

## 2 Notation

$AP^0(H)$ (respectively $AP^0(\mathbb{R})$) denotes the space of the Bohr almost periodic functions from $\mathbb{R}$ into $H$ (respectively $\mathbb{R}$). Endowed with the norm $\|u\|_\infty := \sup_{t \in \mathbb{R}} \|u(t)\|$, $AP^0(H)$ is a Banach space. Recall that, when $u \in AP^0(H)$, its mean value $\mathcal{M}\{u\} = \mathcal{M}_t\{u(t)\} := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u(t) dt$ exists in $H$, [15] p. 78, [19] p. 85. When $E$ is a subset of $H$, we consider $AP^0(E) := \{u \in AP^0(H) : u(\mathbb{R}) \subset E\}$.

When $k \in \mathbb{N}_* := \mathbb{N} \setminus \{0\}$, $C^k(\mathbb{R}, H)$ denotes the space of the $k$-times continuously differentiable functions from $\mathbb{R}$ into $H$ and $AP^k(H) := \{u \in C^k(\mathbb{R}, H) : \forall j = 0, ..., k, u^{(j)} \in AP^0(H)\}$ where $u^{(j)}(t) := \frac{d^j u(t)}{dt^j}$ if $j \geq 1$ and $u^{(0)} = u$. Endowed with the norm $\|u\|_{C^1} := \|u\|_\infty + \|u'\|_\infty$, $AP^1(H)$ is a Banach space, [13] (Corollary 2.12). When $E$ is a subset of $H$, we consider $AP^k(E) = AP^k(H) \cap AP^0(E)$.

If $Z$ is a Banach space, a function $f : H \times \mathbb{R} \to Z$ belongs to $APU(H \times \mathbb{R}, Z)$ when $f$ is continuous and it satisfies the following condition: for all $\epsilon > 0$, for all compact subset $K$ in $H$, there exists $\ell = \ell(\epsilon, K)$ such that, for all $\alpha \in \mathbb{R}$, there exists $\tau \in [\alpha, \alpha + \ell]$ satisfying $\|f(x, t + \tau) - f(x,t)\| \leq \epsilon$ for all $t \in \mathbb{R}$ and for all $x \in K$, [11] p. 45.

$\mathcal{B}^2(H)$ (respectively $\mathcal{B}^1(H)$) denotes the closure of $AP^0(H)$ into the Lebesgue space $L^2_{loc}(\mathbb{R}, H)$ (respectively $L^1_{loc}(\mathbb{R}, H)$) for the semi-norm

$$\mathcal{M}\{\|u\|^2\}^{1/2} := (\lim_{T \to \infty} \frac{1}{2T} \int_{T}^{T} \|u(t)\|^2 dt)^{1/2}$$

(respectively $\mathcal{M}\{\|u\|\} := \lim_{T\to\infty} \frac{1}{2T}\int_T^T \|u(t)\|dt)$, and $B^2(H)$ (respectively $B^1(H)$) is the quotient space $\mathcal{B}^2(H)/\sim_2$ (respectively $\mathcal{B}^1(H)/\sim_1$ where $u \sim_2 v$ (respectively $u \sim_1 v$) means $\mathcal{M}\{\|u-v\|^2\} = 0$ (respectively $\mathcal{M}\{\|u-v\|\} = 0$). Endowed with the inner product $(u \mid v)_{B^2} := \mathcal{M}_t\{\langle u(t), v(t)\rangle\}$, $B^2(H)$ is a Hilbert space; its norm is denoted by $\|u\|_{B^2} := ((u \mid u))^{1/2}$. Endowed with the norm $\|u\|_{B^1} := \mathcal{M}\{\|u\|\}$, $B^1(H)$ is a Banach space. The (classes of) functions of $B^2(H)$ and $B^1(H)$ are Besicovitch almost periodic functions, [6], [17] p. 11-13. As a consequence of the Cauchy-Schwarz-Buniakovski we have: $B^2(H) \subset B^1(H)$ and $\|u\|_{B^1} \leq \|u\|_{B^2}$ for all $u \in B^2(H)$.

$B^{1,2}(H)$ is the space of the $u \in B^2(H)$ such that $\nabla u := \lim_{\tau\to 0} \frac{1}{\tau}(u(.+\tau) - u)$ exists in $B^2(H)$. Endowed with the inner product $(u \mid v)_{B^{1,2}} := (u \mid v)_{B^2} + (\nabla u \mid \nabla v)_{B^2}$, $B^{1,2}(H)$ is an Hilbert space, [7], [10]. This space was used to study the Besicovitch almost periodic solutions of various classes of differential equations in [7], [8], [9], [10], [3], [4].

When $E$ is a subset of $H$, $B^2(E)$ denotes the closure of $AP^0(E)$ into $B^2(H)$, and $B^{1,2}(E)$ denotes the closure of $AP^1(E)$ into $B^{1,2}(H)$.

# 3 Existence theorem

In this section we state the main theorem of the paper. We also need to use the following conditions.

$$\text{There exists } c > 0 \text{ s.t. } \forall x, y \in H, \forall t \in \mathbb{R}, |V(x,t) - V(y,t)| \leq c.\|x-y\|. \tag{3.1}$$

$$\text{There exists } c_1 > 0 \text{ s.t. } \forall x, y \in H, \forall t \in \mathbb{R}, \|D_1V(x,t) - D_1V(y,t)\| \leq c_1.\|x-y\|. \tag{3.2}$$

$$V \text{ is bounded } on\ S \times \mathbb{R}. \tag{3.3}$$

**Theorem 3.1.** *Under (1.2-1.6) and (3.1-3.3), the equation (1.1) possesses a Besicovitch almost periodic solution, i.e. there exists $u_* \in B^{1,2}(S)$ satisfying*
$\nabla^2 u_*(t) + D_1V(u_*(t), t) = 0$ *(equality in $B^2(H)$).*

The dependence with respect to $t$ in the function $V$ is essential to avoid a trivial constant solution of (1.1). Under (1.5-1.6) and (3.3), for all $t \in \mathbb{R}$, since a concave $C^1$ function is weakly upper semi-continuous, we obtain the existence of an $x_*(t) \in S$ such that $V(x_*(t), t) \geq V(y, t)$ for all $y \in B(0, R)$, and consequently we obtain $D_1V(x_*(t), t) = 0$. But $t \mapsto x_*(t)$ is not necessarily constant, and so we are not in the situation where there exists $\overline{x} \in S$ which is a constant solution of (1.1).

In the paper [14] (Theorem 1.1) the author obtains the existence of a Bohr almost periodic solution of a second-order ordinary differential equation in a compact subset of a finite-dimensional space, and this strong compactness is essential in his proof.

# 4 The proof of the theorem

In this section we assume the conditions (1.2-1.6) and (3.1-3.3) fulfilled.

**Lemma 4.1.** *The following assertions hold.*

*(i)* $J \in C^1(B^{1,2}(H), \mathbb{R})$ *and* $DJ(u).h = (\nabla u \mid \nabla h)_{B^2} - (D_1 V(u(.), .) \mid h)_{B^2}$ *for all* $u, h \in B^{1,2}(H)$.

*(ii)* $B^{1,2}(S)$ *is a convex set which is closed into* $B^{1,2}(H)$*, and* $J$ *is convex on* $B^{1,2}(S)$.

*(iii)* $J$ *is weakly lower semi-continuous on* $B^{1,2}(S)$.

*Proof.* **(i)** Since $V \in APU(H \times \mathbb{R}, \mathbb{R})$ and $D_1 V \in APUH \times \mathbb{R}, H)$, we have $t \mapsto V(u(t), t) \in AP^0(\mathbb{R})$ and $t \mapsto D_1 V(u(t), t) \in AP^0(H)$ when $u \in AP^0(H)$, [11] (Lemma 3.4).

Let $u \in B^2(H)$. Then there exists a sequence $(u_m)_m$ in $AP^0(H)$ such that $\lim_{m\to\infty} ||u - u_m||_{B^2} = 0$. After (3.1), for all $t \in \mathbb{R}$, we have

$$|V(u(t), t) - V(u_m(t), t)| \le c.||u(t) - u_m(t)||$$

that implies

$$\limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |V(u(t), t) - V(u_m(t), t)|^2 dt \le c^2.||u - u_m||^2_{B^2}.$$

Taking $m \to \infty$, we obtain $\lim_{m\to\infty} \mathcal{M}_t\{|V(u(t), t) - V(u_m(t), t)|^2\} = 0$ that implies that $t \mapsto V(u(t), t) \in B^2(\mathbb{R})$ since the functions $t \mapsto V(u_m(t), t) \in AP^0(\mathbb{R})$. And so the Nemytski operator

$$N_V : B^2(H) \to B^2(\mathbb{R}), \;\; N_V(u)(t) := V(u(t), t), \tag{4.1}$$

is well defined and by using the previous calculations it satisfies the following inequality for all $u, v \in B^2(h)$.

$$||N_V(u) - N_V(v)||_{B^2} \le c.||u - v||_{B^2}. \tag{4.2}$$

Using a similar reasoning we obtain that the Nemytski operator

$$N_{D_1 V} : B^2(H) \to B^2(H), \;\; N_{D_1 V}(u)(t) := D_1 V(u(t), t), \tag{4.3}$$

is well defined and satisfies the following inequality for all $u, v \in B^2(H)$.

$$||N_{D_1 V}(u) - N_{D_1 V}(v)||_{B^2} \le c.||u - v||_{B^2}. \tag{4.4}$$

Now we can define the functional

$$\Psi : B^2(H) \to \mathbb{R}, \;\; \Psi(u) := \mathcal{M}_t\{V(u(t), t)\}. \tag{4.5}$$

For all $u, v \in B^2(H)$, using (4.2) we have $|\Psi(u) - \Psi(v)| \le \mathcal{M}_t\{|V(u(t), t) - V(v(t), t)|\} \le ||N_V(u) - N_V(v)||_{B^2} \le c.||u - v||_{B^2}$. And so $\Psi$ is Lipschitzean and consequently, it is continuous.

When $u, h \in B^2(H)$, using the mean value inequality, [16] (Corollary 4.4, p. 342), and (4.4) we obtain

$$\begin{aligned} &|V(u(t) + h(t), t) - V(u(t), t) - D_1 V(u(t), t).h(t)| \\ \le& \sup_{z\in[u(t), u(t)+h(t)]} ||D_1 V(z, t) - D_1 V(u(t), t)||.||h(t)|| \\ \le& c_1.(\sup_{z\in[u(t), u(t)+h(t)]} ||z - u(t)||).||h(t)|| \le c_1.||h(t)||^2. \end{aligned}$$

Consequently we have

$$|\Psi(u+h) - \Psi(u) - N_{D_1 V}(u).h|$$
$$\leq \mathcal{M}_t\{|V(u(t)+h(t),t) - V(u(t),t) - D_1 V(u(t),t).h(t)|\} \leq c_1.\|h\|_{B^2},$$

that implies that $\psi$ is Fréchet differentiable at $u$ and

$$D\Psi(u).h = \mathcal{M}_t\{D_1 V(u(t),t).h(t)\}. \tag{4.6}$$

Let $u, v \in B^2(H)$ and $h \in B^2(H)$ such that $\|h\|_{B^2} \leq 1$. Using the Cauchy-Schwarz-Buniakovski we have

$$|(D\Psi(u) - D\Psi(v)).h| \leq \mathcal{M}_t\{|(D_1 V(u(t),t) - D_1 V(v(t),t)).h(t)|\}$$
$$\leq \mathcal{M}_t\{\|(D_1 V(u(t),t) - D_1 V(v(t),t))\|.\|h(t)\|\}$$
$$\leq \mathcal{M}_t\{c_1.\|u(t) - v(t)\|.\|h(t)\|\} \leq c_1.\|u - v\|_{B^2}.\|h\|_{B^2}$$

and so we have proven the following inequality for the norm of linear operators: $\|D\psi(u) - D\Psi(v)\|_{\mathcal{L}} \leq c_1.\|u - v\|_{B^2}$ that implies that $\Psi$ is of class $C^1$.

Note that $in : B^{1,2}(H) \to B^2(H)$, defined by $in(u) := u$, is linear continuous and so it is of class $C^1$. Since $(.\mid .)_{B^2}$ is bilinear continuous, it is of class $C^1$. Since $d : u \mapsto (u,u)$, from $B^2(H)$ into $B^2(H) \times B^2(H)$, is linear continuous, it is of class $C^1$. Since $\nabla : B^{1,2} \to B^2(H)$ is linear continuous, it is of class $C^1$. From the following formula

$$J = \frac{1}{2}(.\mid .)_{B^2} \circ d \circ \nabla - \Psi \circ in \tag{4.7}$$

we see that $J$ is of classs $C^1$ as a composition of $C^1$-mappings. Using the chain rule, the formulas of the differentials of the linear and bilinear mappings of the Fréchet differential calculus in Banach spaces (cf. [16]) and (4.6), we obtain the following formula

$$DJ(u).h = (u \mid h)_{B^2} - \mathcal{M}_t\{D_1 V(u(t),t).h(t)\}. \tag{4.8}$$

for all $u, h \in B^2(H)$.

**(ii)** Let $u, v \in B^{1,2}(S)$ and $\lambda \in (0,1)$. Then there exist two sequences $(u_m)_m$ and $(v_m)_m$ in $AP^1(S)$ such that $\lim_{m\to\infty} \|u - u_m\|_{B^{1,2}} = 0$ and $\lim_{m\to\infty} \|v - v_m\|_{B^{1,2}} = 0$. Since $u_m(\mathbb{R}) \subset S$ and $v_m(\mathbb{R}) \subset S$, and since $S$ is a convex set we have, for all $t \in \mathbb{R}$, $(1-\lambda)u_m(t) + \lambda v_m(t) \in S$, and consequently $(1-\lambda)u_m + \lambda v_m \in AP^1(S)$ for all $m \in \mathbb{N}$. Since $\lim_{m\to\infty} \|(1-\lambda)u + \lambda v - ((1-\lambda)u_m + \lambda v_m)\|_{B^{1,2}} = 0$ we obtain $(1-\lambda)u + \lambda v \in B^{1,2}(S)$. We have proven that

$$B^{1,2}(S) \text{ is convex.} \tag{4.9}$$

Let $u, v \in B^{1,2}(S)$ and $\lambda \in (0,1)$. Let two sequences $(u_m)_m$ and $(v_m)_m$ in $AP^1(S)$ such that $\lim_{m\to\infty} \|u - u_m\|_{B^{1,2}} = 0$ and $\lim_{m\to\infty} \|v - v_m\|_{B^{1,2}} = 0$. Consequently we have $\lim_{m\to\infty} \|u - u_m\|_{B^2} = 0$ and $\lim_{m\to\infty} \|v - v_m\|_{B^2} = 0$. Since $V(.,t)$ is concave on $S$ we have, for all $t \in \mathbb{R}$,

$$V((1-\lambda)u_m(t) + \lambda v_m(t)) \geq (1-\lambda)V(u_m(t),t) + \lambda V(v_m(t),t)$$

that implies

$$N_V((1-\lambda)u_m+\lambda v_m) \geq (1-\lambda)N_V(u_m)+\lambda N_V(v_m).$$

Taking $m \to \infty$, since we have seen in the proof of (i) that $N_V \in C^0(B^2(H), B^2(\mathbb{R}))$, we obtain

$$N_V((1-\lambda)u+\lambda v) \geq (1-\lambda)N_V(u)+\lambda N_V(v)$$

that implies $\Psi((1-\lambda)u+\lambda v) \geq (1-\lambda)\Psi(u)+\lambda\Psi(v)$. And so $\Psi$ is concave on $B^2(S)$. Since $in$ is linear, $-\Psi \circ in$ is convex on $B^2(S)$. Since $\frac{1}{2}\|.\|_{B^2}$ is convex on $B^2(H)$, and since $\nabla$ is linear, $J = \frac{1}{2}\|.\|_{B^2} \circ \nabla - \Psi \circ in$ is convex on $B^2(S)$ as a sum of two convex functionals.

**(iii)** By using (i) and (ii), the characterization of the convexity by level sets, and the equality of the strong and weak closures of the convex sets, we obtain (iii). □

**Lemma 4.2.** *There exists $u_* \in B^{1,2}(S)$ such that $J(u_*) = \inf J(B^{1,2}(S))$.*

*Proof.* Since $V$ is bounded from above on $S \times \mathbb{R}$, $\gamma := \sup V(S \times \mathbb{R}) < \infty$. For all $u \in AP^1(S)$ we have $J(u) \geq \frac{1}{2}\|u'\|^2_{B^2} - \gamma \geq -\gamma > -\infty$. And so $J$ is bounded from below on $AP^1(S)$. Since $J$ is continuous on $B^{1,2}(H)$, we obtain $J(u) \geq -\gamma$ for all $u \in B^{1,2}(S)$, and $\inf J(B^{1,2}(S)) = \inf J(AP^1(S))$. Let $(u_k)_k$ be a minimizing sequence of $J$ in $AP^1(S)$ such that $J(u_k) \leq \inf J(B^{1,2}(S)) + \frac{1}{k}$ for all $k \in \mathbb{N}_*$. Then we have

$$\frac{1}{2}\|u'_k\|^2_{B^2} \leq \mathcal{M}_t\{V(u_k(t),t)\} + \inf J(B^{1,2}(S)) + 1 \leq \gamma + \inf J(B^{1,2}(S)) + 1$$

and so we obtain that $(u'_k)_k$ is bounded in $B^2(H)$. Since $u_k(\mathbb{R}) \subset S \subset B(0,R)$ we have $\|u_k(t)\| \leq R$ for all $t \in \mathbb{R}$ and consequently $\|u_k\|_{B^2} \leq R$ for all $k \in \mathbb{N}_*$. And so $(u_k)_k$ is bounded in $B^{1,2}(H)$. Since $B^{1,2}(H)$ is a Hilbert space, there exists a subsequence $(u_{q_k})_k$ of $(u_k)_k$ and $u_* \in B^{1,2}(H)$ such that $(u_{q_k})_k$ weakly converges to $u_*$ in $B^{1,2}(H)$. Since $B^{1,2}(S)$ is closed convex we have $u_* \in B^{1,2}(S)$. Since $J$ is weakly lower semi-continuous and $(u_{q_k})_k$ is a minimizing sequence, we obtain that $J(u_*) = \inf J(B^{1,2}(S))$. □

**Lemma 4.3.** *Let $a, b \in AP^0(\mathbb{R})$. We set $\mu(t) := \mathcal{M}_s\{a(s)b(s+t)\}$ for all $t \in \mathbb{R}$. Then the following assertions hold.*

*(i) $\mu \in AP^0(\mathbb{R})$, $\mathcal{M}\{\mu\} = \mathcal{M}\{a\}.\mathcal{M}\{b\}$ and $\mu(t) = \mathcal{M}_s\{a(s-t)b(s)\}$ for all $t \in \mathbb{R}$.*

*(ii) Moreover we assume that $a \in AP^1(\mathbb{R})$. Then $\mu \in AP^1(\mathbb{R})$ and $\mu'(t) = \mathcal{M}_s\{a'(s-t)b(s)\}$ for all $t \in \mathbb{R}$.*

*(iii) Moreover we assume that $a \in AP^1(\mathbb{R})$ and that $b$ is locally absolutely continuous on $\mathbb{R}$. Then we have $\mu'(t) = -\mathcal{M}_s\{a(s-t)b'(s)\}$ for all $t \in \mathbb{R}$.*

*Proof.* First note that $\mu(t)$ is well defined since $s \mapsto a(s)b(s+t) \in AP^0(\mathbb{R})$ as a product of two functions of $AP^0(\mathbb{R})$. Secondly note that $a = 0$ or $b = 0$ imply $\mu = 0$ and all the assertions become trivial. And so we assume $a \neq 0$ and $b \neq 0$ for the sequence of the proof.

**(i)** Since $b \in AP^0(\mathbb{R})$, $b$ is uniformly continuous ([12] (Theorem II p. 35)), and so

$$\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall r, r_1 \in \mathbb{R}, |r - r_1| \leq \delta_\epsilon \Longrightarrow |b(r) - b(r_1)| \leq \frac{\epsilon}{\|a\|_\infty}.$$

When $t, t_1 \in \mathbb{R}$ satisfy $|t - t_1| \leq \delta_\epsilon$ we have $|(t+s) - (t_1+s)| \leq \delta_\epsilon$ and then

$$|\mu(t) - \mu(t_1)| \leq \mathcal{M}_s\{|a(s)|.|b(t+s) - b(t_1+s)|\} \leq \frac{\epsilon}{||a||_\infty}\mathcal{M}_s\{|a(s)|\} \leq \epsilon.$$

And so $\mu$ is continuous on $\mathbb{R}$.

Since $b \in AP^0(\mathbb{R})$ we have

$$\forall \epsilon > 0, \exists \ell_\epsilon > 0, \forall \alpha \in \mathbb{R}, \exists \tau \in [\alpha, \alpha + \ell_\epsilon], \forall t \in \mathbb{R}, |b(t+\tau) - b(t)| \leq \frac{\epsilon}{||a||_\infty}$$

and then

$$|\mu(t+\tau) - \mu(t)| \leq \mathcal{M}_s\{|a(s)|.|b(t+\tau+s) - b(t+s)|\} \leq \mathcal{M}_s\{|a(s)|\}.\frac{\epsilon}{||a||_\infty} \leq \epsilon,$$

that proves the almost periodicity of $\mu$.

For all $S > 0$, for all $T > 0$, and for all $t \in \mathbb{R}$, we define

$$\phi_S(t) := \frac{1}{2S}\int_{-S}^{S} a(s)b(s+t)ds, d_{S,T} := \frac{1}{2T}\frac{1}{2S}\int_{-T}^{T}(\int_{-S}^{S} a(s)b(s+t)ds)dt. \tag{4.10}$$

Note that $d_{S,T} = \frac{1}{2T}\int_{-T}^{T}\phi_S(t)dt$. Using a result of [12] p. 43, we have

$$\forall \epsilon > 0, \exists S^1_\epsilon > 0, \forall S \geq S^1_\epsilon, \forall t \in \mathbb{R}, |\phi_S(t) - \mu(t)| \leq \frac{\epsilon}{4}, \tag{4.11}$$

that implies, for all $T > 0$,

$$|d_{S,T} - \frac{1}{2T}\int_{-T}^{T}\mu(t)dt| \leq \sup_{t \in [-T,T]} |\phi_S(t) - \mu(t)| \leq \frac{\epsilon}{4},$$

and by using the definition of the mean value we have

$$\forall \epsilon > 0, \exists T^1_\epsilon > 0, \forall T \geq T^1_\epsilon, |\frac{1}{2T}\int_{-T}^{T}\mu(t)dt - \mathcal{M}\{\mu\}| \leq \frac{\epsilon}{4},$$

and so using the triangular inequality we obtain

$$\forall \epsilon > 0, \exists S^1_\epsilon > 0, \exists T^1_\epsilon > 0, \forall S \geq S^1_\epsilon, \forall T \geq T^1_\epsilon, |d_{S,T} - \mathcal{M}\{\mu\}| \leq \frac{\epsilon}{2}. \tag{4.12}$$

By using a result of [12], p. 43, we have

$$\forall \epsilon > 0, \exists T^2_\epsilon > 0, \forall T \geq T^2_\epsilon, |\frac{1}{2T}\int_{-T}^{T} b(t+s)dt - \mathcal{M}\{b\}| \leq \frac{\epsilon}{4||a||_\infty} \tag{4.13}$$

that implies when $T \geq T^2_\epsilon$, for all $S > 0$,

$$|\frac{1}{2S}\int_{-S}^{S}(\frac{1}{2T}\int_{-T}^{T} a(s)b(t+s)dt)ds - \mathcal{M}\{b\}.\frac{1}{2S}\int_{-S}^{S} a(s)ds| \leq \frac{\epsilon}{4},$$

and by using the Fubini theorem we obtain $|d_{S,T} - \mathcal{M}\{b\}.\frac{1}{2S}\int_{-S}^{S} a(s)ds| \leq \frac{\epsilon}{4}$. By the definition of the mean value we have

$$\forall\epsilon > 0, \exists S_\epsilon^2 > 0, \forall S \geq S_\epsilon^2, |\frac{1}{2S}\int_{-S}^{S} a(s)ds - \mathcal{M}\{a\}| \leq \frac{\epsilon}{4.||b||_\infty},$$

that implies $|\frac{1}{2S}\int_{-S}^{S} a(s)ds.\mathcal{M}\{b\} - \mathcal{M}\{a\}.\mathcal{M}\{b\}| \leq \frac{\epsilon}{4}$, and so using the triangular inequality we obtain

$$\forall\epsilon > 0, \exists S_\epsilon^2 > 0, \exists T_\epsilon^2 > 0, \forall S \geq S_\epsilon^2, \forall T \geq T_\epsilon^2, |d_{S,T} - \mathcal{M}\{a\}.\mathcal{M}\{b\}| \leq \frac{\epsilon}{2}. \tag{4.14}$$

Now, for all $\epsilon > 0$, setting $T_\epsilon := \max\{T_\epsilon^1, T_\epsilon^2\} > 0$, $S_\epsilon := \max\{S_\epsilon^1, S_\epsilon^2\} > 0$, from (4.12) and (4.14) we obtain

$$|\mathcal{M}\{\mu\} - \mathcal{M}\{a\}.\mathcal{M}\{b\}| \leq |\mathcal{M}\{\mu\} - d_{S_\epsilon,T_\epsilon}| + |d_{S_\epsilon,T_\epsilon} - \mathcal{M}\{a\}.\mathcal{M}\{b\}| \leq \epsilon,$$

that implies the equality $\mathcal{M}\{\mu\} = \mathcal{M}\{a\}.\mathcal{M}\{b\}$. Since the mean value of an almost periodic function is invariant under translations, we obtain $\mu(t) = \mathcal{M}_s\{a(s-t).b(s+t-t)\} = \mathcal{M}_s\{a(s-t).b(s)\}$.

**(ii)** Since $AP^1(H) \subset B^{1,2}(H)$ and $f' = \nabla f$ when $f \in AP^1(H)$ Proposition 10 in [10]), setting $a_t(s) := a(s-t)$ we have $a_t \in AP^1(H)$, $a_t'(s) = a'(s-t)$ and then by using the definition of $\nabla$,

$$0 = \lim_{\delta\to 0} \mathcal{M}_s\{|\frac{1}{\delta}(a_t(s+\delta) - a_t(s)) - a_t'(s)|^2\}^{1/2}$$

$$= \lim_{\delta\to 0} \mathcal{M}_s\{|\frac{1}{\delta}(a(s+\delta-t) - a(s-t)) - a'(s-t)|^2\}^{1/2}.$$

We set $\mu_1(t) := \mathcal{M}_s\{a'(s-t).b(s)\}$, and by using the Cauchy-Schwarz-Buniakovski inequality we obtain

$$|\frac{1}{\delta}(\mu(t+\delta) - \mu(t)) - \mu_1(t)| = |\mathcal{M}_s\{[\frac{1}{\delta}(a(s+\delta-t) - a(s-t)) - a'(s-t)].b(s)\}|$$

$$\leq \mathcal{M}_s\{|\frac{1}{\delta}(a(s+\delta-t) - a(s-t)) - a'(s-t)|^2\}^{1/2}.||b||_{B^2}$$

which converges to zero when $\delta \to 0$. And so we have proven that $\mu'(t) = \mu_1(t)$.

**(iii)** Using the formula of the integration by parts for the absolutely continuous functions, [18] p. 54-55, for all $S > 0$, we have

$$\frac{1}{2S}\int_{-S}^{S} a'(s-t)b(s)ds = \frac{1}{2S}(a(S-t)b(S) - a(-S-t)b(-S)) - \frac{1}{2S}\int_{-S}^{S} a(s-t)b(s')ds.$$

Since $a, b \in AP^0(\mathbb{R})$ they are bounded on $\mathbb{R}$ that implies $\lim_{S\to\infty} \frac{1}{2S}(a(S-t)b(S) - a(-S-t)b(-S)) = 0$, and since $\mathcal{M}_s\{a'(s-t)b(s)\}$ exits in $\mathbb{R}$, we obtain the existence of $\mathcal{M}_s\{a(s-t)b'(s)\}$ and the equality $\mathcal{M}_s\{a'(s-t)b(s)\} = -\mathcal{M}_s\{a(s-t)b'(s)\}$. □

**Lemma 4.4.** *The two following assertions hold.*

*(i)* $\inf J(AP^1(S)) = \inf J(AP^1(B(0,R)))$.

*(ii)* $\inf J(B^{1,2}(S)) = \inf J(B^{1,2}(B(0,R)))$.

*Proof.* **(i)** $AP^1(S) \subset B^{1,2}(S)$ implies $\inf J(AP^1(S)) \geq \inf J(AP^1(B(0,R))$. Now we prove the converse inequality. Let $u \in AP^1(B(0,R))$. We consider the best approximation projector on the closed convex set $S$, $P : H \to S$. We set $v(t) := P(u(t))$ for all $t \in \mathbb{R}$. Since $P$ is 1-Lipschitzean ([2] (Proposition 1 p. 16)), $P$ is continuous and we have $v \in AP^0(S)$, [11] (Lemma 3.2). Since $u \in AP^1(H)$, $u'$ is bounded on $\mathbb{R}$ and consequently $u$ is Lipschitzean on $\mathbb{R}$. Therefore $v$ is Lipschitzean on $\mathbb{R}$ as a composition of Lipschitzean mappings, and then $v$ is locally absolutely continuous on $\mathbb{R}$, and consequently $v$ is Lebesgue-almost everywhere differentiable on $\mathbb{R}$, [5] (Corollaire A.2 p.145).

Let $t \in \mathbb{R}$ be a point where $v$ is differentiable. When $\delta \in \mathbb{R} \setminus \{0\}$ we have

$$||\frac{1}{\delta}(v(t+\delta) - v(t))|| = ||\frac{1}{\delta}(P(u(t+\delta)) - P(u(t)))|| = \frac{1}{|\delta|}||P(u(t+\delta)) - P(u(t))||$$

$$\leq \frac{1}{|\delta|}||u(t+\delta) - u(t)|| = ||\frac{1}{\delta}(u(t+\delta) - u(t))||$$

and doing $\delta \to 0$ we obtain

$$||v'(t)|| \leq ||u'(t)|| \text{ a.e. } t \in \mathbb{R} \tag{4.15}$$

We consider the Bochner-Féjer functions $K_m(t) := \sum_{\nu=-m}^{m}(1 - \frac{|\nu|}{m})e^{-i\nu t}$ and $\varphi_m(t) := \prod_{j=1}^{m} K_{(m!)^2}(\frac{\beta_j t}{m})$ for all $m \in \mathbb{N}_*$ and for all $t \in \mathbb{R}$, where $(\beta_j)_j$ is a $\mathbb{Z}$-basis of the set of the Fourier-Bohr exponents of $v$. We know that $K_m \geq 0$, $\mathcal{M}\{K_m\} = 1$, $\varphi_m \geq 0$ and $\mathcal{M}\{\varphi_m\} = 1$ forall $m \in \mathbb{N}_*$, [12] p. 86-88 , [15] p. 115. We define, for all $m \in \mathbb{N}_*$ and for all $t \in \mathbb{R}$,

$$\sigma_m(t) := \mathcal{M}_s\{\varphi_m(s).v(s+t)\}. \tag{4.16}$$

Ever following [15] (p. 116) where we replace $\mathbb{R}$ by $H$, we obtain that $\sigma_m \in AP^1(H)$ and

$$\lim_{m\to\infty} ||\sigma_m - v||_{B^2} = 0. \tag{4.17}$$

Since the functional $\Psi$ defined in the proof of Lemma 4.1 is continuous on $B^2(H)$ we have $\lim_{m\to\infty} \Psi(\sigma_m) = \Psi(v)$. Using the condition (1.6) we have $V(v(t)) \geq V(u(t))$ for all $t \in \mathbb{R}$, that implies $\Psi(v) \geq \Psi(u)$ and consequently we have

$$\forall \epsilon > 0, \exists m_\epsilon \in \mathbb{N}_*, \forall m \geq m_\epsilon, \Psi(\sigma_m) \geq \Psi(u) - \epsilon. \tag{4.18}$$

Using a Mazur theorem, [16] p. 88, [1] (Corollary 5.62 p. 194) , we know that $\overline{co}(v(\mathbb{R})) = \cap_{(p,\alpha)\in\Pi}[p \geq \alpha]$ where $\Pi := \{(p,\alpha) \in H^* \times \mathbb{R} : \forall y \in v(\mathbb{R}), \langle p, y\rangle \geq \alpha\}$ and $[p \geq \alpha] := p^{-1}([\alpha,\infty))$. For all $m \in \mathbb{N}_*$, for all $t \in \mathbb{R}$ and for all $(p,\alpha) \in \Pi$, using $\varphi_m \geq 0$, we have $\langle p, \sigma_m(t)\rangle = \mathcal{M}_s\{\langle p, \varphi_m(s).v(s+t)\rangle\} = \mathcal{M}_s\{\varphi_m(s)\langle p, v(s+t)\rangle\} \geq \mathcal{M}_s\{\varphi_m(s).\alpha\} = \mathcal{M}_s\{\varphi_m(s)\}.\alpha = \alpha$. Therefore we have $\sigma_m(t) \in \overline{co}(v(\mathbb{R})) \subset S$, and consequently we have proven

$$\sigma_m \in AP^1(S) \text{ for all } m \in \mathbb{N}_*. \tag{4.19}$$

Using Lemma 4.3, iii, we have, for all $m \in \mathbb{N}_*$ and for all $t \in \mathbb{R}$,

$$||\sigma_m'(t)|| = ||\mathcal{M}_s\{\varphi_m(s).v'(t+s)\}|| \leq \mathcal{M}_s\{\varphi_m(s).||v'(t+s)||\}$$
$$= \mathcal{M}_s\{\sqrt{\varphi_m(s)}.\sqrt{\varphi_m(s)}.||v'(t+s)||\}$$

and by using the Cauchy-Schwarz-Buniakovski inequality we obtain

$$\leq \mathcal{M}_s\{\varphi_m(s)\}^{1/2}.\mathcal{M}_s\{\varphi_m(s).||v'(t+s)||^2\}^{1/2} = \mathcal{M}_s\{\varphi_m(s).||v'(t+s)||^2\}^{1/2},$$

and by using (4.15) we obtain $||\sigma_m'(t)||^2 \leq \mathcal{M}_s\{\varphi_m(s).||u'(t+s)||^2\}$. From this inequality we obtain $\mathcal{M}_t\{||\sigma_m'(t)||^2\} \leq \mathcal{M}_t\{\mathcal{M}_s\{\varphi_m(s).||u'(t+s)||^2\}\}$. Setting $a = \varphi_m$, $b = ||u'(.)||^2$ and $\mu(t) = \mathcal{M}_s\{\varphi_m(s).||v'(t+s)||^2\}$ and using Lemma 4.3, i, we obtain $\mathcal{M}_t\{\mathcal{M}_s\{\varphi_m(s).||u'(t+s)||^2\}\} = \mathcal{M}\{\varphi_m\}.\mathcal{M}\{||u'||^2\} = \mathcal{M}\{||u'||^2\}$, and consequently we obtain, for all $m \in \mathbb{N}_*$,

$$\mathcal{M}_t\{||\sigma_m'(t)||^2\} \leq \mathcal{M}\{||u'||^2\}. \tag{4.20}$$

We arbitrarily fix $\epsilon > 0$ and we consider $m_\epsilon \in \mathbb{N}_*$ provided by (4.18). Using (4.18) and (4.20) we can write

$$J(\sigma_m) = \frac{1}{2}\mathcal{M}\{||\sigma_m'||^2\} - \Psi(\sigma_m) \leq \frac{1}{2}\mathcal{M}\{||u'||^2\} - \Psi(u) + \epsilon = J(u) + \epsilon,$$

that implies $\inf J(AP^1(S)) \leq J(u) + \epsilon$. Doing $\epsilon \to 0$, we obtain $\inf J(AP^1(S)) \leq J(u)$. And so we have proven that $\inf J(AP^1(S)) \leq \inf J(AP^1(B(0,R)))$ and the proof of (i) is complete.

**(ii)** $AP^1(S) \subset B^{1,2}(S)$ implies $\inf J(B^{1,2}(S)) \leq \inf J(AP^1(S))$. Now we prove the converse inequality. If $u \in B^{1,2}(S)$ there exists a sequence $(u_m)_m$ in $AP^1(S)$ such that $\lim_{m\to\infty} ||u - u_m||_{B^{1,2}} = 0$, and since $J$ is continuous on $B^{1,2}(H)$ we obtain $\inf J(AP^1(S)) \leq \lim_{m\to\infty} J(u_m) = J(u)$. And so we obtain $\inf J(AP^1(S)) \leq \inf J(B^{1,2}(S))$ that implies $\inf J(AP^1(S)) = \inf J(B^{1,2}(S))$. By doing a similar reasoning we obtain $\inf J(AP^1(B(0,R))) = \inf J(B^{1,2}(B(0,R)))$. And then (ii) becomes a consequence of (i). □

Let $h \in AP^1(H)$, $h \neq 0$. We set $\lambda := \frac{R-R_0}{||h||_\infty} \in (0,\infty)$ where $R$ and $R_0$ are defined in (1.6). Since $u_*$, provided by Lemma 4.2, belongs to $B^{1,2}(S)$, there exists a sequence $(u_j)_j$ in $AP^1(S)$ such that $\lim_{j\to\infty} ||u_* - u_j||_{B^{1,2}} = 0$. For all $j \in \mathbb{N}$, for all $t \in \mathbb{R}$, and for all $\theta \in (-\lambda, \lambda)$, wr have

$$||u_j(t) + \theta h(t)| \leq ||u_j(t)|| + |\theta|.||h(t)|| \leq R_0 + \lambda.||h|_\infty \leq R_0 + (R - R_0) = R,$$

and so $u_j + \theta.h \in AP^1(B(0,R))$, and since $\lim_{j\to\infty}(u_j + \theta.h) = u_* + \theta.h$ in $B^{1,2}(H)$, we have $u_* + \theta.h \in B^{1,2}(B(0,R))$ for all $\theta \in (-\lambda,\lambda)$. By using Lemma 4.2 and Lemma 4.4 we obtain $J(u_*) \leq J(u_* + \theta.h)$ for all $\theta \in (-\lambda,\lambda)$. After Lemma 4.1, i, $J$ is of class $C^1$ therefore $\theta \mapsto J(u_* + \theta.h)$ is differentiable and we obtain $DJ(u_*).h = \frac{d}{d\theta}_{|\theta=0} J(u_* + \theta.h) = 0$. Since $Dj(u_*)$ is linear continuous and $AP1(H)$ is dense into $B^{1,2}(H)$ we obtain that $DJ(u_*).h = 0$ for all $h \in B^{1,2}(H)$. Now by using the formula in Lemma 4.1, i, we obtain, for all $h \in B^{1,2}(H)$,

$$0 = DJ(u_*).h = (\nabla u_* \mid \nabla h)_{B^2} + \mathcal{M}_t\{D_1V(u_*(t),t).h(t)\}$$

that implies $\mathcal{M}_t\langle\nabla u_* \mid \nabla h\rangle\} = -\mathcal{M}_t\{\langle D_1V(u_*(t),t) \mid h(t)\rangle\}$, and then using Proposition 10 in [10], we obtain that $\nabla^2 u_* = \nabla(\nabla u_*)$ exists into $B^2(H)$ and $\nabla^2 u_* = D_1V(u_*(.),.)$, i.e. $\nabla^2 u_*(t) + D_1V(u_*(t),t) = 0$ (equality in $B^2(H)$. And so the proof of the theorem 3.1 is complete.

# References

[1] C. D. Aliprantis and K. C. Border, *Infinite dimensional analysis*, Second edition, Springer-Verlag, Berlin, 1999.

[2] J. P. Aubin, *Applied functional analysis*, John Wiley & Sons, New York, 1979.

[3] M. Ayachi and J. Blot, A variational approach for almost periodic solutions in retarded functional differential equations, *Differ. Equ. Appl.* **1**(1) (2009), 67-84.

[4] M. Ayachi, J. Blot, and P. Cieutat, Almost periodic solutions of monotone second-order differential equations, *Adv. Nonlinear Stud.* **11** (2011), 541-554.

[5] P. Benilan, *Fonctions vectorielles d'une variable réelle*, Appendix in the book of H. Brezis, *Opérateurs maximaux monotones, et semi-groupes de contraction dans les espaces de Hilbert*, North-Holland Publishing Company, Amsterdam 1973.

[6] A. S. Besicovitch, *Almost periodic functions*, Cambridge University Press, Cambridge 1932.

[7] J. Blot, Une approche hilbertienne pour les trajectoires presque-périodiques, *C.R. Math. Acad. Sci. Paris, Série I*, **313** (1991), 487-490.

[8] J. Blot, Almost periodic solutions of forced second-order Hamiltonian systems, *Ann. Fac. Sci. Toulouse Math. (6)* **XIII**(3) (1991), 351-363.

[9] J. Blot, Almost periodically forced pendulum, *Funkcial. Ekvac*. **36**(2) (1993), 235-250.

[10] J. Blot, Oscillations presque-périodiques forcées d'équations d'Euler-Lagrange, *Bull. Soc. Math. France* **122**(2) (1994), 285-304.

[11] J. Blot, P. Cieutat, G. M. N'Guérékata, and D. Pennequin, Superposition operators between various almost periodic function spaces and applications, *Commun. Math. Anal.* **6**(1) (2009), 42-70.

[12] H. Bohr, *Almost periodic functions*, Julius Springer, Berlin 1933.

[13] D. Bugajewski and G. M. N'Guérékata, On some classes of almost periodic functions in abstract spaces, *Intern. J. Math. and Math. Sci.* **61** (2004), 3237-3247.

[14] P. Cieutat, Almost periodic solutions of second-order systems with monotone fields on a compact subset, *Nonlinear Anal.* **53** (2003), 751-763.

[15] C. Corduneanu, *Almost periodic oscillations and waves*, Springer Science+Businesss Media, LLC, New York 2009.

[16] S. Lang, *Real and functional analysis*, Third edition, Springer-Verlag, New York, Inc. 1993.

[17] A.A. Pankov, *Bounded and almost periodic solutions of nonlinear operator differential equations*, English edition, Kluwer Academic Publishers, Dordrecht 1990.

[18] F. Riesz and B. Sz Nagy, *Analyse fonctionnelle*, 6-th French edition, Gauthier-Villars, Paris 1975.

[19] S. Zaidman, *Almost periodic functions in abstract spaces*, Pitman Publishing Inc., Marshfield, Mass. 1985.