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Existence of Positive Almost Automorphic Solutions to a Class of Integral Equations

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Abstract

This paper is concerned with positive almost automorphic solutions to a class of nonlinear infinite delay integral equation. By using a fixed point theorem in partially ordered Banach spaces, we establish an existence theorem about positive almost automorphic solutions to the addressed integral equation. Our theorem extend some earlier results to a more general class of integral equations.

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1 Introduction

The aim of this paper is to study the existence of positive almost automorphic solutions to the following nonlinear infinite delay integral equation

$$x(t) = \int_{-\infty}^{t} a(t, t-s) f(s, x(s)) ds + h(t, x(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where a, f, h satisfy some conditions recalled in Section 3.

In [12], A. M. Fink and J. A. Gatica initiated the study on the existence of positive almost periodic solution to a kind of model for the spread of some infectious disease, i.e., the following delay integral equation.

$$x(t) = \int_{t-\tau}^{t} f(s, x(s)) ds. \tag{1.2}$$

Since then, the existence of positive almost periodic type solutions and positive almost automorphic type solutions to equation (1.2) and its variants is extensively studied. Many

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Stimulated by the above works, in this paper, we will make further study on this topic, and extend the results in [3, 6–8] to a more general class of integral equation (1.1).

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^+$ the set of nonnegative real numbers, and by Ω a subset of $\mathbb{R}$. First, let's recall some definitions, notations and basic results for almost automorphic functions.

Definition 2.1. Let X be a Banach space. A continuous function $f : \mathbb{R} \to X$ is called almost automorphic if for every real sequence (s_m), there exists a subsequence (s_n) such that

$$g(t) = \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. Denote by $AA(X)$ the set of all such functions.

Remark 2.2. For more details about almost automorphic functions, we refer the reader to N'Guérékata's book [14]. In addition, it is worth to note that the notion of pseudo almost automorphic functions, which is an important and interesting generalization of almost automorphic functions, was introduced recently in [13, 16].

Definition 2.3. A continuous function $f : \mathbb{R} \times \Omega \to \mathbb{R}$ is called almost automorphic in t uniformly for x in compact subsets of Ω if for every compact subset K of Ω and every real sequence (s_m), there exists a subsequence (s_n) such that

$$g(t, x) = \lim_{n\to\infty} f(t + s_n, x)$$

is well defined for each $t \in \mathbb{R}$, $x \in K$ and

$$\lim_{n\to\infty} g(t - s_n, x) = f(t, x)$$

for each $t \in \mathbb{R}$, $x \in K$. Denote by $AA(\mathbb{R} \times \Omega, \mathbb{R})$ the set of all such functions.

Lemma 2.4. *Assume that f, $g \in AA(\mathbb{R})$. Then the following hold true:*

(a) The range $\mathcal{R}_f = \{f(t) : t \in \mathbb{R}\}$ is precompact in $\mathbb{R}$, and so f is bounded.

(b) $f + g$, $f \cdot g \in AA(\mathbb{R})$.

(c) *Equipped with the sup norm*

$$\|f\| = \sup_{t\in\mathbb{R}} |f(t)|,$$

$AA(\mathbb{R})$ turns out to be a Banach space.

Proof. see [14, §2.1]. □

Lemma 2.5. [5, Lemma 2.2] *Assume that $x \in AA(\mathbb{R})$, $K = \overline{\{x(t), t \in \mathbb{R}\}}$, $f \in AA(\mathbb{R}\times K,\mathbb{R})$, and $\{f(t,\cdot)\}_{t\in\mathbb{R}}$ are equi-continuous at every $x \in K$. Then $f(\cdot, x(\cdot)) \in AA(\mathbb{R})$.*

Lemma 2.6. [3, Lemma 4.4] *Let $f \in AA(\mathbb{R})$ and $a : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $t \mapsto a(t,\cdot)$ is in $AA(L^1(\mathbb{R}^+))$. Then $F \in AA(\mathbb{R})$, where*

$$F(t) = \int_{-\infty}^{t} a(t,t-s) f(s)ds, \quad t \in \mathbb{R}.$$

Next, let us recall a fixed point theorem in partially ordered Banach spaces. Let X be a real Banach space. A closed convex set P in X is called a convex cone if the following conditions are satisfied:

(i) if $x \in P$, then $\lambda x \in P$ for any $\lambda \geq 0$;

(ii) if $x \in P$ and $-x \in P$, then $x = 0$.

A cone P induces a partial ordering $\leq$ in X by

$$x \leq y \quad \text{if and only if} \quad y - x \in P.$$

For any given $u, v \in P$,

$$[u,v] := \{x \in X | u \leq x \leq v\}.$$

A cone P is called normal if there exists a constant $k > 0$ such that

$$0 \leq x \leq y \quad \text{implies that} \quad \|x\| \leq k\|y\|,$$

where $\|\cdot\|$ is the norm on X. We denote by P^o the interior of P. A cone P is called a solid cone if $P^o \neq \emptyset$.

The following theorem will be used in Section 3:

Theorem 2.7. [9, Theorem 2.1] *Let P be a normal and solid cone in a real Banach space X. Suppose that the operator $A : P^o \times P^o \times P^o \to P^o$ satisfies*

(S1) *for each $x, y, z \in P^o$, $A(\cdot, y, z)$ is increasing, $A(x,\cdot,z)$ is decreasing, and $A(x, y,\cdot)$ is decreasing;*

(S2) *there exists a function $\phi : (0,1)\times P^o \times P^o \to (0,+\infty)$ such that for each $x, y, z \in P^o$ and $t \in (0,1)$, $\phi(t, x, y) > t$ and*

$$A(tx, t^{-1}y, z) \geq \phi(t, x, y) A(x, y, z);$$

(S3) there exist $x_0, y_0 \in P^o$ *such that* $x_0 \le y_0$, $x_0 \le A(x_0, y_0, x_0)$, $A(y_0, x_0, y_0) \le y_0$ *and*

$$\inf_{x,y\in[x_0,y_0]} \phi(t,x,y) > t, \quad \forall t \in (0,1);$$

(S4) there exists a constant $L > 0$ *such that for all* $x, y, z_1, z_2 \in P^o$ *with* $z_1 \ge z_2$*,*

$$A(x,y,z_1) - A(x,y,z_2) \ge -L\cdot(z_1 - z_2).$$

Then A has a unique fixed point x^* *in* $[x_0, y_0]$*, i.e.,* $A(x^*, x^*, x^*) = x^*$.

3 Main results

For convenience, we first list some assumptions:

(H1) The function f in (1.1) admits the following decomposition:

$$f(t,x) = \sum_{i=1}^{n} f_i(t,x) g_i(t,x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^+ \tag{3.1}$$

for some $n \in \mathbb{N}$.

(H2) $f_i, g_i, h \in AA(\mathbb{R}\times\mathbb{R}^+, \mathbb{R})$ $(i = 1, 2, ..., n)$ are all nonnegative functions satisfying that for each $t \in \mathbb{R}$ and $i \in \{1, 2, \ldots, n\}$, $f_i(t, \cdot)$ is increasing in $\mathbb{R}^+$, $g_i(t, \cdot)$ is decreasing in $\mathbb{R}^+$, and $h(t, \cdot)$ is decreasing in $\mathbb{R}^+$. In addition, there exists a constant $L > 0$ such that

$$h(t,z_1) - h(t,z_2) \ge -L(z_1 - z_2), \quad \forall t \in \mathbb{R},\ \forall z_1 \ge z_2 \ge 0. \tag{3.2}$$

(H3) There exist $\varphi_i, \psi_i : (0,1)\times(0,+\infty) \to (0,1]$ such that $\varphi_i(\lambda, x) > \lambda$, $\psi_i(\lambda, y) > \lambda$ and

$$f_i(t,\lambda x) \ge \varphi_i(\lambda,x) f_i(t,x), \quad g_i(t,\lambda^{-1} y) \ge \psi_i(\lambda,y) g_i(t,y),$$

for all $x, y > 0$, $\lambda \in (0,1)$, $t \in \mathbb{R}$ and $i \in \{1, 2, \ldots, n\}$; moreover, for all $a, b \in (0, +\infty)$ with $a \le b$,

$$\inf_{x,y\in[a,b]} \varphi_i(\lambda,x)\psi_i(\lambda,y) > \lambda, \quad \lambda \in (0,1),\ i = 1,2,\ldots,n.$$

(H4) $a : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying that $t \mapsto a(t,\cdot)$ is in $AA(L^1(\mathbb{R}^+))$.

(H5) There exist constants $d \ge c > 0$ such that

$$\inf_{t\in\mathbb{R}} \int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{n} f_i(s,c) g_i(s,d)\,ds \ge c.$$

and

$$\sup_{t\in\mathbb{R}} \left[\int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{n} f_i(s,d) g_i(s,c)\,ds + h(t,d) \right] \le d.$$

Now, let us establish our existence result.

Theorem 3.1. *Assume that* (H1)-(H5) *hold. Then equation* (1.1) *has a unique almost automnrophic solution with positive infimum.*

Proof. Let

$$P = \{x \in AA(\mathbb{R}) : x(t) \geq 0, \forall t \in \mathbb{R}\}.$$

It is not difficult to verify that P is a normal and solid cone in $AA(\mathbb{R})$ and

$$P^o = \{x \in AA(\mathbb{R}) : \exists \varepsilon > 0 \text{ such that } x(t) \geq \varepsilon,\ \forall t \in \mathbb{R}\}.$$

We define a nonlinear operator A on $P^o \times P^o \times P^o$ by

$$A(x,y,z)(t) = \int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{n} f_i(s,x(s))g_i(s,y(s))ds + h(t,z(t)), \quad t \in \mathbb{R}.$$

We first show that A is an operator from $P^o \times P^o \times P^o$ to P^o. Let $x,y,z \in P^o$. Then $x,y,z \in AA(\mathbb{R})$. By a similar proof to [3, Lemma 3.3], one can prove that for each $i \in \{1,2,...,n\}$ and each $[a,b] \subset (0,+\infty)$, there exists $L \geq 0$ such that

$$|f_i(t,u) - f_i(t,v)| \leq L|u - v|, \quad \forall t \in \mathbb{R},\ \forall u,v \in [a,b].$$

Thus, $\{f_i(t,\cdot)\}_{t\in\mathbb{R}}$ are equi-continuous at each $x > 0$. Then, by Lemma 2.5, we know that

$$f_i(\cdot,x(\cdot)) \in AA(\mathbb{R}), \quad i = 1,2,...,n.$$

By using a similar idea to the above proof, we can also get

$$g_i(\cdot,y(\cdot)) \in AA(\mathbb{R}), \quad i = 1,2,...,n.$$

In addition, by (H2), we can deduce that

$$|h(t,z_1) - h(t,z_2)| \leq L|z_1 - z_2|, \quad \forall t \in \mathbb{R},\ \forall z_1,z_2 \geq 0.$$

Then, also by Lemma 2.5, we get $h(\cdot,z(\cdot)) \in AA(\mathbb{R})$. Now, combing (H4), Lemma 2.6 and (b) of Lemma 2.4, we conclude that $A(x,y,z) \in AA(\mathbb{R})$. On the other hand, there exist $\varepsilon, M > 0$ such that $x(t) \geq \varepsilon$ and $y(t) \leq M$ for all $t \in \mathbb{R}$. Let

$$\varepsilon' = \min\left\{\frac{c}{2},\varepsilon\right\}, \quad M' = \max\{d+1, M\}.$$

Then, $x(t) \geq \varepsilon'$, $y(t) \leq M'$ for all $t \in \mathbb{R}$. Moreover, $\varepsilon' < c$ and $M' > d$. Now, by (H2), (H3) and (H5), we have

$$\begin{aligned} A(x,y,z)(t) &= \int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{n} f_i(s,x(s))g_i(s,y(s))ds + h(t,z(t)) \\ &\geq \int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{n} f_i(s,\varepsilon')g_i(s,M')ds \\ &= \int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{n} f_i\left(s,\frac{\varepsilon'}{c}\cdot c\right)g_i\left(s,\frac{M'}{d}\cdot d\right)ds \end{aligned}$$

$$
\begin{aligned}
&\geq \int_{-\infty}^{t} a(t,t-s)\sum_{i=1}^{n}\varphi_i\left(\frac{\varepsilon'}{c},c\right)f_i(s,c)\psi_i\left(\frac{d}{M'},d\right)g_i(s,d)ds\\
&\geq \frac{d\varepsilon'}{cM'}\int_{-\infty}^{t} a(t,t-s)\sum_{i=1}^{n} f_i(s,c)g_i(s,d)ds\\
&\geq \frac{d\varepsilon'}{cM'}\cdot c=\frac{d\varepsilon'}{M'}>0,\quad \forall t\in\mathbb{R}.
\end{aligned}
$$

Next, let us verify that the assumptions (S1)-(S4) of Theorem 2.7 hold. It is not difficult to see from (H2) that (S1) and (S4) hold.

Let $x,y\in P^o$ and $\lambda\in(0,1)$. Let

$$a(x,y)=\min\{\inf_{t\in\mathbb{R}}x(t),\inf_{t\in\mathbb{R}}y(t)\},\quad b(x,y)=\max\{\sup_{t\in\mathbb{R}}x(t),\sup_{t\in\mathbb{R}}y(t)\}.$$

Then $0<a(x,y)\leq b(x,y)<+\infty$ and $x(t),y(t)\in[a(x,y),b(x,y)]$ for all $t\in\mathbb{R}$. Define

$$\phi_i(\lambda,x,y)=\inf_{u,v\in[a(x,y),b(x,y)]}\varphi_i(\lambda,u)\psi_i(\lambda,v),\quad i=1,2,\ldots,n$$

and

$$\phi(\lambda,x,y)=\min_{i=1,2,\ldots,n}\phi_i(\lambda,x,y).$$

By (H3), it is easy to see that $\phi_i(\lambda,x,y)>\lambda$, $i=1,2,\ldots,n$, for each $x,y\in P^o$ and $\lambda\in(0,1)$, which gives that $\phi(\lambda,x,y)>\lambda$ for each $x,y\in P^o$ and $\lambda\in(0,1)$. Now, We deduce by (H3) that

$$
\begin{aligned}
A(\lambda x,\lambda^{-1}y,z)(t)-h(t,z(t)) &= \int_{-\infty}^{t} a(t,t-s)\sum_{i=1}^{n} f_i[s,\lambda x(s)]g_i[s,\lambda^{-1}y(s)]ds\\
&\geq \int_{-\infty}^{t} a(t,t-s)\sum_{i=1}^{n}\phi_i(\lambda,x,y)f_i[s,x(s)]g_i[s,y(s)]ds\\
&\geq \phi(\lambda,x,y)\int_{-\infty}^{t} a(t,t-s)\sum_{i=1}^{n} f_i[s,x(s)]g_i[s,y(s)]ds\\
&= \phi(\lambda,x,y)[A(x,y,z)(t)-h(t,z(t))]\\
&\geq \phi(\lambda,x,y)A(x,y,z)(t)-h(t,z(t)),
\end{aligned}
$$

for all $x,y,z\in P^o$, $\lambda\in(0,1)$ and $t\in\mathbb{R}$, which yields that

$$A(\lambda x,\lambda^{-1}y,z)\geq\phi(\lambda,x,y)A(x,y,z),\quad\forall x,y,z\in P^o,\ \forall\lambda\in(0,1),$$

i.e., (S2) holds.

It remains to show that (S3) holds. It follows from (H5) that

$$A(c,d,c)\geq c,\quad A(d,c,d)\leq d.$$

In addition, we have

$$\inf_{x,y\in[c,d]}\phi(\lambda,x,y)=\min_{i=1,\ldots,n}\inf_{x,y\in[c,d]}\phi_i(\lambda,x,y)$$

$$
\begin{aligned}
&= \min_{i=1,\dots,n} \phi_i(\lambda,c,d) \\
&= \phi(\lambda,c,d) > \lambda,
\end{aligned}
$$

for all $\lambda \in (0,1)$.

Now Theorem 2.7 yields that A has a unique fixed point x^* in $[c,d]$, which is just an almost automorphic solution with a positive infimum to Eq. (1.1).

Next, let us show that x^* is the unique almost automorphic solution with a positive infimum to Eq. (1.1), i.e., x^* is the unique fixed point of A in P^o. Let $y^* \in P^o$ be a fixed point of A. Then, there exists $\alpha \in (0,1)$ such that $\alpha c \le x^*, y^* \le \alpha^{-1} d$. Denote $c' = \alpha c$ and $d' = \alpha^{-1} d$. It is not difficult to see that

$$
A(c',d',c') \ge c', \quad A(d',c',d') \le d', \quad \inf_{x,y\in[c',d']} \phi(\lambda,x,y) > \lambda,\ \forall \lambda \in (0,1).
$$

Then, by the above proof, one can conclude that A has a unique fixed point in $[c',d']$, which means that $x^* = y^*$. This completes the proof. □

Next, we present two examples to illustrate our main results.

Example 3.2. Let $n = 1$,

$$
f_1(t,x) = \frac{1 + |\cos \frac{1}{2+\sin t + \sin \pi t}|}{2} \sqrt{x^2 + x},
$$

and

$$
g_1(t,x) \equiv 1, \quad h(t,x) = \frac{\sin^2 t}{1+x}, \quad a(t,s) \equiv \frac{1}{2(1+s^2)}.
$$

By some direct calculations, one can verify that (H1)-(H4) hold. (H5) follows from

$$
\inf_{t\in\mathbb{R}} \int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{n} f_i\left(s, \frac{1}{15}\right) g_i(s,100)ds \ge \frac{\pi \sqrt{\frac{1}{225} + \frac{1}{15}}}{8} \ge \frac{1}{15}
$$

and

$$
\sup_{t\in\mathbb{R}} \left[\int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{n} f_i(s,100) g_i\left(s, \frac{1}{15}\right) ds + h(t,100) \right] \le \frac{\pi}{4} \sqrt{10100} + \frac{1}{101} \le 100.
$$

Thus, Theorem 3.1 yields that the following equation

$$
x(t) = \int_{-\infty}^{t} \frac{1 + |\cos \frac{1}{2+\sin s + \sin \pi s}|}{4[1+(t-s)^2]} \sqrt{x^2(s) + x(s)} ds + \frac{\sin^2 t}{1 + x(t)}
$$

has a unique almost automorphic solution with positive infimum.

Example 3.3. Consider the following equation:

$$
x(t) = \int_{t-1-|\sin t|}^{t} \frac{b(s)\sqrt{\ln(x(s)+1)}}{\sqrt{x(s)+1}} ds + (1+\sin^2 t) e^{-x^2(t)}, \quad t \in \mathbb{R}, \tag{3.3}
$$

where $b(s) = 2 + \sin \frac{1}{2+\cos s+\cos \pi s}$, $s \in \mathbb{R}$.

Let $n = 1$,

$$f_1(t,x) = b(t)\sqrt{\ln(x+1)}, \quad g_1(t,x) = \frac{1}{\sqrt{x+1}}, \quad h(t,x) = (1+\sin^2 t)e^{-x^2},$$

and

$$a(t,s) = \begin{cases} 1 \quad , \quad s \in [0,\tau(t)],\ t \in \mathbb{R}, \\ 0 \quad , \quad s > \tau(t),\ t \in \mathbb{R}, \end{cases} \qquad \text{where } \tau(t) = 1 + |\sin t|.$$

Then, it is not difficult to verify that (H1)-(H4) are satisfied. Let $d = 99$. Then, there exists a sufficiently small $c > 0$ such that

$$\inf_{t\in\mathbb{R}} \int_{-\infty}^{t} a(t,t-s)\sum_{i=1}^{n} f_i(s,c)g_i(s,d)ds \geq \frac{\sqrt{\ln(1+c)}}{10} \geq c;$$

on the other hand, for all $c > 0$,

$$\sup_{t\in\mathbb{R}} \left[\int_{-\infty}^{t} a(t,t-s)\sum_{i=1}^{n} f_i(s,d)g_i(s,c)ds + h(t,d)\right] \leq \frac{6\sqrt{\ln 100}}{\sqrt{1+c}} + 2e^{-99^2} \leq 99 = d.$$

Thus, (H5) holds. By using Theorem 3.1, equation (3.3) has a unique almost automorphic solution with positive infimum.

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