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Some PPF Dependent Random Fixed Point Theorems and Periodic Boundary Value Problems of Random Differential Equations

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Abstract

In this paper some random fixed point theorems with PPF dependence are proved for random operators in separable Banach spaces with different domain and range spaces. The obtained abstract results are applied to prove PPF dependence existence results for first order periodic boundary value problems for functional random differential equations.

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1 Introduction

The study of random fixed point theorems in abstract spaces is initiated by Spacek [12] and Hans [9] and are the stochastic generalizations of the classical fixed point theorems in separable Banach spaces. The research along this line gained momentum after the publication of the article by Bharucha-Reid [2] and the monograph Bharucha-Reid [3] since then several random fixed point theorems have been proved in the literature. It is worthwhile to mention that these randoms fixed point theorems are useful in proving the existence results for random solutions of nonlinear random equations in separable Banach spaces. The details of

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this aspect along with some nice applications to random differential equations appear in an interesting paper of Itoh [10]. The common assumption among all these random fixed point theorems is that the operators in question map abstract spaces into itself, i.e. the domain and the range of the operators are the same. To the best of our knowledge, there is no discussion so far concerning the random fixed point theorems for the operators with different domains and the range spaces. The classical or deterministic fixed point theorems for the operators with respect to domain and range spaces are not same studied in Bernfield *et al.* [1], Drici *et al.* [7], [8] and Dhage [4] are called fixed point theorems with PPF (past, present and future) dependence because they are useful for proving the existence of solutions for certain functional differential equations which may depend upon the past, present and future consideration. Some basic random fixed point theorems with PPF dependence for operators in separable Banach spaces was proved in Dhage [6], by using random contractions, and applied them to prove the existence of PPF dependent random solutions of initial value problems for functional random differential equations.

In this paper we supplement the results proved in Dhage [5], [6], by proving some new random fixed point theorems with PPF dependence for the operators in separable Banach spaces satisfying certain contraction conditions different than that given in Dhage [5], [6]. We apply the obtained new random fixed point theorems with PPF dependence to first order periodic boundary value problems for functional random differential equations and hybrid random functional differential equations.

The rest of the paper is organized as follows: In the following section we present some of the basic terminologies that will be used in the subsequent development of this paper. In Section 3 we prove the basic random fixed point theorems with PPF dependence and in Section 4 we apply them to periodic boundary value problems for random functional differential equations.

2 Preliminaries

Let $(\Omega,\mathcal{A})$ be a measurable space and let E be a separable Banach space with norm $\|\cdot\|_E$. We equip the Banach space E with a σ-algebra β_E of Borel subsets of E so that (E,β_E) becomes a measurable space. A mapping $x:\Omega\to E$ is called measurable if

$$x^{-1}(B)=\{\omega\in\Omega \mid x(\omega)\in B\}\in\mathcal{A} \tag{2.1}$$

for all Borel sets $B\in\beta_E$.

A mapping $Q:\Omega\times E\to E$ is called a random operator if $Q(\omega,x)$ is measurable in ω for all $x\in E$. We also denote a random operator Q on E by $Q(\omega)x=Q(\omega,x)$. A random operator $Q(\omega)$ is called continuous on E if $Q(\omega,x)$ is continuous in x for each $\omega\in\Omega$. Similarly, $Q(\omega)$ is called compact on E if $Q(\omega,E)$ is relatively compact subset of E for each $\omega\in\Omega$.

Given a closed and bounded interval $I=[a,b]$ in $\mathbb{R}$, the set of real numbers, for some $a,b\in\mathbb{R}$, $a<b$, let $E_0=C(I,E)$ denote the Banach space of continuous E-valued functions defined on I equipped with the supremum norm $\|\cdot\|_{E_0}$ defined by

$$\|x\|_{E_0}=\sup_{t\in I}\|x(t)\|_E. \tag{2.2}$$

For a fixed $t \in I$, a **Razumikhin** class of functions in E_0 is defined as

$$\mathcal{R}_c = \{\phi \in E_0 \mid \|\phi\|_{E_0} = \|\phi(c)\|_E\}. \tag{2.3}$$

The class $\mathcal{R}_c$ is algebraically closed with respect to difference if $\phi - \xi \in \mathcal{R}_c$ whenever $\phi, \xi \in \mathcal{R}_c$. Similarly, $\mathcal{R}_c$ is topologically closed if it is closed w.r.t. the topology on E_0 generated by the norm $\|\cdot\|_{E_0}$.

Let $Q : \Omega \times E_0 \to E$ be a random operator. A measurable function $\xi^* : \Omega \to E_0$ is called a PPF dependent random fixed point of the random operator $Q(\omega)$ if

$$Q(\omega, \xi^*(\omega)) = \xi^*(c, \omega)$$

for some $c \in I$. Any mathematical statement that guarantees the existence of PPF dependent random fixed point of the random operator $Q(\omega)$ is a random fixed point theorem with PPF dependence or a PPF dependent random fixed point theorem.

The following theorem is used often in the study of nonlinear discontinuous random differential equations. We also need this result in the subsequent part of this paper.

Theorem 2.1 (Carathéodory)**.** *Let $Q : \Omega \times E \to E$ be a mapping such that $Q(\omega, x)$ is measurable in ω for each $x \in E$ and $Q(\omega, x)$ is continuous in x for each $\omega \in \Omega$. Then the map $(\omega, x) \mapsto Q(\omega, x)$ is jointly measurable.*

The following definitions are introduced in Dhage [5].

Definition 2.2. A random operator $Q : \Omega \times E_0 \to E$ is called a random contraction if for each $\omega \in \Omega$,

$$\|Q(\omega, \xi) - Q(\omega, \eta)\|_E \le \lambda(\omega)\, \|\xi - \eta\|_{E_0} \tag{2.4}$$

for all $\xi, \eta \in E_0$, where $\lambda : \Omega \to \mathbb{R}_+$ is a measurable function satisfying $0 \le \lambda(\omega) < 1$ for all $\omega \in \Omega$.

Definition 2.3. A random operator $Q : \Omega \times E_0 \to E$ is called a strong random contraction if for a given $c \in I$ and for each $\omega \in \Omega$,

$$\|Q(\omega, \xi) - Q(\omega, \eta)\|_E \le \lambda(\omega)\, \|\xi(c, \omega) - \eta(c, \omega)\|_E \tag{2.5}$$

for all $\xi, \eta \in E_0$, where $\lambda : \Omega \to \mathbb{R}_+$ is a measurable function satisfying $0 \le \lambda(\omega) < 1$ for all $\omega \in \Omega$.

Note that every strong random contraction is random contraction, but the converse is not true. Dhage proved in [5] a fundamental random fixed point theorem with PPF dependence for random operators with different domain and range spaces which is further applied to certain nonlinear functional random equations for proving the existence theorem for PPF dependent random solutions. In the following section we prove some random fixed point theorems with PPF dependence for random operators in separable Banach spaces satisfying certain contraction and compactness type conditions and then apply to some periodic boundary value problems of first order functional random differential equations.

3 PPF Dependent Random Fixed Point Theory

We need the following definitions in what follows.

Definition 3.1. A random operator $Q : \Omega \times E_0 \to E$ is called a strong random contraction of Kannan type if for each $\omega \in \Omega$,

$$\|Q(\omega,\xi) - Q(\omega,\eta)\|_E \leq \alpha(\omega)\left[\|\xi(c) - Q(\omega,\xi(\omega))\|_E + \|\eta(c) - Q(\omega,\eta(\omega))\|_E\right]$$

for all $\xi,\eta \in E_0$, where $\alpha : \Omega \to \mathbb{R}_+$ is a measurable function satisfying $0 \leq \alpha(\omega) < 1/2$ for all $\omega \in \Omega$.

Definition 3.2. A random operator $Q : \Omega \times E_0 \to E$ is called a strong random contraction of Riech type if for a given $c \in I$ and for each $\omega \in \Omega$,

$$\begin{aligned}\|Q(\omega,\xi) - Q(\omega,\eta)\|_E \quad &\leq \quad \alpha(\omega)\,\|\xi(c) - Q(\omega,\xi(\omega))\|_E + \beta(\omega)\,\|\eta(c) - Q(\omega,\eta(\omega))\|_E \\ &\qquad + \gamma(\omega)\|\xi(c) - \eta(c)\|_E\end{aligned}$$

for all $\xi,\eta \in E_0$, where $\alpha,\beta,\gamma : \Omega \to \mathbb{R}_+$ are measurable functions satisfying $\alpha(\omega)+\beta(\omega)+\gamma(\omega) < 1$ for all $\omega \in \Omega$.

Definition 3.3. A random operator $Q : \Omega \times E_0 \to E$ is called a random contraction of Riech type if for a given $c \in I$ and for each $\omega \in \Omega$,

$$\begin{aligned}\|Q(\omega,\xi) - Q(\omega,\eta)\|_E \quad &\leq \quad \alpha(\omega)\,\|\xi(c) - Q(\omega,\xi(\omega))\|_E + \beta(\omega)\,\|\eta(c) - Q(\omega,\eta(\omega))\|_E \\ &\qquad + \gamma(\omega)\|\xi - \eta\|_{E_0}\end{aligned}$$

for all $\xi,\eta \in E_0$, where $\alpha,\beta,\gamma : \Omega \to \mathbb{R}_+$ are measurable functions satisfying $\alpha(\omega)+\beta(\omega)+\gamma(\omega) < 1$ for all $\omega \in \Omega$.

Our first random fixed point theorem with PPF dependence is the following result.

Theorem 3.4. *Let $(\Omega,\mathcal{A})$ be a measurable space and let E be a separable Banach space. If $Q : \Omega \times E_0 \to E$ is a continuous random operator satisfying the condition of strong Kannan type random contraction, then the following statements hold in E :*

(a) If $\mathcal{R}_c$ is algebraically closed with respect to difference, then for a given $\xi_0 \in E_0$ and for a given $c \in I$, every sequence $\{\xi_n(\omega)\}$ of measurable functions satisfying

$$Q(\omega,\xi_n(\omega)) = \xi_{n+1}(c,\omega) \tag{3.1}$$

and

$$\|\xi_n(\omega) - \xi_{n+1}(\omega)\|_{E_0} = \|\xi_n(c,\omega) - \xi_{n+1}(c,\omega)\|_E \tag{3.2}$$

converges to a PPF dependent random fixed point of the random operator $Q(\omega)$, i.e. there is a measurable function $\xi^* : \Omega \to E$ such that for each $\omega \in \Omega$,

$$Q(\omega,\xi^*(\omega)) = Q(\omega)\xi^*(\omega) = \xi^*(c,\omega).$$

(b) *Given $\xi_0, \eta_0 \in E_0$, let $\{\xi_n(\omega)\}$ and $\{\eta_n(\omega)\}$ be the sequences of iterates of measurable functions corresponding to ξ_0 and η_0 constructed as in (a). Then,*

$$||\xi_n(\omega) - \eta_n(\omega)||_{E_0} \leq \frac{1}{1-\lambda(\omega)}[||\xi_0 - \xi_1(\omega)||_{E_0} + ||\xi_0 - \xi_1(\omega)||_{E_0}] + ||\xi_0 - \eta_0||_{E_0},$$

where $\lambda(\omega) = \dfrac{\alpha(\omega)}{1-\alpha(\omega)} < 1$ *for all* $\omega \in \Omega$.

If, in particular, $\xi_0 = \eta_0$, *and* $\{\xi_n(\omega)\} \neq \{\eta_n(\omega)\}$, *then*

$$||\xi_n(\omega) - \eta_n(\omega)||_{E_0} \leq \frac{2}{1-\lambda(\omega)}||\xi_0 - \xi_1(\omega)||_{E_0}.$$

(c) *Finally, if $\mathcal{R}_c$ is topologically closed, then for a given $\xi_0 \in E_0$, every sequence $\{\xi_n(\omega)\}$ of iterates of $Q(\omega)$ constructed as in (a), converges to a unique PPF dependent random fixed point $\xi^*(\omega)$ of $Q(\omega)$, i.e. there is a unique measurable function $\xi^* : \Omega \to E_0$ such that $Q(\omega, \xi^*(\omega)) = \xi^*(c, \omega)$ for all $\omega \in \Omega$.*

Proof. Let $\xi_0 \in E_0$ be arbitrary. By hypothesis, $Q(\omega, \xi_0) \in E$. Suppose that $Q(\omega, \xi_0) = x_1(\omega)$, where the function $x_1 : \Omega \to E$ is measurable. Choose a measurable function $\xi_1 : \Omega \to E_0$ such that $x_1(\omega) = \xi_1(c, \omega)$ and that

$$||\xi_1(c, \omega) - \xi_0(c)||_E = ||\xi_1(\omega) - \xi_0||_{E_0}.$$

Define a sequence $\{\xi_n(\omega)\}$ of measurable functions from Ω into E_0 inductively so that

$$Q(\omega, \xi_n(\omega)) = \xi_{n+1}(c, \omega)$$

and

$$||\xi_{n+1}(c, \omega) - \xi_n(c, \omega)||_E = ||\xi_{n+1}(\omega) - \xi_n(\omega)||_{E_0}$$

for all $\omega \in \Omega$.

We claim that $\{\xi_n(\omega)\}$ is a Cauchy sequence in E_0. Now for any $n \in \mathbb{N}$ we have the following estimate for each fixed $\omega \in \Omega$,

$$\begin{aligned}
||\xi_n(\omega) - \xi_{n+1}(\omega)||_{E_0} &= ||\xi_n(c, \omega) - \xi_{n+1}(c, \omega)||_E \\
&= ||Q(\omega, \xi_{n-1}(\omega)) - Q(\omega, \xi_n(\omega))||_{E_0} \\
&\leq \alpha(\omega)[||\xi_{n-1}(c, \omega) - Q(\omega, \xi_{n-1}(\omega))||_E + ||\xi_n(c, \omega) - Q(\omega, \xi_n(\omega))||_E] \\
&\leq \alpha(\omega)[||\xi_{n-1}(c, \omega) - \xi_n(c, \omega)||_E + ||\xi_n(c)(\omega) - \xi_{n+1}(c, \omega)||_E] \\
&\leq \alpha(\omega)||\xi_{n-1}(\omega) - \xi_n(\omega)||_{E_0} + \alpha(\omega)||\xi_n(\omega) - \xi_{n+1}(\omega)||_{E_0}.
\end{aligned}$$

From the above inequality, it follows that

$$||\xi_n(\omega) - \xi_{n+1}(\omega)||_{E_0} \leq \lambda(\omega)||\xi_{n-1}(\omega) - \xi_n(\omega)||_{E_0}$$

for all $n = 1, 2, \ldots$, where $\lambda(\omega) = \dfrac{\alpha(\omega)}{1-\alpha(\omega)} < 1$ for all $\omega \in \Omega$.

By induction,

$$||\xi_n(\omega) - \xi_{n+1}(\omega)|_{E_0} \leq \lambda^n(\omega)||\xi_0 - \xi_1(\omega)||_{E_0}$$

for all $n = 1, 2, \ldots$.

If $m > n$, by triangle inequality, we obtain

$$\begin{aligned}
\|\xi_m(\omega) - \xi_n(\omega)\|_{E_0} &\le \|\xi_n(\omega) - \xi_{n+1}(\omega)\|_{E_0} + \cdots + \|\xi_{m-1}(\omega) - \xi_m(\omega)\|_{E_0} \\
&\le \lambda^n(\omega)\|\xi_0 - \xi_1(\omega)\|_{E_0} + \cdots + \lambda^{m-1}(\omega)\|\xi_0 - \xi_1(\omega)\|_{E_0} \\
&= [\lambda^n(\omega) + \cdots + \lambda^{m-1}(\omega)]\|\xi_0 - \xi_1(\omega)\|_{E_0} \\
&= \left(\frac{\lambda^n(\omega)}{1-\lambda(\omega)}\right)\|\xi_0 - \xi_1(\omega)\|_{E_0}.
\end{aligned}$$

Hence, $\lim_{m,n\to 0} \|\xi_n(\omega) - \xi_m(\omega)\|_{E_0} \to 0$. This shows that $\{\xi_n(\omega)\}$ is a Cauchy sequence of measurable functions on Ω into E_0. Since E_0 is complete and separable Banach space, there is a measurable function $\xi^* : \Omega \to E_0$ such that $\lim_{n\to\infty} \xi_n(\omega) = \xi^*(\omega)$ for all $\omega \in \Omega$. Now, from (2.3) it follows that $\{\xi_n(c,\omega)\}$ is Cauchy and hence converges to a point $\xi^*(c,\omega)$ in view of completeness of E.

It remains to prove that ξ^* is a PPF dependence fixed point of $Q(\omega)$. From continuity of the random operator $Q(\omega)$ it follows that

$$\begin{aligned}
Q(\omega, \xi^*(\omega)) &= Q(\omega, \lim_{n\to\infty} \xi_n(\omega)) \\
&= \lim_{n\to\infty} Q(\omega, \xi_n(\omega)) \\
&= \lim_{n\to\infty} \xi_{n+1}(c,\omega) \\
&= \xi^*(c,\omega)
\end{aligned}$$

for all $\omega \in \Omega$. Hence ξ^* is a random fixed point with PPF dependence of the random operator $Q(\omega)$ on E_0.

(b) Now, let $\{\xi_n(\omega)\}$ and $\{\eta_n(\omega)\}$ be any two sequences of measurable functions as constructed in (a). Then for each $\omega \in \Omega$,

$$\begin{aligned}
\|\xi_n(\omega) - \eta_n(\omega)\|_{E_0} &\le \|\xi_n(\omega) - \xi_{n-1}(\omega)\|_{E_0} + \|\xi_{n-1}(\omega) - \eta_{n-1}(\omega)\|_{E_0} + \|\eta_{n-1}(\omega) - \eta_n(\omega)\|_{E_0} \\
&\le \lambda^{n-1}(\omega)[\|\xi_0 - \xi_1(\omega)\|_{E_0} + \|\eta_0 - \eta_1(\omega)\|_{E_0}] + \|\xi_{n-1}(\omega) - \eta_{n-1}(\omega)\|_{E_0}.
\end{aligned}$$

Therefore, by induction,

$$\begin{aligned}
\|\xi_n(\omega) - \eta_n(\omega)\|_{E_0} &\le \lambda^{n-1}(\omega)[\|\xi_0 - \xi_1(\omega)\|_{E_0} + \|\eta_0 - \eta_1(\omega)\|_{E_0}] \\
&\quad + \|\xi_{n-1}(\omega) - \eta_{n-1}(\omega)\|_{E_0} \\
&\le \left[\lambda^{n-1}(\omega) + \lambda^{n-2}(\omega)\right][\|\xi_0 - \xi_1(\omega)\|_{E_0} + \|\eta_0 - \eta_1(\omega)\|_{E_0}] \\
&\quad + \|\xi_{n-1}(\omega) - \eta_{n-1}(\omega)\|_{E_0} + \|\xi_{n-2}(\omega) - \eta_{n-2}(\omega)\|_{E_0} \\
&\le [\lambda^{n-1}(\omega) + \lambda^{n-2}(\omega) + \cdots + 1][\|\xi_0 - \xi_1(\omega)\|_{E_0} \\
&\quad + \|\eta_0 - \eta_1(\omega)\|_{E_0}] + \|\xi_0 - \eta_0\|_{E_0} \\
&\le \frac{1}{1-\lambda(\omega)}[\|\xi_0 - \xi_1(\omega)\|_{E_0} + \|\eta_0 - \eta_1(\omega)\|_{E_0}] + \|\xi_0 - \eta_0\|_{E_0}.
\end{aligned} \tag{3.3}$$

In particular if, $\xi_0 = \eta_0$, then $Q(\omega,\xi_0) = Q(\omega,\eta_0)$ which implies that $\xi_1(c,\omega) = \eta_1(c,\omega)$. Hence, from (3.3)

$$\|\xi_n(\omega) - \eta_n(\omega)\|_{E_0} \le \frac{2}{1-\lambda(\omega)}\|\xi_0 - \xi_1(\omega)\|_{E_0}.$$

(c) The sequence $\{\xi_n(\omega)\}$ of measurable functions as constructed in (a) converges to a random fixed point $\xi^*(\omega)$ with PPF dependence. As $\mathcal{R}_c$ is topologically closed, $\xi^*(\omega) \in \mathcal{R}_c$. Suppose that $\eta^*(\omega) \neq \xi^*(\omega)$, $\omega \in \Omega$, are two random fixed points of the random operator $Q(\omega)$ in $\mathcal{R}_c$ with PPF dependence. Then,

$$\begin{aligned}
\|\xi^*(\omega) - \eta^*(\omega)\|_{E_0} &= \|\xi^*(c,\omega) - \eta^*(c,\omega)\|_E \\
&= \|Q(\omega,\xi^*(\omega)) - Q(\omega,\eta^*(\omega))\|_E \\
&\le \alpha(\omega)[\|\xi^*(c,\omega) - Q(\omega,\xi^*(\omega))\|_{E_0} + \|\eta^*(c,\omega) - Q(\omega,\eta^*(\omega))\|_{E_0}] \\
&= 0
\end{aligned}$$

and, therefore, $\eta^*(\omega) = \xi^*(\omega)$ for all $\omega \in \Omega$. Hence the random operator $Q(\omega)$ has a unique random fixed point with PPF dependence in $\mathcal{R}_c$. This completes the proof.

Theorem 3.5. *Let $(\Omega,\mathcal{A})$ be a measurable space and let E be a separable Banach space. If $Q : \Omega \times E_0 \to E$ is a continuous random operator satisfying the condition of Reich type random contraction, then the following statements hold in E :*

(a) If $\mathcal{R}_c$ is algebraically closed with respect to difference, then for a given $\xi_0 \in E_0$ and for a given $c \in I$, every sequence $\{\xi_n(\omega)\}$ of measurable functions satisfying (3.1) *and* (3.2) *converges to a PPF dependent random fixed point of the random operator $Q(\omega)$, i.e. there is a measurable function $\xi^* : \Omega \to E$ such that for each $\omega \in \Omega$,*

$$Q(\omega,\xi^*(\omega)) = Q(\omega)\xi^*(\omega) = \xi^*(c,\omega).$$

(b) Given $\xi_0, \eta_0 \in E_0$, let $\{\xi_n(\omega)\}$ and $\{\eta_n(\omega)\}$ be the sequences of iterates of measurable functions corresponding to ξ_0 and η_0 constructed as in (a). Then,

$$\|\xi_n(\omega) - \eta_n(\omega)\|_{E_0} \le \frac{1}{1-\lambda(\omega)}[\|\xi_0 - \xi_1(\omega)\|_{E_0} + \|\xi_0 - \xi_1(\omega)\|_{E_0}] + \|\xi_0 - \eta_0\|_{E_0},$$

where $\lambda(\omega) = \dfrac{\alpha(\omega)+\gamma(\omega)}{1-\beta(\omega)} < 1$ for all $\omega \in \Omega$.

If, in particular, $\xi_0 = \eta_0$, and $\{\xi_n(\omega)\} \neq \{\eta_n(\omega)\}$, then

$$\|\xi_n(\omega) - \eta_n(\omega)\|_{E_0} \le \frac{2}{1-\lambda(\omega)}\|\xi_0 - \xi_1(\omega)\|_{E_0}.$$

(c) Finally, if $\mathcal{R}_c$ is topologically closed, then for a given $\xi_0 \in E_0$, every sequence $\{\xi_n(\omega)\}$ of iterates of $Q(\omega)$ constructed as in (a), converges to a unique PPF dependent random fixed point $\xi^*(\omega)$ of $Q(\omega)$, i.e. there is a unique measurable function $\xi^* : \Omega \to E_0$ such that $Q(\omega,\xi^*(\omega)) = \xi^*(c,\omega)$ for all $\omega \in \Omega$.

Proof. Let $\xi_0 \in E_0$ be arbitrary. By hypothesis, $Q(\omega,\xi_0) \in E$. Suppose that $Q(\omega,\xi_0) = x_1(\omega)$, where the function $x_1 : \Omega \to E$ is measurable. Choose a measurable function $\xi_1 : \Omega \to E_0$ such that $x_1(\omega) = \xi_1(c,\omega)$ and that

$$\|\xi_1(c,\omega) - \xi_0(c)\|_E = \|\xi_1(\omega) - \xi_0\|_{E_0}.$$

Define a sequence $\{\xi_n(\omega)\}$ of measurable functions from Ω into E_0 inductively so that

$$Q(\omega,\xi_n(\omega)) = \xi_{n+1}(c,\omega)$$

and

$$\|\xi_{n+1}(c,\omega) - \xi_n(c,\omega)\|_E = \|\xi_{n+1}(\omega) - \xi_n(\omega)\|_{E_0}$$

for all $\omega \in \Omega$.

We claim that $\{\xi_n(\omega)\}$ is a Cauchy sequence in E_0. Now for any $n \in \mathbb{N}$ we have the following estimate for each fixed $\omega \in \Omega$,

$$\begin{aligned}
\|\xi_n(\omega) - \xi_{n+1}(\omega)\|_{E_0} &= \|\xi_n(c,\omega) - \xi_{n+1}(c,\omega)\|_E \\
&= \|Q(\omega,\xi_{n-1}(\omega)) - Q(\omega,\xi_n(\omega))\|_{E_0} \\
&\le \alpha(\omega)\,\|\xi_{n-1}(c,\omega) - Q(\omega,\xi_{n-1}(\omega))\|_E + \beta(\omega)\|\xi_n(c,\omega) - Q(\omega,\xi_n(\omega))\|_E \\
&\qquad + \gamma(\omega)\|\xi_{n-1}(c,\omega) - Q(\omega,\xi_n(\omega))\|_E \\
&\le \alpha(\omega)\|\xi_{n-1}(c,\omega) - \xi_n(c,\omega)\|_E + \beta(\omega)\|\xi_n(c)(\omega) - \xi_{n+1}(c,\omega)\|_E \\
&\qquad + \gamma(\omega)\|\xi_{n-1}(c,\omega) - Q(\omega,\xi_n(\omega))\|_E \\
&\le [\alpha(\omega) + \gamma(\omega)]\|\xi_{n-1}(\omega) - \xi_n(\omega)\|_{E_0} + \beta(\omega)\|\xi_n(\omega) - \xi_{n+1}(\omega)\|_{E_0}.
\end{aligned}$$

From the above inequality, it follows that

$$\|\xi_n(\omega) - \xi_{n+1}(\omega)\|_{E_0} \le \lambda(\omega)\|\xi_{n-1}(\omega) - \xi_n(\omega)\|_{E_0}$$

for all $n = 1,2,\ldots$, where $\lambda(\omega) = \dfrac{\alpha(\omega) + \gamma(\omega)}{1 - \beta(\omega)} < 1$ foe all $\omega \in \Omega$.

By induction,

$$\|\xi_n(\omega) - \xi_{n+1}(\omega)|_{E_0} \le \lambda^n(\omega)\|\xi_0 - \xi_1(\omega)\|_{E_0}$$

for all $n = 1,2,\ldots$. The rest of the proof is similar to that of Theorem 3.4 and we omit the details.

On taking $\alpha(\omega) = \beta(\omega) = 0$ for all $\omega \in \Omega$ in Theorem 3.5, then we obtain the following result proved in Dhage [5] as corollary.

Corollary 3.6 (Dhage [5]). *Let* $(\Omega,\mathcal{A})$ *be a measurable space and let E be a separable Banach space. If* $Q : \Omega \times E_0 \to E$ *is a continuous random operator satisfying the condition* (2.4) *of random contraction, then the statements (a), (b) and (c) of Theorem 3.4 hold in E.*

Remark 3.7. If the Razumikhin class $\mathcal{R}_c$ of functions in E_0 is not topologically closed, then the sequence $\{\xi_n(\omega)\}$ of measurable functions as constructed in (a) of Theorems 3.4 and 3.5 may converge to a random fixed point with PPF dependence of the random operator $Q(\omega)$ outside the set $\mathcal{R}_c$ which may not be unique. However, we have a nice flexibility in the approximations of the PPF dependent random fixed point of the random operator $Q(\omega)$.

4 Functional Random Differential Equations

In this section, we apply the abstract results of the previous section to periodic boundary value problems (PBVP) of functional random differential equations for proving the existence of PPF dependent random solutions under a Lipschitz type condition.

Given the closed and bounded intervals $I_0 = [-r, 0]$ and $I = [0, T]$ in $\mathbb{R}$, the set of real numbers, for some real numbers $r > 0$, $T > 0$, let C denote the space of continuous real-valued functions defined on I_0. We equip the space C with the supremum norm $\|\cdot\|_C$ defined by

$$\|\xi\|_C = \sup_{\theta \in I_0} |\xi(\theta)|.$$

It is clear that C is a Banach space with this norm, called the history space of the problem under consideration.

For each $t \in I = [0, T]$, define a function $t \to x_t \in C$ by

$$x_t(\theta) = x(t+\theta),\ \theta \in I_0,$$

where the argument θ represents the delay in the argument of solutions.

Now we are equipped with the necessary details to study the nonlinear problems of functional random differential equations for existence and uniqueness results.

4.1 PBVP of functional random differential equations

Let $(\Omega, \mathcal{A})$ be a measurable space and let E be a given Banach space. By a mapping $x : \Omega \to C(J, \mathbb{R})$ we denote a function $x(t, \omega)$ which is continuous in the variable t for each $\omega \in \Omega$. In this case, we also write $x(t, \omega) = x(\omega)(t)$.

Given the measurable functions $\phi : \Omega \to C$ and $x : \Omega \to C(I, \mathbb{R})$, consider a periodic boundary value problem of functional random differential equations of delay type (in short FRDE),

$$\left.\begin{aligned} x'(t,\omega) &= f(t, x_t(\omega), x(t,\omega), \omega) \\ x_0(\omega) &= \phi(\omega) \\ x(0,\omega) &= \phi(0,\omega) = x(T,\omega) \end{aligned}\right\} \tag{4.1}$$

for all $t \in I$ and $\omega \in \Omega$, where $f : I \times C \times \mathbb{R} \times \Omega \to \mathbb{R}$.

By a random solution x of the FRDE (4.1) we mean a measurable function $x : \Omega \to C(J, \mathbb{R})$ that satisfies the equations in (4.1) on J, where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on $J = I_0 \cup I$.

The functional random differential equation (4.1) is not new to the theory of nonlinear functional random differential equations and the existence and uniqueness theorems for FRDE (4.1) are obtained by using the random version of classical fixed point theorems of Schauder and Banach respectively. However, the novelty of the present paper lies in the nice applicability of Theorem 3.6 for proving the existence of random solutions with PPF dependence for the FRDE (4.1) defined on J.

Now, consider the PBVP of first order functional random differential equation

$$\left.\begin{aligned} x'(t,\omega)+h(t)x(t,\omega) &= f_h(t,x_t(\omega),x(t,\omega),\omega)\\ x_0(\omega) &= \phi(\omega)\\ x(0,\omega)=\phi(0,\omega) &= x(T,\omega)\end{aligned}\right\} \tag{4.2}$$

for all $t\in I$ and $\omega\in\Omega$, where $h: I\to\mathbb{R}_+$ is a continuous function and $f_h: I\times C\times\mathbb{R}\times\Omega\to\mathbb{R}$ is a function defined by

$$f_h(t,x,y,\omega) = f(t,x,y,\omega)+h(t)y. \tag{4.3}$$

Remark 4.1. Notice that x is a random solution of the FRDE (4.1) if and only if it is random solution of the FRDE (4.2) on J.

We consider the following hypotheses in what follows.

(H$_1$) The function $\omega\mapsto f_h(t,x,y,\omega)$ is measurable for each $t\in I$, $x\in C$ and $y\in\mathbb{R}$ and the function $(t,x,y)\mapsto f(t,x,y,\omega)$ is jointly continuous for each $\omega\in\Omega$.

(H$_2$) There exists a real number $M_{f_h}>0$ such that for each $\omega\in\Omega$,

$$|f_h(t,x,y,\omega)|\le M_{f_h}$$

for all $t\in I$, $x\in C$ and $y\in\mathbb{R}$.

(H$_3$) There exist real numbers $L_1>0$ and $L_2>0$ such that for each $\omega\in\Omega$,

$$|f_h(t,x_1,x_2,\omega)-f_h(t,y_1,y_2,\omega)|\le L_1\|x_1-y_1\|_C+L_2|x_2-y_2|$$

for all $t\in I$, $x_1,y_1\in C$ and $x_2,y_2\in\mathbb{R}$.

Using the variation of constant formula for FRDE (4.2), we obtain

Lemma 4.2. *If the hypotheses (H$_1$)-(H$_2$) hold, then the FRDE* (4.2) *is equivalent to the functional random integral equation (in short FRIE)*

$$x(t,\omega)=\begin{cases}\displaystyle\int_0^T G_h(t,s)f_h(s,x_s(\omega),x(s,\omega),\omega)\,ds, & \text{if } t\in I,\\ \phi(t,\omega), & \text{if } t\in I_0.\end{cases} \tag{4.4}$$

for all $t\in I$ and $\omega\in\Omega$, where G is a Green's function defined by

$$G_h(t,s)=\begin{cases}\dfrac{e^{H(s)-H(t)+H(T)}}{e^{H(T)}-1}, & 0\le s\le t\le T,\\[2ex] \dfrac{e^{H(s)-H(t)}}{e^{H(T)}-1}, & 0\le t<s\le T,\end{cases} \tag{4.5}$$

where $H(t)=\displaystyle\int_0^t h(s)\,ds.$

It is clear that the function G_h is continuous and nonnegative on $I \times I$ and so the number $M_{G_h} = \sup_{t,s \in I} G_h(t,s)$ exists. Our main existence result is the following.

Theorem 4.3. *Assume that the hypotheses* (H_1) *through* (H_3) *hold. Furthermore, if* $(L_1 + L_2)M_{G_h}T < 1$*, then the FRDE* (4.1) *has a a unique PPF dependent random solution* $\xi^*(\omega)$ *defined on J.*

Proof. Set $E = C(J,\mathbb{R})$. Then E is a Banach space with respect to the usual supremum norm $\|\cdot\|_E$ defined by

$$\|x\|_E = \sup_{t \in J} |x(t)|.$$

Clearly, E is a separable Banach space. Given a function $x \in C(J,\mathbb{R})$, define a mapping $\hat{x} : I \to C$ by $\hat{x}(t) = x_t \in C$ so that $\hat{x}(t)(0) = x_t(0) = x(t), t \in J$ and $\hat{x}(0) = x_0$.
Define a set $\widehat{E}$ of functions by

$$\widehat{E} = \{\hat{x} = (x_t)_{t \in I} : x_t \in C, x \in C(I,\mathbb{R}) \text{ and } x_0 = \phi\}.$$

Define a norm $\|\hat{x}\|_{\widehat{E}}$ in $\widehat{E}$ by

$$\|\hat{x}\|_{\widehat{E}} = \sup_{t \in I} \|x_t\|_C.$$

Clearly, $\hat{x} \in C(I_0,\mathbb{R}) = C$. Next we show that $\widehat{E}$ is a Banach space. Consider a Cauchy sequence $\{\hat{x}_n\} = \{(x_t^n)_{t \in I}\}$ in $\hat{E}$. Then, $\{(x_t^n)\}$ is a Cauchy sequence in C for each $t \in I$. This further implies that $\{x_t^n(s)\}$ is a Cauchy sequence in $\mathbb{R}$ for each $s \in [-r,0]$. Then $\{x_t^n(s)\}$ converges to $x_t(s)$ for each $t \in I_0$. Since $\{x_t^n\}$ is a sequence of uniformly continuous functions for a fixed $t \in I$, $x_t(s)$ is also continuous in $s \in [-r,0]$. Hence the sequence $\{\hat{x}_n\}$ converges to $\hat{x} \in \hat{E}$. As a result, $\widehat{E}$ is complete. Moreover, $\widehat{E}$ is a separable Banach space.

Given a measurable function $\hat{x} : \Omega \to \widehat{E}$, consider the operator $Q : \Omega \times \widehat{E} \to \mathbb{R}$ defined by

$$\begin{aligned} Q(\omega, \hat{x}(\omega)) &= Q(\omega, x_t(\omega)) \\ &= \begin{cases} \displaystyle\int_0^T G_h(t,s) f_h(s, x_s(\omega), x(s,\omega), \omega)\, ds, & \text{if } t \in I, \\ \phi(t,\omega), & \text{if } t \in I_0. \end{cases} \end{aligned}$$

Then the FRIE (4.4) is equivalent to the random operator equation

$$Q(\omega, \hat{x}(\omega)) = \hat{x}(0,\omega).$$

Define a sequence $\{\hat{x}_n(\omega)\}$ of measurable functions by

$$\left.\begin{aligned} &(i)\ Q(\omega, \hat{x}_n(\omega)) = \hat{x}_{n+1}(0,\omega), \\ &(ii)\ \|\hat{x}_n(\omega) - \hat{x}_{n+1}(\omega)\|_{E_0} = \|\hat{x}_n(0,\omega) - \hat{x}_{n+1}(0,\omega)\|_E \end{aligned}\right\} \tag{4.6}$$

for $n = 1,2,\ldots$.

We shall show that the operator Q satisfies all the conditions of Theorem 3.4. Firstly, we show that Q is a random operator on $\Omega \times \widehat{E}$. Since hypothesis ($\mathrm{H}_1$) holds, by Carathéodory theorem, the function $\omega \to f(t,x,y,\omega)$ is measurable for all $t \in I$, $x \in C$ and $y \in \mathbb{R}$. As integral is the limit of the finite sum of finite-valued measurable functions, the map

$$\omega \mapsto \int_0^T G_h(t,s) f_h(s, x_s(\omega), x(s,\omega), \omega)\, ds$$

is measurable. Hence, the operator $Q(\omega, \hat{x})$ is measurable in ω for each $\hat{x} \in \widehat{E}$. As a result, $Q(\omega)$ is a random operator on $\widehat{E}$ into E.

Secondly, we show that the random operator $Q(\omega)$ is continuous on $\widehat{E}$. Let $\omega \in \Omega$ be fixed. We show that the continuity of the random operator $Q(\omega)$ in the following two cases:

Case I: Let $t \in [0,T]$ and let $\{\hat{x}_n(\omega)\}$ be a sequence of points in $\widehat{E}$ such that $\hat{x}_n(\omega) \to \hat{x}(\omega)$ as $n \to \infty$. Then, by dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty} Q(\omega, \hat{x}_n(\omega)) &= \lim_{n\to\infty} \int_0^T G_h(t,s) f_h(s, x_s^n(s,\omega), x_n(s,\omega), \omega)\, ds \\
&= \lim_{n\to\infty} \int_0^T G_h(t,s) f_h(s, x_s^n(\omega), x_n(s,\omega), \omega)\, ds \\
&= \int_0^T \lim_{n\to\infty} G_h(t,s) f_h(s, x_s^n(\omega), x_n(s,\omega), \omega)\, ds \\
&= \int_0^T G_h(t,s) G_h(t,s) f_h(s, x_s(\omega), x(s,\omega), \omega)\, ds \\
&= Q(\omega, \hat{x}(\omega))
\end{aligned}$$

for all $t \in [0,T]$ and for each fixed $\omega \in \Omega$.

Case II: Suppose that $t \in [-r, 0]$. Then we have:

$$|Q(\omega, \hat{x}_n(\omega)) - Q(\omega, \hat{x}(\omega))| = |\phi(t,\omega) - \phi(t,\omega)| = 0$$

for each fixed $\omega \in \Omega$. Hence,

$$\lim_{n\to\infty} Q(\omega)\hat{x}_n(\omega) = Q(\omega)\hat{x}(\omega)$$

for all $t \in [-r, 0]$ and $\omega \in \Omega$. Now combining the Case I with Case II, we conclude that $Q(\omega)$ is a pointwise continuous random operator on $\widehat{E}$ into itself.

Next we show that the family of functions $\{Q(\omega, \hat{x}_n(\omega)\}$ is a uniformly continuous set in E for a fixed $\omega \in \Omega$. We consider the following three cases:

Case I: Let $\epsilon > 0$ and let $t_1, t_2 \in [0,T]$ be arbitrary. Then, we have

$$\begin{aligned}
|Q(\omega, x_{t_1}^n(\omega)) - Q(\omega, x_{t_2}^n(\omega))| &\leq \bigg| \int_0^T G_h(t_1,s) f_h(s, x_s^n(\omega), x_n(s,\omega), \omega)\, ds \\
&\qquad - \int_0^T G_h(t_2,s) f_h(s, x_s^n(\omega), x_n(s,\omega), \omega)\, ds \bigg| \\
&\leq \bigg| \int_0^T |G_h(t_1,s) - G_h(t_2,s)| |f_h(s, x_s^n(\omega), x_n(s,\omega), \omega)| ds \bigg| \\
&\leq M_{f_h} \int_0^T |G_h(t_1,s) - G_h(t_2,s)| ds.
\end{aligned}$$

Since G_h is continuous on compact $I \times I$, it is uniformly continuous there. Hence for $\epsilon > 0$, choose $\delta_1 > 0$ such that if $|t_1 - t_2| < \delta_1$, then

$$|Q(\omega, x_{t_1}^n(\omega)) - Q(\omega, x_{t_2}^n(\omega))| < \frac{M_{f_h} T \epsilon}{2(M_{f_h} T + 1)}$$

uniformly for $t_1, t_2 \in I$ and $\hat{x}_n \in E_0$.

Case II: Let $t_1, t_2 \in [-r, 0]$ be arbitrary. Since $t \mapsto \phi(t, \omega)$ is continuous on a compact $[-r, 0]$, it is uniformly continuous there. Hence for above $\epsilon > 0$ there exists a $\delta_2 > 0$ such that $|t_1 - t_2| < \delta_2$ implies

$$|Q(\omega, x_{t_1}^n(\omega)) - Q(\omega, x_{t_2}^n(\omega))| = |\phi(t_1, \omega) - \phi(t_2, \omega)| \le \frac{\epsilon}{2(M_{f_h} T + 1)}$$

uniformly for $t_1, t_2 \in I$ and $\hat{x}_n \in E_0$.

Case III: Let $t_1 \in [-r, 0]$ and $t_2 \in [0, T]$ be arbitrary. Choose $\delta = \min\{\delta_1, \delta_2\}$. Then, $|t_1 - t_2| < \delta$ implies

$$\begin{aligned}
|Q(\omega, x_{t_1}^n(\omega)) - Q(\omega, x_{t_2}^n(\omega))| &\le |Q(\omega, x_{t_1}^n(\omega)) - Q(\omega, x_0^n(\omega))| + |Q(\omega, x_0^n(\omega)) - Q(\omega, x_{t_2}^n(\omega))| \\
&< \frac{\epsilon}{2(M_{f_h} T + 1)} + \frac{\epsilon M_{f_h} T}{2(M_{f_h} T + 1)} \\
&= \epsilon
\end{aligned}$$

uniformly for $\hat{x}_n \in E_0$.

Thus, in all three cases, $|t_1 - t_2| < \delta$ implies

$$|Q(\omega, x_{t_1}^n(\omega)) - Q(\omega, x_{t_2}^n(\omega))| < \epsilon$$

uniformly for all $t_1, t_2 \in J$ and $\hat{x}_n \in E_0$. This shows that $\{Q(\omega, \hat{x}_n)\}$ is a sequence of uniformly continuous functions on J. Hence, it converges uniformly on J. Hence, $Q(\omega, \hat{x})$ is a continuous random operator on E_0 for a fixed $\omega \in \Omega$.

Finally, we show that Q is a strong random contraction on $\widehat{E}$. Let $\omega \in \Omega$ be fixed. Then,

$$\begin{aligned}
\|Q(\omega, \hat{x}(\omega)) - Q(\omega, \hat{y}(\omega))\|_E &= \|Q(\omega, x_t(\omega)) - Q(\omega, y_t(\omega))\|_E \\
&= \sup_{t \in I} \Big| \int_0^T G_h(t, s) f_h(s, x_s(\omega), x(s, \omega), \omega)\, ds \\
&\qquad - \int_0^T G_h(t, s) f_h(s, y_s(\omega), y(s, \omega), \omega)\, ds \Big| \\
&\le \int_0^T L_1 M_{G_h} \|x_s(\omega) - y_s(\omega)\|_C\, ds \\
&\qquad + \int_0^T L_2 M_{G_h} \|x(\omega) - y(\omega)\|_E\, ds \\
&\le \int_0^T (L_1 + L_2) M_{G_h} \|\hat{x}(\omega) - \hat{y}(\omega)\|_{\widehat{E}} ds \\
&\le (L_1 + L_2) M_{G_h} T \|\hat{x}(\omega) - \hat{y}(\omega)\|_{\widehat{E}}
\end{aligned}$$

for all $\hat{x}(\omega), \hat{y}(\omega) \in \widehat{E}$. Hence, Q is a random contraction on $\widehat{E}$ with contraction constant $\alpha = (L_1 + L_2)M_{G_h}T < 1$.

Thus, the condition (a) of Theorem 3.4 is satisfied. Hence, an application of Theorem 3.4(a) yields that the functional random integral equation (4.1) has a random solution with PPF dependence defined on J. This further implies that the FRDE (4.1) has a PPF dependent random solution ξ^* defined on J and the sequence $\{\xi_n(\omega)\}$ of measurable functions constructed as in (4.6) converges to ξ^*. Moreover, here the Razumikhin class $\mathcal{R}_0, 0 \in [-r, T]$ is $C([0,T],\mathbb{R})$ which is topologically and algebraically closed with respect to difference, and thus by Theorem 3.4(c), ξ^* is a unique random solution with PPF dependence for the the FRDE (4.1) defined on J. This completes the proof.

4.2 PBVP of hybrid random differential equations

Given the functions $\phi : \Omega \to C$ and $x : \Omega \to C(I,\mathbb{R})$, consider a boundary value problem of functional random hybrid differential equations of delay type (in short FRDE),

$$\left.\begin{aligned} x'(t,\omega) &= f(t, x_t(\omega), x(t,\omega), \omega) + g(t, x(t,\omega), \omega) \\ x_0(\omega) &= \phi(\omega) \\ x(0,\omega) &= \phi(0,\omega) = x(T,\omega) \end{aligned}\right\} \tag{4.7}$$

for all $t \in I$ and $\omega \in \Omega$, where $f : I \times C \times \mathbb{R} \times \Omega \to \mathbb{R}$ and $g : I \times \mathbb{R} \times \Omega \to \mathbb{R}$.

By a random solution x of the FRDE (4.7) we mean a measurable function $x : \Omega \to C(J,\mathbb{R})$ that satisfies the equations in (4.7) on J.

The functional random differential equation (4.7) is not new to the theory of nonlinear functional differential equations and the details of the classifications of different types of nonlinear differential equations appear in Dhage [5]. The existence theorems for the FRDE (4.7) are generally proved by using the hybrid fixed point theorems of Krasnoselskii and Dhage type. See for example, Krasnoselskii [11], Dhage [5] and the references given therein. We apply the following PPF dependent random fixed point theorem of Dhage [6] in what follows.

Theorem 4.4 (Dhage [6])**.** *Let $(\Omega, \mathcal{A})$ be a measurable space and let E be a separable Banach space. Suppose that $A : \Omega \times E_0 \to E$ and $B : \Omega \times E \to E$ are two continuous random operators satisfying for each $\omega \in \Omega$*

(a) $A(\omega)$ is strong random contraction, and

(b) B is compact on $\Omega \times E$.

If $\mathcal{R}_c$ is a topologically and algebraically closed with respect to difference, then for a given $c \in I$, the random operator equation

$$A(\omega, \xi(\omega)) + B(\omega, \xi(\omega, c)) = \xi(\omega, c)$$

has a random solution with PPF dependence, i.e. for a given $c \in I$, there is a measurable function $\xi^* : \Omega \to E_0$ such that

$$A(\omega, \xi^*(\omega)) + B(\omega, \xi^*(c, \omega)) = \xi^*(c, \omega)$$

for all $\omega \in \Omega$.

In the following we prove an existence of PPF dependent random solutions for the FRDE (4.7) defined on J under the mixed Lipschitz and compactness type conditions on the nonlinearities involved in (4.7).

Now, consider the PBVP of first order functional random hybrid differential equation,

$$\left.\begin{aligned} x'(t,\omega)+h(t)x(t,\omega) &= f_h(t,x_t(\omega),x(t,\omega),\omega)+g(t,x(t,\omega),\omega)\\ x_0(\omega) &= \phi(\omega)\\ x(0,\omega)=\phi(0,\omega) &= x(T,\omega)\end{aligned}\right\} \tag{4.8}$$

for all $t\in I$ and $\omega\in\Omega$, where $h: I\to\mathbb{R}_+$ is a continuous function and the function $f_h: I\times C\times\mathbb{R}\times\Omega\to\mathbb{R}$ is defined by (4.3).

Remark 4.5. Notice that x is a random solution of the FRDE (4.7) if and only if it is a random solution of the FRDE (4.8) on J.

We consider the following hypothesis in what follows.

(H_4) There exist real number $L_1>0$ and $L_2>0$ such that for each $\omega\in\Omega$,

$$|f_h(t,x_1,x_2,\omega)-f_h(t,y_1,y_2,\omega)|\le L_1\|x_1(0)-y_1(0)\|_C+L_2|x_2-y_2|$$

for all $t\in I$, $x_1,y_1\in C$ and $x_2,y_2\in\mathbb{R}$, for all $t\in I$ and $x,y\in\mathbb{R}$.

(H_5) The function $\omega\mapsto g(t,x,\omega)$ is measurable for each $t\in I$ and $x\in\mathbb{R}$ and the function $(t,x)\mapsto g(t,x,\omega)$ is jointly continuous for each $\omega\in\Omega$.

(H_6) There exists a real number $M_g>0$ such that for each $\omega\in\Omega$,

$$|g(t,x,\omega)|\le M_g$$

for all $t\in I$ and $x\in\mathbb{R}$.

Note that hypothesis (H_4) is stronger than the hypothesis (H_3) in the sense that (H_4) implies (H_3), however, the converse is not true.

Theorem 4.6. *Assume that the hypotheses (H_1)-(H_2) and (H_4)-(H_6) hold. Furthermore, if $(L_1+L_2)M_{G_h}T<1$, then the FRDE* (4.7) *has a PPF dependent random solution defined on J.*

Proof. The FRDE (4.7) is equivalent to the nonlinear functional random integral equation (in short FRIE)

$$x(t,\omega)=\begin{cases}\displaystyle\int_0^T G_h(t,s)f_h(s,x_s(\omega),x(s,\omega),\omega)\,ds & \\ \displaystyle\qquad+\int_0^T G_h(t,s)g(s,x(s,\omega),\omega)\,ds, & \text{if } t\in I,\\ \phi(t,\omega), & \text{if } t\in I_0.\end{cases} \tag{4.9}$$

where the Green's function G_h is defined by (4.5).

Define two separable Banach spaces E and $E_0 = \widehat{E}$ as in the proof of Theorem 4.3. Given a measurable function $\hat{x} : \Omega \to \widehat{E}$, consider the operators $A : \Omega \times \widehat{E} \to \mathbb{R}$ and $B : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} A(\omega, \hat{x}(\omega)) &= A(\omega, x_t(\omega)) \\ &= \begin{cases} \displaystyle\int_0^T G_h(t,s) f(s, x_s(\omega), x(s,\omega), \omega)\, ds, & \text{if } t \in I, \\ \phi(t,\omega), & \text{if } t \in I_0. \end{cases} \end{aligned}$$

and

$$B(\omega, x(t,\omega)) = \begin{cases} \displaystyle\int_0^T G_h(t,s) g(s, x(s,\omega), \omega)\, ds, & \text{if } t \in I, \\ 0, & \text{if } t \in I_0. \end{cases}$$

Then the FRIE (4.9) is equivalent to the operator equation

$$A(\omega, \hat{x}(\omega)) + B(\omega, \hat{x}(0,\omega)) = \hat{x}(0,\omega).$$

We shall show that the operators A and B satisfy all the conditions of Theorem 4.4. It can be shown on the similar lines as in the proof of Theorem 4.3 that $A(\omega)$ and $B(\omega)$ are continuous random operators on $\widehat{E}$ and E respectively. Next, we prove that $A(\omega)$ is a strong contraction random operator on E_0. Let $\omega \in \Omega$ be fixed. Then,

$$\begin{aligned} \|A(\omega, \hat{x}(\omega)) - A(\omega, \hat{y}(\omega))\|_E &= \|A(\omega, x_t(\omega)) - A(\omega, y_t(\omega))\|_E \\ &\le \sup_{t\in I} \bigg| \int_0^T G_h(t,s) f_h(s, x_s(\omega), x(s,\omega), \omega)\, ds \\ &\qquad - \int_0^T G_h(t,s) f_h(s, y_s(\omega), y(s,\omega), \omega)\, ds \bigg| \\ &\le \int_0^T M_{G_h} L_1 \|x_s(0,\omega) - y_s(0,\omega)\|_C\, ds \\ &\qquad + \int_0^T M_{G_h} L_2 |x(s,\omega) - y(s,\omega)|\, ds \\ &\le \int_0^T (L_1 + L_2) M_{G_h} \|\hat{x}(0,\omega) - \hat{y}(0,\omega)\|_E\, ds \\ &\le (L_1 + L_2) M_{G_h} T\, \|\hat{x}(0,\omega) - \hat{y}(0,\omega)\|_E \end{aligned}$$

for all $\hat{x}(\omega), \hat{y}(\omega) \in \widehat{E}$. Hence, A is a strong random contraction on $\widehat{E}$ with contraction constant $\alpha = (L_1 + L_2) M_{G_h} T < 1$. Next, we show that $B(\omega)$ is a compact random operator on E. Let $\{x_n(\omega)\}$ be a sequence of measurable functions on Ω into E To finish, it is enough to show that $\{B(\omega, x_n(\omega))\}$ has a convergent subsequence. Now, using the standard arguments, it is shown that $\{B(\omega, x_n(\omega))\}$ is a uniformly bounded and equicntinuous set in E. Therefore, we apply Arzelá-Ascoli theorem and conclude that B is a compact random operator on $\Omega \times E$ into E. Thus, $A(\omega)$ and $B(\omega)$ satisfy all the conditions of Theorem 4.4. Again, the Razumikhin class $\mathcal{R}_0, 0 \in [-r, T]$ is $C([0,T], \mathbb{R})$ which is topologically and algebraically closed with respect to difference. Hence, by Theorem 4.4, the FRIE (4.9) and consequently FRDE (4.7) has a random solution with PPF dependence defined on J. This completes the proof.

5 Conclusion

Finally, we conclude this paper with the remark that the random fixed point theorems with PPF dependence proved here are very fundamental in random fixed point theory involving geometric hypothesis of distance between the images and objects in question. However, using the principle that has been formulated in Theorems 3.4 and 3.5 several random fixed point theorems with PPF dependence can be proved in separable Banach space. In a forthcoming paper, we plan to prove some PPF dependent random fixed point theorems for the random operators satisfying certain generalized contraction conditions and apply them to some random differential equations different from those considered in this paper.

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