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LOCAL EXISTENCE AND GLOBAL CONTINUATION FOR SOME PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS

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Abstract

In this work, we study the local existence of mild solutions for some partial functional integrodifferential equations. We suppose that the linear part has a resolvent operator in the sense of Grimmer [13]. The non linear part is just assumed to be continuous. We have also that the solution may blow up in finite time. An application is provided to illustration.

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1 Introduction

In this paper, we study the existence results for partial functional integrodifferential equation with finite delay in the following form

$$\begin{cases} u'(t) = Au(t) + \int_0^t B(t-s)u(s)ds + f(t,u_t) \quad \text{for } t \geq 0 \\ u_0 = \varphi \in C = C([-r,0];X), \end{cases} \tag{1.1}$$

where $A : D(A) \to X$ is a closed linear operator on a Banach space X, for $t \geq 0, B(t)$ is a closed linear operator on X with domain $D(B) \supset D(A)$ is time-independent, $C([-r,0];X)$ is the Banach space of all continuous functions from $[-r,0]$ to X endowed with the uniform

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norm topology, and f is a given function defined on $[0,+\infty)\times C$ with values in X. For $u \in C([-r,+\infty),X)$ and for every $t \geq 0$, u_t denotes the history function of C defined by

$$u_t(\theta) = u(t+\theta) \text{ for } \theta \in [-r,0].$$

Partial functional differential equations arise in a variety of areas of biological, physical, and engineering applications, see, for example, the books of Hale [14], Hale and Verduyn Lunel [15], Kolmanovskii and Myshkis [17] and Wu [22], and the papers of Gurtin and Pipkin [12], Miller [19], and the references therein.

In, the recent years, many authors have attracted much attention to the study of existence problems for differential and integrodifferential equations. We refer to [1, 2, 5, 7, 8, 9, 10, 11, 16] and [20] where numerous approaches that are commonly used: the contraction mapping principle, Leray-Schauder alternative, Schauder and Sadovskii fixed point theorems. In [10, 11], Ezzinbi and al., by using the contraction principle have established the local existence and regularity of solutions for Eq.(1.1) with finite and infinite delay. Recently, in [8, 9], Eq.(1.1) has been also studied extensively in neutral case. The authors obtained some results on the existence and regularity of solutions using the theory of resolvent operators and the Banach fixed point theorem.

The present work can be viewed as a continuation of the recent results on this issue. Here we compose the above results and prove the existence of mild solutions for problem (1.1), relying on a Schauder fixed point theorem. We suppose that the strongly continuous semigroup generated by A is compact and that the nonlinear part is continuous.

The paper is organized as follows. In Section 2, we recall some necessary preliminaries. In Section 3, we study the local existence of the mild solutions of Eq. (1.1), we show the global continuation of solutions and we prove that the solutions may blow up in finite time. Our approach is based on resolvent operator theory and Schauder fixed point theorem. Finally in Section 4, an example is presented which illustrates the main results.

2 Preliminaries

In this section, we introduce some definitions and preliminary facts which are used throughout this paper. Let Z and W be Banach spaces. We denote by $\mathcal{L}(Z,W)$ the Banach space of bounded linear operators from Z into W endowed with the operator norm and we abbreviate this notation to $\mathcal{L}(Z)$ when $Z = W$.

Throughout this paper, we assume that

(**H1**) A is densely defined, closed linear operator in a Banach space $(X,|.|)$. Hence $D(A)$ endowed with the graph norm $\|x\| := |Ax| + |x|$ is a Banach space which will be denoted by $(Y,\|.\|)$.

(**H2**) $(B(t))_{t\geq 0}$ is a family of linear operators on X so that $B(t)$ is continuous when regarded as a linear map from Y into X for almost all $t \geq 0$. Moreover, there is a locally integrable function $b:\mathbb{R}^+ \to \mathbb{R}^+$ so that $B(t)y$ is measurable and $|B(t)y| \leq b(t)\|y\|$ for all $y \in Y$ and $t \geq 0$.

(**H3**) For any $y \in Y$, the map $t \rightarrow B(t)y$ belongs to $W^{1,1}_{loc}(\mathbb{R}^+, X)$ and

$$\left|\frac{d}{dt}B(t)y\right| \leq b(t)\|y\| \text{ for } y \in Y \text{ and } t \in \mathbb{R}^+.$$

Now, we consider the following integrodifferential equation

$$\begin{cases} y'(t) = Ay(t) + \int_0^t B(t-s)y(s)ds \text{ for } t \geq 0 \\ y(0) = y_0 \in X. \end{cases} \tag{2.1}$$

Definition 2.1. [13]. A resolvent operator for Eq. (2.1) is a bounded linear operator valued function $R(t) \in \mathcal{L}(X)$ for $t \geq 0$ having the following properties:
(**a**) $R(0) = I$ and $|R(t)| \leq Me^{\beta t}$ for some constants M and β.
(**b**) For each $x \in X, R(t)x$ is strongly continuous for $t \geq 0$.
(**c**) $R(t) \in \mathcal{L}(Y)$ for $t \geq 0$. For $x \in Y, R(.)x \in C^1([0,+\infty); X) \cap C([0,+\infty); Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds \text{ for } t \geq 0. \end{aligned}$$

For properties on resolvent operators theory, we refer the interested reader to the papers [4, 13]. The following theorem gives an existence result of the resolvent operator for Eq. (2.1).

Theorem 2.2. *[6]. Assume that* (**H1**)-(**H3**) *hold. Then Eq. (2.1) admits a resolvent operator iff A generates a* C_0*-semigroup.*

For the next, we suppose that

(**H4**) A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X.

Lemma 2.3. *[6]. Assume that* (**H1**)-(**H4**) *hold. Then there exists a constant* $H = H(t)$ *such that*

$$|R(s+h) - R(h)R(s)| \leq Hh \text{ for } 0 < h \leq s \leq t.$$

Theorem 2.4. *[6]. Assume that* (**H1**)-(**H4**) *hold. Let* $T(.)$ *be a compact for* $t > 0$. *Then the corresponding resolvent operator* $R(.)$ *of Eq. (2.1) is also compact for* $t > 0$.

The following theorem gives the continuity of the resolvent operator in the uniform operator topology.

Theorem 2.5. *[18]. Assume that* (**H1**)-(**H4**) *hold. Let* $T(.)$ *be a compact for* $t > 0$. *Then the corresponding resolvent operator* $R(.)$ *of Eq. (2.1) is operator norm continuous for* $t > 0$.

Next, we introduce the Kuratowski measure of noncompactness $\alpha(.)$ defined on each bounded subset B of the Banach space X by

$$\alpha(B) = \inf\{d > 0; B \text{ can be covered by a finite number of sets of diameter } < d\}.$$

Some basic properties of $\alpha(.)$ are given in the following Lemma.

Lemma 2.6. *[3]. Let X be a Banach space and $B,C \subseteq X$ be bounded, Then*

(1) $\alpha(B) = 0$ *if and only if B is relatively compact;*
(2) $\alpha(B) = \alpha(\overline{B}) = \alpha(\overline{co}B)$*, where $\overline{co}B$ is the closed convex hall of B;*
(3) $\alpha(B) \leq \alpha(C)$ *when $B \subseteq C$;*
(4) $\alpha(B+C) \leq \alpha(B) + \alpha(C)$*;*
(5) $\alpha(B \cup C) \leq \max\{\alpha(B), \alpha(C)\}$*;*
(6) $\alpha(B(0,r)) \leq 2r$*, where $B(0,r) = \{x \in X : |x| \leq r\}$.*

The key tool in our approach is Schauder's fixed point theorem [21].

Theorem 2.7. *Let E be a Banach space and K be a nonempty bounded closed convex subset of E. Let f be a continuous mapping of K into itself such that $f(K)$ is relatively compact. Then f has a fixed point in K.*

3 Main results

3.1 Local existence of mild solutions

Definition 3.1. A continuous function $u : [-r, +\infty) \to X$ is said to be a mild solution of Eq. (1.1) if: $u_0 = \varphi$ and

$$u(t) = R(t)\varphi(0) + \int_0^t R(t-s)f(s,u_s)ds \text{ for } t \geq 0.$$

In what follows, we suppose that

(**H5**) The C_0-semigroup $T(.)$ is compact for $t > 0$.

Theorem 3.2. *Assume that* (**H1**)-(**H5**) *hold. Let $a > 0$, Ω an open subset of the Banach space C and $f : [0,a] \times \Omega \to X$ be continuous. Then for each $\varphi \in \Omega$, there exists $b \in (0,a]$ and a mild solution $u = u(.,\varphi)$ of Eq. (1.1) on $[-r,b]$.*

Proof. Let $\varphi \in \Omega$. Then there exist constants $\rho > 0$, $b \in (0,a]$ and $N \geq 0$ such that $\overline{B_\rho(\varphi)} = \{\psi \in C : |\psi - \varphi| \leq \rho\} \subseteq \Omega$ and $|f(s,\psi)| \leq N$ for all $s \in [0,\rho]$ and $\psi \in \overline{B_\rho(\varphi)}$. Define the function $y : [-r, +\infty) \to X$ by

$$y(t) = \begin{cases} R(t)\varphi(0) & \text{for } t \in [0,+\infty) \\ \varphi(t) & \text{for } t \in [-r,0]. \end{cases}$$

Then $y_t \in C$. For $\rho_1 \in (0,\rho)$, there exists $b_1 \in (0,\rho)$ such that $|y_t - \varphi| \leq \rho_1$ for all $t \in [0,b_1]$. Now, we choose b such that

$$0 < b \leq \min(b_1, \frac{\rho - \rho_1}{M_a N}),$$

where $M_a := \sup_{0 \leq t \leq a} |R(t)|$. Let $\mathbb{S}_b$ be the space defined by

$$\mathbb{S}_b := \{u \in C([0,b];X) : u(0) = \varphi(0)\},$$

equipped with the uniform norm topology.
For $u \in \mathbb{S}_b$, we define its extension $\tilde{u} : [-r,b] \to X$ by

$$\tilde{u}(t) = \begin{cases} u(t) & \text{for } t \in [0,b] \\ \varphi(t) & \text{for } t \in [-r,0]. \end{cases}$$

Let us introduce the subset

$$\mathbb{S}_b(\rho) := \{u \in \mathbb{S}_b : |\tilde{u}_t - \varphi| \leq \rho \text{ for } t \in [0,b]\}.$$

$\mathbb{S}_b(\rho)$ is bounded nonempty subset of $\mathbb{S}_b$. We claim also that $\mathbb{S}_b(\rho)$ is a convex closed subset of $\mathbb{S}_b$. In fact, by using the triangular inequality, we can see that $\lambda u_1 + (1-\lambda)u_2 \in \mathbb{S}_b(\rho)$, for any $u_1, u_2 \in \mathbb{S}_b(\rho)$ and $\lambda \in (0,1)$ and then $\mathbb{S}_b(\rho)$ is convex. To show that $\mathbb{S}_b(\rho)$ is closed, let us consider a convergent sequence $(u^n)_{n\geq 1}$ of $\mathbb{S}_b(\rho)$ such that $\lim_{n\to+\infty} u^n = u$ in $\mathbb{S}_b$. Then, for any $n \in \mathbb{N}$ and $t \in [0,b]$, we have

$$\begin{aligned} |\tilde{u}_t - \varphi| &\leq |\tilde{u}_t - \tilde{u}^n_t| + |\tilde{u}^n_t - \varphi| \\ &\leq |u - u^n| + |u^n - \varphi| \\ &\leq |u - u^n| + \rho. \end{aligned}$$

Letting n go to $+\infty$, we obtain $|\tilde{u}_t - \varphi| \leq \rho$ and consequently $u \in \mathbb{S}_b(\rho)$.
Now we define the mapping Γ on $\mathbb{S}_b(\rho)$ by

$$(\Gamma u)(t) = R(t)\varphi(0) + \int_0^t R(t-s)f(s,\tilde{u}_s)ds \quad \text{for } t \in [0,b]$$

We will prove that Γ has a fixed point by using Theorem 2.7.

Firstly, we claim that $\Gamma\mathbb{S}_b(\rho) \subseteq \mathbb{S}_b(\rho)$. In fact, for any $u \in \mathbb{S}_b(\rho)$, by the continuity of f and the Definition 2.1, the maps $t \to R(t)\varphi(0)$, $t \to f(t,\tilde{u}_t)$ are continuous on $[0,b]$ and so the function $t \to \int_0^t R(t-s)f(s,\tilde{u}_s)ds$ is continuous on $[0,b]$. Then $v := \Gamma u \in \mathbb{S}_b$. To show that $v := \Gamma u \in \mathbb{S}_b(\rho)$, we define $z = v - y$. Then, for any $t \in [0,b]$, we have

$$\begin{aligned} |\tilde{v}_t - \varphi| &\leq |\tilde{z}_t| + |\tilde{y}_t - \varphi| \\ &\leq M_a N b + \rho_1 \\ &\leq \rho \end{aligned}$$

which prove that $\Gamma\mathbb{S}_b(\rho) \subseteq \mathbb{S}_b(\rho)$.

Secondly, we prove that the mapping Γ is continuous on $\mathbb{S}_b(\rho)$. For this purpose, let $(u^n)_{n\geq 1}$ be a sequence in $\mathbb{S}_b(\rho)$ with $\lim_{n\to\infty} u^n = u$ in $\mathbb{S}_b(\rho)$. Then the set $\{u\} \cup \{u^n : n \geq 1\}$ is compact in $\mathbb{S}_b$. Let $\mathcal{K} = \{(s,u_s),(s,u^n_s) : n \geq 1, s \in [0,b]\}$. Then $\mathcal{K}$ is a compact set in $[0,b] \times C$ and f is uniformly continuous in $\mathcal{K}$. Thus for $s \in [0,b]$, we have

$$|\Gamma u^n - \Gamma u| \leq M_a \int_0^b |f(s,\tilde{u}^n_s) - f(s,\tilde{u}_s)|ds.$$

So by using the dominated convergence theorem, we get that $\lim_{n\to\infty} \Gamma u^n = \Gamma u$ in $\mathbb{S}_b(\rho)$.

Now, use Ascoli-Arzela's theorem to prove that $\Gamma\mathbb{S}_b(\rho) = \{\Gamma u : u \in \mathbb{S}_b(\rho)\}$ is relatively compact. We will prove that $(\Gamma\mathbb{S}_b(\rho))(t) = \{(\Gamma u)(t) : u \in \mathbb{S}_b(\rho)\}$ is relatively compact in X for every $t \in [0,b]$.

For $t = 0$; $(\Gamma\mathbb{S}_b(\rho))(0) = \{\varphi(0)\}$, is compact.

For $0 < t \leq b$;

$$(\Gamma\mathbb{S}_b(\rho))(t) = \Big\{R(t)\varphi(0) + \int_0^t R(t-s)f(s,\tilde{u}_s)ds : u \in \mathbb{S}_b(\rho)\Big\}.$$

According to (**H5**) and Theorem 2.4, we need only to show that the set

$$\Big\{\int_0^t R(t-s)f(s,\tilde{u}_s)ds : u \in \mathbb{S}_b(\rho)\Big\}$$

is relatively compact in X. Let $0 < \varepsilon < t$. For $u \in \mathbb{S}_b(\rho)$, we define

$$(\tilde{\Gamma}^{\varepsilon}u)(t) = R(\varepsilon)\int_0^{t-\varepsilon} R(t-s-\varepsilon)f(s,\tilde{u}_s)ds,$$

$$(\Gamma^{\varepsilon}u)(t) = \int_0^{t-\varepsilon} R(t-s)f(s,\tilde{u}_s)ds$$

and

$$(\bar{\Gamma}^{\varepsilon}u)(t) = \int_{t-\varepsilon}^{t} R(t-s)f(s,\tilde{u}_s)ds.$$

Since $R(\varepsilon)$ is compact, then $(\tilde{\Gamma}^{\varepsilon}\mathbb{S}_b(\rho))(t)$ is relatively compact in X. By Lemma 2.6, we infer that

$$\alpha\Big((\tilde{\Gamma}^{\varepsilon}\mathbb{S}_b(\rho))(t)\Big) = 0. \tag{3.1}$$

Using Lemma 2.3, we find that

$$\begin{aligned}
|(\Gamma^{\varepsilon}u)(t) - (\tilde{\Gamma}^{\varepsilon}u)(t)| &= |\int_0^{t-\varepsilon} R(t-s)f(s,\tilde{u}_s)ds - \int_0^{t-\varepsilon} R(\varepsilon)R(t-s-\varepsilon)f(s,\tilde{u}_s)ds| \\
&\leq \int_0^{t-\varepsilon} |R(t-s) - R(\varepsilon)R(t-s-\varepsilon)||f(s,\tilde{u}_s)|ds \\
&\leq \varepsilon H\int_0^{t-\varepsilon} |f(s,\tilde{u}_s)|ds \\
&\leq \varepsilon(b-\varepsilon)HN.
\end{aligned}$$

Thus by Lemma 2.6, we get

$$\alpha\Big((\Gamma^{\varepsilon}\mathbb{S}_b(\rho))(t) - (\tilde{\Gamma}^{\varepsilon}\mathbb{S}_b(\rho))(t)\Big) \leq 2\varepsilon(b-\varepsilon)HN. \tag{3.2}$$

Further, we have

$$\begin{aligned}
|(\bar{\Gamma}^{\varepsilon}u)(t)| &\leq \int_{t-\varepsilon}^{t} |R(t-s)||f(s,\tilde{u}_s)|ds \\
&\leq \varepsilon M_a N
\end{aligned}$$

which implies, by Lemma 2.6, that

$$\alpha\Big((\bar{\Gamma}^{\varepsilon}\mathbb{S}_b(\rho))(t)\Big) \leq 2\varepsilon M_a N. \tag{3.3}$$

Combining (3.1), (3.2) and (3.3) and using Lemma 2.6, we obtain

$$\begin{aligned}\alpha\Big((\Gamma S_b(\rho))(t)\Big) &\leq \alpha\Big((\Gamma^{\varepsilon} S_b(\rho))(t)\Big) + \alpha\Big((\bar{\Gamma}^{\varepsilon} S_b(\rho))(t)\Big)\\ &\leq 2\varepsilon(b-\varepsilon)HN + 2\varepsilon M_a N.\end{aligned}$$

Letting ε go to zero, we deduce that

$$\alpha\Big((\Gamma S_b(\rho))(t)\Big) = 0.$$

Consequently $(\Gamma S_b(\rho))(t)$ is relatively compact in X for all $t \in [0,b]$.

Now, let us show that $\Gamma\mathbb{S}_b(\rho)$ is equicontinuous on $[0,b]$. Let $t = 0$ and $t_1 > 0$. Then

$$\begin{aligned}|(\Gamma u)(t_1) - (\Gamma u)(0)| &\leq |(R(t_1)\varphi(0) - \varphi(0)| + \int_0^{t_1} |R(t_1 - s) f(s, \tilde{u}_s)| ds\\ &\leq |R(t_1)\varphi(0) - \varphi(0)| + M_a N |t_1|.\end{aligned}$$

Using the strong continuity of the resolvent operator $R(.)$, we can deduce that

$$\lim_{t_1 \to 0} |(\Gamma u)(t_1) - (\Gamma u)(0)| = 0,$$

uniformly in $u \in \mathbb{S}_b(\rho)$, and consequently $\Gamma\mathbb{S}_b(\rho)$ is equicontinuous at $t = 0$.
For $0 < t_1 < t_2 \leq b$, we have

$$\begin{aligned}|(\Gamma u)(t_2) - (\Gamma u)(t_1)| &\leq |(R(t_2) - R(t_1))\varphi(0)| + \int_{t_1}^{t_2} |R(t_2 - s) f(s, \tilde{u}_s)| ds\\ &+ \int_0^{t_1} |(R(t_2 - s) - R(t_1 - s)) f(s, \tilde{u}_s)| ds\\ &\leq |R(t_2) - R(t_1)||\varphi(0)| + M_a N |t_2 - t_1| +\\ &+ \int_0^{t_1} |R(t_2 - s) - R(t_1 - s)| N ds.\end{aligned}$$

Since

$$|R(t_2 - s) - R(t_1 - s)| \to 0 \text{ as } t_2 \to t_1 \text{ almost all } s \neq t_1$$

and

$$|R(t_2 - s) - R(t_1 - s)| \leq 2M_a \in L^1([0,t_1]),$$

then the Lebesgue Dominated Convergence Theorem ensures that

$$\int_0^{t_1} |R(t_2 - s) - R(t_1 - s)| ds \to 0 \text{ as } t_2 \to t_1.$$

Moreover, since $R(.)$ is the continuous in the uniform operator topology by Theorem 2.5, then we can see that

$$|(\Gamma u)(t_2) - (\Gamma u)(t_1)| \to 0 \text{ as } t_2 \to t_1,$$

uniformly in $u \in \mathbb{S}_b(\rho)$, which means that $\Gamma\mathbb{S}_b(\rho)$ is equicontinuous. Thus $\Gamma\mathbb{S}_b(\rho)$ is relatively compact by Ascoli Arzela theorem. Therefore, Schauder fixed point theorem yields that Γ has a fixed point in $\mathbb{S}_b(\rho)$, which is the mild solution of the Eq.(1.1) on $[-r,b]$. □

3.2 Global existence and blowing up of the mild solutions

Theorem 3.3. *Assume that* (**H1**)-(**H5**) *hold. Let* $f : [0,+\infty)\times C \to X$ *a continuous mapping and takes bounded sets of* $[0,+\infty)\times C$ *into bounded sets of* X. *Then for each* $\varphi \in C$, *Eq. (1.1) has a mild solution* $u = u(.,\varphi)$ *on a maximal interval of existence* $[-r, b_\varphi)$ *and either*

$$b_\varphi = +\infty \text{ or } \limsup_{t\to b_\varphi^-} |u(t,\varphi)| = +\infty.$$

Proof. Let $u(.,\varphi)$ be the mild solution of Eq.(1.1) defined on $[0,b]$. A similar arguments to that used in the local existence results, can be used for the existence of $b_1 > b$ and a function $u(.,u_b(.,\varphi)) : [b,b_1] \to X$ which satisfies

$$u(t,u_b(.,\varphi)) = R(t-b)u(b,\varphi) + \int_b^t R(t-s)f(s,u_s(.,u_b(.,\varphi)))ds \text{ for } t \in [b,b_1].$$

By a similar proceeding, we can show that $u(.,\varphi)$ can be extended to a maximal interval of existence $[-r,b_\varphi)$. If we assume that $b_\varphi < +\infty$ and $\limsup_{t\to b_\varphi^-} |u(t,\varphi)| < +\infty$, then there exists a constant $c > 0$ such that $|u(t,\varphi)| \le c$ for all $t \in [0,b_\varphi)$. Consequently there exists $N > 0$ such that $|f(t,u_t(.,\varphi))| \le N$ for all $t \in [0,b_\varphi)$. As before let $M_{b_\varphi} = \sup_{0\le t\le b_\varphi} |R(t)|$ and $u : [t_0,b_\varphi) \to X, t_0 \in (t_0,b_\varphi)$, be the restriction of $u(.,\varphi)$ to $[t_0,b_\varphi)$. We will show that $u(.,\varphi)$ is uniformly continuous. Let $t_0 \le t < t+h < b_\varphi$. For $u \in \mathbb{S}_b(\rho)$, we have

$$\begin{aligned} u(t+h)-u(t) &= R(t+h)\varphi(0) - R(t)\varphi(0) + \int_t^{t+h} R(t+h-s)f(s,u_s(.,\varphi))ds \\ &+ \int_0^t (R(t+h-s)-R(t-s))f(s,u_s(.,\varphi))ds \\ &= R(t+h)\varphi(0) - R(t)\varphi(0) + \int_t^{t+h} R(t+h-s)f(s,u_s(.,\varphi))ds \\ &+ \int_0^t (R(t+h-s)-R(h)R(t-s))f(s,u_s(.,\varphi))ds \\ &+ \int_0^t (R(h)-I)R(t-s)f(s,u_s(.,\varphi))ds. \end{aligned}$$

As the map $t \to R(t)\varphi(0)$ is uniformly continuous,

$$\sup_{t \text{ such that } t+h\in[0,b_\varphi)} |R(t+h)\varphi(0) - R(t)\varphi(0)| \to 0 \quad \text{as} \quad h \to 0.$$

Moreover, from Lemma 2.3, we get

$$\Big|\int_0^t (R(t+h-s)-R(h)R(t-s))f(s,u_s(.,\varphi))ds\Big| \le hHNb_\varphi.$$

On the other hand,

$$\int_0^t (R(h)-I)R(t-s)f(s,u_s(.,\varphi))ds = (R(h)-I)\int_0^t R(t-s)f(s,u_s(.,\varphi))ds.$$

We will show that $\Lambda = \{\int_0^t R(t-s)f(s,u_s(.,\varphi))ds : t \in [0,b_\varphi)\}$ is compact. Let $(t_n)_n$ a sequence of real number in $[0,b_\varphi)$. By Bolzano-Weirstrass Theorem, (t_n) has a convergent subsequence, say $(t_{n_k})_k$. Let $t = \lim_{k\to\infty} t_{n_k}$. Then, by the dominated convergence theorem we get

$$\lim_{k\to\infty}\int_0^{t_{n_k}} R(t_{n_k}-s)f(s,u_s(.,\varphi))ds = \int_0^t R(t-s)f(s,u_s(.,\varphi))ds,$$

which implies that Λ is compact. Thus, by Banach-Steinhaus Theorem,

$$|(R(h)-I)\int_0^t R(t-s)f(s,u_s(.,\varphi))ds| \to 0 \quad \text{as} \quad h\to 0,$$

uniformly for $t \geq 0$. Consequently,

$$|u(t+h)-u(t)| \to 0 \quad \text{as} \quad h \to 0,$$

uniformly for $t \geq 0$ such that $t+h \in [0,b_\varphi)$. Using the same reasoning, one can show a similar result for $h<0$. This implies that $u(.,\varphi)$ is uniformly continuous. Therefore

$$\lim_{t\to b_\varphi^-} |u(t,\varphi)| \text{ exists in } X$$

and consequently $u(.,\varphi)$ can be extended to b_φ, which contradicts the maximality of $[0,b_\varphi)$. This end the proof. □

The following result provides sufficient conditions for global solution of Eq.(1.1).

Theorem 3.4. *Under the same assumptions as in Theorem 3.3 and if there exists* $k_1 \in C(\mathbb{R}^+,\mathbb{R}^+)$ *and* $k_2 \in L^1(\mathbb{R}^+,\mathbb{R}^+)$ *such that*

$$|f(t,\varphi)| \leq k_1(t)|\varphi| + k_2(t) \text{ for } t\geq 0 \text{ and } \varphi \in C.$$

Then Eq.(1.1) has a mild global solution.

Proof. Let $[-r,b_\varphi)$ be the maximal interval of existence of the mild solution $u_t(.,\varphi)$ of Eq. (1.1). Then

$$b_\varphi = +\infty \text{ or } \limsup_{t\to b_\varphi^-}|u(t,\varphi)| = +\infty.$$

Assume that $b_\varphi < +\infty$, then $\limsup_{t\to b_\varphi^-}|u_t(.,\varphi)| = +\infty$. For $t \in [0,b_\varphi)$, we have

$$\begin{aligned} |u_t(.,\varphi)| &\leq |R(t)\varphi(0)| + \left|\int_0^t R(t-s)f(s,u_s(.,\varphi))ds\right| \\ &\leq \gamma + M_{b_\varphi}\int_0^t k_1(s)|u_s(.,\varphi)|ds. \end{aligned}$$

where

$$\gamma = M_{b_\varphi}|\varphi| + M_{b_\varphi}\int_0^{b_\varphi} k_2(s)ds.$$

By Gronwall's lemma, we deduce that

$$|u_t(.,\varphi)| \quad \leq \quad \gamma\, e^{M_{b_\varphi}\int_0^t k_1(s)ds} < +\infty \text{ for } t \in [0, b_\varphi).$$

Then

$$\limsup_{t\to b_\varphi^-} |u_t(.,\varphi)| < +\infty,$$

which contradicts our assumption. Hence the mild solution is global. □

4 Application

To illustrate the previous results, we propose the following partial integrodifferential equation

$$\begin{cases} \dfrac{\partial}{\partial t}w(t,\xi) = \dfrac{\partial^2}{\partial \xi^2}w(t,\xi) + \displaystyle\int_0^t h(t-s)\dfrac{\partial^2}{\partial \xi^2}w(s,\xi)ds + \int_{-r}^0 F(t, w(t+\theta,\xi))d\theta \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \geq 0 \text{ and } \xi \in [0,\pi] \\ w(t,0) = w(t,\pi) = 0 \quad \text{ for } t \geq 0 \\ w(\theta,\xi) = w_0(\theta,\xi) \quad \text{ for } \theta \in [-r,0] \text{ and } \xi \in [0,\pi], \end{cases} \tag{4.1}$$

where $w_0 : [-r,0]\times[0,\pi] \to \mathbb{R}$, $F : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions.

We choose $X = L^2([0,\pi])$ and define the operator $A : D(A) \subset X \to X$ by

$$\begin{cases} D(A) = \{z \in X : z'' \in X \text{ and } z(0) = z(\pi)\} \\ Az = z''. \end{cases}$$

Then A generates a strongly continuous semigroup on X which is compact. Thus (**H1**), (**H4**) and (**H5**) are true. Let $B : D(A) \subset X \to X$ be the operator defined by

$$B(t)(z) = h(t)Az \quad \text{ for } t \geq 0 \text{ and } z \in D(A).$$

Let $C = C([-r,0];X)$, we set

$$\begin{aligned} &u(t) = w(t,\xi) \quad \text{ for } t \geq 0 \text{ and } \xi \in [0,\pi] \\ &\varphi(\theta)(\xi) = w_0(\theta,\xi) \quad \text{ for } \theta \in [-r,0] \text{ and } \xi \in [0,\pi] \\ &f(t,\phi)(\xi) = \int_{-r}^0 F(t,\phi(\theta)(\xi))d\theta \quad \text{ for } t \geq 0, \xi \in [0,\pi] \text{ and } \phi \in C. \end{aligned}$$

Then Eq. (4.1) takes the following abstract form

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + \displaystyle\int_0^t B(t)u(s)ds + f(t,u_t) \text{ for } t \geq 0 \\ y_0 = \varphi. \end{cases} \tag{4.2}$$

We assume that h is C^1 function, which implies that the operator $B(t)$ satisfies (**H2**) and (**H3**). Therefore all conditions of Theorem 3.3 are satisfied and consequently Eq.(4.2) has at least one mild solution on a maximal interval $[-r,b]$ for some $b > 0$.

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