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PSEUDO-SZABÓ OPERATORS AND LIGHTLIKE SZABÓ HYPERSURFACES

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Abstract

In this paper, the pseudo-inversion of degenerate metric is considered. We extend Szabó operators associated to algebraic covariant derivative curvature maps (tensors) to lightlike hypersurfaces. Some examples are given with explicit determination of their Szabó operators. Finally, we introduce the notion of lightlike Szabó hypersurfaces and give some characterization results of locally symmetric lightlike hypersurfaces and semi-symmetric lightlike hypersurfaces from Szabó condition.

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1 Introduction

The curvature tensor is a central concept in differential geometry. According to R. Osserman ([6]), one could argue that it is a central one. But the curvature tensor is in general difficult to deal with and the problem which aims to relate algebraic properties of the Riemann curvature tensor to the geometry of the manifold is in general difficult to be solved. Many authors study the geometric consequences that followed if various natural operators defined in terms of the curvature tensor are assumed to have constant eigenvalues on the unit fibre bundle. Osserman has studied the spectral properties of Jacobi operator in ([6]). This operator has been extensively studied in the Riemannian and the pseudo-Riemannian context. In the degenerate geometry, C. Atindogbe and K. L. Duggal have studied Pseudo-Jacobi operators and introduce the Osserman condition on lightlike hypersurfaces in ([2]). The Szabó operator has been also studied in the Riemannian context, since it's introduction by Z. I. Szabo in ([12]) but with more or less interest.

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Let (M,g) be a semi-Riemannian manifold and $u \in M$. An element $F \in \otimes^4 T_u^*M$ is said to be an *algebraic curvature map (tensor)* on T_uM if it satisfies the following symmetries:

$$\begin{aligned} F(x,y,z,w) = -F(y,x,z,w) = F(z,w,x,y) \\ F(x,y,z,w) + F(y,z,x,w) + F(z,x,y,w) = 0 \end{aligned} \tag{1.1}$$

We said that $\mathcal{R} \in \otimes^5 T_u^*M$ is an *algebraic covariant derivative curvature map (tensor)* on T_uM if $\mathcal{R}$ satisfies the symmetries:

$$\begin{aligned} \mathcal{R}(x,y,z,w;v) = \mathcal{R}(z,w,x,y;v) = -\mathcal{R}(y,x,z,w;v) \\ \mathcal{R}(x,y,z,w;v) + \mathcal{R}(y,z,x,w;v) + \mathcal{R}(z,x,y,w;v) = 0 \\ \mathcal{R}(x,y,z,w;v) + \mathcal{R}(x,y,w,v;z) + \mathcal{R}(x,y,v,z;w) = 0 \end{aligned} \tag{1.2}$$

The Szab*ó* operator $S_{\mathcal{R}}(\cdot)$ associated to an algebraic covariant derivative curvature tensor $\mathcal{R} \in \otimes^5 T_u^*M$ is the self-adjoint linear map on T_uM characterized by the identity:

$$g(S_{\mathcal{R}}(x)y,z) = \mathcal{R}(y,x,x,z;x). \tag{1.3}$$

Since $S_{\mathcal{R}}(cx) = c^3 S_{\mathcal{R}}(x)$, the natural domains of Szab*ó* operators $S_{\mathcal{R}}(\cdot)$ are the unit pseudo-sphere of unit spacelike or unit timelike vectors

$$S_u^{\pm}(M) := \{x \in T_uM : g(x,x) = \pm 1\}.$$

The tensor $\mathcal{R} \in \otimes^5 T_u^*M$ is said to be a spacelike (resp. timelike) Szab*ó* tensor on T_uM if its spectrum, $Spec\{S_{\mathcal{R}}\}$, is constant on the pseudo-sphere $S_u^+(M)$ (resp $S_u^-(M)$).

In semi-Riemannian geometry, the Riemann curvature tensor R is an algebraic curvature tensor on the tangent space T_uM for every point $u \in M$. Similarly, by using (1.1) and the second Bianchi identity

$$\nabla_X R(Y,Z) + \nabla_Y R(Z,X) + \nabla_Z R(X,Y) = 0 \tag{1.4}$$

we see that the covariant derivative of the curvature tensor ∇R is an algebraic covariant derivative curvature tensor on T_uM.

In the geometry of lightlike hypersurfaces, the induced connection is not compatible with the induced structure. Indeed, we have

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \quad \forall X,Y \in \Gamma(TM|_{\mathcal{U}}). \tag{1.5}$$

where the local $1-$form η is defined on $\Gamma(TM|_{\mathcal{U}})$ by

$$\eta(\ \cdot\ ) = \bar{g}(\cdot\ ,\ N), \tag{1.6}$$

where $\overline{g}$ is the pseudo-Riemannian metric on the ambient space $\overline{M}$. The covariant derivative of the induced Riemann curvature tensors is not algebraic in general, that is (1.2) does not hold. Therefore, in section 4 we find conditions on lightlike hypersurface to have an algebraic covariant derivative of induced Riemann curvature tensor.

The semi-Riemannian manifold (M,g) is said to be Szab*ó* manifold if the covariant derivative ∇R of its associated Riemann curvature is a Szab*ó* tensor on T_uM, for any $u \in M$;

or equivalently if the eigenvalues of Szabó operator $S_{\nabla R}(\cdot)$ on T_uM are constant on the pseudo-spheres of unit timelike and spacelike vectors $S_u^{\pm}(M)$, for any $u \in M$. In section 5 and 6 we have extended this operator on lightlike hypersurfaces, spaces of signature $(p,q,1)$ by using non-degenerate metric $\widetilde{g}$ associated to the degenerate metric g (see [1]). We give some examples.

A natural condition to impose on semi-Riemannian manifold of signature (p,q) is that its Riemann curvature tensor R be parallel, that is, have vanishing covariant differential. Such a manifold is said to be *locally symmetric*. This class of manifolds contains one of manifolds of constant curvature and can be find from Szabó condition. Szabó ([12]) showed in the Riemannian setting $(q=0)$ that if $S_{\nabla R}(\cdot)$ has constant eigenvalues on $S_u^{+}(M)$, for any $u \in M$, then ∇R vanish identically. Gilkey and Stavrov ([7]) extended this result to the Lorentzian setting $(q=1)$ and showed that for $p,q \geqslant 2$, there exists an algebraic covariant derivative curvature tensor so that $S^2_{\nabla R}(\cdot) = 0$ and $S_{\nabla R}(\cdot)$ does not vanish identically.

In this paper, we introduce in section 7, the notion of lightlike Szabó hypersurfaces of a semi-Riemannian manifolds. We extend results of Szabó, Gilkey and Stavrov to spaces of signature $(p,q,1)$ (theorem 7.4 and 7.5). Also, we show that in integrable screen distributions setting, Szabó condition at a point on lightlike hypersurface sometimes reduces to be one for the semi-Riemannian screen leaf through this point (theorem 7.6). In section 8, We study locally symmetric lightlike hypersurfaces and semi-symmetric lightlike hypersurfaces under Szabó condition. We prove some characterizations.

2 Preliminaries

In this section, we will give a brief review of lightlike hypersurfaces of semi-Riemannian manifolds. A full discussion of the content of this section can be found in [4]. Note that, except the contrary mention, in this paper, we consider semi-Riemannian manifolds $(\overline{M},\overline{g})$ of signature (p,q) and constant index q with $1 \leqslant q \leqslant p$, that is

$$sign(\overline{g}) = \{\underbrace{-,\ldots,-}_{q},\underbrace{+,\ldots,+}_{p}\}.$$

Also, we consider degenerate manifolds or lightlike hypersurfaces (M,g) of signature $(p,q,1)$, and constant index q, that is

$$sign(g) = \{0,\underbrace{-,\ldots,-}_{q},\underbrace{+,\ldots,+}_{p}\}.$$

Let $(\overline{M},\overline{g})$ be an $(m+2)$-dimensional semi-Riemannian manifold of constant index q, $1 \leqslant q < n+2$ and M be a hypersurface of $\overline{M}$. We denote the tangent space at $u \in M$ by T_uM. Then

$$T_uM^{\perp} = \{X_u \in T_u\overline{M} : \overline{g}(X_u,Y_u) = 0,\ \forall Y_u \in T_uM\}$$

and

$$RadT_uM = T_uM \cap T_uM^{\perp}$$

Then, M is called a lightlike hypersurface of $\overline{M}$ if $RadT_uM \neq \{0\}$ for any $u \in M$. Thus $TM^{\perp} = \cap_{u\in M} T_uM^{\perp}$ becomes a one-dimensional distribution on M. We denote $\mathcal{F}(M)$ the algebra of differential functions on M and by $\Gamma(E)$ the $\mathcal{F}(M)$-module of differentiable sections of a vector bundle E over M.

Definition 2.1 ([4], p.78)**.** Let M be a lightlike hypersurface of a semi-Riemannian manifold $\overline{M}$. A complementary vector subbundle $S(TM)$ to $TM^{\perp}$ in TM is called a *screen distribution* of M.

It is known from ([4], Proposition 2.1, p.5) that $S(TM)$ is non-degenerate. Thus we have the orthogonal direct sum

$$TM = S(TM) \perp TM^{\perp}, \tag{2.1}$$

A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. From (2.1), we observe that $TM^{\perp}$ lies in the tangent bundle of the lightlike hypersurface M. Thus a vital problem of this theory is to replace the intersecting part by a vector subbundle of $T\overline{M}_{|M}$ whose sections are nowhere tangent to M. Next theorem shows that there exists a such complementary (non-orthogonal) vector bundle to TM in $T\overline{M}$.

Theorem 2.2. *([4], p.79) Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \overline{g})$. Then there exists a unique vector bundle $tr(TM)$ of rank 1 over M such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exist a unique section N of $tr(TM)$ on $\mathcal{U}$ satisfying*

$$\overline{g}(N,\xi) = 1, \quad \overline{g}(N,N) = \overline{g}(N,W) = 0, \qquad \forall W \in \Gamma(S(TM)_{|\mathcal{U}}) \tag{2.2}$$

It follows from (2.2) that $tr(TM)$ is a lightlike vector bundle such that $tr(TM)_u \cap T_uM = \{0\}$, it is called the *lightlike transversal bundle* of M with respect to screen distribution $S(TM)$. Thus from (2.1) and (2.2) we have

$$T\overline{M}_{|M} = S(TM) \perp (TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM). \tag{2.3}$$

Suppose M is a lightlike hypersurface of $\overline{M}$ and $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$. Then according to the decomposition (2.3) we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \text{and} \quad \overline{\nabla}_X V = -A_N X + \nabla^{\perp}_X V, \tag{2.4}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\nabla_X Y$ and $A_V X$ belong to $\Gamma(TM)$, $h(X,Y)$ and $\nabla^{\perp}_X V$ belong to $\Gamma(tr(TM))$. We note that it is easy to see that ∇ is a torsion free connection, h is a $tr(TM)$ valued, symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$, A_V is a $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$ and ∇^t is a linear connection on $tr(TM)$. h and A_V are called the *second fundamental form* and *shape operator* of the lightlike hypersurface M, respectively.

Locally suppose $\{\xi, N\}$ is a normalizing pair of vector fields on $\mathcal{U} \subset M$ in Theorem 2.2. Then we define a symmetric bilinear form B and 1-form τ on $\mathcal{U} \subset M$ by

$$B(X,Y) = \overline{g}(h(X,Y),\xi) \quad \text{and} \quad \tau(X) = \overline{g}(\nabla^{\perp}_X N, \xi),$$

for any $X, Y \in \Gamma(TM)$. Thus (2.4) becomes,

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \quad \text{and} \quad \overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.5}$$

for any $X,Y\in\Gamma(TM)$. Let P denote the projection morphism of TM on $S(TM)$ with respect to the orthogonal decomposition (2.1). We obtain

$$\nabla_X PY = \overset{*}{\nabla}_X PY + C(X,PY)\xi \quad \text{and} \quad \nabla_X \xi = -\overset{*}{A}_\xi X - \tau(X)\xi, \tag{2.6}$$

for any $X,Y\in\Gamma(TM)$, where $\overset{*}{\nabla}_X PY$ and $\overset{*}{A}_E X$ belong to $\Gamma(S(TM))$ and C is locally, a $\Gamma(TM^\perp)$-valued $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)\times\Gamma(S(TM))$ defined by

$$C(X,PY) = \overline{g}(\nabla_X PY, N),$$

for any $X,Y\in\Gamma(TM)$. C, $\overset{*}{A}_E$ and $\overset{*}{\nabla}$ are called the *local second fundamental form* of $S(TM)$, the *shape operator* of $S(TM)$ and the *induced connection* on $S(TM)$, respectively.

By direct calculation, using (2.5) and (2.6) and since $\overline{g}(\overline{\nabla}_X\xi,\xi)=0$, we obtain

$$B(X,PY) = g(\overset{*}{A}_\xi X, PY) \quad \text{and} \quad B(X,\xi) = 0 \tag{2.7}$$

$$C(X,PY) = g(A_N X, PY) \quad \text{and} \quad g(A_N Y, N) = 0, \tag{2.8}$$

for any $X,Y\in\Gamma(TM)$. It is important to mention that the local second fundamental form B is independent of the choice of screen distribution ([4], Proposition 2.1, p.83).

Definition 2.3 ([4], p.107)**.** A lightlike hyprsurface $(M,g,S(TM))$ of a semi-Riemannian manifold $(\overline{M},\overline{g})$ is said to be *totally umbilical*, if and only if, locally, on each $\mathcal{U}$ there exists a smooth function ρ such that

$$B(X,Y) = \rho g(X,Y), \qquad \forall X,Y\in\Gamma(TM_{|\mathcal{U}}). \tag{2.9}$$

Definition 2.4 ([3])**.** A lightlike hypersurface $(M,g,S(TM))$ of a semi-Riemannian manifold $\overline{M}$ is *screen conformal* if on any coordinate neighborhood $\mathcal{U}\subseteq M$ and for any normalizing pair $\{\xi,N\}$ there exists a non-vanishing smooth function φ on $\mathcal{U}$ such that

$$A_N = \varphi \overset{*}{A}_\xi \,. \tag{2.10}$$

Denote by $\overline{R}$ and R the Riemann curvature tensors of $\overline{\nabla}$ and ∇, respectively. Recall the following Gauss equation ([4]), for all $X,Y,Z\in\Gamma(TM)$, we have

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X + (\nabla_X B)(Y,Z)N \\ &\quad + B(Y,Z)\tau(X)N - (\nabla_Y B)(X,Z)N - B(X,Z)\tau(Y)N,\end{aligned} \tag{2.11}$$

where

$$(\nabla_X B)(Y,Z) = X.B(Y,Z) - B(\nabla_X Y,Z) - B(Y,\nabla_X Z). \tag{2.12}$$

3 Pseudo-inversion of degenerate metrics

In this section, we recall from [1] the following results. Consider on M a normalizing pair $\{E,N\}$ satisfying the Theorem 2.2 and the one-form

$$\eta(\ \cdot\ ) = \overline{g}(\ \cdot\ ,N).$$

For all $X \in \Gamma(TM)$, $X = PX + \eta(X)\xi$ and $\eta(X) = 0$ if and only if $X \in \Gamma(S(TM))$. Now, we define $\flat_g$ by

$$\begin{aligned} \flat_g : \Gamma(TM) &\longrightarrow \Gamma(T^*M) \\ X &\longmapsto X^{\flat_g} = g(X,\ \cdot\ ) + \eta(X)\eta(\ \cdot\ ). \end{aligned} \tag{3.1}$$

Clearly, such a $\flat_g$ is an isomorphism of $\Gamma(TM)$ onto $\Gamma(T^*M)$, and generalize the usual non-degenerate theory. In the latter case, $\Gamma(S(TM))$ coincides with $\Gamma(TM)$, and as a consequence the 1-form η vanishes identically and the projection morphism P becomes the identity map on $\Gamma(TM)$. We let $\sharp_g$ denote the inverse of the isomorphism $\flat_g$ given by (3.1). For $X \in \Gamma(TM)$ (resp. $\omega \in \Gamma(T^*M)$), $X^{\flat_g}$ (resp. $\omega^{\sharp_g}$) is called the *dual 1-form* of X (resp. the *dual vector field* of ω) with respect to the degenerate metric g. It follows from (3.1) that if ω is a 1-form on M, we have for $X \in \Gamma(TM)$,

$$\omega(X) = g(\omega^{\sharp_g}, X) + \omega(E)\eta(X). \tag{3.2}$$

Define a $(0,2)$-tensor $\tilde{g}$ by

$$\tilde{g}(X,Y) = X^{\flat_g}(Y) = g(X,Y) + \eta(X)\eta(Y), \qquad \forall X,Y \in \Gamma(TM). \tag{3.3}$$

Clearly, $\tilde{g}$ defines a non-degenerate metric on M which plays an important role in defining the usual differential operators *gradient, divergence, Laplacian* with respect to degenerate metric g on lightlike hypersurfaces, see [1]. Also, observe that $\tilde{g}$ coincides with g if the latter is non-degenerate. The (0,2)-tensor $g^{[\cdot,\cdot]}$, inverse of $\tilde{g}$ is called the *pseudo-inverse* of the degenerate metric g. With respect to the quasi orthogonal local frame field $\{\xi, X_1, ..., X_n\}$ adapted to the decomposition (2.1) we have

$$\begin{aligned} \tilde{g}(\xi,\xi) = 1, \qquad \tilde{g}(\xi,X_i) = \eta(X_i) = 0 \\ \tilde{g}(X_i,X_j) = g(X_i,X_j) = g_{ij}, \qquad 1 \leqslant i,j \leqslant n. \end{aligned} \tag{3.4}$$

The matrics of $\widetilde{g}$, $g^{[\cdot,\cdot]}$ and $g.g^{[\cdot,\cdot]}$ are given by:

$$\widetilde{g} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & (g_{ij}) & \\ 0 & & & \end{pmatrix}, \quad g^{[\cdot,\cdot]} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & (g_{ij})^{-1} & \\ 0 & & & \end{pmatrix} \tag{3.5}$$

$$g.g^{[\cdot,\cdot]} = g^{[\cdot,\cdot]}.g = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & I_n & \\ 0 & & & \end{pmatrix} \tag{3.6}$$

4 Algebraic covariant derivative of curvature tensors.

Contrary to non-degenerate hypersurface, the induced Riemann curvature on lightlike hypersurface $(M,g,S(TM))$ may not have an algebraic covariant derivative. For this, we have the following.

Proposition 4.1. *Let $(M,g,S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M},\overline{g})$, R the algebraic induced Riemann curvature tensor of M. The covariant derivative curvature tensor ∇R is algebraic, if and only if the torsion-free induced connection on M satisfies*

$$(\nabla_V g)(R(Z,W)X,Y)+(\nabla_Z g)(R(W,V)X,Y)+(\nabla_W g)(R(V,Z)X,Y)=0, \tag{4.1}$$

for all $X,Y,Z,W,V\in\Gamma(TM)$.

Proof. By derivation of algebraic symmetries (1.1) of R, we have the two first algebraic symmetries in (1.2) for the tensor ∇R. The induced Riemann curvature R with respect to the torsion-free ∇ verifies the second Bianchi identity, so for all $X,Y,Z,W,V\in\Gamma(TM)$,

$$g((\nabla_V R)(Z,W)X,Y)+g((\nabla_Z R)(W,V)X,Y)+g((\nabla_W R)(V,Z)X,Y)=0. \tag{4.2}$$

Also by calculation we can check that

$$(\nabla_V R)(X,Y,Z,W)=g((\nabla_V R)(X,Y)Z,W)+(\nabla_V g)(R(X,Y)Z,W). \tag{4.3}$$

Using (4.2), (4.3) and the first algebraic symmetry in (1.2) for ∇R , we have

$$\begin{aligned}&(\nabla_V R)(X,Y,Z,W)-(\nabla_V g)(R(Z,W)X,Y)+(\nabla_Z R)(X,Y,W,V)\\ &-(\nabla_Z g)(R(W,V)X,Y)+(\nabla_W R)(X,Y,V,Z)-(\nabla_W g)(R(V,Z)X,Y) \quad = \quad 0\end{aligned}$$

Thus, the third algebraic symmetry in (1.2) for the tensor ∇R is satisfied if and only if the condition (4.1) holds. □

Let $\overline{\nabla}$ and ∇ be the Levi-Civita connection on $(\overline{M},\overline{g})$ and the induced connection on (M,g), respectively. Denote by $\overline{R}$ and R the curvature tensors of $\overline{\nabla}$ and ∇, respectively. Since $(\overline{\nabla}_V\overline{R})(X,Y,Z,W)=\overline{g}(\overline{\nabla}_V\overline{R}(X,Y)Z,W)$, using relation (4.3) and lemma 3.2 of ([9]), we obtain, for any $V,X,Z\in\Gamma(TM)$, $\xi\in\Gamma(TM^{\perp})$ and $W\in\Gamma(S(TM))$

$$\begin{aligned}(\nabla_V R)(X,\xi,Z,W)=(\overline{\nabla}_V\overline{R})(X,\xi,Z,W)-\big\{&(\nabla_V B)(X,Z)C(\xi,W)\\ &+B(X,Z)g((\nabla_V A_N)\xi,W)-(\nabla_V B)(\xi,Z)C(X,W)+(\nabla_\xi B)(X,Z)C(V,W)\\ &-(\nabla_X B)(\xi,Z)C(V,W)+B(X,Z)\tau(\xi)C(V,W)-B(V,X)\overline{R}(N,\xi,Z,W)\\ &-B(V,Z)\overline{R}(X,\xi,N,W)-B(V,W)\overline{R}(X,\xi,Z,N)\big\}\end{aligned} \tag{4.4}$$

In the following, we show that, a lightlike hypersurface of a Lorentzian space form (of signature (p,q), $q=1$) admits algebraic tensor ∇R if and only if it is totaly geodesic.

Theorem 4.2. *Let $(M,g,S(TM))$ be a lightlike hypersurface of a Lorentzian space form $(\overline{M}(c),\overline{g})$, with $A_N\xi\neq 0$. Then the covariant derivative ∇R of induced Riemann curvature tensor R on M is algebraic if and only if (M,g) is totally geodesic.*

Proof. Assume that ∇R defines an algebraic covariant derivative curvature tensor. The Riemann curvature tensor $\overline{R}$ on $\overline{M}$ is given by

$$\overline{R}(X,Y)Z = c\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\}, \quad \forall X,Y,Z \in \Gamma(T\overline{M}). \tag{4.5}$$

By direct calculation, using relations (4.4) and (4.5), we obtain, for any $V,X \in \Gamma(TM)$, $W \in \Gamma(S(TM))$ and $\xi \in \Gamma(TM^{\perp})$,

$$(\nabla_V R)(X,\xi,\xi,W) = -(\nabla_V B)(X,\xi)C(\xi,W) = -B(X,\overset{*}{A}_\xi V)g(A_N\xi,W). \tag{4.6}$$

Also, since the tensor ∇R is algebraic, we have

$$(\nabla_V R)(X,\xi,\xi,W) = -(\nabla_V R)(X,\xi,W,\xi) = -R(X,\xi,W,\overset{*}{A}_\xi V).$$

Thus, by using relation (2.11), we obtain, for any $V,X \in \Gamma(TM)$, $W \in \Gamma(S(TM))$ and $\xi \in \Gamma(TM^{\perp})$,

$$(\nabla_V R)(X,\xi,\xi,W) = B(X,W)g(A_N\xi,\overset{*}{A}_\xi V). \tag{4.7}$$

From relations (4.6) and (4.7), we get

$$B(X,W)g(A_N\xi,\overset{*}{A}_\xi V) = -B(X,\overset{*}{A}_\xi V)g(A_N\xi,W). \tag{4.8}$$

By taking $W = \overset{*}{A}_\xi V$ into (4.8), we get

$$B(X,\overset{*}{A}_\xi V)g(A_N\xi,\overset{*}{A}_\xi V) = 0, \quad \forall V,X \in \Gamma(TM). \tag{4.9}$$

Since $A_N\xi \neq 0$, if $g(A_N\xi,\overset{*}{A}_\xi V) = 0$, we have $\overset{*}{A}_\xi V = 0, \forall V \in \Gamma(TM)$. Now suppose that $g(A_N\xi,\overset{*}{A}_\xi V_0) \neq 0$, for some $V_0 \in \Gamma(TM)$. From (4.9), we obtain

$$g(\overset{*}{A}_\xi X,\overset{*}{A}_\xi V) = B(X,\overset{*}{A}_\xi V) = 0.$$

Then, by taking $X = V$, we get $g(\overset{*}{A}_\xi X,\overset{*}{A}_\xi X) = 0$. On the other hand, any screen distribution $S(TM)$ of a lightlike hypersurface of Lorentzian manifold is Riemannian. Thus we have $\overset{*}{A}_\xi X = 0$, $\forall X \in \Gamma(TM)$, that is (M,g) is totally geodesic. Conversely, suppose that $B = 0$. Then, using relation (2.11), we have $\nabla R = \overline{\nabla}\overline{R}_{|TM}$, that is ∇R is algebraic. □

Lemma 4.3. *Let $(M,g,S(TM))$ be a locally screen conformal lightlike hypersurface of a semi-Riemannian manifold $(\overline{M},\overline{g})$ with ambient holonomy condition*

$$\overline{R}(X,PY)(RadTM) \subset RadTM \; \forall X,Y \in \Gamma(TM).$$

Then, the induced Riemann curvature R of M defines an algebraic curvature tensor.

Proof. Consider M to be locally screen conformal. Since $C(X,PY) = \varphi B(X,PY)$, from (2.11) we have for any $X,Y,Z,W \in \Gamma(TM)$,

$$\begin{aligned} R(X,Y,Z,PW) &= \overline{R}(X,Y,Z,PW) - B(X,Z)C(Y,PW) + B(Y,Z)C(X,PW) \\ &= \overline{R}(X,Y,Z,PW) - \varphi\big(B(X,Z)B(Y,PW) - B(Y,Z)B(X,PW)\big) \\ &= \overline{R}(X,Y,Z,PW) - \varphi\mathcal{A}(X,Y,Z,PW), \end{aligned}$$

where $\mathcal{A}(X,Y,Z,W) = B(X,Z)B(Y,W) - B(Y,Z)B(X,W)$, $\forall X,Y,Z,W \in \Gamma(TM)$. It is straightforward that $\mathcal{A}$ verifies the algebraic symmetries of (1.1). So, $R(X,Y,Z,PW)$ has the required symmetries. On the other hand, for any $X,Y,Z \in \Gamma(TM)$ and $\xi \in \Gamma(TM^{\perp})$, we have $R(X,Y,Z,\xi) = -R(Y,X,Z,\xi) = 0$. Also, $R(Z,\xi,X,Y) = R(Z,\xi,X,PY) = \overline{R}(Z,\xi,X,PY) - B(Z,X)C(\xi,PY) + B(\xi,X)C(Z,PY) = -\overline{R}(X,PY,\xi,Z) = 0$. This completes the proof. □

Assume that a lightlike hypersurface (M,g) is totally geodesic, from relation (2.11), we have $\nabla R = \overline{\nabla}\overline{R}|_{TM}$. Then, the covariant derivative ∇R of the induced Riemann curvature R of M defines an algebraic covariant derivative curvature tensor. For the non-totally geodesic setting, in virtue of Proposition 4.1 and the above lemma, the following holds.

Theorem 4.4. *Let $(M,g,S(TM))$ be a locally screen conformal lightlike hypersurface of a semi-Riemannian manifold $(\overline{M},\overline{g})$, R the induced Riemann curvature tensor of M. Then the covariant derivative ∇R defines an algebraic covariant derivative curvature tensor, if the following conditions are satisfied*
(a) $\overline{R}(X,PY)\xi \in \Gamma(TM^{\perp})$ and
(b) $(\nabla_V g)(R(Z,W)X,Y) + (\nabla_Z g)(R(W,V)X,Y) + (\nabla_W g)(R(V,Z)X,Y) = 0$,
for all $X,Y,Z,W,V \in \Gamma(TM)$ and $\xi \in \Gamma(TM^{\perp})$.

Corollary 4.5. *Let $(M,g,S(TM))$ be a locally screen conformal lightlike hypersurface of a semi-Euclidean space, R the induced Riemann curvature tensor of M. Then the covariant derivative ∇R defines an algebraic covariant derivative curvature tensor.*

5 Pseudo-Szab*ó operators*.

Let us start by intrinsic interpretation of relation (1.3) which in pseudo-Riemannian setting characterizes the Szab*ó* operator $S_{\mathcal{R}}(\cdot)$ associated to an algebraic covariant derivative curvature map $\mathcal{R} \in \otimes^5 T_u^* M$, $u \in M$. Indeed, for $x \in S_u^{\pm}(M)$, y, w in $T_u M$, we have,

$$(S_{\mathcal{R}}(x)y)^{\flat}(w) = \mathcal{R}(y,x,x,w;x) \tag{5.1}$$

that is

$$S_{\mathcal{R}}(x)y = \mathcal{R}(y,x,x,\bullet;x)^{\sharp} \tag{5.2}$$

where $\flat$ and $\sharp$ are the usual natural isomorphisms between $T_u M$ and its dual $T_u^* M$, for non-degenerate metric g. For degenerate setting, let's consider the associate non-degenerate metric $\widetilde{g}$ of g, defined by relation (3.3) and denote by $\flat_g$ and $\sharp_g$ the above natural isomorphisms for the metric $\widetilde{g}$. Thus equivalently, relation (5.1) can be written in the form:

$$\widetilde{g}(S_{\mathcal{R}}(x)y,w) = \mathcal{R}(y,x,x,w;x) \tag{5.3}$$

in which $S_{\mathcal{R}}(x)y$ is well defined. This leads to the following definition.

Definition 5.1 (Pseudo-Szab*ó* Operator)**.** Let $(M,g,S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M},\overline{g})$, $u \in M$, $x \in S_u^{\pm}(M)$ and $\mathcal{R} \in \otimes^5 T_u^* M$ an algebraic covariant derivative curvature map on $T_u M$. By *pseudo-Szabó operator* associated to $\mathcal{R}$ with respect to x, we call the self-adjoint linear map $S_{\mathcal{R}}(x)$ on $T_u M$ defined by

$$S_{\mathcal{R}}(x)y = \mathcal{R}(y,x,x,\bullet;x)^{\sharp_g} \tag{5.4}$$

or equivalently

$$(S_{\mathcal{R}}(x)y)^{\flat_g}(w) = \mathcal{R}(y,x,x,w;x) \tag{5.5}$$

where $\flat_g$ and $\sharp_g$ denote the natural isomorphisms between T_uM and its dual T_u^*M, for associate non-degenerate metric $\widetilde{g}$ of g.

By using a covariant derivative ∇R of induced Riemann curvature tensor R on M, for any $X \in S^{\pm}(M)$, $Y \in \Gamma(TM)$, the pseudo-Szabó operator associated to ∇R with respect to X is defined by

$$S_{\nabla R}(X)Y = (\nabla_X R)(Y,X,X,\bullet)^{\sharp_g}, \tag{5.6}$$

or equivalently, for any $X \in S^{\pm}(M)$ and $Y,W \in \Gamma(TM)$

$$\widetilde{g}(S_{\nabla R}(X)Y,W) = (\nabla_X R)(Y,X,X,W). \tag{5.7}$$

Thus, by direct calculation we obtain, for $X \in S^{\pm}(M)$, $Y,W \in \Gamma(TM)$,

$$\widetilde{g}(S_{\nabla R}(X)Y,W) = (\nabla_X g)(R(Y,X)X,W) + g((\nabla_X R)(Y,X)X,W) \tag{5.8}$$

Note that for any $X \in S^{\pm}(M)$, $Y \in \Gamma(TM)$, we have $S_{\nabla R}(X)Y \in X^{\perp}$ and $S_{\nabla R}(X)X = 0$, where

$$X^{\perp} = \{Y \in \Gamma(TM) : g(X,Y) = 0\} \tag{5.9}$$

6 Some basic examples

Example 1. (lightlike cone Λ_0^{n+1})
Let's consider the lightlike cone Λ_0^{n+1} at the origin of $\mathbb{R}_1^{n+2}$ endowed with the semi-Euclidean metric $\overline{g}(x,y) = -x^0y^0 + \sum_{a=1}^{n+1} x^a y^a$, $(x = \sum_{A=0}^{n+1} x^A \frac{\partial}{\partial x^A})$. The lightlike cone Λ_0^{n+1} is given by the equation

$$-(x^0)^2 + \sum_{a=1}^{n+1} (x^a)^2 = 0, \quad x \neq 0$$

It is known that Λ_0^{n+1} is a lightlike hypersurface of $\mathbb{R}_1^{n+2}$ and the radical distribution is spanned by a global vector field $\xi = \sum_{A=0}^{n+1} x^A \frac{\partial}{\partial x^A}$ on Λ_0^{n+1}. The lightlike transversal vector bundle $tr(T\Lambda_0^{n+1})$ is spanned by the section $N = \frac{1}{2(x^0)^2}\{-x^0 \frac{\partial}{\partial x^0} + \sum_{a=1}^{n+1} x^a \frac{\partial}{\partial x^a}\}$, it is also globally defined. Next, any $X \in \Gamma(S(T\Lambda_0^{n+1}))$ is expressed by $X = \sum_{a=1}^{n+1} X^a \frac{\partial}{\partial x^a}$, where $X^1, \ldots, X^{n+1}$ satisfy

$$\sum_{a=1}^{n+1} x^a X^a = 0. \tag{6.1}$$

By direct calculation, for any $X \in \Gamma(T\Lambda_0^{n+1})$, we have

$$\nabla_X \xi = \overline{\nabla}_X \xi = X \tag{6.2}$$

The lightlike cone Λ_0^{n+1} is totally umbilical, since by direct calculation the local second fundamental form is given by

$$B(X,Y) = -g(X,Y), \qquad \forall X,Y \in \Gamma(T\Lambda_0^{n+1}). \tag{6.3}$$

So its algebraic induced Riemann curvature is given by, (see [4], p.114),

$$R(X,Y)Z = \frac{1}{2(x^0)^2}\{g(Y,Z)PX - g(X,Z)PY\} \tag{6.4}$$

where P is the projection morphism on the screen associated to $\{\xi, N\}$. It easy to check that ∇R is an algebraic covariant derivative curvature tensor.
By taking $X = \eta(X)\xi + (X^1\frac{\partial}{\partial x^1} + X^{n+1}\frac{\partial}{\partial x^{n+1}})$ with $\sum_{a=1}^{n+1} x^a X^a = 0$ and $\xi = \sum_{A=0}^{n+1} x^A \frac{\partial}{\partial x^A}$, the spaces $S^{\pm}(\Lambda_0^{n+1})$ and $X^{\perp}$ are given by

$$\begin{aligned} S^{\pm}(\Lambda_0^{n+1}) &= \{X \in \Gamma(T\Lambda_0^{n+1}),\ g(X,X) = \pm 1\} \\ &= \{X \in \Gamma(T\Lambda_0^{n+1}),\ \sum_{a=1}^{n+1}(X^a)^2 = \pm 1\} \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} X^{\perp} &= \{Y \in \Gamma(T\Lambda_0^{n+1}),\ g(X,Y) = 0\} \\ &= \{Y \in \Gamma(T\Lambda_0^{n+1}),\ \sum_{a=1}^{n+1} X^a Y^a = 0\}. \end{aligned} \tag{6.6}$$

Now let compute $S_{\nabla R}(X)$ for $X \in S^+(\Lambda_0^{n+1})$. Let $X \in S^+(\Lambda_0^{n+1})$, $Y, W \in X^{\perp}$. By using (6.2), we have

$$P\nabla_X Y = P\nabla_X(PY + \eta(Y)\xi) = P\nabla_X PY + P(X\cdot\eta(Y)\xi + \eta(Y)X).$$

Thus, using relations (6.3) and (6.4), we obtain

$$(\nabla_X g)(R(Y,X)X,W) = \frac{1}{2(x^0)^2}[B(X,PY)\eta(W) + B(X,W)\eta(PY)] = 0$$

and

$$g((\nabla_X R)(Y,X)X,W) = [X(\frac{1}{2(x^0)^2}) - \frac{1}{(x^0)^2}\eta(X)]g(PY,W).$$

Therefore,

$$S_{\nabla R}(X)Y = [\frac{1}{2}x^0\eta(X)\frac{\partial}{\partial x^0}(\frac{1}{(x^0)^2}) - \frac{1}{(x^0)^2}\eta(X)]PY.$$

Since ∇R is algebraic, it is easy to see that $\widetilde{g}(S_{\nabla R}(X)X,W) = (\nabla_X R)(X,X,X,W) = 0$, that is $S_{\nabla R}(X)X = 0$. Therefore the pseudo-Szab$\acute{o}$ operator associated to ∇R is given, for $X \in S^+(\Lambda_0^{n+1})$, by

$$S_{\nabla R}(X)Y = \begin{cases} -[\frac{2}{(x^0)^2}\eta(X)]PY & Y \in X^{\perp}, \\ \\ 0 & Y = X. \end{cases} \tag{6.7}$$

Also, we can check that for $X \in S^-(\Lambda_0^{n+1})$,

$$S_{\nabla R}(X)Y = \begin{cases} [\frac{2}{(x^0)^2}\eta(X)]PY & Y \in X^{\perp}, \\ \\ 0 & Y = X. \end{cases} \tag{6.8}$$

The found expressions $S_{\nabla R}(X)$ verify, for any $X \in S^{\pm}(\Lambda_0^{n+1})$, $Y \in T\Lambda_0^{n+1} = \{X\} \perp X^{\perp}$, $S_{\nabla R}(X)X = 0$ and

$$\widetilde{g}(S_{\nabla R}(X)Y,X) = \pm\frac{2}{(x^0)^2}\eta(X)g(PY,X) = 0, \text{ ie } S_{\nabla R}(X)Y \in X^{\perp}.$$

Example 2.
Let $\overline{M} = \mathbb{R}^6$ be a 6-dimensional real number space. We consider $\{x^i\}_{0\leqslant i\leqslant 5}$ as Cartesian coordinates on $\overline{M}$ and define with respect to the natural field of frames $\{\frac{\partial}{\partial x^i}\}$ a metric $\overline{g}_f$ on $\overline{M}$ by

$$\begin{aligned}\overline{g}_f &= dx^0dx^3 + dx^1dx^4 + dx^2dx^5 + f(x^0,x^1,x^2)[(dx^0)^2 + (dx^1)^2 + (dx^2)^2 \\ &\quad + dx^0dx^1 + dx^0dx^2 + dx^1dx^2]. \qquad (6.9)\end{aligned}$$

where $f = f(x^0,x^1,x^2)$ is smooth real-valued function. It is easy to check that $\overline{g}_f$ is a semi-Riemannian metric of signature $(3,3)$. Now let consider a hypersurface M of $(\overline{M},\overline{g}_f)$ defined by

$$M = \left\{(x^0,...,x^5) \in \mathbb{R}^6 : x^5 = x^4\right\}$$

FACT 1. According to the proof of lemma 2.1 in ([8]), where authors considered the local coordinates $(x^1,...,x^p,y^1,...,y^p)$ and the local field of frames $\{\partial_1^x,...,\partial_p^x,\partial_1^y,...,\partial_p^y\}$ and the symmetric application(tensor) $\psi_{ij}(x)$. In this Example we take $p = 3$, the local coordinates $(x^0,...,x^5)$, the local field of frames $\{\partial_{x^0},...,\partial_{x^5}\}$ and $\psi_{ij}(x) = f(x^0,x^1,x^2)$. Thus, from relation

$$\nabla_{\partial_i^x}\partial_i^x = \frac{1}{2}\sum_k(\psi_{ij/j} + \psi_{ij/i} - \psi_{ij/k})\partial_k^y$$

given in ([8]), we obtain that, non-vanishing components of Levi-Civita connection $\overline{\nabla}$ are given by

$$\overline{\nabla}_{\partial_j}\partial_i = \frac{1}{2}\sum_{k=3}^{5}(\partial_j f + \partial_i f - \partial_{k-3}f)\partial_k, \quad 0 \le i,j \le 2, \qquad (6.10)$$

where $\partial_i = \frac{\partial}{\partial x^i}$. Thus, by straightforward calculation the tangent space TM is spanned by $\{U_i\}_{0\leqslant i\leqslant 4}$, where $U_0 = \frac{\partial}{\partial x^0}$, $U_1 = \frac{\partial}{\partial x^1}$, $U_2 = \frac{\partial}{\partial x^2}$, $U_3 = \frac{\partial}{\partial x^3}$, $U_4 = \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5}$, and the radical distribution $TM^{\perp}$ on M of rank 1 is spanned by $\xi = -\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}$. It follows that $TM^{\perp} \subset TM$. Then M is a 5-dimensional lightlike hypersurface of $\overline{M}$. Also the transversal vector bundle $tr(TM)$ is spanned by $N = -\frac{\partial}{\partial x^4}$. It follows that the corresponding screen distribution $S(TM)$ is spanned by $\{W_i\}_{1\leqslant i\leqslant 4}$, where $W_1 = \frac{\partial}{\partial x^0}$, $W_2 = \frac{\partial}{\partial x^2}$, $W_3 = \frac{\partial}{\partial x^3}$, $W_4 = \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5}$.

FACT 2. Let's consider on $\overline{M}$ a local field of frames $\{E_0 = \xi, E_i = W_i, N\}_{1\le i\le 4}$ such that $\{E_0 = \xi, E_i\}$ is a local field of frames on M with respect to the decomposition (2.1). By Gauss equation, we have $\overline{\nabla}_{E_\beta}E_\alpha = \Gamma_{\alpha\beta}^{\gamma}E_\gamma + B_{\alpha\beta}N$, $\alpha,\beta,\gamma \in \{0,...,4\}$,
where $B_{\alpha\beta} = B(E_\alpha,E_\beta)$ and $\Gamma_{\alpha\beta}^{\gamma}$ are the coefficients of the induced connection ∇ with respect to $\{E_\alpha\}$, ie $\nabla_{E_\beta}E_\alpha = \Gamma_{\alpha\beta}^{\gamma}E_\gamma$. Then, by direct calculations, the only non-vanishing components of induced connection on M are given by
$\nabla_{E_1}E_0 = \nabla_{E_0}E_1 = \frac{1}{2}(\partial_2 f - \partial_1 f)(E_3 + E_4)$, $\quad \nabla_{E_2}E_0 = \nabla_{E_0}E_2 = \frac{1}{2}(\partial_2 f - \partial_1 f)(E_3 + E_4)$,

$\nabla_{E_1}E_1 = \frac{1}{2}(\partial_0 f)E_3 + (\partial_0 f - \frac{1}{2}\partial_2 f)E_4, \quad \nabla_{E_2}E_1 = \nabla_{E_1}E_2 = \frac{1}{2}(\partial_2 f)E_3 + \frac{1}{2}(\partial_0 f)E_4,$
$\nabla_{E_2}E_2 = (\partial_2 f - \frac{1}{2}\partial_0 f)E_3 + \frac{1}{2}(\partial_2 f)E_4.$
Also the local bilinear form B is given by

$$B(E_1,E_1) = B(E_1,E_2) = B(E_2,E_2) = \frac{1}{2}(\partial_1 f - \partial_2 f)$$

FACT 3. By direct calculations, the only non-vanishing components of the induced Riemannian curvature tensor on M are given by :

$$R(E_0,E_1)E_0 = \alpha(E_3+E_4), \quad R(E_0,E_1)E_1 = \beta E_4, \quad R(E_0,E_1)E_2 = -\beta E_3$$

$$R(E_0,E_2)E_0 = \alpha(E_3+E_4), \quad R(E_0,E_2)E_1 = \beta E_4, \quad R(E_0,E_2)E_2 = -\beta E_3$$

$$R(E_1,E_2)E_0 = \beta(E_3+E_4), \quad R(E_1,E_2)E_1 = \gamma E_4, \quad R(E_1,E_2)E_2 = -\gamma E_3$$

Where α, β, and γ are defined by :

$$\begin{aligned} \alpha &= \frac{1}{2}(\partial_1^2 f + \partial_2^2 f - 2\partial_2\partial_1 f), \quad \beta = \frac{1}{2}(-\partial_1\partial_0 f + \partial_2\partial_0 f - \partial_2^2 f + \partial_2\partial_1 f) \\ \gamma &= \frac{1}{2}\partial_0^2 f - \partial_2\partial_0 f + \frac{1}{2}\partial_2^2 f \end{aligned} \tag{6.11}$$

FACT 4. Let $X = \sum_{i=0}^{4} X^i E_i$ the tangent vector field on M, by straightforward calculations we obtain :

$$\begin{aligned} (\nabla_X R)(E_0,X)X &= [(X^0X^1 + X^0X^2)\mathcal{D}(\alpha) - (X^1X^2 + X^2X^2)\mathcal{D}(\beta)]E_3 \\ &\quad +[(X^0X^1 + X^0X^2)\mathcal{D}(\alpha) - (X^1X^1 + X^1X^2)\mathcal{D}(\beta)]E_4 \end{aligned}$$

$$\begin{aligned} (\nabla_X R)(E_1,X)X &= [-X^0X^0\mathcal{D}(\alpha) + 2X^0X^2\mathcal{D}(\beta) - X^2X^2\mathcal{D}(\gamma)]E_3 \\ &\quad +[-X^0X^0\mathcal{D}(\alpha) + (X^0X^2 - X^0X^1)\mathcal{D}(\beta) + X^1X^2\mathcal{D}(\gamma)]E_4 \end{aligned}$$

$$\begin{aligned} (\nabla_X R)(E_2,X)X &= [-X^0X^0\mathcal{D}(\alpha) + (X^0X^2 - X^0X^1)\mathcal{D}(\beta) + X^1X^2\mathcal{D}(\gamma)]E_3 \\ &\quad [-X^0X^0\mathcal{D}(\alpha) - 2X^0X^1\mathcal{D}(\beta) - X^1X^1\mathcal{D}(\gamma)]E_4 \end{aligned}$$

where $\mathcal{D}$ is defined by $\mathcal{D} = \sum_{i=0}^{2} X^i E_i = X^1\partial_0 - X^0\partial_1 + (X^0 + X^2)\partial_2$

FACT 5. On the local field of frames $\{E_0, E_i\}$, the induced metric $g = \overline{g}_{|\{E_0,E_i\}}$ has matrix form given by

$$g = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & f & f & 1 & 0 \\ 0 & f & f & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{6.12}$$

Thus, for any tangent vector field $X = \sum_{i=0}^{4} X^i E_i$ on M, the spaces $S^{\pm}(M)$ and $X^{\perp}$ are given by

$$S^{\pm}(M) = \{X \in \Gamma(TM), \ \ f(X^1)^2 + f(X^2)^2 + fX^1X^2 + 2X^1X^3 + 2X^2X^4 = \pm 1\} \tag{6.13}$$

$$\begin{aligned} X^{\perp} &= \{Y \in \Gamma(TM),\ fX^1Y^1 + fX^2Y^2 + fX^1Y^2 + fX^2Y^1 + X^1Y^3 + X^3Y^1 \\ &\quad + X^2Y^4 + X^4Y^2 = 0\}. \end{aligned} \tag{6.14}$$

(1) For the totally geodesic case ie $\partial_1 f = \partial_2 f$, the covariant derivative ∇R is algebraic. By (6.11), we have $\alpha = \beta = 0$. Also as per relation (5.8), for $X \in S^{\pm}(M)$ and $W \in X^{\perp}$ we have

$$\widetilde{g}(S_{\nabla R}(X)E_i, W) = g((\nabla_X R)(E_i, X)X, W).$$

(2) For the non-totally geodesic case, let consider for example the case where

$$f = (x^0)^2(x^1)^2 + (x^0)^2(x^2)^2 + 2(x^0)^2x^1x^2 + x^1$$

By (6.11), we have $\alpha = \beta = 0$. Using the local components of induced Riemann curvature R given in Fact 3, we can easy check that the covariant derivative ∇R is algebraic. Also by calculation, using Fact 3 and the components of bilinear form B, we get $(\nabla_X g)(R(E_i, X)X, W) = 0$. Thus, for the both cases (1) and (2), we get as per Fact 4

$$(\nabla_X R)(E_1, X)X = -X^2X^2\mathcal{D}(\gamma)E_3 + X^1X^2\mathcal{D}(\gamma)E_4$$

$$(\nabla_X R)(E_2, X)X = X^1X^2\mathcal{D}(\gamma)E_3 - X^1X^1\mathcal{D}(\gamma)E_4$$

Therefore, by using (5.8) and the above results, with respect to the local field of frame $\{E_0 = \xi, E_i = W_i\}_{1 \leqslant i \leqslant 4}$ on M, the pseudo-Szab*ó* operator associated to ∇R is given, for $X \in S^{\pm}(M)$, by

$$S_{\nabla R}(X) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -X^2X^2\mathcal{D}(\gamma) & X^1X^2\mathcal{D}(\gamma) & 0 & 0 \\ 0 & X^1X^2\mathcal{D}(\gamma) & -X^1X^1\mathcal{D}(\gamma) & 0 & 0 \end{pmatrix}$$

By direct calculation, using found matrix of $S_{\nabla R}(X)$, we obtain, for any $X \in S^{\pm}(M)$, $Y \in \Gamma(TM)$, $S_{\nabla R}(X)X = 0$ and

$$\widetilde{g}(S_{\nabla R}(X)Y, X) = g(aE_3 + bE_4, \sum_{i=0}^{4} X^i E_i) = aX^1 + bX^2 = 0, \tag{6.15}$$

where $a = \mathcal{D}(\gamma)[-(X^2)^2Y^1 + X^1X^2Y^2]$ and $b = \mathcal{D}(\gamma)[X^1X^2Y^1 - (X^1)^2Y^2]$. Thus, we infer that $S_{\nabla R}(X)Y \in X^{\perp}$.

7 Lightlike Szab*ó* hypersurfaces

It is known by approach developed in ([4]) that, the extrinsic geometry of lightlike hypersurfaces depends on a choice of screen distribution, or equivalently, on the normalization. Since the screen distribution is not uniquely determined, a well defined concept of Szab*ó* condition is not possible for an arbitrary lightlike hypersurface of a semi-Riemannian manifold. Thus, one must look for a class of screen distributions for which the induced Riemann curvature and associated pseudo-Szab*ó* operator have the desired symmetries and properties. In short, we precise the following.

Definition 7.1. A screen distribution $S(TM)$ is said to be ∇-admissible if the covariant derivative ∇R of its associated induced Riemann curvature R is an algebraic covariant derivative curvature tensor.

Examples.
(1) It is obvious that on totally geodesic lightlike hypersurfaces, all screen distributions are ∇-admissible .

(2) Based on corollary 4.5, any locally screen conformal lightlike hypersurface of a semi-Euclidean space $\mathbb{R}^{n+2}_q$ admits a ∇-admissible screen distribution. In particular, the lightlike cone at the origin of Lorentzian space $\mathbb{R}^{n+2}_1$ and the lightlike Monge hypersurfaces of $\mathbb{R}^{n+2}_q$, all of them admit ∇-admissible screen distributions.

(3) Let (M,g) be a (proper) totally umbilical lightlike hypersurface of a semi-Euclidean space $\mathbb{R}^{n+2}_q$. By using the proposition 4.1, we can prove that any totally umbilical screen distribution $S(TM)$ is ∇-admissible.

According to above examples, one can see that, there exist classes of lightlike hypersurfaces of semi-Riemannian manifolds which admit ∇-admissible screen distributions. Using this information, we give the following definition.

Definition 7.2. A lightlike hypersurface (M,g) of a semi-Riemannian manifold $(\overline{M},\overline{g})$ of constant index is called timelike (resp. spacelike) Szab*ó* at $u\in M$ if for each ∇-admissible screen distribution $S(TM)$ and algebraic covariant derivative ∇R of associate induced Riemann curvature R, the characteristic polynomial of $S_{\nabla R}(x)$ is independent of $x\in S^-_u(M)$ (resp. $x\in S^+_u(M)$). Moreover, if this holds at each $u\in M$, then (M,g) is called pointwise Szab*ó* (or Szab*ó*). If this holds independently of the point $u\in M$, (M,g) is called globally Szab*ó*.

Examples
(1) By referring to the example 2, one deducts that for a given ∇-admissible screen on the lightlike hypersurface M, the pseudo-Szab*ó* operator admits the characteristic polynomial given by

$$f_X(t) = -t^5, \qquad \forall X\in S^{\pm}(M) \tag{7.1}$$

which is independent neither the points of M, neither admissible screen distributions, nor $X\in S^{\pm}(M)$. Thus M is globally lightlike Szab*ó* hypersurface of $(\mathbb{R}^6,\overline{g}_f)$.

(2) According to example 1, we see that, in adapted quasi-orthonormal basis $\{\xi,X,X_1,...,X_{n-1}\}$ on the lightlike cone Λ^{n+1}_0, the matrix of $S_{\nabla R}(X)$ has the form

$$S_{\nabla R}(X) = \begin{pmatrix} O_2 & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & -\frac{2}{(x^0)^2}\eta(X)I_{n-1} \end{pmatrix}.$$

Then, the characteristic polynomial f_X of $S_{\nabla R}(X)$ is given by

$$f_X(t) = (-1)^{n+1}t^2[\frac{2}{(x^0)^2}\eta(X)+t]^{n-1}, \qquad \forall X\in S^{+}(\Lambda^{n+1}_0), \tag{7.2}$$

this depend of unit vector field X, so the lightlike cone Λ_0^{n+1} of $\mathbb{R}_1^{n+2}$ is not lightlike Szab*ó* hypersurface.

By using the following lemma whose result is due in the Riemannian setting $(q=0)$ to Szab*ó* ([12]) and in the Lorentzian setting $(q=1)$ to Gilkey and Stavrov ([7]), we prove that there is not non-vanishing algebraic covariant derivative ∇R of an induced Riemann curvature tensor R on the totally geodesic lightlike Szab*ó* hypersurface of signature $(p,q,1)$ with $q=0$ or $q=1$.

Lemma 7.3. *Let ∇R be a Szabó tensor on a hypersurface $(M,g,S(TM))$ of signature $(p,q,1)$. If $q=0$ or if $q=1$, then for any $u\in M$, on screen vector space $S_u(TM)$, the tensor ∇R vanishes identically.*

Proof: If ∇R is a Szab*ó* tensor associated to a ∇-admissible screen $S(TM)$, for any $u\in M$, $\nabla R_{|S_u(TM)}$ is Szab*ó* tensor on semi-Euclidean space $S_u(TM)$ of signature (p,q). For case of $q=0$ or $q=1$, we have result in virtue of results given by Szab*ó* ([12]) and Gilkey and Stavrov ([7]). $\square$

Note that the algebraic covariant derivative ∇R of the Riemann curvature R on a manifold M is called Szab*ó* tensor if the manifold M is pointwise Szab*ó*.

Theorem 7.4. *Let (M,g) be a totally geodesic lightlike hypersurface of a semi-Riemannian manifold $(\overline{M},\overline{g})$ of constant index $q=1$ or 2. If (M,g) is pointwise Szabó, then the covariant derivative ∇R of the induced Riemann curvature R, associated to any screen distribution $S(TM)$ vanishes identically on T_uM, for any $u\in M$. Therefore the pseudo-Szabó operator $S_{\nabla R}(\cdot)$ vanishes on $S_u^{\pm}(M)$, for all $u\in M$.*

Proof. Consider a screen distribution $S(TM)$ with associate algebraic curvature R. The manifold $(\overline{M},\overline{g})$ being of constant index $q=1$ or 2, for all $u\in M$, T_uM is degenerate vector space of signature $(0,+,...,+)$ or $(0,-,+,...,+)$. Also

$$x\in S_u^{\pm}(M)\Longleftrightarrow Px\in S^{\pm}(S(T_uM))$$

where $S(T_uM)$ is Euclidean or Lorentzian. By hypothesis, for all $u\in M$, the pseudo-Szab*ó* operator $S_{\nabla R}(\cdot)$ has constant eigenvalues on $S_u^{\pm}(M)$, therefore $S_{\nabla R_{|S(T_uM)}}(\cdot)$ has constant eigenvalues on $S^{\pm}(S(T_uM))$. By the lemma 7.3, the tensor ∇R vanishes identically on the screen space $S(T_uM)$, that is

$$(\nabla_v R)(x,y,z,w)=0,\quad \forall x,y,z,w,v\in S(T_uM).$$

By using algebraic tensors R and ∇R, the theorem 2.2 (see [4], p.88) and considering $x=\eta(x)\xi+Px$ with respect to the decomposition (2.1), for all $x,y,z,w,v\in T_uM$, we have

$$\begin{aligned}(\nabla_\xi R)(x,y,z,w) &= -(\nabla_z R)(x,y,w,\xi)-(\nabla_w R)(x,y,\xi,z)\\ &= -R(x,y,w,\nabla_z\xi)-R(x,y,\nabla_w\xi,z)\\ &= 0.\end{aligned}$$

Thus,

$$
\begin{aligned}
(\nabla_v R)(x,y,z,w) &= \eta(v)(\nabla_\xi R)(x,y,z,w) + (\nabla_{Pv} R)(x,y,z,w) \\
&= \eta(x)(\nabla_{Pv} R)(\xi,Py,Pz,Pw) + \eta(y)(\nabla_{Pv} R)(Px,\xi,Pz,Pw) \\
&\quad \eta(z)(\nabla_{Pv} R)(Px,Py,\xi,Pw) + \eta(w)(\nabla_{Pv} R)(Px,Py,Pz,\xi) \\
&= -\eta(x)R(\nabla_{Pv}\xi,Py,Pz,Pw) - \eta(y)R(Px,\nabla_{Pv}\xi,Pz,Pw) \\
&\quad -\eta(z)R(Px,Py,\nabla_{Pv}\xi,Pw) - \eta(w)R(Px,Py,Pz,\nabla_{Pv}\xi) \\
&= 0.
\end{aligned}
$$

where P is the projection of T_uM on $S(T_uM)$. It follow that the tensor ∇R vanishes identically on T_uM. □

Refering to the theorem 7.4 we see that for the signature $(p,q,1)$ with $q=0$ or $q=1$, none of totally geodesic lightlike Szab*ó* hypersurface admits non-vanishing pseudo-Szab*ó* operator. This can fail in the higher signature setting. The proof of the following is the example 2 of section 6, note that the domain $S_u^{\pm}(M)$ of $S_{\nabla R}(\cdot)$ can be extended to T_uM, since $S_{\nabla R}(cx) = c^3 S_{\nabla R}(x)$. This result extend the theorem 1.7 of ([7]).

Theorem 7.5. *Let (M,g) be a lightlike Szabó hypersurface of signature $(p,q,1)$ with $q \geqslant 2$. There exists an algebraic covariant derivative tensor ∇R associated to a ∇-admissible screen distribution $S(TM)$ so that $S^2_{\nabla R}(x) = 0$, for any $x \in T_uM$ and so that $S_{\nabla R}(\cdot)$ does not vanish identically on T_uM.*

In the following example we consider lightlike hypersurfaces of 3-dimensional semi-Riemannian manifolds of signatures $\{-,+,+\}$ and $\{-,-,+\}$. We obtain that in totally geodesic case, the lightlike surfaces, spaces of signature $(1,0,1)$ or $(0,1,1)$ are Szab*ó*, its pseudo-Szab*ó* operators $S_{\nabla R}(\cdot)$ vanish identically on $S^{\pm}(M)$ and its Szab*ó* tensors ∇R vanish identically on TM.

Example (Lightlike surfaces)

Consider a lightlike surface $(M,g,S(TM))$ of a 3-dimensional semi-Riemannian manifold $(\overline{M},\overline{g})$. Let $\overline{\nabla}$ and ∇ be the Levi-civita connection on $\overline{M}$ and the induced connection on M, respectively. Consider on $\overline{M}$ a local frame $\{E_0=\xi,E_1,N\}$ such that $\{E_0=\xi,E_1\}$ be a frame on M with respect to the decomposition (2.1). Let ∇R, the covariant derivative of the induced Riemann curvature R associated to the ∇ admissible screen $S(TM)$. By Gauss equation, we have

$$\overline{\nabla}_{E_\beta} E_\alpha = \Gamma^\gamma_{\alpha\beta} E_\gamma + B_{\alpha\beta} N, \qquad \alpha,\beta,\gamma \in \{0,1\}$$

where $B_{\alpha\beta} = B(E_\alpha,E_\beta)$ and $\Gamma^\gamma_{\alpha\beta}$ are the coefficients of the induced connection ∇ with respect to $\{E_\alpha\}$, ie $\nabla_{E_\beta}E_\alpha = \Gamma^\gamma_{\alpha\beta}E_\gamma$. Note that
$B_{00} = B_{01} = B_{10} = 0$. Then by straightforward calculations, we obtain :

$$
\begin{aligned}
R(E_0,E_1)E_0 &= \nabla_{E_0}\nabla_{E_1}E_0 - \nabla_{E_1}\nabla_{E_0}E_0 - \nabla_{[E_0,E_1]}E_0 \\
&= \alpha^0 E_0 + \alpha^1 E_1 \qquad (7.3)
\end{aligned}
$$

where

$$\alpha^0 = E_0(\Gamma^0_{01}) - E_1(\Gamma^0_{00}) + \Gamma^0_{00}\Gamma^0_{01} + \Gamma^0_{01}\Gamma^1_{01} + \Gamma^0_{10}\Gamma^1_{01} - \Gamma^0_{00}\Gamma^0_{10} - \Gamma^1_{00}\Gamma^0_{11} - \Gamma^0_{01}\Gamma^1_{10},$$

$$\alpha^1 = E_0(\Gamma^1_{01}) - E_1(\Gamma^1_{00}) + 2\Gamma^1_{00}\Gamma^0_{01} + \Gamma^1_{01}\Gamma^1_{01} - \Gamma^0_{00}\Gamma^1_{01} - \Gamma^1_{00}\Gamma^0_{10} - \Gamma^1_{00}\Gamma^1_{11}.$$

Similarly, we have

$$R(E_0,E_1)E_1 = \beta^0 E_0 + \beta^1 E_1 \tag{7.4}$$

where

$$\beta^0 = E_0(\Gamma^0_{11}) - E_1(\Gamma^0_{10}) + \Gamma^0_{00}\Gamma^0_{11} + \Gamma^1_{01}\Gamma^0_{11} + \Gamma^0_{10}\Gamma^1_{11} - 2\Gamma^1_{10}\Gamma^0_{11} - \Gamma^0_{10}\Gamma^0_{10},$$

$$\beta^1 = E_0(\Gamma^1_{11}) - E_1(\Gamma^1_{10}) + \Gamma^1_{00}\Gamma^0_{11} + \Gamma^0_{01}\Gamma^1_{10} + \Gamma^1_{01}\Gamma^1_{11} - \Gamma^1_{01}\Gamma^0_{10} - \Gamma^0_{10}\Gamma^1_{10} - \Gamma^1_{10}\Gamma^1_{11}.$$

Now by using (7.3) and (7.4), for any $X = X^0E_0 + X^1E_1 \in TM$, we have

$$\begin{aligned} R(E_0,X)X &= X^1X^0R(E_0,E_1)E_0 + (X^1)^2R(E_0,E_1)E_1 \\ &= (\alpha^0X^0X^1 + \beta^0(X^1)^2)E_0 + (\alpha^1X^0X^1 + \beta^1(X^1)^2)E_1. \end{aligned} \tag{7.5}$$

Similarly, we have

$$R(E_1,X)X = -(\alpha^0(X^0)^2 + \beta^0X^0X^1)E_0 - (\alpha^1(X^0)^2 + \beta^1X^0X^1)E_1. \tag{7.6}$$

The spaces $S^{\pm}(M)$ and $X^{\perp}$ are given by

$$S^{\pm}(M) = \{X = X^0E_0 + X^1E_1,\ \ (X^1)^2g_{11} = \pm 1\}$$

and

$$X^{\perp} = \{Y = Y^0E_0 + Y^1E_1,\ \ X^1Y^1 = 0\}$$

Thus, since $g \neq 0$, for any $X \in S^{\pm}(M)$, we have $X^1 \neq 0$ and if $W = W^0E_0 + W^1E_1 \in X^{\perp}$, then $W^1 = 0$ that is $W = W^0E_0$. Using (5.8) leads to

$$\begin{aligned} \widetilde{g}(S_{\nabla R}(X)E_0,W) &= B(X,R(E_0,X)X)\eta(W) + B(X,W)\eta(R(E_0,X)X) \\ &= B_{11}(\alpha^1X^0(X^1)^2 + \beta^1(X^1)^3)\widetilde{g}(E_0,W). \end{aligned}$$

Hence,

$$S_{\nabla R}(X)E_0 = B_{11}(\alpha^1X^0(X^1)^2 + \beta^1(X^1)^3)E_0 \tag{7.7}$$

Similarly, we can check that

$$S_{\nabla R}(X)E_1 = -B_{11}(\alpha^1(X^0)^2X^1 + \beta^1X^0(X^1)^2)E_0. \tag{7.8}$$

Therefore the pseudo-Szab*ó* operator associated to ∇R is given by

$$S_{\nabla R}(X) = \begin{pmatrix} B_{11}(\alpha^1X^0(X^1)^2 + \beta^1(X^1)^3) & -B_{11}(\alpha^1(X^0)^2X^1 + \beta^1X^0(X^1)^2) \\ 0 & 0 \end{pmatrix} \tag{7.9}$$

We can check that $S_{\nabla R}(X)X = 0$ and for any $Y \in \Gamma(TM)$, $\widetilde{g}(S_{\nabla R}(X)Y,X) = 0$. For totally geodesic case, that is $B_{11} = 0$, we obtain, for any $X \in S^{\pm}(M)$,

$$S_{\nabla R}(X) = 0. \tag{7.10}$$

Also, for this case, since R and ∇R are algebraic tensors, by direct calculation, we get, for any $V,X,Y,Z,W \in \{E_0,E_1\}$,

$$(\nabla_V R)(X,Y,Z,W) = 0,$$

that is the Szab*ó* tensor of M vanish identically on $\Gamma(TM)$.

It is known that lightlike submanifold whose screen distribution is integrable have interesting properties in degenerate geometry. From theorem 2.3 (see [4], p.89), the screen distribution is integrable if and only if its second fundamental form C is symmetric. This is verified for lightlike hypersurfaces with totally umbilical or totally geodesic screen distribution. For this later case, the following result hold.

Theorem 7.6. *Let (M,g) lightlike hypersurface of a semi-Riemannian manifold $(\overline{M},\overline{g})$, totally geodesic on a neighbourhood $\mathcal{U} \subset M$, of a $u \in M$. If all ∇-admissible screen distributions are totally geodesic on $\mathcal{U}$, then, (M,g) is Szabó at u if and only if the semi-Riemannian screen leaves is Szabó at this point.*

Proof. Consider a generic totally geodesic ∇-admissible screen distribution $S(TM)$ on $\mathcal{U} \in M$. Let ∇R the algebraic covariant derivative of associate induced Riemann curvature R. Let R' and $\overset{*}{R}$ denote the restriction of R on $S(TM)$ and the Riemann curvature tensor with respect to the Levi-Civita connection $\overset{*}{\nabla}$ on the screen distribution, respectively. Let $x,y,z \in S(T_uM)$, by straightforward calculation using equations (2.6), we have

$$\begin{aligned} R'(x,y)z = R(x,y)z = &\overset{*}{R}(x,y)z + [C(x,z)\overset{*}{A}_\xi y - C(y,z)\overset{*}{A}_\xi x] + [\nabla_x C(y,z) - \nabla_y C(x,z) \\ &+ \tau(y)C(x,z) - \tau(x)C(y,z)]\xi. \end{aligned}$$

Thus, we get $R'(x,y)z = \overset{*}{R}(x,y)z$ from $C = 0$. Also, $x \in S_u^+(M)$ if and only if $\overset{*}{x} \in S_u^+(M^*)$, with $\overset{*}{x} = Px$ and M^* the leaf of $S(TM)$ through u. Moreover for any $x \in T_uM$, $x^\perp = (Px)^\perp$. Let $\eta(x)\xi$ denote the projection of x on $T_uM^\perp$. By hypothesis and theorem 2.2 (see [4], p.88), we infer that the induced curvature tensors R is algebraic on $TM_{|\mathcal{U}}$ and $\nabla_\xi R = 0$. For any $y,w \in x^\perp$, we have

$$\begin{aligned} \widetilde{g}(S_{\nabla R}(Px)y,w) &= (\nabla_x R)(y, x-\eta(x)\xi, x-\eta(x)\xi, w) \\ &= (\nabla_x R)(y,x,x,w) - (\nabla_x R)(y,\eta(x)\xi,x,w) \\ &\quad -(\nabla_x R)(y,x,\eta(x)\xi,w) - (\nabla_x R)(y,\eta(x)\xi,\eta(x)\xi,w) \\ &= \widetilde{g}(S_{\nabla R}(x)y,w) + \eta(x)R(y,\nabla_x\xi,x,w) + \eta(x)R(y,x,\nabla_x\xi,w) \\ &= \widetilde{g}(S_{\nabla R}(x)y,w). \end{aligned}$$

Thus,

$$S_{\nabla R}(\overset{*}{x}) = S_{\nabla R}(x).$$

We infer that $S_{\overset{*}{\nabla R}}(\overset{*}{x})$ is the restriction of $S_{\nabla R}(x)$ to $\overset{*}{x}^{\perp_{S(T_uM)}} = \{y \in S(T_uM) : g(\overset{*}{x},y) = 0\}$. On the other hand, observe that

$$x^\perp = \overset{*}{x}^{\perp_{S(T_uM)}} \overset{\perp}{\oplus} T_uM^\perp,$$

and we can check that $S_{\nabla R}(x)\xi = 0$. Then, let $h_{\overset{*}{x}}(t)$ and $f_x(t)$ denote the characteristic polynomials of $S_{\overset{*}{\nabla R}}(\overset{*}{x})$ ($\overset{*}{x} \in S_u^+(M^*)$) and $S_{\nabla R}(x)$ ($x \in S_u^+(M)$), respectively. We have finally

$$f_x(t) = t h_{\overset{*}{x}}(t)$$

which shows that the characteristic polynomial of $S_{\overset{*}{\nabla R}}(\overset{*}{x})$ is independent of $(\overset{*}{x} \in S_u^+(M^*))$ if and only if the characteristic polynomial of $S_{\nabla R}(x)$ is independent of $(x \in S_u^+(M))$. Hence, (M,g) is spacelike Szabǿ at a point u if and only if M^* is spacelike Szabǿ at u. Similar is the case for $(x \in S_u^-(M))$ and $(\overset{*}{x} \in S_u^-(M^*))$. $\square$

8 Symmetry properties on lightlike Szabǿ hypersurfaces

In this section, we give some characterizations of locally symmetric lightlike hypersurfaces and semi-symmetric lightlike hypersurfaces under Szabǿ condition.
A lightlike hypersurface $(M,g,S(TM))$ of a semi Riemannian manifold $(\overline{M},\overline{g})$ is said locally symmetric if and only if for any $X,Y,Z,W,V \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$ the following hold ([9])

$$g((\nabla_V R)(X,Y)Z,PW) = 0 \quad \text{and} \quad \overline{g}((\nabla_V R)(X,Y)Z,N) = 0. \tag{8.1}$$

That is $(\nabla_V R)(X,Y)Z = 0$.
Using the lemma 3.2 of ([9]), for any $V,X,Y,Z \in \Gamma(TM)$, $W \in \Gamma(S(TM))$ and $N \in \Gamma(tr(TM))$, we have

$$\begin{aligned} \overline{g}((\overline{\nabla}_V \overline{R})(X,Y)Z,W) &= g((\nabla_V R)(X,Y)Z,W) + (\nabla_V B)(X,Z)C(Y,W) \\ &\quad + B(X,Z)g((\nabla_V A_N)Y,W) - (\nabla_V B)(Y,Z)C(X,W) \\ &\quad - B(Y,Z)g((\nabla_V A_N)X,W) - B(Y,Z)\tau(X)C(V,W) + (\nabla_Y B)(X,Z)C(V,W) \\ &\quad - (\nabla_X B)(Y,Z)C(V,W) + B(X,Z)\tau(Y)C(V,W) - B(V,X)\overline{R}(N,Y,Z,W) \\ &\quad - B(V,Y)\overline{R}(X,N,Z,W) - B(V,Z)\overline{R}(X,Y,N,W) \end{aligned} \tag{8.2}$$

and,

$$\begin{aligned} \overline{g}((\overline{\nabla}_V \overline{R})(X,Y)Z,N) &= g((\nabla_V R)(X,Y)Z,N) + B(X,Z)g((\nabla_V (A_N Y),N) \\ &\quad - B(Y,Z)g((\nabla_V (A_N X),N) - B(V,X)\overline{R}(N,Y,Z,N) - B(V,Y)\overline{R}(X,N,Z,N) \end{aligned} \tag{8.3}$$

Referring to the lemma 7.3, it is easy to see that in the non-degenerate case, an hypersurface of a Lorentzian manifold is locally symmetric if and only if it is Szabǿ. Also for the higher signature setting, if a non-degenerate hypersurface of a semi-Riemannian manifold is locally symmetric then it is Szabǿ. Now, for the degenerate case we have the following propositions.

Proposition 8.1. *Let $(\overline{M},\overline{g})$ be a locally symmetric semi-Riemannian manifold and (M,g) be a totally geodesic lightlike hypersurface of $\overline{M}$. Then $(M,g,S(TM))$ with a given screen distribution $S(TM)$ is locally symmetric. Moreover (M,g) is lightlike Szabó hypersurface.*

Proof. Since $(\overline{\nabla}_V \overline{R})(X,Y)Z = 0$, $\forall X,Y,Z,V \in \Gamma(TM)$, the relations (8.1) are verified from (8.2) and (8.3). Since (M,g) is totally geodesic, using the relation (4.3), we see that the tensor ∇R vanish identically on M. This complete the proof. $\square$

Corollary 8.2. *Let $(\overline{M},\overline{g})$ be a Szabó Lorentzian manifold and (M,g) be a totally geodesic lightlike hypersurface of $\overline{M}$. Then $(M,g,S(TM))$ with a given screen distribution $S(TM)$ is locally symmetric. Moreover (M,g) is lightlike Szabó hypersurface.*

By using theorem 3.1 in ([9]) we have the following.

Proposition 8.3. *Let $(\overline{M},\overline{g})$ be a locally symmetric semi-Riemannian manifold and (M,g) be a lightlike hypersurface of $\overline{M}$ such that $A_N\xi$ is not a null vector field. If $(M,g,S(TM))$ with a given screen distribution $S(TM)$ is locally symmetric then (M,g) is a lightlike Szabó hypersurface.*

Corollary 8.4. *Let $(\overline{M},\overline{g})$ be a Szabó Lorentzian manifold and (M,g) be a lightlike hypersurface of $\overline{M}$ such that $A_N\xi$ is not a null vector field. If $(M,g,S(TM))$ with a given screen distribution $S(TM)$ is locally symmetric then (M,g) is a lightlike Szabó hypersurface.*

In the following result, we obtain that under Szab*ó* condition, the locally symmetric lightlike hypersurfaces of semi-Riemannian manifolds of index $q=1$ or 2 are totally geodesic.

Theorem 8.5. *Let $(M,g,S(TM))$ be a lightlike Szabó hypersurface of a semi-Riemannian manifold $(\overline{M},\overline{g})$ of index 1 or 2, such that $A_N\xi$ is a non-null vector field and $(\overline{\nabla}_V\overline{R})(X,Y)Z \in \Gamma(S(TM))$, for any $V,X,Y,Z \in \Gamma(TM)$. Then $(M,g,S(TM))$ is locally symmetric if and only if it is totally geodesic.*

Proof. Assume that (M,g) is totally geodesic. In virtue of theorem 7.4 and using relation (4.3), we obtain $g(\nabla_V R(X,Y)Z,PW)=0, \forall V,X,Y,Z,W \in \Gamma(TM)$. By hypothesis and using relation (8.3), we obtain $g(\nabla_V R(X,Y)Z,N)=0, \forall V,X,Y,Z \in \Gamma(TM)$. Thus $(M,g,S(TM))$ is locally symmetric. Conversely suppose that $(M,g,S(TM))$ is locally symmetric. By hypothesis and taking $V=Y=\xi$ into (8.3), we obtain

$$B(X,Z)g(\nabla_\xi(A_N\xi),N)=0, \quad \forall X,Z \in \Gamma(TM),$$

that is $B(X,Z)g(A_N\xi,A_N\xi)=0$. Since $A_N\xi$ is non-null, we infer that $B=0$. □

In what follows, we consider curvature operator on a smooth manifold defined by

$$R(X,Y)=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}. \tag{8.4}$$

A lightlike hypersurface $(M,g,S(TM))$ of a semi-Riemannian manifold $(\overline{M},\overline{g})$ is said to be semi-symmetric if the following condition is stisfied (see [11])

$$(R(V_1,V_2)\cdot R)(X,Y,Z,W)=0 \quad \forall V_1,V_2,X,Y,Z,W \in \Gamma(TM) \tag{8.5}$$

where R is the induced Riemann curvature on M. This is equivalent to

$$-R(R(V_1,V_2)X,Y,Z,W)-\ldots-R(X,Y,Z,R(V_1,V_2)W)=0.$$

In general the condition (8.5) is not equivalent to $(R(V_1,V_2)\cdot R)(X,Y)Z=0$ as in the non-degenerate setting. Indeed, by direct calculation we have for any $V_1,V_2,X,Y,Z,W \in \Gamma(TM)$,

$$(R(V_1,V_2)\cdot R)(X,Y,Z,W)=g((R(V_1,V_2)\cdot R)(X,Y)Z,W)+(R(V_1,V_2).g)(R(X,Y)Z,W). \tag{8.6}$$

It is straightforward to see that, any totally geodesic lightlike hypersurface M of a semi-symmetric pseudo-Riemannian manifold $\overline{M}$ is semi-symmetric. This is not true in general in the non semi-symmetric ambient space $\overline{M}$. By using Szab*ó* condition, the following result hold.

Theorem 8.6. *Let (M,g) be a lightlike Szabó hypersurface of a semi-Riemannian manifold $(\overline{M},\overline{g})$ of index $q=1$ or 2. If M is totally geodesic then $(M,g,S(TM))$ with a given screen distribution $S(TM)$ is semi-symmetric.*

Proof. Assume that $(M,g,S(TM))$ is totally geodesic, by assumption and using Theorem 7.4, we get

$$(\nabla_V R)(X,Y,Z,W)=0, \quad \forall V,X,Y,Z,W \in \Gamma(TM).$$

Thus, using relation (8.4), we obtain

$$(R(V_1,V_2).R)(X,Y,Z,W)=0, \quad \forall V_1,V_2,X,Y,Z,W \in \Gamma(TM),$$

that is $(M,g,S(TM))$ is semi-symmetric. $\square$

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