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ADMISSIBLE MANNHEIM CURVES IN PSEUDO-GALILEAN SPACE G_3^1

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Abstract

In this paper, admissible Mannheim partner curves are defined in pseudo-Galilean space G_3^1. Moreover, it is proved that the distance between the reciprocal points of admissible Mannheim curve and the torsions of these curves are constant. Furthermore, the relations for curvatures and torsions of these curves and Schell Theorem are obtained. Finally, a result about these curves being general helix is given.

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1 Introduction

The fundamental theory and characterization of space curves are an interesting and important study field in the realm of differential geometry. Every space curve is uniquely determined by its curvature torsions up to rigid motions. The well known Bertrand curve is characterized as a kind of such corresponding relation between the two curves, whose principal normal is the principal normal of an another curve. Bertrand curves are studied in different space, by many authors [1], [2]. Another kind of associated curves is Mannheim curve. In the recent works, Liu and Wang are very curious about Mannheim curves in

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Euclidean and Minkowski space. They have given the definition of Mannheim mate as follows: Let α and α^* be two space curves. The curve α is said to be a Mannheim partner curve of α^* if there exists a one to one correspondence between their points such that the principal normal vector of α is the binormal vector of α^*. The necessary and sufficient conditions between the curvature and torsion for a curve to be the Mannheim partner curve are obtained in [3]. Also, in 3-dimensional Euclidean space, the characterization of Mannheim curves are obtained in [4]. The detailed discussion concerned with the Mannheim curves can be found in literature.

The pseudo-Galilean geometry is one of the real Cayley-Klein geometries (of projective signature (0,0,+,-), explained in [5]). In [6, 7, 8], the pseudo-Galilean space G_3^1 has been studied by Divjak. Ogrenmis and Ergut are studied the properties of the curves in the pseudo-Galilean space, [9].

In this study, we are concerned with Mannheim curves in 3-dimensional pseudo Galilean space, but the fundamental theorem of pseudo Galilean theory of curves differs crucially from analogous theorem in Euclidean, Lorentzian and Galilean space. In [7], the fundamental theorem of curves in pseudo Galilean space is proved that there are two admissible Mannheim curves that pass through a given point whose Frenet trihedron coincides with a given positively oriented trihedron. In this regard, we introduce the notion of Mannheim curves in 3-dimensional pseudo Galilean space and obtain the characterization of the Mannheim curve.

2 Preliminaries

We will use the same notations and terminologies as in [7] unless otherwise stated. The pseudo-Galilean Space G_3^1 is a three dimensional projective space in which the absolute consist of real plane w (the absolute plane), a real line $f \subset w$ (the absolute line) and a hyperbolic involution of f. The group of motions of G_3^1 is a six-parameter group given (in affine coordinates) by

$$\begin{aligned} \overline{x} &= a + x \\ \overline{y} &= b + cx + y\cosh\varphi + z\sinh\varphi \\ \overline{z} &= d + ex + y\sinh\varphi + z\cosh\varphi. \end{aligned}$$

The scalar product of two vectors $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$ in G_3^1 is defined

$$< v_1, v_2 >= \begin{cases} x_1 x_2 & , \; if \;\; x_1 \neq 0 \vee x_2 \neq 0 \\ y_1 y_2 - z_1 z_2 \, , & if \;\; x_1 = 0 \wedge x_2 = 0. \end{cases}$$

The scalar product is an invariant under the group of motion of G_3^1. The pseudo-Galilean length of the vector $v = (x, y, z)$ is given by [10]

$$\|v\| = \begin{cases} x & , x \neq 0 \\ \sqrt{|y^2 - z^2|} & , x = 0. \end{cases}$$

A vector $v = (x, y, z)$ in G_3^1 is said to be non-isotropic if $x \neq 0$, otherwise it is isotropic. All unit non-isotropic vectors are of the form $(1, y, z)$. Isotropic vectors are of the following types: spacelike ($y^2 - z^2 > 0$), timelike ($y^2 - z^2 < 0$) and two types of ligthlike ($y = \pm z$) vectors. A non-lightlike isotropic vector is unit vector if $y^2 - z^2 = \pm 1$ (Figure1).

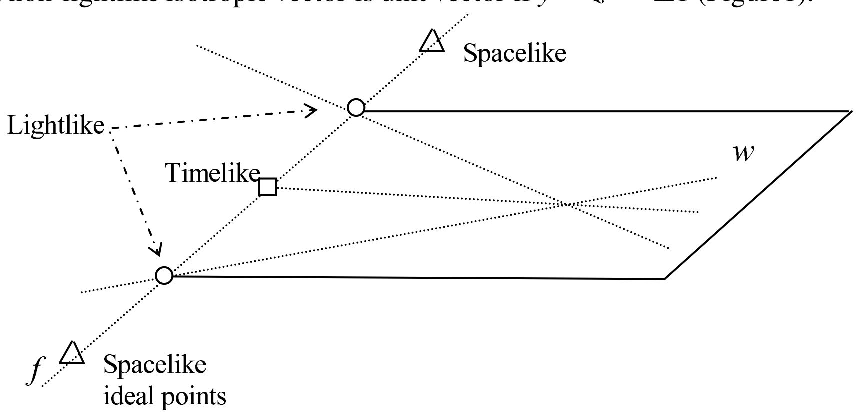

Figure1:The points in pseudo-Galilean space, [7]

A trihedron $(T_0; e_1, e_2, e_3)$, with a proper origin $T_0(x_0, y_0, z_0) \sim (1 : x_0 : y_0 : z_0)$, is orthonormal in pseudo-Galilean sense if and only if the vectors e_1, e_2, e_3 have the following form:

$$e_1 = (1, y_1, z_1),\ e_2 = (0, y_2, z_2),\ e_3 = (0, \varepsilon z_2, \varepsilon y_2)$$

with $y_2^2 - z_2^2 = \delta$, where each of ε, δ is +1 or -1. An above trihedron $(T_0; e_1, e_2, e_3)$ is called positively oriented if for its vectors $\det(e_1, e_2, e_3) = 1$, i.e., if $y_2^2 - z_2^2 = \varepsilon$ stand.

A curve $\alpha : I \to G_3^1$ given as

$$\alpha(t) = (x(t), y(t), z(t)) \tag{2.1}$$

where $x(t), y(t), z(t) \in C^3$, $I \subseteq \mathbb{R}$, is said to be an admissible curve if

$$\begin{array}{l} i)\ \ \dot{\alpha} \times \dot{\alpha} = 0 \\ ii)\ \ \dot{x} \neq 0 \\ iii)\ \ \dot{y} \neq \pm \dot{z}. \end{array}$$

An admissible curve parameterized by the parameter of arc length $s = t$ is given in the coordinate form by

$$\alpha(s) = (x(s), y(s), z(s)). \tag{2.2}$$

The curvature $\kappa(s)$ and the torsion $\tau(s)$ of an admissible curve are given by

$$\begin{array}{l} \kappa(s) = \sqrt{|y''^2(s) - z''^2(s)|} \\ \tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\kappa^2(s)}. \end{array} \tag{2.3}$$

Furthermore, the associated moving trihedron is given by

$$\begin{array}{l} T(s) = \alpha'(s) = (1, y'(s), z'(s)) \\ N(s) = \frac{1}{\kappa(s)}\alpha''(s) = \frac{1}{\kappa(s)}(0, y''(s), z''(s)) \\ B(s) = \frac{1}{\kappa(s)}(0, z''(s), y''(s)) \end{array} \tag{2.4}$$

where the vectors T, N and B are called the vectors of tangent, principal normal and binormal line of α, respectively. The α curve given by (2.2) is timelike (resp. spacelike) if $N(s)$ is a spacelike (resp. timelike) vector. Consequently, the following Frenet's formulas are true [11]

$$\begin{aligned} T'(s) &= \kappa(s)N(s) \\ N'(s) &= \tau(s)B(s) \\ B'(s) &= \tau(s)N(s). \end{aligned} \tag{2.5}$$

3 Admissible Mannheim Curves in Pseudo-Galilean Space G_3^1

In [3], Liu and Wang have studied Mannheim partner curves in 3-dimensional Euclidean space E^3. In this section, we introduce the notion of Mannheim partner curves in G_3^1 for admissible curves in the following way:

Definition 3.1. Let α and α^* be two admissible curves with non-zero $\kappa(s)$, $\kappa^*(s)$, $\tau(s)$, $\tau^*(s)$ for each $s \in I$ and $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ be Frenet frame in G_3^1 along α and α^*, respectively. If there exists a corresponding relationship between the admissible curves α and α^* such that, at the corresponding points of the admissible curves, the principal normal lines N of α coincides with the binormal lines B^* of α^*, then α is called an admissible Mannheim curves and α^* is an admissible Mannheim partner curve of α. The pair (α, α^*) is said to be an admissible Mannheim pair in G_3^1 (see Figure 2).

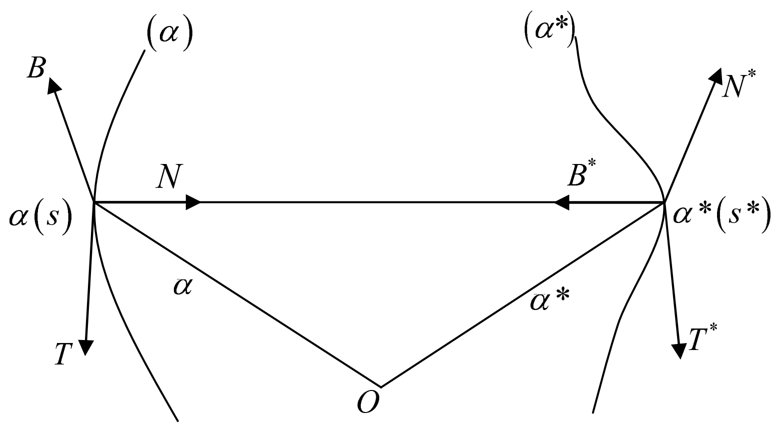

Figure2:The admissible Mannheim partner curves

The fundamental theorem of the pseudo-Galilean theory of curves differs crucially from the analogous theorem in Euclidean, isotropic and Galilean space. Actually, the uniqueness in this theorem is not fulfilled and the reason for this is existence of pseudo-Euclidean planes in pseudo-Galilean space. As it is well known pseudo-Euclidean plane geometry, uniqueness in the fundamental theorem of plane curves does not hold yet. Then, κ and τ are functions such that $0 < \kappa$ and $\tau \neq 0$, there are two admissible curves that one is spacelike (timelike) and the other is timelike (spacelike).

Throughout this paper, we assume that α is a timelike curve and its Mannheim partner α^* is a spacelike curve, i.e., N, the principal normal of curve α, is spacelike vector and N^*, the principal normal of curve α^*, is timelike vector.
Moreover the characterization of a curve α is timelike or spacelike whenever the vector field N is timelike or spacelike. Since $N = B \wedge T$, characteristic of N is related with the inverse characteristic of plane which is generated by B and T. In this case, the plane in which B and T lies is related to the characteristic of Mannheim curves.

Theorem 3.2. *Let (α, α^*) be an admissible Mannheim pair in G_3^1. Then the distance between the corresponding points of admissible Mannheim curves α and α^* is constant.*

Proof. Let (α, α^*) be an admissible Mannheim pair and the Frenet frame be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ on G_3^1 along α and α^*, respectively. A point on α^* corresponding to a point on α is then given by

$$\alpha^*(s) = \alpha(s) + \lambda(s)N(s). \tag{3.1}$$

By taking the derivative of equation (3.1) with respect to s and applying the Frenet formulas, we get

$$T^*(s)\frac{ds^*}{ds} = T(s) + \lambda'(s)N(s) + \lambda(s)\tau(s)B(s). \tag{3.2}$$

Since N is coincident with B^* in the same direction, we have

$$\lambda'(s) = 0. \tag{3.3}$$

This gives $\lambda(s) = \text{constant}$. □

In other words, the theorem proves that the distance between corresponding points of curve α and its Mannheim pair α^* is constant. It should be noted that the relationship between a curve and its Mannheim pair is a reciprocal one, i.e., if a curve α^* is a Mannheim pair of a curve α, then α is also Mannheim pair of α^*.

Theorem 3.3. *Let α be an admissible curve with arc length parameter s. If α is an admissible Mannheim curve, then the torsion τ of admissible curve α is constant.*

Proof. Let (α, α^*) be an admissible Mannheim curves. Then we have

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \cosh\theta & -\sinh\theta & 0 \\ 0 & 0 & 1 \\ -\sinh\theta & \cosh\theta & 0 \end{bmatrix} \begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} \tag{3.4}$$

where θ is the angle between T and T^* at the corresponding points of α and α^* (see Figure 2).

On the other hand, from the last equation, we get

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \begin{bmatrix} \cosh\theta & 0 & \sinh\theta \\ \sinh\theta & 0 & \cosh\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{3.5}$$

By taking the derivative of the equation (3.5) with respect to s we get

$$\begin{aligned}\tau^*(s)B^*(s)\tfrac{ds^*}{ds} = \tfrac{d(\sinh\theta)}{ds}T(s) + \sinh\theta\kappa(s)N(s) \\ +\cosh\theta\tau(s)N(s) + \tfrac{d(\cosh\theta)}{ds}B(s).\end{aligned} \tag{3.6}$$

Since the principal normal vector N of the curve α and the binormal vector B^* of its Mannheim partner curve is linearly dependent. From the equation (3.6), it is easily seen that $\frac{d(\sinh\theta)}{ds} = 0$. So, $\langle B^*, T\rangle = 0$. This last expression becomes:

$$\theta = constant.$$

From equation (3.2), it follows that

$$T^*(s)\frac{ds^*}{ds} = T(s) + \lambda\tau(s)B(s). \tag{3.7}$$

If we consider equations (3.5) and (3.7), then we obtain

$$\lambda\tau(s)\coth\theta = 1. \tag{3.8}$$

Taking $u = \lambda\coth\theta$ and using Theorem 3.2., we reach

$$\tau(s) = \frac{1}{u}. \tag{3.9}$$

The fact that τ is a constant and this completes the proof. □

Theorem 3.4. *(Schell's Theorem). Let (α, α^*) be an admissible Mannheim pair in G_3^1. The product of torsions τ and τ^* at the corresponding points of the admissible Mannheim curves is constant, where the torsion τ belong to α and the torsion τ^* belong to α^*.*

Proof. If we take α^* instead of α, then we can rewrite the equation (3.1) as follow:

$$\alpha(s) = \alpha^*(s) - \lambda B^*(s). \tag{3.10}$$

By taking derivative of equation (3.10) with respect to s and using equation (3.4), we get

$$\tau^*(s) = \frac{1}{\lambda}\tanh\theta. \tag{3.11}$$

Multiplying both sides of equation (3.8) by the corresponding sides of equation (3.11), we have

$$\tau(s)\tau^*(s) = \frac{\tanh^2\theta}{\lambda^2} = constant. \tag{3.12}$$

Hence the proof is completed. □

Theorem 3.5. *Let (α, α^*) be an admissible Mannheim pair in G_3^1 and $\kappa, \tau, \kappa^*, \tau^*$ be a curvatures and torsions of α and α^*, respectively. Then their curvatures and torsions satisfy the following relations:*

$$
\begin{aligned}
&i)\ \ \kappa^*(s) = -\tfrac{d\theta}{ds^*}\\
&ii)\ \ \kappa(s) = \tau^*(s)\sinh\theta\tfrac{ds^*}{ds}\\
&iii)\ \ \tau(s) = -\tau^*(s)\cosh\theta\tfrac{ds^*}{ds}.
\end{aligned}
$$

Proof. i) If we consider equation (3.4), we get

$$< T(s), T^*(s) >= \cosh\theta. \tag{3.13}$$

By taking the derivative of the last equation with respect to s^* and using the Frenet formula for α and α^*, we reach

$$< \kappa(s)N(s)\frac{ds}{ds^*}, T^*(s) > + < T(s), \kappa^*(s)N^*(s) >= \sinh\theta\frac{d\theta}{ds^*} \tag{3.14}$$

By considering the principal normal vector N of α and binormal vector B^* of α^* are linearly dependent and using equations (3.4) and (3.14), we obtain

$$\kappa^*(s) = -\frac{d\theta}{ds^*}. \tag{3.15}$$

If we consider the equations (2.5), (3.4), (3.5) and the scalar products of $< T, B^* >$, $< B, B^* >$ it is easily proved $ii)$, $iii)$ of the theorem, respectively. □

From equations $ii)$ and $iii)$ of the last theorem, we get

$$\frac{\kappa(s)}{\tau(s)} = -\tanh\theta = constant.$$

Hence, we give following result.

Result 3.6. If θ is constant where θ is angle between T and T^* at the corresponding points of α and α^*. An admissible Mannheim curve α in G_3^1 is a general helix.

We give the following example (helices as Mannheim partner curves).

Result 3.7. Let (α, α^*) be an admissible Mannheim pair with arc length parameter in G_3^1. If α is a generalized helix, then α^* is a straight line, [3].

Remark 3.8. The method used above can be considered for spacelike curve α and its Mannheim partner timelike curve α^*. So we can easily obtain the similar theorems and results for these curves as above theorems and results. Moreover, Bertrand partner curves can be studied in pseudo-Galilean space, too.

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