Advances in Differential Equations Volume 12, Number 5 (2007), 515–540

ON A RESULT OF LEIZAROWITZ AND MIZEL

Alexander J. Zaslavski
Department of Mathematics, The Technion-Israel Institute of Technology
32000 Haifa, Israel

(Submitted by: Moshe Marcus)

Abstract. Leizarowitz and Mizel (1989) studied a class of one-dimensional infinite horizon variational problems arising in continuum mechanics and established that these problems possess periodic solutions. They considered a one-parameter family of integrands and show the existence of a constant c such that if a parameter is larger than or equal to c, then the corresponding variational problem has a solution which is a constant function, while if a parameter is less than c, then the corresponding variational problem possesses only non-constant periodic solutions. In this paper we generalize this result for a large class of families of integrands.

1. Introduction

In this paper we consider a class of one-dimensional variational problems arising in continuum mechanics which was introduced in [4] and was studied in [3–10]. Given $x \in R^2$ we study the infinite horizon problem of minimizing the expression $\int_0^T f(w(t), w'(t), w''(t))dt/T$ as T grows to infinity, where

$$w \in A_x = \{v \in W^{2,1}_{loc}([0,\infty))\colon (v(0), v'(0)) = x\}.$$

Here $W^{2,1}_{loc}([0,\infty)) = \{v :\ [0,\infty) \to R^1 :\ v \in W^{2,1}[0,T],\ \forall T > 0\}$ [1] and f belongs to a space of functions to be described below. Namely, we study the following variational problem:

$$\text{Minimize } \liminf_{T\to\infty} \int_0^T f(w(t), w'(t), w''(t))dt/T,\ w \in A_x, \qquad (P_\infty)$$

where $x \in R^2$.

The interest in variational problems of the form (P_∞) stems from the theory of thermodynamical equilibrium for second-order materials developed in [3, 4]. In [4] under some assumptions on integrands f Leizarowitz and Mizel established the existence of a periodic solution of the problem (P_∞).

Accepted for publication: January 2007.
AMS Subject Classifications: 49J99.

In their work they also studied the problem (P_∞) with integrands of the form

$$f_c(w,p,r) = \psi(w) + cp^2 + br^2,\ (w,p,r) \in R^3,$$

where b is a positive constant, a real number c is a parameter and ψ is a smooth function such that

$$\psi(w) \geq a|w|^\alpha - d \text{ for all } w \in R^1 \text{ with some constants } \alpha > 2,\ a,d > 0,$$

there are at most two absolute minimizers of ψ and such that $\psi''(w) > 0$ for any absolute minimizer w of ψ.

Leizarowitz and Mizel [4] established the existence of a constant $c_0 < 0$ such that if $c \geq c_0$, then the problem (P_∞) with $f = f_c$ has a solution which is a constant function, while if $c < c_0$, then the problem (P_∞) possesses only non-constant periodic solutions. Note that in [4] the proof of this result was strongly based on the special form of the integrands f_c. In this paper we generalize this result for a large class of one-parameter families of integrands. Namely we show that the result of Leizarowitz and Mizel above is valid for a one-parameter family of integrands

$$f_c(w,p,r) = \psi(w) + c\phi(p) + h(r),\ (w,p,r) \in R^3,$$

where a real number c is a parameter, and ψ is a smooth function with a finite number of absolute minimizers which satisfies the growth condition of [4] and such that $\psi''(w) > 0$ for any absolute minimizer w of ψ. Here ϕ and h belong to large classes of smooth functions (see Theorem 1.2). Note that in our proofs we use tools and techniques developed in [5–10].

In this paper we use the following notation. We denote by $\|\cdot\|$ the Euclidean norm of the space R^n and by $\langle\cdot,\cdot\rangle$ the scalar product in R^n. (Here n is a natural number.) We denote by $\mathrm{int}(\Omega)$ the interior of Ω, which is a subset of a metric space. Denote by $\mathrm{mes}(\Omega)$ the Lebesgue measure of a Lebesgue-measurable set $\Omega \subset R^1$. For each function $h : X \to R^1$ where X is non-empty, set $\inf(h) = \inf\{h(x) :\ x \in X\}$.

Now we describe a space of integrands introduced in [4] and studied in [3–10].

Let α, β, and γ be positive numbers such that $1 \leq \beta < \alpha$, $\beta \leq \gamma$, and $\gamma > 1$. For each $a = (a_1, a_2, a_3, a_4) \in (0,\infty)^4$ denote by $\mathfrak{M}(\alpha,\beta,\gamma,a)$ the set of all functions $f : R^3 \to R^1$ such that

$$f(w,p,r) \geq a_1|w|^\alpha - a_2|p|^\beta + a_3|r|^\gamma - a_4,\ (w,p,r) \in R^3, \tag{1.1}$$

$$f,\ \partial f/\partial p \in C^2,\ \partial f/\partial r \in C^3,\ \partial^2 f/\partial r^2(w,p,r) > 0 \text{ for all } (w,p,r) \in R^3,$$

and there is an increasing function $M_f : [0, \infty) \to [0, \infty)$ such that for every $(w, p, r) \in R^3$

$$\max\{f(w,p,r),\ |\partial f/\partial w(w,p,r)|, |\partial f/\partial p(w,p,r)|,\ |\partial f/\partial r(w,p,r)|\} \\ \le M_f(|w| + |p|)(1 + |r|^\gamma).$$

Set

$$\mathfrak{M}(\alpha, \beta, \gamma) = \cup\{\mathfrak{M}(\alpha, \beta, \gamma, a) :\ a \in (0, \infty)^4\}.$$

We consider functionals of the form

$$I^f(T_1, T_2, w) = \int_{T_1}^{T_2} f(w(t), w'(t), w''(t))dt,$$

where $-\infty < T_1 < T_2 < \infty$, $w \in W^{2,1}([T_1, T_2])$ and $f \in \mathfrak{M}(\alpha, \beta, \gamma)$. Consider any $f \in \mathfrak{M}(\alpha, \beta, \gamma)$. Of special interest is the minimal long-run average cost growth rate

$$\mu(f) = \inf\{\liminf_{T\to\infty} T^{-1} \int_0^T f(w(t), w'(t), w''(t))dt :\ w \in A_x\}. \tag{1.2}$$

It was shown in [4] that $\mu(f)$ is well-defined and is independent of the initial vector x. A function $w \in W^{2,1}_{loc}([0, \infty))$ is called (f)-good if the function

$$T \to \int_0^T [f(w(t), w'(t), w''(t)) - \mu(f)]dt,\ T \in (0, \infty),$$

is bounded.

Leizarowitz and Mizel [4] established that for every $f \in \mathfrak{M}(\alpha, \beta, \gamma)$ satisfying $\mu(f) < \inf\{f(w, 0, s) : (w, s) \in R^2\}$ there exists a periodic (f)-good function. In [8] it was shown that this result is valid for every $f \in \mathfrak{M}(\alpha, \beta, \gamma)$.

Let $f \in \mathfrak{M}(\alpha, \beta, \gamma)$. It is easy to see that $\mu(f) \le \inf\{f(t, 0, 0) :\ t \in R^1\}$. If

$$\mu(f) = \inf\{f(t, 0, 0) : t \in R^1\}, \tag{1.3}$$

then there is an (f)-good function which is constant. If

$$\mu(f) < \inf\{f(t, 0, 0) : t \in R^1\}, \tag{1.4}$$

then there exist only non-constant periodic (f)-good functions.

The following theorem is our first main result.

Theorem 1.1. *Let $b_i > 0,\ i = 1, 2, 3, 4, 5,$ and let a function $G : R^2 \to R^1$ satisfy*

$$G(w, r) \ge b_1|w|^\alpha + b_2|r|^\gamma - b_3 \text{ for all } (w, r) \in R^2, \tag{1.5}$$

$$G \in C^2,\ \partial G/\partial r \in C^3,\ (\partial^2 G/\partial r^2)(w, r) > 0 \text{ for all } (w, r) \in R^2, \tag{1.6}$$

$$\inf\{G(w, r) : (w, r) \in R^2\} = \inf\{G(w, 0) : w \in R^1\}, \tag{1.7}$$

and let there exist an increasing function $M_G : [0,\infty) \to [0,\infty)$ *such that for all* $(w,r) \in R^2$,

$$\max\{G(w,r),\ |\partial G/\partial w(w,r)|,\ |\partial G/\partial r(w,r)|\} \\ \le M_G(|w|)(1+|r|^\gamma) \text{ for all } (w,r) \in R^2. \tag{1.8}$$

Assume that $\phi \in C^3(R^1)$ *satisfies*

$$|\phi(t)| \le b_4|t|^\beta + b_5 \text{ for all } t \in R^1 \text{ and } \phi''(0) > 0, \tag{1.9}$$

and that at least one of the following properties hold:

(*i*) $\phi(t) \ge \phi(0)$ *for all* $t \in R^1$;

(*ii*) $\phi''(t) \ge 0$ *for all* $t \in R^1$.

Then for each $c \in R^1$ *the function* $f_c : R^3 \to R^1$ *defined by*

$$f_c(w,p,r) = G(w,r) + c\phi(p),\ (w,p,r) \in R^3, \tag{1.10}$$

belongs to $\mathfrak{M}(\alpha,\beta,\gamma)$, *and the following property holds:*

($P1$) *There is a real number* $c_0 \le 0$ *such that* $\mu(f_c) < \inf\{f_c(t,0,0) : t \in R^1\}$ *for all* $c < c_0$ *and that* $\mu(f_c) = \inf\{f_c(t,0,0) : t \in R^1\}$ *for all* $c \ge c_0$.

The next theorem is our second main result.

Theorem 1.2. *Let* κ *and* b_i, $i = 1,2,3,4,5$ *be positive real numbers, and let functions* g, h, *and* $\phi : R^1 \to R^1$ *satisfy*

$$\phi \in C^3(R^1),\ g \in C^2(R^1),\ h \in C^4(R^1), \\ h''(t) > \kappa \text{ for all } t \in R^1 \text{ and } h(0) = \inf(h), \tag{1.11}$$

$$h(r) \ge b_1|r|^\gamma - b_3 \text{ for all } r \in R^1 \text{ and } |h'(r)|, h(r) \le b_2|r|^\gamma + b_3 \text{ for all } r \in R^1, \tag{1.12}$$

$$g(w) \ge b_1|w|^\alpha - b_3 \text{ for all } w \in R^1, \tag{1.13}$$

$$|\phi(t)| \le b_4|t|^\beta + b_5 \text{ for all } t \in R^1 \text{ and } \phi''(0) > 0. \tag{1.14}$$

Assume that there exists a finite number of real numbers $\xi_1, \dots, \xi_k$ *such that*

$$\xi_1 < \cdots < \xi_k,\ g''(\xi_i) > 0,\ i = 1,\dots,k \text{ and} \\ \{\xi_1,\dots,\xi_k\} = \{t \in R^1 :\ g(t) = \inf(g)\}, \tag{1.15}$$

and that at least one of the following properties holds:

(i) $\phi(t) \ge \phi(0)$ *for all* $t \in R^1$;

(ii) $\phi''(t) \ge 0$ *for all* $t \in R^1$.

Then for each $c \in R^1$ *the function* $f_c : R^3 \to R^1$ *defined by*

$$f_c(w,p,r) = g(w) + h(r) + c\phi(p),\ (w,p,r) \in R^3 \tag{1.16}$$

belongs to $\mathfrak{M}(\alpha,\beta,\gamma)$, *and the following property holds:*

($P2$) *There is a real number* $c_0 < 0$ *such that* $\mu(f_c) < \inf\{f_c(t,0,0) : t \in R^1\}$ *for all* $c < c_0$ *and* $\mu(f_c) = \inf\{f_c(t,0,0) : t \in R^1\}$ *for all* $c \ge c_0$.

The proof of Theorem 1.1 has the following structure. First, in Section 3 (see Proposition 3.3) we obtain sufficient conditions which guarantee that for the problem (P_∞) with an integrand of some form there exist only non-constant periodic solutions. Then we show the existence of a constant c_1 such that the sufficient conditions of Proposition 3.3 hold for f_{c_1}. This implies that for the problem (P_∞) with the integrand f_{c_1} there exist only non-constant periodic solutions, or, in other words, $\mu(f_{c_1}) < \inf\{f_{c_1}(t,0,0) : t \in R^1\}$. On the other hand, it is not difficult to verify that $\mu(f_0) = \inf\{f_0(t,0,0) : t \in R^1\}$. Then we show that the following property holds:

If $c \in R^1$, $\tilde{c} > c$ and if $\mu(f_c) = \inf\{f_c(t,0,0) : t \in R^1\}$, then $\mu(f_{\tilde{c}}) = \inf\{f_{\tilde{c}}(t,0,0) : t \in R^1\}$.

Combining all these facts mentioned above, we obtain that there exists a number $c_0 \le 0$ such that $\mu(f_c) < \inf\{f_c(t,0,0) : t \in R^1\}$ for all $c < c_0$ and that $\mu(f_c) = \inf\{f_c(t,0,0) : t \in R^1\}$ for all $c > c_0$. Using the continuity of the functional $g \to \mu(g)$ (see Proposition 2.3) we obtain that $\mu(f_{c_0}) = \inf\{f_{c_0}(t,0,0) : t \in R^1\}$.

Let us now describe the structure of the proof of Theorem 1.2. First note that we can apply Theorem 1.1 to the family of integrands of Theorem 1.2. Let a number $c_0 \le 0$ be as guaranteed by Theorem 1.1. In order to prove Theorem 1.2 we need only to show that $c_0 < 0$. For this goal we choose a small positive constant δ and show that if w is a periodic $(f_{-\delta})$-good function, then it is constant. The proof of this assertion is rather complicated. We assume that w is a non-constant periodic $(f_{-\delta})$-good function with a period T_w. We construct a finite number of intervals $\mathcal{I}_j \subset [0, T_w]$, $j = 1, \dots, q$ such that $\mathcal{I}_j \cap \mathcal{I}_s = \emptyset$ for all pairs of integers j, s such that $j \ne s$ and denote by $\mathcal{D}$ the complement of the union of all these intervals in $[0, T_w]$. It turns out that the following properties hold:

For each $t \in \mathcal{D}$ and each $i \in \{1, \dots, k\}$ we have $|w(t) - \xi| \ge \Delta_0$.

For each $j \in \{1, \dots, q\}$, there is $s \in \{1, \dots, k\}$ such that $w(\mathcal{I}_j) \subset [\xi_s - 2\Delta_0, \xi_s + 2\Delta_0]$.

Here, Δ_0 is some positive constant. Using these two properties we estimate the integrals with the integrand $f_{-\delta}$ on the sets $\mathcal{I}_j$, $j = 1, \dots, q$ and $\mathcal{D}$. We obtain that for any $t \in \mathcal{D}$

$$f_{-\delta}(w(t), w'(t), w''(t)) \ge \mu(f_{-\delta}) + m,$$

where m is some positive constant. For the estimations of the integrals over the sets $\mathcal{I}_j$, $j = 1, \dots, q$ we use the interpolation inequality [1]. As a result we obtain that $(T_w)^{-1} I^f(0, T_w, w) > \mu(f_{-\delta})$, a contradiction. The paper is organized as follows. Some preliminary results are collected in Section 2. An auxiliary result for Theorem 1.1 is proved in Section 3, while Theorem 1.1 is proved in Section 4. Section 5 contains an auxiliary result for Theorem 1.2, which is proved in Section 6.

2. Preliminaries

In the sequel we use the following auxiliary results and notation.

Proposition 2.1. [6, Lemma 3.1]. *Assume that* $f \in \mathfrak{M}(\alpha,\beta,\gamma)$, $w \in W^{2,1}_{loc}(R^1)$, $\tau > 0$, $w(t+\tau) = w(t)$ *for all* $t \in R^1$, $I^f(0,\tau,w) = \tau\mu(f)$, $w(0) = \inf\{w(t) : t \in R^1\}$ *and that* $w'(t) \neq 0$ *for some* $t \in R^1$. *Then there exist* $\tau_1 > 0$ *and* $\tau_2 > \tau_1$ *such that the function* w *is strictly increasing in* $[0,\tau_1]$, w *is strictly decreasing in* $[\tau_1,\tau_2]$ *and*

$$w(\tau_1) = \sup\{w(t) : t \in R^1\},\ w(t+\tau_2) = w(t),\ t \in R^1.$$

Let $f \in \mathfrak{M}(\alpha,\beta,\gamma)$. For each $T > 0$ define a function $U^f_T : R^2 \times R^2 \to R^1$ by

$$U^f_T(x,y) = \inf\Big\{ \int_0^T f(w(t),w'(t),w''(t))dt : w \in A^T_{x,y}\Big\}, \tag{2.1}$$

where

$$A^T_{x,y} = \{v \in W^{2,1}([0,T]) : (v(0),v'(0)) = x,\ (v(T),v'(T)) = y\},\ x,y \in R^2. \tag{2.2}$$

In [4], analyzing the problem (P_∞), Leizarowitz and Mizel studied the function $U^f_T : R^2 \times R^2 \to R^1$ with $T > 0$ and established the following representation formula:

$$U^f_T(x,y) = T\mu(f) + \pi^f(x) - \pi^f(y) + \theta^f_T(x,y),\ x,y \in R^2,\ T > 0, \tag{2.3}$$

where $\pi^f : R^2 \to R^1$ and $(T,x,y) \to \theta^f_T(x,y)$ (where $x,y \in R^2$, $T > 0$) are continuous functions,

$$\pi^f(x) = \inf\Big\{ \liminf_{T\to\infty} \int_0^T [f(w(t),w'(t),w''(t)) - \mu(f)]dt : w \in A_x\Big\},\ x \in R^2, \tag{2.4}$$

$\theta^f_T(x,y) \geq 0$ for all $T > 0$ and all $x,y \in R^2$, and for every $T > 0$ and every $x \in R^2$ there is $y \in R^2$ satisfying $\theta^f_T(x,y) = 0$.

For any $\tau > 0$ and $v \in W^{2,1}([0,\tau])$ we define $X_v : [0,\tau] \to R^2$ as

$$X_v(t) = (v(t),v'(t)),\ t \in [0,\tau]. \tag{2.5}$$

We also use this notation for $v \in W^{2,1}_{loc}([0,\infty))$. For a function $w \in W^{2,1}_{loc}([0,\infty))$ denote by $\Omega(w)$ the set of all points $z \in R^2$ such that $X_w(t_j) \to z$ as $j \to \infty$ for some sequence of numbers $t_j \to \infty$.

Let $a = (a_1, a_2, a_3, a_4) \in (0, \infty)^4$. The set $\mathfrak{M}(\alpha, \beta, \gamma, a)$ is equipped with the uniformity which is determined by the following base:

$$E(N, \epsilon, \Gamma) = \{(h, g) \in \mathfrak{M}(\alpha, \beta, \gamma, a) \times \mathfrak{M}(\alpha, \beta, \gamma, a) :$$
$$|h(x_1, x_2, x_3) - g(x_1, x_2, x_3)| \le \epsilon$$
$$\text{for all } (x_1, x_2, x_3) \in R^3 \text{ satisfying } |x_i| \le N,\ i = 1, 2, 3\}$$
$$\cap\{(h, g) \in \mathfrak{M}(\alpha, \beta, \gamma, a) \times \mathfrak{M}(\alpha, \beta, \gamma, a) :$$
$$(|h(x_1, x_2, x_3)| + 1)(|g(x_1, x_2, x_3)| + 1)^{-1} \in [\Gamma^{-1}, \Gamma]$$
$$\text{for all } (x_1, x_2, x_3) \in R^3 \text{ satisfying } |x_,|x_2| \le N\},$$

where $N, \epsilon > 0$ and $\Gamma > 1$. Clearly the uniform space $\mathfrak{M}(\alpha, \beta, \gamma, a)$ is metrizable (by a metric ρ).

Proposition 2.2. [10, Proposition 5.1] *Let $a \in (0, \infty)^4$ and $g \in \mathfrak{M}(\alpha, \beta, \gamma, a)$. Then there exist a neighborhood $\mathcal{U}$ of g in $\mathfrak{M}(\alpha, \beta, \gamma, a)$ and a number $S > 0$ such that for each $h \in \mathcal{U}$ and each (h)-good function $v \in W^{2,1}_{loc}([0, \infty))$ the inequality $|(v(t), v'(t))| \le S$ holds for all sufficiently large numbers t.*

Proposition 2.3. [10, Proposition 5.2] *Let $a \in (0, \infty)^4$. Then the function $f \to \mu(f)$ is continuous for $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$.*

Proposition 2.4 [10, Proposition 3.1] *Let $a \in (0, \infty)^4$, $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$, $0 < c_1 < c_2 < \infty$, $\epsilon > 0$ and $D > 0$. Then there exists a neighborhood V of f in $\mathfrak{M}(\alpha, \beta, \gamma, a)$ such that for every $g \in V$, every $T \in [c_1, c_2]$ and every $w \in W^{2,1}([0, T])$ satisfying $\min\{I^f(0, T, w),\ I^g(0, T, w)\} \le D$ the relation $|I^f(0, T, w) - I^g(0, T, w)| \le \epsilon$ holds.*

Proposition 2.5 [10, Proposition 3.2] *Let $a \in (0, \infty)^4$, $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$, $0 < c_1 < c_2 < \infty$, $c_3 > 0$ and $\epsilon \in (0, 1)$. Then there exists a neighborhood V of f in $\mathfrak{M}(\alpha, \beta, \gamma, a)$ such that for every $g \in V$, every $T \in [c_1, c_2]$ and every $x, y \in R^2$ satisfying $|x|, |y| \le c_3$ the relation $|U^f_T(x, y) - U^g_T(x, y| \le \epsilon$ holds.*

Proposition 2.6 [8, Proposition 3.5] *Let $f \in \mathfrak{M}(\alpha, \beta, \gamma)$ and $w \in W^{2,1}_{loc}([0, \infty)$. Then w is either (f)-good or $\int_0^T f(w(t), w'(t), w''(t))dt - T\mu(f) \to \infty$ as $T \to \infty$. Moreover, if w is (f)-good then $\sup\{|(w(t), w'(t))| : t \in [0, \infty)\} < \infty$.*

Proposition 2.3 of [6] implies the following result.

Proposition 2.7. *Let $f \in \mathfrak{M}(\alpha, \beta, \gamma)$, $w \in W^{2,1}_{loc}([0, \infty))$, $\sup\{|X_w(t)| : t \in [0, \infty)\} < \infty$ and $I^f(0, T, u) = U^f_T(X_w(0), X_w(T))$ for every $T > 0$. Then w is an (f)-good function.*

Proposition 2.7 and Lemma 2.5 of [6] imply the following result.

Proposition 2.8. *Let $f \in \mathfrak{M}(\alpha, \beta, \gamma)$, $u \in W^{2,1}_{loc}([0, \infty))$, $\sup\{|X_u(t)| : t \in [0, \infty)\} < \infty$ and $I^f(0, T, u) = U^f_T(X_u(0), X_u(T))$ for every $T > 0$. Assume* *that either $u'(t) \geq 0$ for all $t \geq 0$ or $u'(t) \leq 0$ for all $t \geq 0$. Then there exists $d_0 \in R^1$ such that $(u(t), u'(t)) \to (d_0, 0)$ as $t \to \infty$ and $\mu(f) = f(d_0, 0, 0)$.*

3. An auxiliary result for Theorem 1.1

Proposition 3.1. *Let $f \in \mathfrak{M}(\alpha, \beta, \gamma)$, $t_0 \in R^1$, $\theta > 0$,*

$$f(t_0, 0, 0) = \inf\{f(t, 0, 0) : t \in R^1\}, \tag{3.1}$$

and let

$$(\partial^2 f/\partial x_1^2)(t_0, 0, 0) + \theta(\partial^2 f/\partial x_2^2)(t_0, 0, 0) + \theta^2(\partial^2 f/\partial x_3^2)(t_0, 0, 0) \\ -2\theta(\partial^2 f/\partial x_1 \partial x_3)(t_0, 0, 0) < 0. \tag{3.2}$$

Then

$$\mu(f) < \inf\{f(t, 0, 0) : t \in R^1\}. \tag{3.3}$$

Proof. Choose $\delta_0 \in (0, 1)$ such that

$$\delta_0 3(1+\theta)^2 < -4^{-1}[(\partial^2 f/\partial x_1^2)(t_0, 0, 0) + \theta(\partial^2 f/\partial x_2^2)(t_0, 0, 0) \\ +\theta^2(\partial^2 f/\partial x_3^2)(t_0, 0, 0) - 2\theta(\partial^2 f/\partial x_1 \partial x_3)(t_0, 0, 0)]. \tag{3.4}$$

By (3.1)

$$(\partial f/\partial x_1)(t_0, 0, 0) = 0. \tag{3.5}$$

By Taylor's theorem

$$\lim_{z \to 0} \|z\|^{-2}[f((t_0, 0, 0) + z) - f(t_0, 0, 0) - \langle \nabla f(t_0, 0, 0), z\rangle \\ -2^{-1} \sum_{i,j=1}^{3} (\partial^2 f/\partial x_i \partial x_j)(t_0, 0, 0) z_i z_j] = 0. \tag{3.6}$$

There is $\Delta \in (0, 1)$ such that for each $z = (z_1, z_2, z_3) \in R^3$ such that

$$|z_1|, |z_2|, |z_3| \leq (2\Delta(1+\theta)), \tag{3.7}$$

the following inequality holds:

$$|f((t_0, 0, 0) + z) - f(t_0, 0, 0) - \langle \nabla f(t_0, 0, 0), z\rangle \\ -2^{-1} \sum_{i,j=1}^{3} (\partial^2 f/\partial x_i \partial x_j)(t_0, 0, 0) z_i z_j| \leq (\delta_0/4)\|z\|^2. \tag{3.8}$$

Let us consider the function

$$v(t) = t_0 + \Delta \cos(t\theta^{1/2}),\ t \in R^1. \tag{3.9}$$

For any $t \in R^1$ we have

$$v(t + (2\pi)\theta^{-1/2}) = t_0 + \Delta \cos(t\theta^{1/2} + 2\pi) = t_0 + \Delta \cos(t\theta^{1/2}) = v(t), \tag{3.10}$$

$$v'(t) = \Delta\theta^{1/2}(-\sin(t\theta^{1/2})),\ v''(t) = \Delta\theta(-\cos(t\theta^{1/2})), \tag{3.11}$$

$$|t_0 - v(t)|,\ |v'(t)|,\ |v''(t)| \le \Delta \max\{1, \theta\}. \tag{3.12}$$

For each $t \in R^1$ set

$$z(t) = (v(t), v'(t), v''(t)) - (t_0, 0, 0). \tag{3.13}$$

By (3.13), (3.12) and the definition of Δ (see (3.7) and (3.8)), for each $t \in R^1$

$$|f(v(t), v'(t), v''(t)) - f(t_0, 0, 0) - \langle \nabla f(t_0, 0, 0), z(t)\rangle$$

$$-2^{-1} \sum_{i,j=1}^{3} (\partial^2 f/\partial x_i \partial x_j)(t_0, 0, 0) z_i(t) z_j(t)| \le (\delta_0/4) 3\Delta^2 (\max\{1, \theta\})^2. \tag{3.14}$$

By (3.14), (3.5) and (3.13), for each $t \in R^1$

$$\begin{aligned} &f(v(t), v'(t), v''(t)) \le f(t_0, 0, 0) + (\partial f/\partial x_2)(t_0, 0, 0) v'(t) \\ &+ (\partial f/\partial x_3)(t_0, 0, 0) v''(t) + 2^{-1}[(\partial^2 f/\partial x_1^2)(t_0, 0, 0)(v(t) - t_0)^2 \\ &+ (\partial^2 f/\partial x_2^2)(t_0, 0, 0)(v'(t))^2 + (\partial^2 f/\partial x_3^2)(t_0, 0, 0)(v''(t))^2 \\ &+ 2(\partial^2 f/\partial x_1 \partial x_2)(t_0, 0, 0)(v(t) - t_0) v'(t) \\ &+ 2(\partial^2 f/\partial x_1 \partial x_3)(t_0, 0, 0)(v(t) - t_0) v''(t) \\ &+ 2(\partial^2 f/\partial x_2 \partial x_3)(t_0, 0, 0) v'(t) v''(t)] + (\delta_0/4) 3\Delta^2 (\max\{1, \theta\})^2. \end{aligned} \tag{3.15}$$

By (3.9) and (3.10),

$$\mu(f) \le (2\pi)^{-1}\theta^{1/2} \int_0^{2\pi\theta^{-1/2}} f(v(t), v'(t), v''(t)) dt. \tag{3.16}$$

By (3.15), (3.9) and (3.11),

$$(2\pi)^{-1}\theta^{1/2} \int_0^{2\pi\theta^{-1/2}} f(v(t), v'(t), v''(t)) dt$$

$$\le f(t_0, 0, 0) + (2\pi)^{-1}\theta^{1/2} \int_0^{2\pi\theta^{-1/2}} (\partial f/\partial x_2)(t_0, 0, 0) v'(t) dt$$

$$+(2\pi)^{-1}\theta^{1/2} \int_0^{2\pi\theta^{-1/2}} (\partial f/\partial x_3)(t_0, 0, 0) v''(t) dt$$

$$+2^{-1}\theta^{1/2}(2\pi)^{-1} \int_0^{2\pi\theta^{-1/2}} (\partial^2 f/\partial x_1)^2 (t_0, 0, 0) \Delta^2 \cos^2(t\theta^{1/2}) dt$$

$$+2^{-1}(2\pi)^{-1}\theta^{1/2} \int_0^{2\pi\theta^{-1/2}} (\partial^2 f/\partial x_2^2)(t_0, 0, 0) \Delta^2 \theta (\sin(t\theta^{1/2}))^2 dt$$

$$+2^{-1}(2\pi)^{-1}\theta^{1/2}\int_0^{2\pi\theta^{-1/2}}(\partial^2 f/\partial x_3^2)(t_0,0,0)\Delta^2\theta^2\cos^2(t\theta^{1/2})dt$$
$$+(2\pi)^{-1}\theta^{1/2}\int_0^{2\pi\theta^{-1/2}}(\partial^2 f/\partial x_1\partial x_2)(t_0,0,0)\Delta\cos(t\theta^{1/2})\Delta\theta^{1/2}(-\sin(t\theta^{1/2}))dt$$
$$+(2\pi)^{-1}\theta^{1/2}\int_0^{2\pi\theta^{-1/2}}(\partial^2 f/\partial x_1\partial x_3)(t_0,0,0)\Delta\cos(t\theta^{1/2})\Delta\theta(-\cos(t\theta^{1/2}))dt$$
$$+(2\pi)^{-1}\theta^{1/2}\int_0^{2\pi\theta^{-1/2}}(\partial^2 f/\partial x_2\partial x_3)(t_0,0,0)\Delta\theta^{1/2}\sin(t\theta^{1/2})\Delta\theta\cos(t\theta^{1/2}))dt$$
$$+(\delta_0/4)3\Delta^2(\max\{1,\theta\})^2. \tag{3.17}$$

By (3.10)

$$\int_0^{2\pi\theta^{-1/2}} v'(t)dt = v(2\pi\theta^{-1/2}) - v(0) = 0,$$
$$\int_0^{2\pi\theta^{-1/2}} v''(t)dt = v'(2\pi\theta^{-1/2}) - v'(0) = 0. \tag{3.18}$$

It is easy to see that

$$\theta^{1/2}(2\pi)^{-1}\int_0^{2\pi\theta^{-1/2}}\cos^2(t\theta^{1/2})dt$$
$$= 2^{-1}(2\pi)^{-1}\theta^{1/2}\int_0^{2\pi\theta^{-1/2}}(1+\cos(2t\theta^{1/2}))dt = 2^{-1}, \tag{3.19}$$
$$\theta^{1/2}(2\pi)^{-1}\int_0^{2\pi\theta^{-1/2}}\sin^2(t\theta^{1/2})dt$$
$$= \theta^{1/2}(2\pi)^{-1}2^{-1}\int_0^{2\pi\theta^{-1/2}}(1-\cos(2t\theta^{1/2}))dt = 1/2, \tag{3.20}$$
$$\int_0^{2\pi\theta^{-1/2}}\cos(t\theta^{1/2})\sin(t\theta^{1/2})dt = 2^{-1}\int_0^{2\pi\theta^{-1/2}}\sin(2t\theta^{1/2})dt = 0. \tag{3.21}$$

It follows from (3.17)–(3.21) and (3.4) that

$$(2\pi)^{-1}\theta^{\frac{1}{2}}\int_0^{2\pi\theta^{-\frac{1}{2}}} f(v(t),v'(t),v''(t))dt$$
$$\leq f(t_0,0,0) + 2^{-1}(\partial^2 f/\partial x_1^2)(t_0,0,0)\Delta^2 2^{-1} + 2^{-1}\Delta^2\theta(\partial^2 f/\partial x_2^2)(t_0,0,0)2^{-1}$$
$$+ 2^{-1}(\partial^2 f/\partial x_3^2)(t_0,0,0)\Delta^2\theta^2 2^{-1}$$
$$- \Delta^2\theta(\partial^2 f/\partial x_1\partial x_3)(t_0,0,0)2^{-1} + (\delta_0/4)3\Delta^2(\max\{1,\theta\})^2$$

$$= f(t_0,0,0) + \Delta^2\Big[4^{-1}(\partial^2 f/\partial x_1^2)(t_0,0,0) + 4^{-1}\theta(\partial^2 f/\partial x_2^2)(t_0,0,0)$$
$$+ 4^{-1}\theta^2(\partial^2 f/\partial x_3^2)(t_0,0,0) - 2^{-1}\theta(\partial^2 f/\partial x_1\partial x_3)(t_0,0,0)$$
$$+ \delta_0(\tfrac{3}{4})(\max\{1,\theta\}^2)\Big]$$
$$< f(t_0,0,0) + \Delta^2(-3/4)\delta_0 3(1+\theta)^2 < f(t_0,0,0).$$

Combined with (3.16) this inequality implies that $\mu(f) < f(t_0,0,0)$. The proposition is proved. □

Proposition 3.2. *Let a function $G : R^2 \to R^1$ satisfy*

$$G \in C^3(R^2),\ \partial G/\partial r \in C^3,\ (\partial^2 G/\partial r^2)(p,r) > 0$$
$$\text{for all } (p,r) \in R^2,\ (\partial^2 G/\partial p^2)(0,0) < 0,$$
$$G(p,r) \ge -b_1|p|^\beta + b_2|r|^\gamma - b_0 \text{ for all } (p,r) \in R^2, \tag{3.22}$$

with positive constants b_0, b_1, and b_2, and let there exist an increasing function $M_G : [0,\infty) \to [0,\infty)$ such that

$$\max\{G(p,r),\ |\partial G/\partial p(p,r)|,\ |\partial G/\partial r(p,r)\} \le M_G(|p|)(1+|r|^\gamma) \tag{3.23}$$

for all $(p,r) \in R^2$. Then there exists $\Delta > 0$ such that if $\phi \in C^2(R^1)$ satisfies

$$\phi(t) \ge b_3|t|^\alpha - b_4 \text{ for all } t \in R^1, \tag{3.24}$$

where b_3 and b_4 are positive constants, and if $t_0 \in R^1$ satisfies

$$\phi''(t_0) < \Delta,\ \phi(t_0) = \min\{\phi(t) :\ t \in R^1\}, \tag{3.25}$$

then the function $f : R^3 \to R^1$ defined by

$$f(x_1,x_2,x_3) = \phi(x_1) + G(x_2,x_3),\ (x_1,x_2,x_3) \in R^3, \tag{3.26}$$

belongs to $\mathfrak{M}(\alpha,\beta,\gamma)$ and

$$\mu(f) < \inf\{\phi(t) :\ t \in R^1\}. \tag{3.27}$$

Proof. By (3.22) there is $\theta > 0$ such that

$$\theta(\partial^2 G/\partial p^2)(0,0) + \theta^2(\partial^2 G/\partial r^2)(0,0) < 0. \tag{3.28}$$

Choose a positive number Δ such that

$$\Delta + \theta(\partial^2 G/\partial p^2)(0,0) + \theta^2(\partial^2 G/\partial r^2)(0,0) < 0. \tag{3.29}$$

Assume that $\phi \in C^2(R^1)$ satisfies (3.24) with positive constants b_3 and b_4, and that $t_0 \in R^1$ satisfies (3.25). Let a function $f : R^3 \to R^1$ be defined by (3.26). It is not difficult to see that $f \in \mathfrak{M}(\alpha,\beta,\gamma)$. It follows from (3.25), (3.29) and Proposition 3.1 that (3.27) holds. Proposition 3.2 is proved. □

Proposition 3.3. *Let a function* $G : R^2 \to R^1$ *satisfy*

$$G \in C^2(R^2),\ \partial G/\partial r \in C^3,\ (\partial^2 G/\partial r^2)(w,r) > 0 \text{ for all } (w,r) \in R^2,$$
$$\inf\{G(w,r) : (w,r) \in R^2\} = \inf\{G(w,0) : w \in R^1\},$$
$$G(w,r) \ge b_1|w|^\alpha + b_2|r|^\gamma - b_3 \text{ for all } (w,r) \in R^2, \tag{3.30}$$

where b_1, b_2, *and* b_3 *are positive constants, and let there exist an increasing function* $M_G : [0,\infty) \to [0,\infty)$ *such that*

$$\max\{G(w,r),\ |\partial G/\partial w(w,r)|,\ |\partial G/\partial r(w,r)|\} \le M_G(|w|)(1+|r|^\gamma), \tag{3.31}$$

for all $(w,r) \in R^2$. *Then there exists* $\Delta > 0$ *such that for each* $\phi \in C^3(R^1)$ *satisfying*

$$\phi''(0) < -\Delta, \tag{3.32}$$
$$\phi(t) \ge -b_4|t|^\beta - b_5 \text{ for all } t \in R^1,$$

where b_4 *and* b_5 *are positive constants, the function* $f : R^3 \to R^1$ *defined by*

$$f(x_1,x_2,x_3) = G(x_1,x_3) + \phi(x_2),\ (x_1,x_2,x_3) \in R^3, \tag{3.33}$$

belongs to $\mathfrak{M}(\alpha,\beta,\gamma,a)$ *and satisfies*

$$\mu(f) < \inf\{f(t,0,0) : t \in R^1\}. \tag{3.34}$$

Proof. By (3.30) there is $t_0 \in R^1$ such that

$$G(t_0,0) = \inf\{G(t,0) : t \in R^1\}. \tag{3.35}$$

Choose a positive number Δ such that

$$(\partial^2 G/\partial w^2)(t_0,0) - \Delta + (\partial^2 G/\partial r^2)(0,0) - 2(\partial^2 G/\partial w \partial r)(t_0,0) < 0. \tag{3.36}$$

Assume that $\phi \in C^3(R^1)$ satisfies (3.32) with $b_4, b_5 > 0$. Consider the function f defined by (3.33). It is not difficult to see that $f \in \mathfrak{M}(\alpha,\beta,\gamma)$. By (3.32) and (3.36), the relation (3.2) is valid with $\theta = 1$. Relations (3.33), (3.35), Proposition 3.1 and (3.2) with $\theta = 1$ imply (3.34). Proposition 3.3 is proved. □

4. Proof of Theorem 1.1

It is clear that for each $c \in R^1$ the function f_c belongs to $\mathfrak{M}(\alpha,\beta,\gamma)$. Assume that the property (i) holds. By Proposition 3.3 there exists $c_1 < 0$ such that

$$\mu(f_{c_1}) < \inf\{f_{c_1}(t,0,0) : t \in R^1\}. \tag{4.1}$$

Consider the function

$$f_0(x_1,x_2,x_3) = G(x_1,x_3),\ (x_1x_2,x_3) \in R^3.$$

By (3.30) there exists $t_0 \in R^1$ such that

$$G(t_0, 0) = \inf\{G(w, 0) : w \in R^1\} = \inf\{G(w, z) : (w, z) \in R^2\}. \tag{4.2}$$

By the equality above, for each $(x_1, x_2, x_3) \in R^3$ we have $f_0(x_1, x_2, x_3) \geq G(t_0, 0)$. This implies that $\mu(f_0) \geq G(t_0, 0)$. On the other hand, $\mu(f_0) \leq G(t_0, 0)$ by definition. Thus,

$$\mu(f_0) = G(t_0, 0) = \inf\{f_0(t_0, 0, 0) : t \in R^1\}. \tag{4.3}$$

Assume that $c \in R^1$ and

$$\mu(f_c) = \inf\{f_c(t, 0, 0) : \ t \in R^1\}. \tag{4.4}$$

In view of (4.4), (1.10) and (4.2),

$$\mu(f_c) = \inf\{G(w, 0) : w \in R^1\} + c\phi(0) = G(t_0, 0) + c\phi(0). \tag{4.5}$$

Let $\tilde{c} > c$. By (1.10) and property (i), for each $x = (x_1, x_2, x_3) \in R^3$

$$f_{\tilde{c}}(x) = f_c(x) + (\tilde{c} - c)\phi(x_2) \geq f_c(x) + (\tilde{c} - c)\phi(0).$$

Together with (4.5) this implies that

$$\mu(f_{\tilde{c}}) \geq \mu(f_c) + (\tilde{c} - c)\phi(0) = G(t_0, 0) + \tilde{c}\phi(0). \tag{4.6}$$

Put $v(t) = t_0, \ t \in [0, \infty)$. Clearly,

$$\mu(f_{\tilde{c}}) \leq \liminf_{T \to \infty} (\int_0^T f_{\tilde{c}}(v(t), 0, 0) dt) T^{-1} = f_{\tilde{c}}(t_0, 0, 0) = G(t_0, 0) + \tilde{c}\phi(0).$$

Together with (4.6), (1.10) and (4.2) this implies that

$$\mu(f_{\tilde{c}}) = G(t_0, 0) + \tilde{c}\phi(0) = \inf\{f_{\tilde{c}}(t_0, 0, 0) : t \in R^1\}.$$

Thus, we have shown that the following property holds:

(P3) If $c \in R^1$, $\tilde{c} > c$ and $\mu(f_c) = \inf\{f_c(t, 0, 0) : t \in R^1\}$, then

$$\mu(f_{\tilde{c}}) = \inf\{f_{\tilde{c}}(t, 0, 0) : t \in R^1\}.$$

Set

$$c_0 = \inf\{c \in R^1 : \mu(f_c) = \inf\{f_c(t, 0, 0) : t \in R^1\}\}. \tag{4.7}$$

By (4.3), (P3) and (4.1), c_0 is well-defined, finite, satisfies $c_1 \leq c_0 \leq 0$, for each $c \in R^1$ satisfying $c > c_0$

$$\mu(f_c) = \inf\{f_c(t, 0, 0) : t \in R^1\}, \tag{4.8}$$

and for each real number $c < c_0$

$$\mu(f_c) < \inf\{f_c(t, 0, 0) : t \in R^1\}. \tag{4.9}$$

We show that

$$\mu(f_{c_0}) = \inf\{f_{c_0}(t, 0, 0) : t \in R^1\}. \tag{4.10}$$

It is easy to see that there exists $a = (a_1, a_2, a_3, a_4) \in (0, \infty)^4$ such that

$$f_c \in \mathfrak{M}(\alpha, \beta, \gamma, a) \text{ for each } c \in [c_0 - 1, c_0 + 1]. \tag{4.11}$$

By Proposition 2.3 the function $g \to \mu(g)$, $g \in \mathfrak{M}(\alpha, \beta, \gamma, a)$ is continuous. It is not difficult to see that $f_c \to f_{c_0}$ as $c \to c_0$ in in $\mathfrak{M}(\alpha, \beta, \gamma, a)$. Thus

$$\mu(f_c) \to \mu(f_{c_0}) \text{ as } c \to c_0.$$

Together with the equality (4.8) (which holds for all $c > c_0$), (4.2) and (1.10), the relation above implies that

$$\mu(f_{c_0}) = \lim_{c \to c_0^+} \mu(f_c) = \lim_{c \to c_0^+} \inf\{f_c(t, 0, 0) : t \in R^1\}$$
$$= \lim_{c \to c_0^+} (G(t_0, 0) + c\phi(0)) = G(t_0, 0) + c_0\phi(0) = \inf\{f_{c_0}(t, 0, 0) : t \in R^1\}.$$

Therefore, the assertion of Theorem 1.1 is proved if property (i) holds.

Assume that property (ii) holds. Let $t \in R^1 \setminus \{0\}$. By Taylor's theorem there is $\xi_t \in R^1$ such that

$$\phi(t) = \phi(0) + \phi'(0)t + 2^{-1}\phi''(\xi_t)t^2.$$

Together with property (ii) this implies that

$$\phi(t) \geq \phi(0) + \phi'(0)t \text{ for all } t \in R^1. \tag{4.12}$$

Set

$$\tilde{\phi}(t) = \phi(t) - \phi(0) - \phi'(0)t,\ t \in R^1. \tag{4.13}$$

For each $c \in R^1$ set

$$\tilde{f}_c(x_1, x_2, x_3) = G(x_1, x_3) + c\tilde{\phi}(x_2),\ (x_1, x_2, x_3) \in R^3. \tag{4.14}$$

Clearly, $\tilde{f}_c \in \mathfrak{M}(\alpha, \beta, \gamma)$ for all $c \in R^1$. By (4.12) and (4.13) $\tilde{\phi}(t) \geq 0 = \tilde{\phi}(0)$ for all $t \in R^1$. Clearly property (i) holds with $\tilde{\phi}$, (1.9) holds with $\tilde{\phi}$ and the assertion of Theorem 1.1 holds for $\tilde{\phi}$. Therefore, there is $c_0 \in R^1$ such that $c_0 \leq 0$ and

$$\mu(\tilde{f}_c) = \inf\{\tilde{f}_c(t, 0, 0) : t \in R^1\} \text{ for each } c \geq c_0,$$
$$\mu(\tilde{f}_c) < \inf\{\tilde{f}_c(t, 0, 0) : t \in R^1\} \text{ for each } c < c_0. \tag{4.15}$$

Let $c \in R^1$. Clearly by (1.10), (4.2) and (4.13),

$$\begin{aligned}\inf\{\tilde{f}_c(t, 0, 0) : t \in R^1\} &= G(t_0, 0) + c\tilde{\phi}(0) = G(t_0, 0) \\ &= \inf\{f_c(t, 0, 0) : t \in R^1\} - c\phi(0).\end{aligned} \tag{4.16}$$

Denote by $\mathcal{P}$ the set of all $w \in W^{2,1}_{loc}([0,\infty))$ for which there is $T > 0$ such that $w(t+T) = w(t)$ for all $t \geq 0$. For any $w \in \mathcal{P}$ denote by T_w a positive number such that

$$w(t + T_w) = w(t) \text{ for all } t \in [0, \infty). \tag{4.17}$$

Since there is a periodic (f_c)-good (respectively, $(\tilde{f}_c)$-good) function

$$\mu(f_c) = \inf\{(T_w)^{-1} \int_0^{T_w} f_c(w(t), w'(t), w''(t))dt : w \in \mathcal{P}\}, \tag{4.18}$$

$$\mu(\tilde{f}_c) = \inf\{(T_w)^{-1} \int_0^{T_w} \tilde{f}_c(w(t), w'(t), w''(t))dt : w \in \mathcal{P}\}. \tag{4.19}$$

Let $w \in \mathcal{P}$. By (4.14), (4.13), (1.10) and (4.17)

$$\begin{aligned}
&(T_w)^{-1} \int_0^{T_w} \tilde{f}_c(w(t), w'(t), w''(t))dt \\
&= (T_w)^{-1} \int_0^{T_w} G(w(t), w''(t))dt + (T_w)^{-1} c \int_0^{T_w} \tilde{\phi}(w'(t))dt \\
&= (T_w)^{-1} \int_0^{T_w} G(w(t), w''(t))dt \\
&\qquad + (T_w)^{-1} c \int_0^{T_w} [\phi(w'(t)) - \phi(0) - \phi'(0)w'(t)]dt \\
&= (T_w)^{-1} \int_0^{T_w} G(w(t), w''(t))dt + (T_w)^{-1} c \int_0^{T_w} \phi(w'(t))dt - c\phi(0) \\
&\qquad - (T_w)^{-1} c\phi'(0) \int_0^{T_w} w'(t)dt \\
&= (T_w)^{-1} \int_0^{T_w} f_c(w(t), w'(t), w''(t))dt - c\phi(0).
\end{aligned}$$

Together with (4.18) and (4.19) this implies that $\mu(\tilde{f}_c) = \mu(f_c) - c\phi(0)$. Combined with (4.16) the equality above implies that for all $c \in R^1$

$$\mu(f_c) - \inf\{f_c(t,0,0) : t \in R^1\} = \mu(\tilde{f}_c) - \inf\{\tilde{f}_c(t,0,0) :\ t \in R^1\}.$$

Together with (4.15) this implies that $\mu(f_c) = \inf\{f_c(t,0,0) : t \in R^1\}$ for each $c \geq c_0$, $\mu(f_c) < \inf\{f_c(t,0,0) : t \in R^1\}$ for all $c < c_0$. The assertion of Theorem 1.1 is true if the property (ii) holds. Theorem 1.1 is proved.

5. An auxiliary result for Theorem 1.2

For each $f \in \mathfrak{M}(\alpha, \beta, \gamma)$ denote by $\mathcal{P}(f)$ the set of all periodic (f)-good functions w such that $w(0) = \inf(w)$ and denote by $\mathcal{P}_0(f)$ the set of all non-constant $w \in \mathcal{P}(f)$.

Let $f \in \mathfrak{M}(\alpha, \beta, \gamma)$ and $w \in \mathcal{P}_0(f)$. By Proposition 2.1 there exist numbers $T_w > T_{w,0} > 0$ such that

$$w(t + T_w) = w(t) \text{ for all } t \geq 0, \tag{5.1}$$

$$w \text{ is strictly increasing on } [0, T_{w.0}] \text{ and strictly decreasing on } [T_{w,0}, T_w]. \tag{5.2}$$

If $w \in \mathcal{P}(f) \setminus \mathcal{P}_0(f)$, then

$$T_w = 1. \tag{5.3}$$

Proposition 5.1. *Let $a \in (0, \infty)^4$ and $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$, and let $\xi_1, \dots, \xi_q \in R^1$ satisfy $\xi_1 < \dots < \xi_q$,*

$$\{\xi_1, \dots, \xi_q\} = \{s \in R^1 : f(s, 0, 0) = \inf\{f(t, 0, 0) : t \in R^1\}\}. \tag{5.4}$$

Assume that each element of $\mathcal{P}(f)$ is a constant function and that $\Delta > 1$ and $\delta > 0$. Then there exists a neighborhood $\mathcal{U}$ of f in $\mathfrak{M}(\alpha, \beta, \gamma, a)$ such that if $g \in \mathcal{U}$, $w \in \mathcal{P}(g)$, $T_w \leq \Delta$, then there is $i \in \{1, \dots, q\}$ such that

$$|X_w(t) - (\xi_i, 0)| \leq \delta \textit{ for all } t \in [0, \infty).$$

Proof. Let us assume the converse. Then for each integer $k \geq 1$ there exist $g_k \in \mathfrak{M}(\alpha, \beta, \gamma, a)$ such that

$$\rho(f, g_k) \leq 1/k, \tag{5.5}$$

$w_k \in \mathcal{P}(g_k)$ such that $T_{w_k} \leq \Delta$ and for each $i \in \{1, \dots, q\}$

$$\sup\{|X_{w_k}(t) - (\xi_i, 0)| : t \in [0, \infty)\} > \delta. \tag{5.6}$$

By (5.5), Propositions 2.2 and 2.6 and the inclusion $w_k \in \mathcal{P}(g_k)$, $k = 1, 2, \dots$

$$\sup\{|(w_k(t), w_k'(t))| : t \in [0, \infty),\ k = 1, 2 \dots\} < \infty. \tag{5.7}$$

By the continuity of the function U_T^f and (5.7) for each $T > 0$

$$\sup\{|U_T^f(w_k(s), w_k'(s)), (w_k(s+T), w_k'(s+T)))| : \\ s \in [0, \infty),\ k = 1, 2, \dots\} < \infty. \tag{5.8}$$

In view of (5.7), (5.5) and Proposition 2.5, for any $T > 0$

$$|U_T^f(X_{w_k}(s), X_{w_k}(s+T)) - U_T^{f_k}(X_{w_k}(s), X_{w_k}(s+T)))| \to 0 \text{ as } k \to \infty, \tag{5.9}$$

uniformly in $s \in [0,\infty)$. By the inclusion $w_k \in \mathcal{P}g_k)$, $k = 1, 2, \dots$, (5.7), (5.8), (5.9) and the continuity of $U_T^{f_k}$ for each $T > 0$, the set

$$\{I^{f_k}(s, s+T, w_k) : s \in [0,\infty),\ k = 1, 2, \dots\}$$
$$= \{U_T^{f_k}(X_{w_k}(s), X_{w_k}(s+T)) : s \in [0,\infty),\ k = 1, 2, \dots\} \tag{5.10}$$

is bounded. By (5.10), (5.5) and Proposition 2.4, for any $T > 0$

$$|I^f(s, s+T, w_k) - I^{f_k}(s, s+T, w_k)| \to 0 \text{ as } k \to \infty, \tag{5.11}$$

uniformly in $s \in [0,\infty)$ and

$$\sup\{I^f(s, s+T, w_k) : s \in [0,\infty),\ k = 1, 2, \dots\} < \infty. \tag{5.12}$$

By (5.12), (1.1) and (5.7), for each $T > 0$

$$\sup\{\int_s^{s+T} |w_k''(t)|^\gamma dt : s \in [0,\infty),\ k = 1, 2, \dots\} < \infty. \tag{5.13}$$

By (5.13), (5.7) and the relation $T_{w_k} \le \Delta$, $k = 1, 2, \dots$ there are a subsequence $\{w_{k_j}\}_{j=1}^\infty$ and $w \in W_{loc}^{2,\gamma}([0,\infty))$ such that there exists

$$T_* = \lim_{j\to\infty} T_{w_{k_j}}, \tag{5.14}$$

and for any $T > 0$

$$(w_{k_j}(t), w_{k_j}'(t)) \to (w(t), w'(t)) \text{ as } j \to \infty \text{ uniformly on } [0, T], \tag{5.15}$$

$$w_{k_j}'' \to w'' \text{ weakly in } L^\gamma[0, T] \text{ as } j \to \infty. \tag{5.16}$$

By the lower semicontinuity of integral functionals [2], (5.15), (5.16), (5.11) and the inclusion $w_k \in \mathcal{P}(g_k)$, $k = 1, 2, \dots$, for each $\tau_1 \ge 0$ and $\tau_2 > \tau_1$

$$I^f(\tau_1, \tau_2, w) \le \liminf_{j\to\infty} I^f(\tau_1, \tau_2, w_{k_j}) = \liminf_{j\to\infty} I^{f_{k_j}}(\tau_1, \tau_2, w_{k_j})$$
$$= \liminf_{j\to\infty} U_{\tau_2-\tau_1}^{f_{k_j}}(X_{w_{k_j}}(\tau_1), X_{w_{k_j}}(\tau_2))$$
$$= \liminf_{j\to\infty} U_{\tau_2-\tau_1}^{f}(X_{w_{k_j}}(\tau_1), X_{w_{k_j}}(\tau_2)) = \liminf_{j\to\infty} U_{\tau_2-\tau_1}^{f}(X_w(\tau_1), X_w(\tau_2)).$$

Thus,

$$I^f(\tau_1, \tau_2, w) = U_{\tau_2-\tau_1}^f(X_w(\tau_1), X_w(\tau_2)) \text{ for each } \tau_1 \ge 0,\ \tau_2 > \tau_1. \tag{5.17}$$

Clearly, $T_* \le \Delta$. Assume that $T_* > 0$. Then by (5.15), (5.14) and the definition of T_{w_k}, $k = 1, 2, \dots$ (see (5.1)), $w(t + T_*) = w(t)$ for all $t \ge 0$, and in view of (5.17) and Proposition 2.3 w is a periodic (f)-good function. Since any element of $\mathcal{P}(f)$ is constant, it follows from (5.4) that there is $i \in \{1, \dots, q\}$ such that $w(t) = \xi_i$, $t \in [0,\infty)$.

Assume that $T = 0$. Then by (5.14), (5.7) and (5.15), $\lim_{j\to\infty} T_{k_j} = 0$,

$$\sup\{w_{k_j}(t) : t \in R^1\} - \inf\{w_{k_j}(t) : t \in R^1\} \to 0 \text{ as } j \to \infty,$$

and $w(t) = w(0)$ for all $t \in [0, \infty)$. By (5.17) and Proposition 2.7 the function w is (f)-good. Together with (5.4) this implies that there is $i \in \{1, \dots, q\}$ such that $w(t) = \xi_i$ for all $t \in [0, \infty)$. Thus, we have shown that in both cases there is $i \in \{1, \dots, q\}$ such that

$$w(t) = \xi_i, \ t \in [0, \infty). \tag{5.18}$$

In view of (5.15) and (5.18) there is an integer $m \geq 1$ such that

$$|(\xi_i, 0) - (w_{k_m}(t), w'_{k_m}(t))| \leq \delta/8 \text{ for all } t \in [0, 4\Delta + 4].$$

Since w_{k_m} is periodic with a period $T_{k_m} \leq \Delta$, the relation above implies that

$$|(\xi_i, 0) - (w_{k_m}(t), w'_{k_m}(t))| \leq \delta/8,$$

for all $t \in [0, \infty)$. This inequality contradicts (5.6). The contradiction we have reached proves Proposition 5.1. □

6. Proof of Theorem 1.2

By Theorem 1.1, for each $c \in R^1$ the function f_c belongs to $\mathfrak{M}(\alpha, \beta, \gamma)$ and there is a real number $c_0 \leq 0$ such that

$$\mu(f_c) < \inf\{f_c(t, 0, 0) : t \in R^1\} \text{ for all } c < c_0, \tag{6.1}$$

$$\mu(f_c) = \inf\{f_c(t, 0, 0) : t \in R^1\} \text{ for all } c \geq c_0. \tag{6.2}$$

We need only to show that $c_0 < 0$. We may assume without loss of generality that

$$\inf(g) = 0. \tag{6.3}$$

Assume that property (i) holds. Clearly there exists $a = (a_1, a_2, a_3, a_4) \in (0, \infty)^4$ such that

$$f_c \in \mathfrak{M}(\alpha, \beta, \gamma, a) \text{ for all } c \in [-1, 1], \tag{6.4}$$

and

$$f_c \to f_0 \text{ as } c \to 0 \text{ in } \mathfrak{M}(\alpha, \beta, \gamma, a). \tag{6.5}$$

By (6.5) and Proposition 2.2 there are $c_1 \in (0, 1)$ and $M > 0$ such that for all $c \in [-c_1, c_1]$ and each (f_c)-good function u

$$|(u(t), u'(t))| \leq M \text{ for all large enough } t. \tag{6.6}$$

In view of (1.15) there is $m_1 \in (0, 1)$ such that

$$g''(\xi_i) \geq 2m_1, \ i = 1, \dots, k. \tag{6.7}$$

If $k = 1$, set

$$r_0 = 1/2. \tag{6.8}$$

If $k > 1$, choose $r_0 > 0$ such that

$$r_0 < (\xi_{i+1} - \xi_i)/8, \ i = 1, \dots, k-1. \tag{6.9}$$

By (6.7) there exists

$$\Delta_0 \in (0, \min\{1, r_0\}/8), \tag{6.10}$$

such that

$$g''(z) \geq m_1 \text{ for all } z \in \cup_{i=1}^k [\xi_i - 2\Delta_0, \xi_i + 2\Delta_0]. \tag{6.11}$$

By (6.3), (1.15) and (1.13) there is $m_2 > 0$ such that

$$g(z) \geq m_2 \text{ for each } z \in R^1 \setminus \cup_{i=1}^k [\xi_i - \Delta_0/2, \xi_i + \Delta_0/2]. \tag{6.12}$$

By the interpolation inequality [1, Lemma 4.10] there exists $d_3 > 0$ such that for each $T \geq \min\{8^{-1}, 8^{-1}(M+1)^{-1}\Delta_0\}$ and each $u \in C^2([0,T])$

$$\int_0^T (u'(t))^2 dt \leq \Delta_0 \int_0^T (u''(t))^2 dt + d_3 \int_0^T (u(t))^2 dt. \tag{6.13}$$

Choose

$$\delta_0 \in (0, c_1), \tag{6.14}$$

such that

$$\begin{aligned} &(d_3+1)\delta_0 \sup\{|\phi(z)| + |\phi'(z)| + |\phi''(z)| : \\ &z \in [-M, M]\}(\Delta_0 + 1) < \min\{\kappa/8, \ m_1/8, m_2/8\}. \end{aligned} \tag{6.15}$$

Choose a number

$$\tau_0 \geq r_0 + 4. \tag{6.16}$$

By (1.11) and (1.15) any element of $\mathcal{P}(f_0)$ is constant. In view of Proposition 5.1, (6.4) and (1.15), there exists a neighborhood $\mathcal{U}$ of f_0 in $\mathfrak{M}(\alpha, \beta, \gamma, a)$ such that the following property holds:

(P4) If $g \in \mathcal{U}$ and if $w \in \mathcal{P}_g$ satisfies $T_w \leq 2\tau_0$, then there is $i \in \{1, \dots, k\}$ such that $|X_w(t) - (\xi_i, 0)| \leq \Delta_0/16$ for all $t \in [0, \infty)$.

By (6.4) and (6.5) there exists $\delta \in (0, \delta_0)$ such that

$$f_{-\delta} \in \mathcal{U}. \tag{6.17}$$

Assume that

$$w \in \mathcal{P}_{f_{-\delta}}. \tag{6.18}$$

By Proposition 4.1 of [8], $w \in C^4([0, \infty))$. In view of the choice of c_1 and M (see (6.6)), (6.18), (6.17) and (6.14)

$$|(w(t), w'(t))| \leq M \text{ for all } t \in [0, \infty). \tag{6.19}$$

By the definition of $\mathcal{P}(f_{-\delta})$

$$w(0) = \inf\{w(t) : t \in R^1\}. \tag{6.20}$$

We show that w is a constant function. Let us assume the converse. We show that the following property holds:

(P5) If $i \in \{1, \dots, k\}$ and if $t \in [0, \infty)$ satisfies $|w(t) - \xi_i| \le 2\Delta_0$, then

$$g(w(t)) \ge g(\xi_i) + 2^{-1} m_1 (w(t) - \xi_i)^2, \tag{6.21}$$

$$\phi(w'(t)) \le \phi(0) + 2^{-1} \sup\{|\phi''(z)| : z \in [-M, M]\}(w'(t))^2, \tag{6.22}$$

$$h(w''(t)) \ge h(0) + 2^{-1} \kappa (w''(t))^2. \tag{6.23}$$

Let

$$t \in [0, \infty),\ i \in \{1, \dots, k\},\ |w(t) - \xi_i| \le 2\Delta_0. \tag{6.24}$$

By Taylor's theorem, there exist real numbers z_1, z_2 and z_3 such that

$$z_1 \in [\min\{w(t), \xi_i\}, \max\{w(t), \xi_i\}], \tag{6.25}$$

$$g(w(t)) = g(\xi_i) + g'(\xi_i)(w(t) - \xi_i) + 2^{-1} g''(z_1)(w(t) - \xi_i)^2, \tag{6.26}$$

$$z_2 \in [\min\{0, w'(t)\},\ \max\{0, w'(t)\}], \tag{6.27}$$

$$\phi(w'(t)) = \phi(0) + \phi'(0) w'(t) + 2^{-1} \phi''(z_2)(w'(t))^2, \tag{6.28}$$

$$h(w''(t)) = h(0) + h'(0) w''(t) + +2^{-1} h''(z_3)(w''(t))^2. \tag{6.29}$$

By (6.24) and (6.25), $|z_1 - \xi_i| \le |w(t) - \xi_i| \le 2\Delta_0$. Together with (6.11), (6.26) and (1.15), this implies that $g''(z_1) \ge m_1$ and

$$\begin{aligned} g(w(t)) &\ge g(\xi_i) + g'(\xi_i)(w(t) - \xi_i) + 2^{-1} m_1 (w(t) - \xi_i)^2 \\ &\ge g(\xi_i) + 2^{-1} m_1 (w(t) - \xi_i)^2. \end{aligned}$$

By (6.27) and (6.19) $|z_2| \le M$. Together with (6.28) and property (i) this implies that

$$\phi(w'(t)) \le \phi(0) + 2^{-1} \sup\{|\phi''(z)| : z \in [-M, M]\}(w'(t))^2.$$

Relations (6.29) and (1.1) imply that

$$h(w''(t)) \ge h(0) + 2^{-1} \kappa (w''(t))^2.$$

Thus property (P5) holds.

Assume that $i \in \{1 \dots, k\}$ and

$$|w(t) - \xi_i| \le 2\Delta_0 \text{ for all } t \in [0, \infty). \tag{6.30}$$

There is $T \ge 4$ such that

$$w(t + T) = w(t) \text{ for all } t \in [0, \infty). \tag{6.31}$$

In view of (6.30), (6.31) and (P5),

$$T^{-1}\int_0^T g(w(t))dt \ge g(\xi_i) + 2^{-1}m_1T^{-1}\int_0^T (w(t)-\xi_i)^2dt,$$

$$T^{-1}\delta\int_0^T \phi(w'(t))dt$$

$$\le \delta\phi(0) + T^{-1}\delta 2^{-1}\sup\{|\phi''(z)| : z\in[-M,M]\}\int_0^T (w'(t))^2dt$$

$$= \delta\phi(0) + T^{-1}\delta 2^{-1}\sup\{|\phi''(z)| : z\in[-M,M]\}\int_0^T (w'(t))^2dt,$$

$$T^{-1}\int_0^T h(w''(t))dt \ge h(0) + 2^{-1}\kappa T^{-1}\int_0^T (w''(t))^2dt.$$

By (6.18), (6.31), (1.16) and the relation above,

$$\mu(f_{-\delta}) = T^{-1}\int_0^T f_{-\delta}(w(t),w'(t),w''(t))dt$$

$$= T^{-1}\int_0^T g(w(t))dt - T^{-1}\delta\int_0^T \phi(w'(t))dt + T^{-1}\int_0^T h(w''(t))dt$$

$$\ge g(\xi_i) + (2T)^{-1}m_1\int_0^T (w(t)-\xi_i)^2dt - \delta\phi(0) - (2T)^{-1}\delta\sup\{|\phi''(z)| :$$

$$z\in[-M,M]\}\int_0^T (w'(t))^2dt + h(0) + (2T)^{-1}\kappa\int_0^T (w''(t))^2dt.$$

It follows from the inequality above, the inequality $T\ge 4$, the choice of d_3, (6.13) applied to $u = w-\xi_i$, (1.16), (6.15) and (6.17) that

$$\mu(f_{-\delta}) \ge g(\xi_i) - \delta\phi(0) + h(0) + (2T)^{-1}m_1\int_0^T (w(t)-\xi_i)^2dt$$

$$+ (2T)^{-1}\kappa\int_0^T (w''(t))^2dt - (2T)^{-1}\delta\sup\{|\phi''(z)| :$$

$$z\in[-M,M]\}[\Delta_0\int_0^T (w''(t))^2dt + d_3\int_0^T (w(t)-\xi_i)^2dt]$$

$$\ge f_{-\delta}(\xi_i,0,0) + (4T)^{-1}\kappa\int_0^T (w''(t))^2dt + (4T)^{-1}m_1\int_0^T (w(t)-\xi_i)^2dt.$$

Since w is a non-constant function, $\mu(f_{-\delta}) > f_{-\delta}(\xi_i,0,0)$. This inequality contradicts the definition of $\mu(f_{-\delta})$. The contradiction we have reached proves that the following property holds:

(P6) There is no $i \in \{1, \dots, k\}$ such that $|w(t)-\xi_i| \le 2\Delta_0$ for all $t \in [0, \infty)$.

Since the function w is non-constant, relation (6.18) implies that $w \in \mathcal{P}_0(f_{-\delta})$. This inclusion implies (see (5.1) and (5.2)) that there exist positive numbers $T_w > T_{w,0} > 0$ such that

$$w(t + T_w) = w(t) \text{ for all } t \in [0, \infty), \tag{6.32}$$

$$w \text{ is strictly increasing on } [0, T_{w,0}] \text{ and strictly decreasing on } [T_{w,0}, T_w]. \tag{6.33}$$

By (6.20), (6.32) and (6.33), there exist functions $w_1 : w([0, \infty)) \to [0, T_{w,0}]$, $w_2 : w([0, \infty)) \to [T_{w,0}, T_w]$ such that

$$w(w_1(t)) = t,\ w(w_2(t)) = t \text{ for all } t \in w([0, \infty)). \tag{6.34}$$

Property (P6) implies that the following property holds:

(P7) If $i \in \{1, \dots, k\}$ satisfies $[\xi_i - \Delta_0, \xi_i + \Delta_0] \cap w([0, \infty)) \ne \emptyset$, then at least one of the following relations holds:

$$w(0) < \xi_i - 2\Delta_0,\ w(T_w) > \xi_i + 2\Delta_0.$$

Assume that $T_w \le 2\tau_0$. By (P4), (6.17) and (6.18), there is $i \in \{1, \dots, k\}$ such that $|(w(t), w'(t)) - (\xi_i, 0)| \le \Delta_0/16$ for all $t \in [0, \infty)$. This contradicts (P6). The contradiction we have reached proves that

$$T_w > 2\tau_0. \tag{6.35}$$

Assume that

$$[w(0), w(T_w)] \cap [\xi_i - \Delta_0, \xi_i + \Delta_0] = \emptyset \text{ for all } i = 1, \dots, k. \tag{6.36}$$

In view of (6.36) and (6.12)

$$g(w(t)) \ge m_2 \text{ for all } t \in [0, \infty). \tag{6.37}$$

By (6.18), (1.16), (6.37), (6.19), (1.11), (1.15), (6.3), (6.15), (6.17) and the definition of $\mu(f_{-\delta})$,

$$\begin{aligned}
&\mu(f_{-\delta}) = (T_w)^{-1} \int_0^{T_w} f_{-\delta}(w(t), w'(t), w''(t)) = (T_w)^{-1} \int_0^{T_w} g(w(t))dt \\
&- \delta(T_w)^{-1} \int_0^{T_w} \phi(w'(t))dt + (T_w)^{-1} \int_0^{T_w} h(w''(t))dt \\
&\ge m_2 - \delta \sup\{|\phi(z)| : z \in [-M, M]\} + \inf(h) \\
&= m_2 - \delta \sup\{|\phi(z)| : z \in [-M, M]\} + h(0) \\
&\ge m_2 + g(\xi_i) - \delta\phi(0) + h(0) - 2\delta \sup\{|\phi(z)| :\ z \in [-M, M]\} \\
&\ge f_{-\delta}(\xi_i, 0, 0) + m_2 - 2\delta \sup\{|\phi(z)| : z \in [-M, M]\} > f_{-\delta}(\xi_i, 0, 0) \ge \mu(f_{-\delta}).
\end{aligned}$$

The contradiction we have reached proves that there is $i \in \{1, \dots, k\}$ such that

$$[w(0), w(T_w)] \cap [\xi_i - \Delta_0, \xi_i + \Delta_0] \neq \emptyset. \tag{6.38}$$

Denote by E the set of all $i \in \{1, \dots, k\}$ such that (6.38) holds. We have shown that $E \neq \emptyset$. For each $i \in E$ set

$$\mathcal{L}_i = [\max\{w(0), \xi_i - 2\Delta_0\}, \min\{w(T_w),\ \xi_i + 2\Delta_0\}]. \tag{6.39}$$

By (6.39), (6.38) and (6.32), for any $i \in E$

$$\mathcal{L}_i \neq \emptyset \text{ and } \mathcal{L}_i \subset [w(0), w(T_w)] = w([0, \infty)). \tag{6.40}$$

Relations (6.38) and (6.39) and property (P7) imply that for any $i \in E$

$$\operatorname{mes}(\mathcal{L}_i) \geq \Delta_0. \tag{6.41}$$

In view of (6.40) and the definition of w_1 and w_2 (see (6.34)), for all $i \in E$

$$w_1(\mathcal{L}_i) \subset [0, T_{w,0}],\ w_2(\mathcal{L}_i) \subset [T_{w,0}, T_w]. \tag{6.42}$$

By (6.39), (6.10) and the choice of r_0 (see (6.9)),

$$\mathcal{L}_i \cap \mathcal{L}_j \text{ for each pair } i, j \in E \text{ such that } i \neq j. \tag{6.43}$$

Since w_1 is strictly increasing and w_2 is strictly decreasing, relations (6.40) and (6.43) imply that

$$w_1(\mathcal{L}_i) \cap w_1(\mathcal{L}_j) = \emptyset, \quad w_2(\mathcal{L}_i) \cap w_2(\mathcal{L}_j) = \emptyset \tag{6.44}$$

for each pair of $i, j \in E$ such that $i \neq j$. Let $i \in E$. By (6.34)

$$\mathcal{L}_i = w(w_1(\mathcal{L}_i)),\ \mathcal{L}_i = w(w_2(\mathcal{L}_i)).$$

Together with (6.19) this implies that

$$\operatorname{mes}(\mathcal{L}_i) \leq M \operatorname{mes}(w_1(\mathcal{L}_i)),\ \operatorname{mes}(\mathcal{L}_i) \leq M \operatorname{mes}(w_2(\mathcal{L}_i)).$$

It follows from the inequalities above and (6.41) that

$$\operatorname{mes}(w_1(\mathcal{L}_i)),\ \operatorname{mes}(w_2(\mathcal{L}_i)) \geq M^{-1} \operatorname{mes}(\mathcal{L}_i) \geq M^{-1}\Delta_0 \text{ for all } i \in E. \tag{6.45}$$

Set $\mathcal{D} = [0, T_w] \setminus \cup_{i \in E}(w_1(\mathcal{L}_i) \cup w_2(\mathcal{L}_i))$. By (6.18), (6.46), (6.45), (6.44) and (6.42),

$$\begin{aligned}
\mu(f_{-\delta}) &= (T_w)^{-1} \int_0^{T_w} f_{-\delta}(w(t), w'(t), w''(t))dt \\
&= (T_w)^{-1} \int_{\mathcal{D}} f_{-\delta}(w(t), w'(t), w''(t))dt \\
&+ \sum_{i \in E} \sum_{j=1}^{2} (T_w)^{-1} \int_{w_j(\mathcal{L}_i)} f_{-\delta}(w(t), w'(t), w''(t))dt.
\end{aligned} \tag{6.47}$$

Assume that

$$t \in \mathcal{D}. \tag{6.48}$$

In view of (6.48), (6.46) and the definitions of w_1 and w_2 (see (6.33) and (6.34)), $w(t) \notin \cup_{i \in E} \mathcal{L}_i$. Together with (6.39) this relation implies that $|w(t) - \xi_i| \geq \Delta_0$, $i = 1, \dots, k$. Together with (6.12) this implies that $g(w(t)) \geq m_2$. Combined with (1.16), (6.19), (1.11), (6.3), (6.15) and (6.17) the inequality above implies that

$$\begin{aligned} &f_{-\delta}(w(t), w'(t), w''(t)) = g(w(t)) - \delta\phi(w'(t)) + h(w''(t)) \\ &\geq m_2 - \delta \sup\{|\phi(z)| : z \in [-M, M]\} + h(0) \\ &\geq \inf(g) - \delta\phi(0) + h(0) - 2\delta \sup\{|\phi(z)| : z \in [-M, M]\} + m_2 \\ &\geq \inf\{f_{-\delta}(z, 0, 0) : z \in R^1\} + m_2/2 \geq \mu(f_{-\delta}) + m_2/2. \end{aligned}$$

Therefore, we have shown that for each $t \in \mathcal{D}$

$$f_{-\delta}(w(t), w'(t), w''(t)) \geq \mu(f_{-\delta}) + m_2/2. \tag{6.49}$$

Let

$$i \in E, \ j \in \{1, 2\}, \tag{6.50}$$

and

$$t \in w_j(\mathcal{L}_i). \tag{6.51}$$

By (6.50) and the definition of w_j (see (6.33) and (6.34)), $w(t) \in \mathcal{L}_i$. Together with (6.39) this implies that $|w(t) - \xi_i| \leq 2\Delta_0$. Combined with (P5) this implies that (6.21), (6.22) and (6.23) hold for all $t \in w_j(\mathcal{L}_i)$. In view of (1.16) and (6.21)–(6.23), which hold for all $t \in w_j(\mathcal{L}_i)$,

$$\begin{aligned} &\int_{w_j(\mathcal{L}_i)} f_{-\delta}(w(t), w'(t), w''(t))dt = \int_{w_j(\mathcal{L}_i)} g(w(t))dt - \delta \int_{w_j(\mathcal{L}_i)} \phi(w'(t))dt \\ &+ \int_{w_j(\mathcal{L}_i)} h(w'(t))dt \geq g(\xi_i)\text{mes}(w_j(\mathcal{L}_i)) + 2^{-1}m_1 \int_{w_j(\mathcal{L}_i)} (w(t) - \xi_i)^2 dt \\ &-\delta\phi(0)\text{mes}(w_j(\mathcal{L}_i)) - \delta 2^{-1} \sup\{|\phi'(z)| : z \in [-M, M]\} \int_{w_j(\mathcal{L}_i)} (w'(t))^2 dt \\ &+h(0)\text{mes}(w_j(\mathcal{L}_i)) + 2^{-1}\kappa \int_{w_j(\mathcal{L}_i)} w''(t))^2 dt. \end{aligned} \tag{6.52}$$

It follows from (6.52), (6.45), the interpolation inequality (6.13) applied to $u = w - \xi_i$, (1.16), (6.15) and (6.17) that

$$\int_{w_j(\mathcal{L}_i)} f_{-\delta}(w(t), w'(t), w''(t))dt \geq \ \text{mes}(w_j(\mathcal{L}_i))(g(\xi_i) - \delta\phi(0) + h(0))$$

$$
\begin{aligned}
&+ 2^{-1}m_1 \int_{w_j(\mathcal{L}_i)} (w(t)-\xi_i)^2 dt + 2^{-1}\kappa \int_{w_j(\mathcal{L}_i)} (w''(t))^2 dt \\
&- 2^{-1}\delta \sup\{|\phi'(z)| : z \in [-M, M]\} \\
&\times \Big(\Delta_0 \int_{w_j(\mathcal{L}_i)} (w''(t))^2 dt + d_3 \int_{w_j(\mathcal{L}_i)} (w(t)-\xi_i)^2 dt\Big) \qquad (6.53) \\
&\geq f_{-\delta}(\xi_i, 0, 0)\mathrm{mes}(w_j(\mathcal{L}_i)) + \tfrac{m_1}{4} \int_{w_j(\mathcal{L}_i)} (w(t)-\xi_i)^2 dt + \tfrac{\kappa}{4} \int_{w_j(\mathcal{L}_i)} (w''(t))^2 dt.
\end{aligned}
$$

By (6.47), (6.49), (6.53), (6.46), (6.44) and (6.42),

$$
\begin{aligned}
\mu(f_{-\delta}) &\geq (T_w)^{-1}\mathrm{mes}(\mathcal{D})(\mu(f_{-\delta}) + m_2/2) + \sum_{i\in E}\sum_{j=1}^{2} (T_w)^{-1}[\mu(f_{-\delta})\mathrm{mes}(w_j(\mathcal{L}_i)) \\
&\quad + 4^{-1}m_1 \int_{w_j(\mathcal{L}_i)} (w(t)-\xi_i)^2 dt + 4^{-1}\kappa \int_{w_j(\mathcal{L}_i)} (w''(t))^2 dt] \\
&= (T_w)^{-1}[\mathrm{mes}(\mathcal{D}) + \sum_{i\in E}\sum_{j=1}^{2} \mathrm{mes}(w_j(\mathcal{L}_i))]\mu(f_{-\delta}) + (T_w)^{-1}\,\mathrm{mes}(\mathcal{D})m_2/2 \\
&\quad + (T_w)^{-1}\sum_{i\in E}\sum_{j=1}^{2}\Big[4^{-1}m_1 \int_{w_j(\mathcal{L}_i)} (w(t)-\xi_i)^2 dt + 4^{-1}\kappa \int_{w_j(\mathcal{L}_i)} (w''(t))^2 dt\Big] \\
&= \mu(f_{-\delta}) + (T_w)^{-1}\mathrm{mes}(\mathcal{D})m_2/2 \\
&\quad + (T_w)^{-1}\sum_{i\in E}\sum_{j=1}^{2}\Big[4^{-1}m_1 \int_{w_j(\mathcal{L}_i)} (w(t)-\xi_i)^2 dt + 4^{-1}\kappa \int_{w_j(\mathcal{L}_i)} (w''(t))^2 dt\Big].
\end{aligned}
$$

This relation implies that for any $i \in E$ and $j = 1, 2$ we have $w(t) = \xi_i$ for all $t \in w_j(\mathcal{L}_i)$. This equality, (6.42) and (6.45) contradict (6.33). The contradiction we have reached proves that w is constant. This implies that $\mu(f_{-\delta}) = \inf\{f_{-\delta}(t, 0, 0) : t \in R^1\}$. In view of (6.1) and (6.2), $c_0 \leq -\delta < 0$. Thus the assertion of Theorem 1.2 is true if property (i) holds.

Assume that property (ii) holds. Let $t \in R^1$. By Taylor's theorem there exist $\xi_t \in R^1$ such that $\phi(t) = \phi(0) + \phi'(0)t + 2^{-1}\phi''(\xi_t)t^2$, and by (ii) $\phi(t) \geq \phi(0) + \phi'(0)t$. Set $\tilde{\phi}(t) = \phi(t) - \phi(0) - \phi'(0)t$ for all $t \in R^1$. For any $c \in R^1$ put

$$\tilde{f}_c(x_1, x_2, x_3) = g(x_1) + h(x_3) + c\tilde{\phi}(x_2),\ x = (x_1, x_2, x_3) \in R^3.$$

It is clear that there exist $\tilde{b}_4, \tilde{b}_5 > 0$ such that $|\tilde{\phi}(t)| \leq \tilde{b}_4|t|^\beta + \tilde{b}_5$ for all $t \in R^1$, $\tilde{\phi}''(0) = \phi''(0) > 0$ and that for all $t \in R^1$ we have $\tilde{\phi}(t) \geq 0 = \tilde{\phi}(0)$.

Thus, property (i) holds for the function $\tilde{\phi}$ and the assertion of the theorem is true for $\tilde{\phi}$. Therefore, there is a number $c_0 < 0$ such that

$$\mu(\tilde{f}_c) < \inf\{\tilde{f}_c(t,0,0) : t \in R^1\} \text{ for all } c < c_0, \tag{6.54}$$
$$\mu(\tilde{f}_c) = \inf\{\tilde{f}_c(t,0,0) : t \in R^1\} \text{ for all } c \geq c_0.$$

Arguing as in the proof of Theorem 1.1 we can show that for any $c \in R^1$

$$\mu(f_c) - \inf\{f_c(t,0,0) : t \in R^1\} = \mu(\tilde{f}_c) - \inf\{\tilde{f}_c(t,0,0) :\ t \in R^1\}.$$

Together with (6.54) this implies that the assertion of Theorem 1.2 is true if property (ii) holds. Theorem 1.2 is proved.

REFERENCES

[1] R.A. Adams, "Sobolev Spaces," Academic Press, New York, 1984.

[2] L.D. Berkovitz, *Lower semicontinuity of integral functionals,* Trans. Amer. Math. Soc., 192 (1974), 51–57.

[3] B.D. Coleman, M. Marcus, and V.J. Mizel, *On the thermodynamics of periodic phases,* Arch. Rational Mech. Anal., 117 (1992), 321–347.

[4] A. Leizarowitz and V.J. Mizel, *One dimensional infinite horizon variational problems arising in continuum mechanics,* Arch. Rational Mech. Anal., 106 (1989), 161–194.

[5] M. Marcus and A.J. Zaslavski, *On a class of second order variational problems with constraints,* Israel Journal of Mathematics, 111 (1999), 1–28.

[6] M. Marcus and A.J. Zaslavski, *The structure of extremals of a class of second order variational problems,* Ann. Inst. H. Poincare, Anal. Non Lineaire, 16 (1999), 593–629.

[7] M. Marcus and A.J. Zaslavski, *The structure and limiting behavior of locally optimal minimizers,* Ann. Inst. H. Poincare, Anal. Non Lineaire, 19 (2002), 343–370.

[8] A.J. Zaslavski, *The existence of periodic minimal energy configurations for one dimensional infinite horizon variational problems arising in continuum mechanics,* Journal of Mathematical Analysis and Applications, 194 (1995), 459–476.

[9] A.J. Zaslavski, *The existence and structure of extremals for a class of second order infinite horizon variational problems,* Journal of Mathematical Analysis and Applications, 194 (1995), 660–696.

[10] A.J. Zaslavski, *Structure of extremals for one-dimensional variational problems arising in continuum mechanics,* Journal of Mathematical Analysis and Applications, 198 (1996), 893–921.