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INFINITE-DIMENSIONAL ELLIPTIC EQUATIONS WITH HÖLDER-CONTINUOUS COEFFICIENTS

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Abstract. Infinite-dimensional elliptic equations, with Hölder-continuous coefficients are here studied by purely analytic methods. In particular, Schauder estimates for solutions of such equations are derived.

1. Introduction. We are here concerned with the elliptic equation

$$\mu u(x) - \tfrac{1}{2} \operatorname{Tr} [Q(x)D^2u(x)] = f(x), \quad x \in H, \tag{1.1}$$

where $\mu > 0$ is fixed. H is a real separable Hilbert space and $f : H \mapsto \mathbb{R}$ is uniformly continuous and bounded.

There is an increasing interest in studying elliptic equations in infinitely many variables, due to applications to several domains as Field Theory, Dirichlet forms and Statistical Mechanics; see, e.g., Z.M. Ma and M. Röckner ([9]), and D.W. Stroock ([12]). Another typical motivation is the well-known connection with the stochastic differential equation

$$dX(t) = Q^{1/2}(X(t))dW(t). \tag{1.2}$$

When $Q(x) = Q$, Q being a positive self-adjoint trace-class operator in H, problem (1.1) has been studied by L. Gross, [6], and Yu. Daleckij; see [4]. Using fundamental solutions, A. Piech, [11], has considered the case of

$$Q(x) = Q^{1/2}(1 + F(x))Q^{1/2}, \quad x \in H,$$

where $F(x),\ x \in H$, is a family of trace-class operators satisfying suitable smoothness assumptions.

We are interested in treating problem (1.1) when the coefficients $F(x),\ x \in H$, are only Hölder continuous. We note that in this case no existence and uniqueness theory for problem (1.1) seems to be available. Also the probabilistic approach, based on solving the stochastic differential equation (1.2), requires, in infinite dimensions, that

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coefficients be Lipschitz continuous. Moreover, solving directly equation (1.1) can in turn be useful in studying (1.2).

Our approach consists of two steps:

(i) Using some recent results on parabolic equations with constant coefficients in Hilbert spaces ([2]), we prove Schauder-type estimates for equation (1.1) when Q is constant.

(ii) We extend to infinite-dimensional equations some techniques that are well known in finite dimensions, such as the Localization and Continuity Method.

The main tools we are using here are some precise estimates on the behavior at 0 of the infinite-dimensional heat semigroup together with a characterization of interpolation spaces between $C_b(H)$ and $C_b^1(H)$, given in Proposition 2.1.

In fact, the usual probabilistic approach does not apply when the coefficient $Q(x)$ is merely Hölder continuous. The coefficient $Q(x)$, $x \in H$, is of the form

$$Q(x) = Q^{1/2}(1 + F(x))Q^{1/2}, \quad x \in H,$$

where Q is a positive self-adjoint trace-class operator in H and $F(x)$, $x \in H$, is also trace-class and fulfills a Hölder continuity condition.

2. Notation. Throughout the paper H is a separable Hilbert space with norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$. We denote by $\mathcal{L}(H)$ (norm $\|\cdot\|$) the space of all bounded linear operators in H and by $\mathcal{L}_1(H)$ the space of all trace-class operators in $\mathcal{L}(H)$. Moreover $\mathcal{L}_1^+(H)$ represents the cone of all symmetric nonnegative operators in $\mathcal{L}_1(H)$. For any $T \in \mathcal{L}_1(H)$ we denote by Tr (T) the trace of T.

We define $C_b(H)$ to be the space of all functions $f : H \mapsto \mathbb{R}$ which are uniformly continuous and bounded in H, and set

$$\|f\|_0 = \sup_{x\in H} |f(x)|_X, \quad \forall\, f \in C_b(H).$$

We denote by $C_b^1(H)$ the set of all bounded uniformly continuous functions $f : H \mapsto \mathbb{R}$ which are Fréchet differentiable on H, with a bounded uniformly continuous gradient Df, and set

$$\|f\|_1 = \|f\|_0 + \sup_{x\in H} |Df(x)|.$$

Moreover, we denote by $C_b^{1,1}(H)$ the space of all functions $f \in C_b^1(H)$ that possess a bounded Lipschitz-continuous gradient Df, and set

$$\|f\|_{1,1} = \|f\|_1 + \sup_{x\neq y} \frac{|Df(x) - Df(y)|}{|x-y|}\ .$$

We note that $C_b^{1,1}(H)$ is *dense* in $C_b(H)$; see A.S. Nemirowski and S.N. Semenov ([10]), J.M. Lasry and P.L. Lions ([7]).

Remark 2.1. In a similar way one could define spaces $C_b^k(H)$ for any $k \in \mathbb{N}$, requiring f to possess bounded uniformly continuous Fréchet derivatives up to the order k. It is well known that, if H is finite dimensional, then all the spaces $C_b^k(H)$ are dense in $C_b(H)$. When H is infinite dimensional, on the contrary, the density of $C_b^k(H)$ in $C_b(H)$ fails to be true for $k \geq 2$; see A.S. Nemirovski and S.N. Semenov ([10], Section 7). Therefore, in this paper we will *not* use the above notion of $C_b^k(H)$ for $k \geq 2$.

Finally $C_b(H,H)$ is the set of all mappings $H \mapsto H,\ x \mapsto F(x)$ that are uniformly continuous and bounded. It is a Banach space with the norm

$$\|F\|_0 = \sup_{x \in H} |F(x)|.$$

Throughout the paper an operator $Q \in \mathcal{L}_1^+(H)$ is fixed. For the sake of simplicity we assume that the kernel of Q is equal to $\{0\}$.

Then there exists a complete orthonormal system $\{e_k\}_{k=1}^\infty$ in H, consisting of eigenvectors of Q. We denote by $\{\lambda_k\}_{k=1}^\infty$ the corresponding eigenvalues. Then $\lambda_k > 0$, $\forall k \in \mathbb{N}$, and

$$Qe_k = \lambda_k e_k,\ k \in \mathbb{N}, \qquad \mathrm{Tr}\ Q = \sum_{k=1}^\infty \lambda_k < \infty.$$

For any $n \in \mathbb{N}$ and $x \in H$ we define $x_n = \langle x, e_n \rangle$ and

$$R_n : H \mapsto H\ , \quad R_n x\ =\ \sum_{k=n+1}^\infty x_k e_k\ ,\ \ \forall x \in H.$$

Moreover, we denote by Π_n the projection

$$\Pi_n : H \mapsto \mathbb{R}^n\ , \quad \Pi_n x = (x_1, \dots, x_n)\ ,\ \ \forall x \in H,$$

and by Σ_n the immersion

$$\Sigma_n : \mathbb{R}^n \mapsto H\ , \quad \Sigma_n \xi = \sum_{k=1}^n \xi_k e_k\ ,\ \ \forall \xi \in \mathbb{R}^n.$$

Given $\varphi \in C_b(H)$ and $v \in H$ we say that φ is *differentiable in the direction of* v if

(i) For any $x \in H$ there exists the limit

$$\lim_{t \to 0} \frac{1}{t}(\varphi(x+tv) - \varphi(x)) =: D_v\varphi(x).$$

(ii) The mapping $H \mapsto \mathbb{R},\ \ x \mapsto D_v\varphi(x)$ belongs to $C_b(H)$.

We set

$$D_k\varphi = D_{e_k}\varphi, \quad k \in \mathbb{N}.$$

2.1. Spaces $C_Q^1(H)$ and $C_Q^2(H)$. In this subsection we will define some spaces that we will use in the following. We start with the space $C_Q^1(H)$. Let $\varphi \in C_b(H)$ such that:

(i) φ is differentiable in all directions $Q^{1/2}v,\ v \in H$.

(ii) For all $x \in H$ the mapping $H \mapsto \mathbb{R}$, $v \mapsto D_{Q^{1/2}v}\varphi(x)$ is linear and continuous.

Thus for all $x \in H$ there exists a vector that we denote by $D_Q\varphi(x)$ such that

$$D_{Q^{1/2}v}\varphi(x) = \langle D_Q\varphi(x), v\rangle, \quad \forall\, v \in H.$$

(iii) The mapping

$$H \mapsto H, \quad x \mapsto D_Q\varphi(x)$$

is uniformly continuous and bounded.

We shall denote by $C^1_Q(H)$ the set of all functions in $C_b(H)$ fulfilling (i)–(iii). $C^1_Q(H)$ is a Banach space with the norm

$$\|\varphi\|_{1,Q} = \|\varphi\|_0 + \sup_{x\in H} |D_Q\varphi(x)|.$$

Clearly $C^1_b(H) \subset C^1_Q(H)$ and if $\varphi \in C^1_b(H)$ we have

$$\langle D_Q\varphi(x), v\rangle = \langle D\varphi(x), Q^{1/2}v\rangle, \quad \forall\, v \in H.$$

Let us define now the space $C^2_Q(H)$. Let $\varphi \in C^1_Q(H)$ such that

(i) For all $v_1 \in H$, $\langle D_Q\varphi(\cdot), v_1\rangle$ is differentiable in all directions $Q^{1/2}v_2,\ v_2 \in H$.

(ii) For all $x \in H$ the mapping

$$H \times H \mapsto \mathbb{R}, \quad (v_1, v_2) \mapsto D_{Q^{1/2}v_1}[\langle D_Q\varphi(x), v_2\rangle]$$

is continuous.

Thus for all $x \in H$ there exists a linear bounded operator that we denote by D^2_Q such that

$$D_{Q^{1/2}v_1}[\langle Q^{1/2}D\varphi(x), v_2\rangle] = \langle D^2_Q\varphi(x)v_1, v_2\rangle, \quad \forall\, v_1,\ v_2 \in H.$$

(iii) The mapping $H \mapsto \mathcal{L}(H), \quad x \mapsto D^2_Q\varphi(x)$ is uniformly continuous and bounded.

We shall denote by $C^2_Q(H)$ the set of all functions in $C^1_Q(H)$ fulfilling (i)–(iii). $C^2_Q(H)$ is a Banach space with the norm

$$\|\varphi\|_{2,Q} = \|\varphi\|_{2,Q} + \sup_{x\in H} |D^2_Q\varphi(x)|.$$

Clearly we have $C^2_b(H) \subset C^2_Q(H)$ and if $\varphi \in C^2_b(H)$ we have

$$\langle D^2_Q\varphi(x)h_1, h_2\rangle = \langle D^2\varphi(x)Q^{1/2}h_1, Q^{1/2}h_2\rangle, \quad \forall\, h_1, h_2 \in H.$$

2.2. Spaces of Hölder-continuous functions. Let $\theta \in (0,1)$. We denote by $C^\theta_Q(H)$ the space of all functions $f \in C_b(H)$ such that

$$|f|_{\theta,Q} = \sup_{x,y\in H} \frac{|f(Q^{1/2}x) - f(Q^{1/2}y)|}{|x-y|^\theta} < +\infty.$$

The space $C_Q^\theta(H)$, endowed with the norm

$$\|f\|_{\theta,Q} = \|f\|_0 + |f|_{\theta,Q},$$

is a Banach space.

Moreover, we denote by $C_Q^{1+\theta}(H)$ the space of all functions $f \in C_Q^1(H)$ such that

$$|f|_{1+\theta,Q} := \sup_{x,x' \in H} \frac{\left|D_Q f(Q^{1/2}x) - D_Q f(Q^{1/2}x')\right|}{|x-x'|^\theta} < +\infty.$$

The space $C_Q^{1+\theta}(H)$, endowed with the norm

$$\|f\|_{1+\theta,Q} = \|f\|_{1,Q} + |f|_{1+\theta,Q},$$

is a Banach space.

Finally we denote by $C_Q^{2+\theta}(H)$ the space of all functions $f \in C_Q^2(H)$ such that

$$|f|_{2+\theta,Q} := \sup_{x,x',y \in H, y \neq 0} \frac{\|D_Q^2 f(Q^{1/2}x) - D_Q^2 f(Q^{1/2}x')\|_{\mathcal{L}(H)}}{|x-x'|^\theta} < +\infty.$$

The space $C_Q^{2+\theta}(H)$, endowed with the norm

$$\|f\|_{2+\theta,Q} = \|f\|_{1,Q} + |f|_{2,Q} + |f|_{2+\theta,Q},$$

is a Banach space.

We now recall some useful results from interpolation theory. Let $X, \|\cdot\|_X$ and $Y, \|\cdot\|_Y$ be Banach spaces such that $Y \subset X$ and

$$\|y\|_X \leq c\|y\|_Y\ , \quad \forall y \in Y$$

for some constant $c > 0$.

Let $\theta \in (0,1]$. We denote by $(X,Y)_{\theta,\infty}$ the real interpolation space that consists of all points $x \in X$ such that

$$\|x\|_{(X,Y)_{\theta,\infty}} = \sup_{t>0} \frac{1}{t^\theta}\, K(t,x,X,Y) < +\infty$$

where

$$K(t,x,X,Y) = \inf\big\{\|a\|_X + t\|b\|_Y : x = a+b, a \in X, b \in Y\big\}.$$

It is well known that $(X,Y)_{\theta,\infty}$ is a Banach space with norm $\|\cdot\|_{(X,Y)_{\theta,\infty}}$.

An equivalent definition of the real interpolation spaces can be given in terms of trace spaces following the approach of Lions; see e.g. [1].

A useful class of intermediate spaces between X and Y is the class $J_\alpha(X,Y)$, whose definition we recall next. Let $\alpha \in [0,1]$. A Banach space E such that $Y \subset E \subset X$ is said to be of class $J_\alpha(X,Y)$ if there exists a constant $C > 0$ such that

$$\|x\|_E \leq C\|x\|_X^{1-\alpha}\|x\|_Y^\alpha, \ \ \forall\, x \in Y. \tag{2.1}$$

Let now E_0, E_1 be Banach spaces such that $Y \subset E_1 \subset E_0 \subset X$ and

$$E_i \in J_{\alpha_i}(X,Y), \quad \alpha_i \in [0,1], \quad i = 0,1.$$

Let $\alpha \in (0,1)$ and define $\theta_\alpha = (1-\alpha)\alpha_0 + \alpha\alpha_1$. Then, the Reiteration Theorem guarantees that

$$(X,Y)_{\theta_\alpha,\infty} \subset (E_0,E_1)_{\alpha,\infty}; \tag{2.2}$$

see [13]. The following result is proved in Appendix B.

Proposition 2.1. *Let $\theta \in (0,1)$. Then*

$$(C_b(H), C^1_Q(H))_{\theta,\infty} = C^\theta_Q(H).$$

We conclude this section recalling some interpolatory inequalities that will be used later on. The proof is standard and is left to the reader.

Proposition 2.2. *Let $\theta \in (0,1)$. Then for all $u \in C^{2+\theta}_Q(H)$ we have*

$$\|u\|_{2,Q} \le C_\theta \|u\|^{\frac{2}{2+\theta}}_{2+\theta,Q}\ \|u\|_0^{\frac{\theta}{2+\theta}}, \tag{2.3}$$

and

$$\|u\|_{1+\theta,Q} \le C^1_\theta \|u\|^{\frac{1}{2+\theta}}_{2+\theta,Q}\ \|u\|_0^{\frac{1+\theta}{2+\theta}}. \tag{2.4}$$

3. The heat semigroup P_t. In this section we consider the semigroup approach to the linear problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\,\mathrm{Tr}\,[QD^2_x u(t,x)]\,,\ t\ge 0,\ x\in H \\ u(0,x) = \varphi(x), \end{cases} \tag{3.1}$$

where $\varphi \in C_b(H)$ and $Q \in \mathcal{L}(H)$ is a positive nuclear operator in H with kernel equal to $\{0\}$, developed in [2]. However, in the present paper we study problem (3.1) in the space $C_b(H)$, whereas in [2] we took φ in the less natural space consisting of the closure of $C^\infty_b(H)$.

Theorem 3.1. *For any $n \in \mathbb{N}$ and $\varphi \in C_b(H)$ let*

$$P^n_t \varphi = \prod_{k=1}^n e^{\frac{t}{2}\lambda_k D^2_k}\varphi\,,\quad t\ge 0, \tag{3.2}$$

where $e^{tD^2_k}$ is the strongly continuous semigroup defined by

$$e^{tD^2_k}\varphi(x) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{2t}}\varphi(y)\,dy,\quad \varphi\in C_b(H). \tag{3.3}$$

Then

$$P^n_t\varphi(x) = \int_{\mathbb{R}^n}\varphi\big(x-\Sigma_n\xi\big)\rho_n(t,\xi)\,d\xi = \int_{\mathbb{R}^n}\varphi\big(\Sigma_n\eta + R_n x\big)\rho_n(t,\Pi_n x-\eta)\,d\eta \tag{3.4}$$

where $\rho_n(t,\xi)$ denotes the n-dimensional heat potential

$$\rho_n(t,\xi) = \frac{1}{\sqrt{(2\pi t)^n\lambda_1\cdots\lambda_n}}\,e^{-\sum_{k=1}^n\frac{\xi_k^2}{2t\lambda_k}}\,,\ t>0\,,\ \xi\in\mathbb{R}^n. \tag{3.5}$$

Moreover, the limit

$$P_t\varphi := \lim_{n\to\infty} P_t^n\varphi \tag{3.6}$$

exists in $C_b(H)$, uniformly for t in the bounded subsets of $[0,+\infty)$. Furthermore, P_t is a strongly continuous semigroup of contractions in $C_b(H)$.

Proof. First we note that formula (3.4) is well known. It follows immediately from (3.3) and (3.2).

Next, let us show the existence of the limit in (3.6) for functions $\varphi \in C_b^{1,1}(H)$. Since $e^{tD_k^2}$ is a contraction semigroup for any $k \in \mathbb{N}$, we have

$$\left\|P_t^n\varphi - P_t^{n-1}\varphi\right\|_0 \le \left\|e^{\frac{t}{2}\lambda_n D_n^2}\varphi - \varphi\right\|_0. \tag{3.7}$$

Moreover, recalling (3.3), for any $x \in H$ we have

$$\left|e^{\frac{t}{2}\lambda_n D_n^2}\varphi(x) - \varphi(x)\right| = \frac{1}{\sqrt{2\pi t\lambda_n}}\Big|\int_{-\infty}^{+\infty}\left[\varphi\big(x-\xi_n e_n\big) - \varphi(x)\right]e^{-\frac{\xi_n^2}{2t\lambda_n}}\,d\xi_n\Big|$$

$$= \frac{1}{\sqrt{2\pi t\lambda_n}}\Big|\int_{-\infty}^{+\infty}\xi_n e^{-\frac{\xi_n^2}{2t\lambda_n}}\,d\xi_n\int_0^1 D_n\varphi\big(x - s\xi_n e_n\big)\,ds\Big|.$$

Since

$$\int_{-\infty}^{+\infty}\xi_n e^{-\frac{\xi_n^2}{2t\lambda_n}}D_n\varphi\big(x\big)\,d\xi_n = 0,$$

we can subtract this term from the right-hand side of the above equality to obtain

$$\left|e^{\frac{t}{2}\lambda_n D_n^2}\varphi(x) - \varphi(x)\right| \le \frac{|\varphi|_{1,1}}{\sqrt{2\pi t\lambda_n}}\int_{-\infty}^{+\infty}\xi_n^2 e^{-\frac{\xi_n^2}{2t\lambda_n}}\,d\xi_n = t\lambda_n|\varphi|_{1,1}.$$

The last inequality and (3.7) yield

$$\left\|P_t^{n+p}\varphi - P_t^n\varphi\right\|_0 \le t|\varphi|_{1,1}\sum_{k=n+1}^{n+p}\lambda_k$$

for all $n, p \ge 1$. Therefore, $\{P_t^n\varphi\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C_b(H)$, uniformly for t in the bounded sets of $[0,+\infty)$. Hence, the limit (3.2) exists for all $\varphi \in C_b^{1,1}(H)$.

Finally, to show that the limit exists for all $\varphi \in C_b(H)$, it suffices to note that $C_b^{1,1}(H)$ is dense in $C_b(H)$ as we recalled in Section 2. The proof of the remaining part of the conclusion is routine. □

Remark 3.1. It is easy to see that P_t commutes with all derivatives $D_k, k \in \mathbb{N}$, in the sense that, if $D_k\varphi \in C_b(H)$, then $D_k(P_t\varphi) \in C_b(H)$, and

$$D_k(P_t\varphi) = P_t(D_k\varphi)\ , \quad t \ge 0. \tag{3.8}$$

Indeed, if $D_k\varphi \in C_b(H)$, then by differentiating the integral in (3.4) we obtain

$$D_k(P_t^n\varphi) = P_t^n(D_k\varphi)\ , \ \ t \ge 0$$

for any $n \in \mathbb{N}$. Since D_k is closed, (3.8) follows taking the limit as $n \to \infty$ in the equation above.

We now proceed to derive some smoothing properties of the infinite-dimensional heat semigroup constructed above. We begin with a first-order differentiability result.

Theorem 3.2. *Let $\varphi \in C_b(H)$. Then $P_t\varphi \in C^1_Q(H)$ for any $t > 0$, and*

$$\|D_Q(P_t\varphi)\|_0 \le \frac{1}{\sqrt{t}}\|\varphi\|_0. \tag{3.9}$$

Proof. We will work out all details of the present proof, so that in the analogous proofs of the next propositions we can omit similar details.

First, let us prove that $D_h(P_t\varphi) \in C_b(H)$ for any $h \in \mathbb{N}$ and $t > 0$. Let $h \in \mathbb{N}$ be fixed and define, for $n \ge h$,

$$P_t^{n,(h)}\varphi = \prod_{h\neq k=1}^{n} e^{\frac{t}{2}\lambda_k D_k^2}\varphi\,, \quad t \ge 0,$$

for any $\varphi \in C_b(H)$. Then, by the same argument used in the proof of Theorem 3.1, we conclude that the limit

$$P_t^{(h)}\varphi = \lim_{n\to\infty} P_t^{n,(h)}\varphi$$

exists in $C_b(H)$, uniformly for t in the bounded subsets of $[0, +\infty)$. Then, $P_t^{(h)}$ is a strongly continuous semigroup of contractions in $C_b(H)$. Moreover, it is easy to check that

$$P_t^n\varphi = P_t^{n,(h)} e^{\frac{t}{2}\lambda_h D_h^2}\varphi, \quad \varphi \in C_b(H).$$

Now, by a well-known smoothing property of the heat semigroup that can be easily derived from (3.3), we have

$$D_h\big(e^{\frac{t}{2}\lambda_h D_h^2}\varphi\big) \in C_b(H)$$

for any $t > 0$. Therefore, by Remark 3.1, which applies to $P_t^{n,(h)}$ just as it does to P_t^n, it follows that

$$D_h(P_t^n\varphi) = P_t^{n,(h)} D_h\big(e^{\frac{t}{2}\lambda_h D_h^2}\varphi\big), \quad \varphi \in C_b(H)$$

for any $t > 0$. Taking the limit as $n \to \infty$ in the equation above, we conclude that

$$D_h(P_t\varphi) = \lim_{n\to\infty} D_h(P_t^n\varphi) = P_t^{(h)} D_h\big(e^{\frac{t}{2}\lambda_h D_h^2}\varphi\big) \in C_b(H) \tag{3.10}$$

for any $t > 0$.

Next, we proceed to prove (3.9). Let $n, m \in \mathbb{N}, n \ge m$ and $y \in H$. Differentiating the convolution product in (3.4), we have

$$\sum_{h=1}^{m} \sqrt{\lambda_h}\, y_h\, D_h(P_t^n\varphi)(x) = \int_{\mathbb{R}^n} \sum_{h=1}^{m} \frac{x_h - \xi_h}{t\sqrt{\lambda_h}}\, y_h \varphi\big(\Sigma_n\xi + R_n x\big)\rho_n(t, \Pi_n x - \xi)\, d\xi \tag{3.11}$$

for any $t > 0$. Since $\int_{\mathbb{R}^n} \rho_n(t,\xi)\,d\xi = 1$, by Hölder's inequality it follows that

$$\Big|\sum_{h=1}^{m} \sqrt{\lambda_h} y_h D_h(P_t^n\varphi)(x)\Big|^2 \le \|\varphi\|_0^2 \int_{\mathbb{R}^n} \Big(\sum_{h=1}^{m} \frac{x_h - \xi_h}{t\sqrt{\lambda_h}} y_h\Big)^2 \rho_n(t, \Pi_n x - \xi)\,d\xi$$
$$= \frac{\|\varphi\|_0^2}{t} \int_{\mathbb{R}^n} \Big(\sum_{h=1}^{m} \frac{\eta_h y_h}{\sqrt{\lambda_h}}\Big)^2 \rho_n(1,\eta)\,d\eta,$$

where we have used the change of variable $\Pi_n x - \xi = \eta\sqrt{t}$. Evaluating the last integral

$$\int_{\mathbb{R}^n} \Big(\sum_{h=1}^{m} \frac{\eta_h y_h}{\sqrt{\lambda_h}}\Big)^2 \rho_n(1,\eta)\,d\eta = \int_{\mathbb{R}^n} \sum_{h=1}^{m} \frac{\eta_h^2 y_h^2}{\lambda_h} \rho_n(1,\eta)\,d\eta = \sum_{h=1}^{m} y_h^2 \tag{3.12}$$

we obtain

$$\Big|\sum_{h=1}^{m} \sqrt{\lambda_h} y_h D_h(P_t^n\varphi)(x)\Big|^2 \le \frac{1}{t}\|\varphi\|_0^2 |y|^2.$$

Since y is arbitrary,

$$\sum_{h=1}^{m} \lambda_h \big|D_h(P_t^n\varphi)(x)\big|^2 \le \frac{1}{t}\|\varphi\|_0^2$$

for any $x \in H$. Hence, taking the limit as $n \to \infty$ and recalling (3.10), we finally get

$$\sum_{h=1}^{m} \lambda_h \big|D_h(P_t\varphi)(x)\big|^2 \le \frac{1}{t}\|\varphi\|_0^2, \quad \forall x \in H, \quad \forall m \in \mathbb{N}.$$

Now, (3.9) follows as $m \to \infty$.

Finally we show that $D_Q P_t\varphi \in C_b(H,H)$. Indeed, by (3.11) and the Hölder inequality we obtain, for any $x, x' \in H$ and $n \in \mathbb{N}$,

$$\Big|\sum_{h=1}^{m} \sqrt{\lambda_h} y_h [D_h(P_t^n\varphi)(x) - D_h(P_t^n\varphi)(x')]\Big|^2 \le \frac{\omega_\varphi^2(|x-x'|)}{t} \int_{\mathbb{R}^n} \Big(\sum_{h=1}^{m} \frac{\eta_h y_h}{\sqrt{\lambda_h}}\Big)^2 \rho_n(1,\eta)\,d\eta,$$

where ω_φ denotes the continuity modulus of φ. Arguing as before we conclude that

$$|D_Q P_t\varphi(x) - D_Q P_t\varphi(x')| \le \frac{\omega_\varphi(|x-x'|)}{\sqrt{t}}.$$

We now give a similar estimate for second-order derivatives.

Theorem 3.3. *Let $\varphi \in C_b(H)$. Then $P_t\varphi \in C_Q^2(H)$ for all $t > 0$, and*

$$|P_t\varphi|_{2,Q} \le \frac{\sqrt{2}}{t}\|\varphi\|_0. \tag{3.13}$$

Moreover, if $\varphi \in C^1_Q(H)$, *then*

$$|P_t\varphi|_{2,Q} \leq \frac{1}{\sqrt{t}}\|\varphi\|_{1,Q} \tag{3.14}$$

for any $t > 0$. *Furthermore, if* $\varphi \in C^{1,1}_b(H)$, *then*

$$\sup_{h,k\geq 1} \|D_hD_k(P_t\varphi)\|_0 \leq |\varphi|_{1,1} \tag{3.15}$$

for any $t > 0$.

Proof. Arguing as in the first part of the proof of Theorem (3.2), it follows that all second-order partial derivatives $D_hD_k(P_t\varphi) \in C_b(H)$, $h, k \in \mathbb{N}$ for any $t > 0$, and

$$D_hD_k(P_t\varphi) = \lim_{n\to\infty} D_hD_k(P^n_t\varphi). \tag{3.16}$$

Let us then derive (3.13). Differentiating the convolution integral (3.4) with respect to h, k, we obtain

$$\begin{aligned} D_hD_k(P^n_t\varphi)(x) &= \frac{1}{t}\int_{\mathbb{R}^n}\Big[\frac{(x_h-\xi_h)(x_k-\xi_k)}{t\lambda_h\lambda_k} - \frac{\delta_{hk}}{\lambda_k}\Big]\varphi\big(\Sigma_n\xi + R_nx\big)\rho_n(t,\Pi_nx-\xi)\,d\xi \\ &= \frac{1}{t}\int_{\mathbb{R}^n}\Big[\frac{\eta_h\eta_k}{\lambda_h\lambda_k} - \frac{\delta_{hk}}{\lambda_k}\Big]\varphi\big(x-\sqrt{t}\Sigma_n\eta\big)\rho_n(1,\eta)\,d\eta, \end{aligned} \tag{3.17}$$

where δ_{hk} denotes the Kronecker symbol. Notice that here we have used the change of variable $\Pi_nx - \xi = \eta\sqrt{t}$. Since $\int_{\mathbb{R}^n}\rho_n(1,\eta)\,d\eta = 1$, the above identity and Hölder's inequality yield

$$\begin{aligned} &\Big[\sum_{h,k=1}^m \sqrt{\lambda_h\lambda_k}D_hD_k(P^n_t\varphi)(x)y_hy_k\Big]^2 \\ &\quad= \frac{1}{t^2}\Big\{\int_{\mathbb{R}^n}\Big[\Big(\sum_{k=1}^m\frac{\eta_ky_k}{\sqrt{\lambda_k}}\Big)^2 - \sum_{k=1}^m y_k^2\Big]\varphi\big(x-\sqrt{t}\Sigma_n\eta\big)\rho_n(1,\eta)\,d\eta\Big\}^2 \\ &\quad\leq \frac{\|\varphi\|_0^2}{t^2}\int_{\mathbb{R}^n}\Big[\Big(\sum_{k=1}^m\frac{\eta_ky_k}{\sqrt{\lambda_k}}\Big)^2 - \sum_{k=1}^m y_k^2\Big]^2\rho_n(1,\eta)\,d\eta \end{aligned} \tag{3.18}$$

for any $n, m \in \mathbb{N}$, $n \geq m$, $t > 0$, and any $x, y \in H$. Now, the rightmost term in (3.18) can be estimated using the identities (3.12) and

$$\int_{\mathbb{R}^n}\Big(\sum_{k=1}^m\frac{\eta_ky_k}{\sqrt{\lambda_k}}\Big)^4\rho_n(1,\eta)\,d\eta = \frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m}\Big(\sum_{k=1}^m\zeta_ky_k\Big)^4e^{-\frac{1}{2}\sum_{j=1}^m\zeta_j^2}\,d\zeta = 3\Big(\sum_{k=1}^m y_k^2\Big)^2, \tag{3.19}$$

that can be shown by elementary computations. We then obtain

$$\Big[\sum_{h,k=1}^{m}\sqrt{\lambda_h\lambda_k}D_hD_k(P_t^n\varphi)(x)y_hy_k\Big]^2\le\frac{2\|\varphi\|_0^2}{t^2}\Big(\sum_{k=1}^{m}y_k^2\Big)^2.$$

Hence, passing to the limit as $n\to\infty$ and recalling (3.16), we deduce

$$\Big|\sum_{h,k=1}^{m}\sqrt{\lambda_h\lambda_k}D_hD_k(P_t\varphi)(x)y_hy_k\Big|\le\frac{\sqrt{2}\|\varphi\|_0}{t}\sum_{k=1}^{m}y_k^2,\quad\forall x,y\in H$$

and then (3.13), as $m\to\infty$.

To show that $P_t\varphi\in C_Q^2(H)$ it remains to check that $D_hD_kP_t\varphi\in C_b(H)$, $\forall h,k\in\mathbb{N}$. By (3.17) we have indeed

$$\begin{aligned}&D_hD_k(P_t^n\varphi)(x)-D_hD_k(P_t^n\varphi)(x')\\&\quad=\frac{1}{t}\int_{\mathbb{R}^n}\Big[\frac{\eta_h\eta_k}{\lambda_h\lambda_k}-\frac{\delta_{hk}}{\lambda_k}\Big]\big[\varphi\big(x-\sqrt{t}\Sigma_n\eta\big)-\varphi\big(x'-\sqrt{t}\Sigma_n\eta\big)\big]\rho_n(1,\eta)\,d\eta.\end{aligned}$$

Now the conclusion follows from the uniform continuity of φ.

The proof of (3.14) is similar. Since $\varphi\in C_Q^1(H)$, we can replace (3.17) by the following:

$$D_hD_k(P_t^n\varphi)(x)=-\int_{\mathbb{R}^n}\frac{x_h-\xi_h}{t\lambda_h}D_k\varphi\big(\Sigma_n\xi+R_nx\big)\rho_n(t,\Pi_nx-\xi)\,d\xi.\tag{3.20}$$

Therefore, by the change of variable $\Pi_nx-\xi=\eta\sqrt{t}$ and Hölder's inequality,

$$\begin{aligned}&\Big[\sum_{h,k=1}^{m}\sqrt{\lambda_h\lambda_k}D_hD_k(P_t^n\varphi)(x)y_hy_k\Big]^2\\&\times\Big[\int_{\mathbb{R}^n}\sum_{h,k=1}^{m}\sqrt{\lambda_k}y_hy_k\frac{x_h-\xi_h}{t\sqrt{\lambda_h}}D_k\varphi\big(\Sigma_n\xi+R_nx\big)\rho_n(t,\Pi_nx-\xi)\,d\xi\Big]^2\\&\le\frac{1}{t}\int_{\mathbb{R}^n}\Big(\sum_{h=1}^{m}\frac{\eta_hy_h}{\sqrt{\lambda_h}}\Big)^2\rho_n(1,\eta)\,d\eta\int_{\mathbb{R}^n}\Big[\sum_{k=1}^{m}\sqrt{\lambda_k}D_k\varphi\big(x-\sqrt{t}\Sigma_n\eta\big)y_h\Big]^2\rho_n(1,\eta)\,d\eta\end{aligned}$$

for any $n,m\in\mathbb{N}$, $n\ge m$, $t>0$, and any $x,y\in H$. Now, recalling the identity (3.12), we get

$$\Big[\sum_{h,k=1}^{m}\sqrt{\lambda_h\lambda_k}D_hD_k(P_t^n\varphi)(x)y_hy_k\Big]^2\le\frac{1}{t}\|Q^{1/2}D\varphi\|_0^2\Big(\sum_{k=1}^{m}y_k^2\Big)^2$$

and (3.14) follows as $n,m\to\infty$.

Finally, to prove (3.15) we note that

$$D_hD_k(P_t^n\varphi)(x)=\lim_{s\to0}\int_{\mathbb{R}^n}\frac{1}{s}[D_k\varphi(x-\Sigma_n\xi+se_h)-D_k\varphi(x-\Sigma_n\xi)]\rho_n(t,\xi)\,d\xi.$$

Therefore,

$$|D_hD_k(P_t^n\varphi)(x)|\le|\varphi|_{1,1}$$

and (3.15) follows recalling (3.16). □

Interpolating between (3.13) and (3.14) it follows:

Corollary 3.1. *Let $\varphi \in C_Q^\theta(H)$ for some $\theta \in (0,1]$. Then, there exists a constant $C_\theta > 0$ such that*

$$|P_t\varphi|_{2,Q} \le C_\theta t^{\frac{\theta}{2}-1}\|\varphi\|_{\theta,Q}, \tag{3.21}$$

for any $t > 0$.

4. The infinitesimal generator of P_t. In (3.6) we have introduced the heat semigroup P_t. In this section we describe some properties of its infinitesimal generator

$$\mathcal{A} : D(\mathcal{A}) \subset C_b(H) \mapsto C_b(H).$$

As is well known, the domain of $\mathcal{A}$, $D(\mathcal{A})$, is the subspace of $C_b(H)$ consisting of all functions $\varphi \in C_b(H)$ such that the limit

$$\lim_{t\downarrow 0} \frac{P_t\varphi - \varphi}{t} \tag{4.1}$$

exists in $C_b(H)$. Moreover $\mathcal{A}\varphi$ is equal to the above limit for any $\varphi \in D(\mathcal{A})$.

We would like to relate the operator $\mathcal{A}$ to the infinite-dimensional heat equation (3.1). For this purpose we introduce the following operator $\mathcal{A}_0$.

Definition 4.1. We define $D(\mathcal{A}_0)$ to be the subspace of $C_b(H)$ that consists of all functions $f \in C_b^{1,1}(H)$ such that $D_hD_kf \in C_b(H)$ for any $h, k \ge 1$, and

$$|f|_{D(\mathcal{A}_0)} := \sup_{h,k\ge 1} \|D_hD_kf\|_0 < \infty.$$

Moreover, we define

$$\mathcal{A}_0 f = \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k D_k^2 f = \frac{1}{2}\mathrm{Tr}[QD^2f], \quad \forall f \in D(\mathcal{A}_0).$$

Since the above series is totally convergent in $C_b(H)$, it follows that

$$\mathcal{A}_0 : D(\mathcal{A}_0) \subset C_b(H) \to C_b(H).$$

Also, $D(\mathcal{A}_0)$ is a Banach space with norm

$$\|f\|_{D(\mathcal{A}_0)} = \|f\|_{1,1} + |f|_{D(\mathcal{A}_0)}.$$

Furthermore, we have the following properties, that will be used in the sequel.

Lemma 4.1. *The space $D(\mathcal{A}_0)$ is dense in $C_b(H)$. Moreover, $D(\mathcal{A}_0)$ is invariant under $P_t, t \ge 0$, and*

$$\|P_tf\|_{D(\mathcal{A}_0)} \le \|f\|_{D(\mathcal{A}_0)}, \quad \forall f \in D(\mathcal{A}_0). \tag{4.2}$$

Proof. Let $f \in C_b(H)$. Since $C_b^{1,1}(H)$ is dense in $C_b(H)$, for any $\varepsilon > 0$ there exists $f_\varepsilon \in C_b^{1,1}(H)$ such that $\|f - f_\varepsilon\|_0 < \varepsilon$. Moreover, by (3.15) we conclude that $P_tf_\varepsilon \in D(\mathcal{A}_0)$ for any $t > 0$. Therefore,

$$\|f - P_tf_\varepsilon\|_0 < \varepsilon + \|f_\varepsilon - P_tf_\varepsilon\|_0 < 2\varepsilon$$

for a sufficiently small t, in view of the strong continuity of P_t. The rest of the proof can be easily verified. $\square$

The following characterization of $\mathcal{A}$ may be useful to clarify the connection of P_t with (3.1).

Theorem 4.1. *The operator $\mathcal{A}$ is the closure of $\mathcal{A}_0$.*

Proof. We argue as in the proof of Theorem 3.1 of [2]. First, we claim that $\mathcal{A}$ extends $\mathcal{A}_0$; i.e.,

$$D(\mathcal{A}_0) \subset D(\mathcal{A}) \quad \text{and} \quad \mathcal{A}\psi = \mathcal{A}_0\psi, \quad \forall \psi \in D(\mathcal{A}_0). \tag{4.3}$$

Indeed, let $\psi \in D(\mathcal{A}_0)$ and set

$$\mathcal{A}_0^n \psi = \frac{1}{2} \sum_{k=1}^{n} \lambda_k D_k^2 \psi.$$

Then, $\mathcal{A}_0^n \psi \to \mathcal{A}_0 \psi$ in $C_b(H)$ as $n \to \infty$. Moreover, differentiating (3.4) with respect to t we obtain

$$\frac{d}{dt}(P_t^n \psi) = \frac{1}{2} \sum_{k=1}^{n} \lambda_k D_k^2 (P_t^n \psi) = P_t^n(\mathcal{A}_0^n \psi).$$

Since

$$\begin{aligned} \|P_t^n(\mathcal{A}_0^n \psi) - P_t(\mathcal{A}_0 \psi)\|_0 &\le \|P_t^n(\mathcal{A}_0^n \psi) - P_t^n(\mathcal{A}_0 \psi)\|_0 + \|P_t^n(\mathcal{A}_0 \psi) - P_t(\mathcal{A}_0 \psi)\|_0 \\ &\le \|\mathcal{A}_0^n \psi - \mathcal{A}_0 \psi\|_0 + \|(P_t^n - P_t)\mathcal{A}_0 \psi\|_0 \to 0 \end{aligned}$$

as $n \to \infty$, we conclude that

$$\frac{d}{dt}(P_t \psi) = P_t(\mathcal{A}_0 \psi)$$

and the claim (4.3) follows.

Let now $\varphi \in D(\mathcal{A})$ and $\mu > 0$. Define

$$f = \mu\varphi - \mathcal{A}\varphi \in C_b(H).$$

By Lemma 4.1 there exists a sequence $\{f_k\} \subset D(\mathcal{A}_0)$ which converges to f in $C_b(H)$. If we set $\varphi_k = (\mu - \mathcal{A})^{-1} f_k$, then

$$\lim_{k \to \infty} \varphi_k = \varphi \quad \text{in } C_b(H).$$

To prove the conclusion it remains to show that

$$\lim_{k \to \infty} \mathcal{A}_0 \varphi_k = \mathcal{A}\varphi.$$

By the Hille-Yosida Theorem we have that

$$\varphi_k = \int_0^\infty e^{-\mu t} P_t f_k \, dt. \tag{4.4}$$

Notice that $P_t\big(D(\mathcal{A}_0)\big) \subset D(\mathcal{A}_0)$ for any $t \ge 0$, as is easy to check. Hence, (4.4) implies that $\varphi_k \in D(\mathcal{A}_0)$. So, recalling (4.3),

$$\mu\varphi_k - \mathcal{A}_0 \varphi_k = f_k.$$

Therefore,

$$\lim_{k\to\infty} \mathcal{A}_0\varphi_k = \lim_{k\to\infty} (\mu\varphi_k - f_k) = \mu\varphi - f = \mathcal{A}\varphi$$

and the proof is complete.

5. Schauder estimates. In this section we prove an infinite-dimensional analogue of the classical Schauder estimates for solutions of the equation

$$\mu u(x) - \frac{1}{2}\,\mathrm{Tr}\,(QD^2u(x)) = f(x), \quad x \in H. \tag{5.1}$$

More precisely, we study the problem

$$\mu u(x) - \mathcal{A}u(x) = f(x), \quad x \in H. \tag{5.2}$$

Here $\mu > 0, f \in C_b(H)$ and $\mathcal{A}$ is the infinitesimal generator of the heat semigroup P_t, that we have studied in the previous section.

Since P_t is a contraction semigroup in $C_b(H)$, for any $f \in C_b(H)$ equation (5.2) has a unique solution $u \in D(\mathcal{A})$ that can be represented, in view of the Hille-Yosida Theorem, as

$$u = \int_0^\infty e^{-\mu t} P_t f\, dt. \tag{5.3}$$

In view of Lemma 4.1, from the representation formula above we conclude that, if $f \in D(\mathcal{A}_0)$, then $u \in D(\mathcal{A}_0)$. Therefore, equation (5.1) holds for such u.

In order to prove Schauder estimates on the solution of (5.2) we need the interpolation spaces

$$D_{\mathcal{A}}(\theta,\infty) := \big(C_b(H), D(\mathcal{A})\big)_{\theta,\infty}, \quad 0 < \theta < 1,$$

that can be defined in terms of the semigroup P_t generated by $\mathcal{A}$; see [8]. More precisely, $f \in D_{\mathcal{A}}(\theta,\infty)$ if and only if

$$\|f\|_{\theta,\mathcal{A}} = \sup_{t\in(0,1]} \frac{\|P_t f - f\|_0}{t^\theta} < +\infty. \tag{5.4}$$

We will prove a maximal regularity result in the space $C^\theta_Q(H), \theta \in (0, 1/2)$ for the solution of (5.2). We begin with an interpolatory estimate.

Lemma 5.1. *For any $u \in D(\mathcal{A})$ we have that $u \in C^1_Q(H)$ and*

$$\|D_Q u\|_0 \le 2\sqrt{\pi}\|u\|_0^{1/2}\|\mathcal{A}u\|_0^{1/2}. \tag{5.5}$$

Proof. Let $u \in D(\mathcal{A})$, let $\mu > 0$ and set $f = \mu u - \mathcal{A}u$. Recalling (5.3), we have that $P_t f \in C^1_Q(H)$ for any $t > 0$ by Theorem 3.2. Moreover, D_Q is clearly a closed operator in $C_b(H)$. Therefore (3.9) yields $u \in C^1_Q(H)$ and

$$\begin{aligned}\|D_Q u\|_0 &= \Big\|\int_0^\infty e^{-\mu t} D_Q(P_t f)\, dt\Big\|_0 \le \|f\|_0 \int_0^\infty \frac{e^{-\mu t}}{\sqrt{t}}\, dt = \sqrt{\frac{\pi}{\mu}}\|f\|_0 \\ &\le \sqrt{\pi}\big(\sqrt{\mu}\|u\|_0 + \frac{\|\mathcal{A}u\|_0}{\sqrt{\mu}}\big).\end{aligned} \tag{5.6}$$

We claim that the above estimate completes the proof. Indeed, (5.5) is trivial if $\mathcal{A}u = 0$, and follows from (5.6) otherwise, taking $\mu = \frac{\|\mathcal{A}u\|_0}{\|u\|_0}$. $\square$

In particular, Lemma 5.1 implies that $C_Q^1(H) \in J_{1/2}(C_b(H), D(\mathcal{A}))$; see (2.1). Now the Reiteration Theorem and Proposition 2.1 yield the following.

Corollary 5.1. *For any* $\theta \in (0,1)$

$$D_{\mathcal{A}}(\theta/2,\infty) \subset (C_b(H), C_Q^1(H))_{\theta,\infty} = C_Q^\theta(H).$$

We now prove the main result.

Theorem 5.1. *Let* $f \in C_Q^\theta(H), \theta \in (0,1)$. *Then the solution* u *of* (5.2) *belongs to* $C_Q^{2+\theta}(H)$, *and*

$$\mu\|u\|_{\theta,Q} + \|u\|_{2+\theta,Q} + \|\mathcal{A}u\|_{\theta,Q} \le C\|f\|_{\theta,Q}, \tag{5.7}$$

where C *depends only on* μ *and* Q.

Proof. Since u is given by (5.3), from (3.21) it follows that, for any $x \in H$,

$$\begin{aligned}|\langle D_Q^2 u(x)y, y\rangle| &\le \Big\|\int_0^\infty e^{-\mu t}\langle D_Q^2 y, y\rangle\, dt\Big\|_0 \\ &\le C_\theta |y|^2\|f\|_{\theta,Q}\int_0^\infty \frac{e^{-\mu t}}{t^{1-\theta/2}}\, dt \le C_{\mu,\theta}|y|^2\|f\|_{\theta,Q}.\end{aligned}$$

We have thus proved that u belongs to $C_Q^2(H)$.

Now, to show the last part of the conclusion, let

$$v(x) = \langle D_Q^2 u(x)y, y\rangle, \quad x \in H.$$

We will estimate the norm $\|v\|_{\theta/2,\mathcal{A}}$, defined in (5.4). We have for all $s > 0$,

$$P_s v - v = \int_0^\infty \big(e^{-\mu(t-s)} - e^{-\mu t}\big)\langle D_Q^2(P_t f)y, y\rangle\, dt - \int_0^s e^{-\mu(t-s)}\langle D_Q^2(P_t f)y, y\rangle\, dt.$$

Therefore, recalling (3.21), we obtain

$$\begin{aligned}\|P_s v - v\|_0 &\le \big(e^{\mu s} - 1\big)\Big\|\int_0^\infty e^{-\mu t}\langle D_Q^2(P_t f)y, y\rangle\, dt\Big\|_0 \\ &\quad + e^{\mu s}\Big\|\int_0^s e^{-\mu t}\langle D_Q^2(P_t f)y, y\rangle\, dt\Big\|_0 \le C(\mu,\theta)|y|^2 s^{\theta/2}\|f\|_{\theta,Q}.\end{aligned}$$

So we have proved that v belongs to $D_{\mathcal{A}}(\theta/2,\infty)$, and so $u \in C_Q^{2+\theta}(H)$ by Corollary 5.1. The proof is thus complete. $\square$

We will now extend the previous result to elliptic equations of the form

$$\mu u(x) - \frac{1}{2}\,\mathrm{Tr}\,[(I+F)D_Q^2 u(x)] = f(x), \quad x \in H, \tag{5.8}$$

where μ is a fixed positive number, and $F \in \mathcal{L}_1^+(H)$.

Equation (5.8) can be rewritten as

$$\mu u(x) - \mathcal{A}u(x) - \frac{1}{2}\,\mathrm{Tr}\,[F D_Q^2 u(x)] = f(x), \quad x \in H. \tag{5.9}$$

Proposition 5.1. *Let* $F \in \mathcal{L}_1^+(H)$ *and set* $S = Q^{1/2}(1+F)Q^{1/2}$. *Then*

$$S^{1/2}(H) = Q^{1/2}(H) \tag{5.10}$$

and the linear operators $S^{1/2}Q^{-1/2}$, $Q^{1/2}S^{-1/2}$, $Q^{-1/2}S^{1/2}$ *and* $S^{-1/2}Q^{1/2}$ *are bounded in* H.

Proof. We first remark that since

$$\langle Sx, x\rangle = \langle (1+F)Q^{1/2}x, Q^{1/2}x\rangle, \quad x \in H,$$

and $F \geq 0$, we have

$$|Q^{1/2}x|^2 \leq |S^{1/2}x|^2 \leq \|1+F\|_{\mathcal{L}(H)}|Q^{1/2}x|^2, \quad x \in H.$$

Hence, by a well-known result, see e.g. [3, Proposition B.1], this implies obviously

$$Q^{1/2}(H) \subset S^{1/2}(H) \subset Q^{1/2}(H).$$

Finally, by the Closed Graph Theorem, (5.10) implies that all operators $S^{1/2}Q^{-1/2}$, $Q^{1/2}S^{-1/2}$, $Q^{-1/2}S^{1/2}$, $S^{-1/2}Q^{1/2}$ are bounded. □

We can prove now the result

Theorem 5.2. *Let* $F \in \mathcal{L}_1^+(H)$, $f \in C_Q^\theta(H)$ *and let* u *be a solution to equation* (5.8). *Then* $u \in C_Q^{2+\theta}(H)$, *and*

$$\mu\|u\|_{\theta,Q} + \|u\|_{2+\theta,Q} + \|\mathcal{A}u\|_{\theta,Q} \leq C\|f\|_{\theta,Q}, \tag{5.11}$$

where C *depends only on* μ, ε, Q, *and the norm* $\|S^{1/2}Q^{-1/2}\|$, *where* $S = Q^{1/2}(1+F)Q^{1/2}$.

Proof. Since the operator $S = Q^{1/2}(I+F)Q^{1/2}$ belongs to $\mathcal{L}_1^+(H)$, then for all $f \in C_b(H)$, equation (5.8) has a unique solution u given by the formula

$$u = \int_0^{+\infty} e^{-\mu t} P_t^S f \, dt,$$

where P_t^S is the strongly continuous semigroup on $C_b(H)$ associated with the linear operator

$$S = Q^{1/2}(I+F)Q^{1/2}.$$

Moreover, by Theorem 5.1 if $f \in C_S^\theta(H)$ one has $u \in C_S^{2+\theta}(H)$, and

$$\mu\|u\|_{\theta,S} + \|u\|_{2+\theta,S} + \|\mathcal{A}u\|_{\theta,S} \leq C\|f\|_{\theta,S}. \tag{5.12}$$

By Proposition 5.1 we obtain (5.11). In fact we have, for example,

$$\begin{aligned} |u(Q^{1/2}x) - u(Q^{1/2}x')| &= |u(S^{1/2}S^{-1/2}Q^{1/2}x) - u(S^{1/2}S^{-1/2}Q^{1/2}x')| \\ &\leq |u|_{\theta,S}\|S^{-1/2}Q^{1/2}\|^\theta |x-x'|^\theta, \end{aligned}$$

for all $x, x' \in H$. Therefore

$$\frac{|u|_{\theta,Q}}{\|S^{-1/2}Q^{1/2}\|^\theta} \le \|Q^{-1/2}S^{1/2}\|^\theta |u|_{\theta,S} \le |u|_{\theta,Q},$$

and the conclusion follows.

6. Elliptic equations with variable coefficients. We are now ready to study an elliptic equation with variable coefficients of the form

$$\mu u(x) - \frac{1}{2}\,\mathrm{Tr}\,[(I + F(x))D_Q^2 u(x)] = f(x), \quad x \in H. \tag{6.1}$$

Throughout this section we assume that $\mu > 0$ and $\theta \in (0,1)$ are fixed, and that $F : H \mapsto \mathcal{L}_1(H)$ is such that

$$\begin{array}{ll} (i) & F(x) = F^*(x) \ge 0, \quad \forall x \in H \\ (ii) & \|F(x)\|_{\mathcal{L}_1(H)} \le M \\ (iii) & \|F(x) - F(y)\|_{\mathcal{L}_1(H)} \le M|x-y|^\theta \end{array} \tag{6.2}$$

for any $x \in H$ and some constants $M > 0$.

Definition 6.1. Let $f \in C_Q^\theta(H)$. A solution to equation (6.1) is a function $u \in D(\mathcal{A}) \cap C_Q^{2+\theta}(H)$ such that

$$\mu u(x) - \mathcal{A}u(x) - \frac{1}{2}\,\mathrm{Tr}\,[F(x)D_Q^2 u(x)] = f(x), \quad x \in H. \tag{6.3}$$

In order to solve equation (6.3) we shall first prove an a priori estimate, and then apply the classical continuity method.

6.1. A priori estimate. For any $x \in H$ and any $r > 0$ we denote by $\rho_{x,r}$ a function in $C_b^\infty(H)$ such that

$$\rho_{x,r}(z) = \begin{cases} 1 & \text{if } z \in B(x,r) \\ 0 & \text{if } z \in B(x,2r). \end{cases}$$

We need a lemma.

Lemma 6.1. *Let $x \in H$, $r > 0$, and let $u \in D(\mathcal{A})$. Then $\rho_{x,r}u \in D(\mathcal{A})$ and*

$$\mathcal{A}(\rho_{x,r}u) = \rho_{x,r}\mathcal{A}(u) + \frac{1}{2}u\mathrm{Tr}[QD^2\rho_{x,r}] + \langle D_Q u, Q^{1/2}D\rho_{x,r}\rangle. \tag{6.4}$$

Proof. Assume first that $u \in D(\mathcal{A}_0)$. Then it is easy to check that $\rho_{x,r}u \in D(\mathcal{A}_0)$ and

$$\mathcal{A}_0(\rho_{x,r}u) = \rho_{x,r}\mathcal{A}_0(u) + \frac{1}{2}u\ \mathrm{Tr}[QD^2\rho_{x,r}] + \langle D_Q u, Q^{1/2}D\rho_{x,r}\rangle.$$

Now if $u \in D(\mathcal{A})$ there exists a sequence $\{u_n\} \subset D(\mathcal{A}_0)$ such that

$$u_n \to u \quad \text{and} \quad \mathcal{A}_0 u_n \to \mathcal{A}u \text{ in } C_b(H).$$

It follows that

$$\mathcal{A}_0(\rho_{x,r}u_n) = \rho_{x,r}\mathcal{A}_0(u_n) + \frac{1}{2}u_n\mathrm{Tr}[QD^2\rho_{x,r}] + \langle D_Q u_n, Q^{1/2}D\rho_{x,r}\rangle.$$

Now the conclusion follows letting n tend to ∞ and recalling that $D(\mathcal{A}_0)$ is included in $C_b^1(H)$ with continuous embedding, by (5.7). $\square$

Theorem 6.1. *Assume that F fulfills* (6.2). *Let $f \in C^\theta_Q(H)$ and let u be a solution to equation* (6.3). *Then $u \in C^{2+\theta}_Q(H)$ and*

$$\mu\|u\|_{\theta,Q} + \|u\|_{2+\theta,Q} + \|\mathcal{A}u\|_{\theta,Q} \le C\|f\|_{\theta,Q}, \tag{6.5}$$

where C depends only on μ, Q, θ, M.

Proof. Step 1. Localization. In all the proof numbers M, θ and μ are fixed. We shall denote by C_i, $i \in \mathbb{N}$ several constants depending only on M, θ, M_θ and μ. Moreover we shall denote by K_ε and E_r constants depending on $\varepsilon > 0$ and $r > 0$. Numbers ε and r will be chosen later.

Let now $f \in C^\theta_Q(H)$ and let u be the solution to equation (6.3). Fix $x_0 \in H$, $r > 0$ and set $v = \rho_{x_0,r}u = \rho u$. Then, by Lemma 6.1 we have

$$\mu v - \mathcal{A}v - \frac{1}{2}\mathrm{Tr}[F(x_0)D^2_Q v] = f_1 + f_2 + f_3,$$

where

$$f_1(x) = \rho(x)f(x), \quad f_2(x) = \frac{1}{2}\,\mathrm{Tr}\,[(F(x) - F(x_0))D^2_Q v(x)]$$
$$f_3(x) = -\langle Q^{1/2}(I + F(x))D_Q u(x), D_x\rho(x)\rangle - \frac{1}{2}u(x)\mathrm{Tr}[(I + F(x))D^2_Q\rho(x)].$$

In view of Theorem 5.2 we find

$$\begin{aligned}\mu\|v\|_{\theta,Q} + \|v\|_{2+\theta,Q} &\le C(\|f_1\|_{\theta,Q} + \|f_2\|_{\theta,Q} + \|f_3\|_{\theta,Q})\\ &\le C_1(\|f\|_{\theta,Q} + \|f_2\|_{\theta,Q} + \|f_3\|_{\theta,Q}).\end{aligned} \tag{6.6}$$

Step 2. Estimate of $\|f_2\|_\theta$. We have

$$\|f_2\|_0 \le C_1\|D^2_Q v\|_0.$$

Since for all $\alpha, \beta \in C^\theta_Q(H)$

$$|\alpha\beta|_{\theta,Q} \le |\alpha|_{\theta,Q}\|\beta\|_0 + \|\alpha\|_0|\beta|_{\theta,Q},$$

then we have

$$\begin{aligned}|f_2|_{\theta,Q} &\le C_2\big[\sup_{x\in P(x_0,2r)}\|F(x) - F(x_0)\||D^2v|_{\theta,Q} + M\|D^2_Q v\|_0\big]\\ &\le C_3\big[r^\theta|D^2v|_{\theta,Q} + \|D^2v\|_0\big].\end{aligned}$$

It follows that

$$\|f_2\|_{\theta,Q} \le C_4\big[r^\theta|D^2v|_{\theta,Q} + \|D^2_Q v\|_0\big]. \tag{6.7}$$

By the interpolatory estimate (2.3) and (6.7) it follows that

$$\|f_2\|_{\theta,Q} \le C_5\big[r^\theta |D^2 v|_{\theta,Q} + \|v\|_0^{\frac{\theta}{2+\theta}} \|v\|_{2+\theta,Q}^{\frac{2}{2+\theta}}\big] \le C_6\big[(\varepsilon + r^\theta)\|v\|_{2+\theta,Q} + K_\varepsilon \|v\|_0\big].$$

By the Maximum Principle, Theorem A.1, we have

$$\|v\|_0 \le \|u\|_0 \le \frac{1}{\mu}\|f\|_0$$

and so

$$\|f_2\|_{\theta,Q} \le C_7\big[(\varepsilon + r^\theta)\|v\|_{2+\theta,Q} + K_\varepsilon \|f\|_0\big]. \tag{6.8}$$

Step 3. Estimate of $\|f_3\|_{\theta,Q}$. We have

$$\|f_3\|_0 \le C_8\big\{\|f\|_{\theta,Q} + E_r\|u\|_{1+\theta,Q}\big\}. \tag{6.9}$$

By (6.8)–(6.9) it follows that

$$\|f\|_{\theta,Q} \le C_9\big[K_\varepsilon\|f\|_{\theta,Q} + (\varepsilon + r^\theta)\|v\|_{2+\theta,Q} + E_r\|u\|_{1+\theta,Q}\big]. \tag{6.10}$$

By Theorem 5.2 it follows that

$$\|v\|_{2+\theta,Q} \le C_{10}\big[K_\varepsilon\|f\|_{\theta,Q} + (\varepsilon + r^\theta)\|v\|_{2+\theta,Q} + E_r\|u\|_{1+\theta,Q}\big]. \tag{6.11}$$

We choose now ε and r such that

$$C_{10}(\varepsilon + r^\theta) \le \frac{1}{2}. \tag{6.12}$$

From now on ε and r are fixed. Now, from (6.11) it follows that

$$\|v\|_{2+\theta,Q} \le C_{11}\big\{\|f\|_{\theta,Q} + \|u\|_{1+\theta,Q}\big\},$$

which yields

$$\|u\|_{C_Q^{2+\theta}(B(x_0,2r))} \le C_{11}\big\{\|f\|_{\theta,Q} + \|u\|_{1+\theta,Q}\big\}. \tag{6.13}$$

Since x_0 is arbitrary, we get

$$\|u\|_{2+\theta,Q} \le C_{11}\big[\|f\|_{\theta,Q} + \|u\|_{1+\theta,Q}\big]. \tag{6.14}$$

Using now the interpolatory estimate (2.4) we find

$$\|u\|_{2+\theta,Q} \le C_{12}\big[\|f\|_{\theta,Q} + \|u\|_0^{\frac{1}{1+\theta}} \|u\|_{2+\theta,Q}^{\frac{1+\theta}{2+\theta}}\big], \tag{6.15}$$

from which

$$\|u\|_{2+\theta,Q} \le C_{12}\big\{E_\eta\|f\|_{\theta,Q} + \eta\|u\|_{2+\theta,Q}\big\}. \tag{6.16}$$

Choosing finally η such that $\eta C_{11} \le \frac{1}{2}$, we get

$$\|u\|_{2+\theta,Q} \le C_{12}\|f\|_{\theta,Q},$$

as required. □

6.2. Continuity method.

Theorem 6.2. *Assume that F fulfills* (6.2) *and let $f \in C_Q^\theta(H)$. Then there exists a unique solution to equation* (6.3).

Proof. Uniqueness is an immediate consequence of the a priori estimate (6.5). In order to prove existence, let us introduce the set Λ consisting of all $\alpha \in [0,1]$ such that the equation

$$\mu u(x) - \mathcal{A}u(x) - \frac{\alpha}{2}\ \mathrm{Tr}\ [F(x)D_Q^2 u(x)] = f(x), \quad x \in H, \tag{6.17}$$

has a unique solution for all $f \in C_Q^\theta(H)$. In view of Theorem 5.1 we have that $0 \in \Lambda$. We now prove that the set Λ is closed. Let $\{\alpha_n\} \subset \Lambda$ be a sequence convergent to some element α_0, and let $\{u_n\}$ be the solutions to the equations

$$\mu u_n(x) - \mathcal{A}u_n(x) - \frac{\alpha_n}{2}\ \mathrm{Tr}\ [F(x)D_Q^2 u_n(x)] = f(x).$$

We claim that $\{u_n\}$ is a Cauchy sequence in $C_Q^{2+\theta}(H)$. In fact, from the identity

$$\mu(u_n - u_m) - \mathcal{A}(u_n - u_m) - \frac{\alpha_n}{2}\,\mathrm{Tr}\,[F(x)D_Q^2(u_n - u_m)]$$
$$= \frac{(\alpha_n - \alpha_m)}{2}\,\mathrm{Tr}\,[F(x)D_Q^2 u_m],$$

and Theorems 5.1 and 6.1, it follows that there exists $C_1 > 0$ such that

$$\|u_n - u_m\|_{2+\theta} \le C_1|\alpha_n - \alpha_m|.$$

Now, it is easy to check, recalling that $\mathcal{A}$ is a closed operator, that the limit $u_0 \in C_Q^{2+\theta}(H)$ of $\{u_n\}$ is the solution to

$$\mu u_0(x) - \mathcal{A}u_0(x) - \frac{\alpha_0}{2}\ \mathrm{Tr}\ [F(x)D_Q^2 u_0(x)] = f(x),$$

so the set Λ is closed.

It remains to prove that Λ is open in $[0,1]$. Let $\alpha_0 \in \Lambda$, and let u_0 be the corresponding solution to (6.17). We are going to show that equation (6.17) has a solution for any α close to α_0. By virtue of Theorem 6.1 the mapping

$$C_Q^{2+\theta}(H) \mapsto C_Q^{2+\theta}(H), \quad v \mapsto \gamma(v),$$

where $\gamma(v)$ is the solution to

$$\mu u - \mathcal{A}u - \frac{\alpha_0}{2}\ \mathrm{Tr}\ [F(x)D_Q^2 u] = \frac{(\alpha_0 - \alpha)}{2}\ \mathrm{Tr}\ [F(x)D_Q^2 v],$$

is well defined. Note that u is a solution to (6.17) if and only if it is a fixed point of γ. Let $u = \gamma(v)$ and $\overline{u} = \gamma(\overline{v})$. Then by Theorem 6.1 there exists $C_2 > 0$ such that

$$\|u - \overline{u}\|_{2+\theta} \le C_2|\alpha_0 - \alpha|\|v - \overline{v}\|_{2+\theta}.$$

So, γ is a contraction whenever $|\alpha_0 - \alpha|$ is small and the conclusion follows. □

We want now to study equation (6.1), under hypothesis (6.2) in the space $C_b(H)$. To this aim we introduce a linear operator $\mathcal{A}_F$ in $C_b(H)$ by setting

$$D(\mathcal{A}_F) = C_Q^{2+\theta}(H) \cap D(\mathcal{A}), \quad \mathcal{A}_F u(x) = \mathcal{A}u(x) - \frac{1}{2}\ \mathrm{Tr}\ [F(x)D_Q^2 u(x)]. \tag{6.18}$$

By the Maximum Principle, Theorem A.1, A_F is a dissipative operator in $C_b(H)$.

Theorem 6.3. *The closure of the operator $\mathcal{A}_F$ in $C_b(H)$ is m-dissipative in $C_b(H)$.*

Proof. Let $\overline{\mathcal{A}_F}$ be the closure of $\mathcal{A}_F$ in $C_b(H)$. Clearly $\overline{\mathcal{A}_F}$ is dissipative in $C_b(H)$. To show that it is m-dissipative, it is sufficient to show that for any $f \in C_b(H)$ there exists $u \in D(\overline{\mathcal{A}_F})$ such that

$$u - \overline{\mathcal{A}_F}u = f. \tag{6.19}$$

In fact let $f \in C_b(H)$ and let $\{f_n\}$ be a sequence in $C_Q^\theta(H)$ convergent to f in $C_b(H)$. Moreover let $u_n \in D(\mathcal{A}_F)$ be the solution to

$$u_n - \mathcal{A}_F u_n = f_n,$$

granted by Theorem 6.2. By the Maximum Principle we have

$$\|u_n - u_m\|_0 \le \|f_n - f_m\|_0,$$

so that the sequence $\{u_n\}$ is Cauchy in $C_b(H)$ and we have

$$u_n \to u, \quad \mathcal{A}_F u_n \to f.$$

So u is a solution to equation (6.19). □

A. Maximum principle in infinite dimensions. In this section we want to generalize the classical maximum principle to infinite-dimensional equations of the form

$$\mu u(x) - \frac{1}{2}\,\mathrm{Tr}\,[Q^{1/2}(I + F(x))Q^{1/2}D^2u(x)] = f(x), \tag{A.1}$$

where $Q \in \mathcal{L}_1(H)$ is a symmetric positive operator in H and F fulfills hypothesis (6.2). We denote by $\mathcal{A}_0$ the infinite-dimensional elliptic operator associated with Q (see Definition 4.1), and by $\mathcal{A}$ the closure of $\mathcal{A}_0$ in $C_b(H)$.

We begin with a general lemma.

Lemma A.1. *Let $u \in C_b(H)$. Then for any $\epsilon > 0$ there exists $u_\epsilon \in C_b(H)$ attaining a maximum in H, such that $u - u_\epsilon \in D(\mathcal{A}_0)$ and*

$$\|u - u_\epsilon\|_{D(\mathcal{A}_0)} \le C\epsilon \tag{A.2}$$

for some constant $C > 0$, independent of u and ϵ.

Proof. First we note that, possibly replacing u by $u - \inf_H u$, we may assume that $u \ge 0$ without loss of generality.

Now, fix $\epsilon > 0$ and let $x_\epsilon \in H$ be such that

$$u(x_\epsilon) > \|u\|_0 - \epsilon.$$

Let $\eta \in C^\infty([0, +\infty))$ be such that

$$0 \le \eta \le 1,\ \ \eta(0) = 1,\ \ \eta(r) = 0,\ \ \forall r \ge 1,\ \ |\eta'(r)| \le 2,\ \ |\eta''(r)| \le 4,\ \ \forall r \ge 0. \tag{A.3}$$

If we define $v_\epsilon(x) = u(x) + 2\epsilon\eta\big(|x - x_\epsilon|^2\big)$, then

$$\sup_{x\in H} v_\epsilon(x) = \sup_{|x-x_\epsilon|\le 1} v_\epsilon(x).$$

Indeed, $v_\epsilon(x_\epsilon) \ge \|u\|_0 + \epsilon$, and $v_\epsilon(x) \le \|u\|_0$ for $|x - x_\epsilon| \ge 1$.

By Asplund's Theorem, we have that there exists $p_\epsilon \in H$ such that the function $x \mapsto v_\epsilon(x) + \langle p_\epsilon, x\rangle$ attains a maximum on the set $|x - x_\epsilon| \le 1$, and

$$|p_\epsilon| \le \frac{\epsilon}{4(2 + |x_\epsilon|)} \ . \tag{A.4}$$

Let $\rho \in C^\infty([0, +\infty))$ be such that

$$0 \le \rho \le 1;\ \rho(r) = 1,\ \forall r \le 1;\ \rho(r) = 0,\ \forall r \ge 2;\ |\rho'(r)| \le 2,\ |\rho''(r)| \le 4,\ \forall r \ge 0 \tag{A.5}$$

and define

$$u_\epsilon(x) = v_\epsilon(x) + \rho\big(|x - x_\epsilon|^2\big)\langle p_\epsilon, x\rangle \quad x \in H.$$

From our previous observations it follows that u_ϵ attains a maximum on $|x - x_\epsilon| \le 1$.

We now claim that

$$\max_{x\in H} u_\epsilon(x) = \max_{|x-x_\epsilon|\le 1} u_\epsilon(x), \tag{A.6}$$

which will complete the proof of the lemma. Indeed, recalling (A.4), we have

$$\max_{|x-x_\epsilon|\le 1} u_\epsilon(x) \ge u_\epsilon(x_\epsilon) \ge \|u\|_0 + \frac{3}{4}\epsilon.$$

On the other hand, if $1 \le |x - x_\epsilon| \le 2$, then

$$u_\epsilon(x) \le \|u\|_0 + |p_\epsilon||x| \le \|u\|_0 + \frac{\epsilon}{4},$$

whereas, for $|x - x_\epsilon| \ge 2$, we even have that $u_\epsilon(x) \le \|u\|_0$. Our claim $(A.6)$ follows.

Lemma A.2. *If a function $u \in D(\mathcal{A}) \cap C^2_Q(H)$ attains a local maximum at a point $x_0 \in H$, then*

$$\text{Tr } [F(x_0)D^2_Q u(x_0)] \le 0 \tag{A.7}$$

$$\mathcal{A}u(x_0) \le 0. \tag{A.8}$$

Proof. First we note that the proof of (A.7) is straightforward. In fact, we have that

$$\text{Tr}[F(x_0)D^2_Q u(x_0)] = \sum_{k=1}^{\infty} \langle D^2_Q u(x_0)F^{1/2}(x_0)e_k, F^{1/2}(x_0)e_k\rangle \le 0$$

at any local maximum point x_0 of u.

We now proceed to prove (A.8). Without loss of generality we may assume that $u \geq 0$ and

$$0 < u(x_0) = \max_{|x-x_0|\leq 1} u(x).$$

Indeed, Lemma 6.1 implies that $\mathcal{A}u(x_0) = 0$ if $u(x) = 0$ for $|x - x_0| \leq 1$.

Fix $0 < \delta < u(x_0)$ and set

$$v(x) = \big(u(x) - \delta|x - x_0|^2\big)\rho(|x - x_0|^2), \quad x \in H,$$

where ρ is chosen as in (A.5). Then, $v \in D(\mathcal{A})$ in light of Lemma 6.1, and v has a strict local maximum at x_0 by construction.

Let $v^n \in D(\mathcal{A}_0)$ be such that

$$\lim_{n\to\infty} v^n = v, \quad \lim_{n\to\infty} \mathcal{A}_0 v^n = \mathcal{A}v \tag{A.9}$$

in $C_b(H)$. By Lemma A.1, for any $\epsilon > 0$ there exists $v^n_\epsilon \in D(\mathcal{A}_0)$ attaining a maximum at a point $x^n_\epsilon \in H$, such that

$$\|v^n - v^n_\epsilon\|_{D(\mathcal{A}_0)} \leq C\epsilon. \tag{A.10}$$

In particular,

$$\lim_{n\to\infty} \mathcal{A}_0 v^n_\epsilon = \mathcal{A}v.$$

We claim that there exist $\nu \in \mathbb{N}$ and $\epsilon_0 > 0$ such that

$$|x_0 - x^n_\epsilon| \leq 1 \tag{A.11}$$

for any $n \geq \nu$ and any $\epsilon \in (0, \epsilon_0)$. Indeed, by (A.10), we have

$$u(x_0) - \|v^n - v\|_0 - C\epsilon \leq v^n_\epsilon(x_0) \leq v^n_\epsilon(x^n_\epsilon) \leq v(x^n_\epsilon) + \|v^n - v\|_0 + C\epsilon. \tag{A.12}$$

On the other hand, if $|x^n_\epsilon - x_0| \geq 1$, then

$$\begin{aligned} v(x^n_\epsilon) &= \big(u(x^n_\epsilon) - \delta|x^n_\epsilon - x_0|^2\big)\rho(|x^n_\epsilon - x_0|^2) \leq \big(u(x_0) - \delta|x^n_\epsilon - x_0|^2\big)\rho(|x^n_\epsilon - x_0|^2) \\ &\leq \big(u(x_0) - \delta\big)\rho(|x^n_\epsilon - x_0|^2) \leq u(x_0) - \delta. \end{aligned} \tag{A.13}$$

Combining the above inequalities we obtain

$$\delta \leq 2(\|v^n - v\|_0 + C\epsilon),$$

which can only hold true for finitely many integers n and ϵ sufficiently large. Our claim follows.

Next, in light of (A.11) we can go back to (A.13) and deduce that

$$v(x^n_\epsilon) \leq u(x_0) - \delta|x^n_\epsilon - x_0|^2.$$

Hence, recalling (A.12), we conclude that

$$\delta|x_\epsilon^n - x_0|^2 \le 2(\|v^n - v\|_0 + C\epsilon) \tag{A.14}$$

for any $n \ge \nu$ and any $\epsilon \in (0, \epsilon_0)$.

Now, taking in the above construction any sequence $\epsilon_n \downarrow 0$ (e.g. $\epsilon_n = 1/n$), we obtain functions $w^n = v_{\epsilon_n}^n \in D(\mathcal{A}_0)$ attaining a maximum at a point $y_n = x_{\epsilon_n}^n \in H$ such that

$$\lim_{n\to\infty} w^n = v, \qquad \lim_{n\to\infty} \mathcal{A}_0 w^n = \mathcal{A}v.$$

Moreover, by (A.14),

$$\lim_{n\to\infty} y_n = x_0.$$

Therefore,

$$\mathcal{A}v(x_0) = \lim_{n\to\infty} \mathcal{A}_0 w^n(y_n) \le 0.$$

Finally, computing $\mathcal{A}v(x_0)$ by Lemma 6.1, we obtain

$$\mathcal{A}u(x_0) - \delta \text{ Tr } Q \le 0.$$

Since $\delta > 0$ is arbitrary, the proof is complete. □

Remark A.1. The reader might have noticed that the proof of (A.8) is much less straightforward than the proof of (A.7). This fact is due to the fact that, in general, $\mathcal{A}u$ is not equal to $\frac{1}{2}$ Tr D^2u.

We can now prove the Maximum Principle for subsolutions of (A.1).

Theorem A.1. *Let $\mu > 0, f \in C_b(H)$ and $u \in D(\mathcal{A}) \cap C_Q^2(H)$ be such that*

$$\mu u(x) - \mathcal{A}u(x) - \frac{1}{2} \text{ Tr } [Q^{1/2}F(x)Q^{1/2}D^2u(x)] \le f(x), \quad \forall x \in H \tag{A.15}$$

where F fulfills hypothesis (6.2). *Then*

$$\sup_H u \le \frac{1}{\mu} \sup_H f. \tag{A.16}$$

Proof. For any $\epsilon > 0$ let u_ϵ be given by Lemma A.1, and let $x_\epsilon \in H$ be such that

$$u_\epsilon(x_\epsilon) = \max_H u_\epsilon.$$

By Lemma A.2 and (A.2),

$$\begin{aligned}
\mu u_\epsilon(x_\epsilon) &\le \mu u_\epsilon(x_\epsilon) - \mathcal{A}u_\epsilon(x_\epsilon) - \frac{1}{2} \text{ Tr } [Q^{1/2}F(x_\epsilon)Q^{1/2}D^2u(x_\epsilon)] \\
&\le f(x_\epsilon) + \mu(u_\epsilon(x_\epsilon) - u(x_\epsilon)) + \mathcal{A}u(x_\epsilon) - \mathcal{A}u_\epsilon(x_\epsilon) \\
&\quad - \frac{1}{2} \text{ Tr } [Q^{1/2}F(x_\epsilon)Q^{1/2}(D^2u_\epsilon(x_\epsilon) - D^2u(x_\epsilon))] \\
&\le \sup_H f + C_1\|u_\epsilon - u\|_{D(\mathcal{A}_0)} \le \sup_H f + C_2\epsilon
\end{aligned}$$

where C_2 depends on μ, M, and Tr Q. Therefore,

$$\mu u(x) \le \mu u_\epsilon(x) + \mu\|u_\epsilon - u\|_0 \le \sup_H f + C_3\epsilon$$

for any $x \in H$. The conclusion follows since ϵ is arbitrary. □

Applying Theorem A.1 to u and $-u$, we immediately obtain the following.

Corollary A.1. *Let $\mu > 0, f \in C_b(H)$ and $u \in D(\mathcal{A}) \cap C^2_Q(H)$ be a solution of*

$$\mu u(x) - \mathcal{A}u(x) - \frac{1}{2}\ Tr\,[Q^{1/2}F(x)Q^{1/2}D^2u(x)] = f(x)$$

where F fulfills hypothesis (6.2). *Then*

$$\|u\|_0 \le \frac{1}{\mu}\|f\|_0.$$

B. Proof of Proposition 2.1. The following lemma was proved in [7] for $Q = I$ and $\varphi \in C_b(H)$. Here we give a version of it for a general self-adjoint trace-class operator $Q > 0$ and $\varphi \in C^\theta_Q(H)$.

Lemma B.1. *Let $\theta \in (0,1]$ and $\varphi \in C^\theta_Q(H)$. Define, for $t > 0$ and $x \in H$,*

$$u(t,x) = \sup_{z\in H}\big\{\inf_{y\in H}\big[\varphi(y) + \frac{1}{2t}|Q^{-1/2}(z-y)|^2\big] - \frac{1}{t}|Q^{-1/2}(z-x)|^2\big\}. \tag{B.1}$$

Then $u(t,\cdot) \in C^1_Q(H)$ and

$$\|u(t,\cdot)\|_0 \le \|\varphi\|_0, \tag{B.2}$$

$$0 \le \varphi(x) - u(t,x) \le \big(2^\theta[\varphi]^2_{\theta,Q}\big)^{\frac{1}{2-\theta}}\, t^{\frac{\theta}{2-\theta}}, \tag{B.3}$$

$$\sup_{x\in H}|D_Q u(t,x)| \le \big(2^{3-\theta}[\varphi]_{\theta,Q}\big)^{\frac{1}{2-\theta}}\, t^{\frac{\theta-1}{2-\theta}}, \tag{B.4}$$

for all $x \in H$ and $t > 0$.

Remark B.1. We note that, even though $Q^{-1/2}$ may be unbounded, formula (B.1) still makes sense for any $x \in H$. Indeed, the unbounded terms in the right-hand side of (B.1) have the right sign to keep the sup–inf envelope bounded.

Remark B.2. In view of (B.3), $u(t,\cdot)$ can be regarded as an approximation from below of φ. One could construct a completely analogous approximation from above of φ taking the inf–sup convolution

$$v(t,x) = \inf_{z\in H}\big\{\sup_{y\in H}\big[\varphi(y) - \frac{1}{2t}|Q^{-1/2}(z-y)|^2\big] + \frac{1}{t}|Q^{-1/2}(z-x)|^2\big\}.$$

Then, $v(t,\cdot)$ still satisfies (B.2), (B.4) and

$$0 \le v(t,x) - \varphi(x) \le C_\theta t^{\frac{\theta}{2-\theta}},$$

for all $x \in H$ and $t > 0$.

Proof. We first show $(B.2)$. By the definition of $u(t,x)$ we immediately obtain

$$u(t,x) \le \varphi(x), \tag{B.5}$$

and

$$u(t,x) \geq \inf_{y\in H}\big[\varphi(y) + \frac{1}{2t}|Q^{-1/2}(y-x)|^2\big].$$

Now, fix $\varepsilon > 0$ and let $y_\varepsilon \in H$ be such that

$$u(t,x) + \varepsilon > \varphi(y_\varepsilon) + \frac{1}{2t}|Q^{-1/2}(y_\varepsilon - x)|^2 \geq \varphi(y_\varepsilon). \tag{B.6}$$

In particular, since ε is arbitrary,

$$u(t,x) \geq -\|\varphi\|_0. \tag{B.7}$$

Hence, $(B.2)$ and the first inequality in $(B.3)$ follow from $(B.5)$ and $(B.7)$.

Let us complete the proof of $(B.3)$. Recalling $(B.6)$ and the assumption $\varphi \in C^\theta_Q(H)$, we obtain

$$\varphi(x) - u(t,x) \leq [\varphi]_{\theta,Q}|Q^{-1/2}(y_\varepsilon - x)|^\theta - \frac{1}{2t}|Q^{-1/2}(y_\varepsilon - x)|^2 + \varepsilon. \tag{B.8}$$

Since the left-hand side above is nonnegative, we find

$$|Q^{-1/2}(y_\varepsilon - x)| \leq M_\varepsilon,$$

where M_ε is the maximum positive number such that

$$M^2 \leq 2t\big([\varphi]_{\theta,Q}M^\theta + \varepsilon\big).$$

Then, $(B.8)$ yields

$$\varphi(x) - u(t,x) \leq [\varphi]_{\theta,Q}M_\varepsilon^\theta + \varepsilon, \forall x \in H,$$

and $(B.3)$ follows as

$$\lim_{\varepsilon\downarrow 0} M_\varepsilon = \big(2t[\varphi]_{\theta,Q}\big)^{\frac{1}{2-\theta}}.$$

Next, we note that the existence of the partial derivatives $D_k u(t,\cdot)$ is a well-known finite-dimensional result; see e.g. [LL]. In order to conclude the proof, it suffices to estimate the Lipschitz norm of $u(t,\cdot)$. For this purpose, let $\varepsilon > 0, x \in H$ be fixed, and let $z_{\varepsilon,x} \in H$ be such that

$$u(t,x) < \inf_{y\in H}\big[\varphi(y) + \frac{1}{2t}|Q^{-1/2}(z_{\varepsilon,x} - y)|^2\big] - \frac{1}{t}|Q^{-1/2}(z_{\varepsilon,x} - x)|^2 + \varepsilon. \tag{B.9}$$

Then $z_{\varepsilon,x} - x \in Q^{1/2}H$, and

$$\frac{1}{t}|Q^{-1/2}(z_{\varepsilon,x} - x)|^2 < \varepsilon + \varphi(x) - u(t,x) + \frac{1}{2t}|Q^{-1/2}(z_{\varepsilon,x} - x)|^2.$$

Recalling (B.3), we obtain

$$\frac{1}{2t}|Q^{-1/2}(z_{\varepsilon,x}-x)|^2 \le \varepsilon + \left(2^\theta [\varphi]^2_{\theta,Q}\right)^{\frac{1}{2-\theta}} t^{\frac{\theta}{2-\theta}}. \tag{B.10}$$

Now, let $x' \in H$ be such that $x - x' \in Q^{1/2}H$. From (B.9) it follows that

$$\begin{aligned} u(t,x) - u(t,x') &< \varepsilon + \frac{1}{t}\left(|Q^{-1/2}(z_{\varepsilon,x}-x')|^2 - |Q^{-1/2}(z_{\varepsilon,x}-x)|^2\right) \\ &= \varepsilon + \frac{1}{t}|Q^{-1/2}(x-x')|^2 + \frac{2}{t}\langle Q^{-1/2}(x-x'), Q^{-1/2}(z_{\varepsilon,x}-x)\rangle. \end{aligned}$$

Hence, by (B.10),

$$u(t,x) - u(t,x') \le \varepsilon + \frac{1}{t}|Q^{-1/2}(x-x')|^2 + \frac{2}{t}|Q^{-1/2}(x-x')|\left[2t\left(\varepsilon + C_\theta t^{\frac{\theta}{2-\theta}}\right)\right]^{1/2}$$

where

$$C_\theta = \left(2^\theta [\varphi]^2_{\theta,Q}\right)^{\frac{1}{2-\theta}}.$$

Therefore,

$$|u(t,x) - u(t,x')| \le \frac{1}{t}|Q^{-1/2}(x-x')|^2 + 2\sqrt{2C_\theta}|Q^{-1/2}(x-x')| t^{\frac{\theta-1}{2-\theta}} \tag{B.11}$$

for any $x, x' \in H$ such that $x - x' \in Q^{1/2}H$. Now, by standard arguments estimate (B.11) implies that

$$\left|\langle D_Q u(t,x), y\rangle\right| \le 2\sqrt{2C_\theta}|y| t^{\frac{\theta-1}{2-\theta}}$$

for any $y \in H$ and $t > 0$. The conclusion (B.4) follows. □

Proof of Proposition 2.1. **Step 1**. $(C_b(H), C^1_Q(H))_{\theta,\infty} \subset C^\theta_Q(H)$.

Let $\varphi \in (C_b(H), C^1_Q(H))_{\theta,\infty}$. Then for any $t > 0$ and $\varepsilon > 0$ there exist functions $f_{t,\varepsilon} \in C_b(H)$ and $g_{t,\varepsilon} \in C^1_b(H)$ such that

$$\varphi(x) = f_{t,\varepsilon}(x) + g_{t,\varepsilon}(x), \quad x \in H,$$

and

$$\|f_{t,\varepsilon}\|_0 + t\|g_{t,\varepsilon}\|_{1,Q} \le t^\theta |\varphi|_{(C_b(H),C^1_Q(H))_{\theta,\infty}} + \varepsilon.$$

Then

$$\begin{aligned} |\varphi(Q^{1/2}x) - \varphi(Q^{1/2}y)| &\le 2\|f_{t,\varepsilon}\|_0 + \|Q^{1/2}Dg_{t,\varepsilon}\|_0|x-y| \\ &\le 2Ct^\theta + Ct^{\theta-1}|x-y| + 2\varepsilon + \frac{\varepsilon}{t}|x-y| \end{aligned}$$

for all $x, y \in H, t > 0$ and some constant $C > 0$. Now, letting $\varepsilon \downarrow 0$ and taking $t = |x-y|$, the conclusion follows.

Step 2. $C^\theta_Q(H) \subset (C_b(H), C^1_Q(H))_{\theta,\infty}$.

Let $\varphi \in C^\theta_Q(H)$, and define $u(t,x)$ as in $(B.1)$. We set, for any $t>0, x\in H$,

$$f_t(x) = \varphi(x) - u\big(t^{2-\theta}, x\big), \quad g_t(x) = u\big(t^{2-\theta}, x\big).$$

Then, from $(B.3)$, we obtain

$$\|f_t\|_0 \le Ct^\theta \tag{B.12}$$

for some constant $C>0$. Similarly $(B.4)$ yields

$$\|D_Q g_t\|_0 \le Ct^{\theta-1}. \tag{B.13}$$

Recalling that $\|g_t\|_0 \le \|\varphi\|_0$, from (3.2) and (3.3) we conclude that

$$\|f_t\|_0 + t\|g_t\|_{1,Q} \le Ct^\theta,$$

and so

$$K(t,\varphi,C_b(H),C_{1,Q}(H)) \le Ct^\theta,$$

for any $t\in(0,1]$. Since the last inequality is trivial for $t\ge 1$, the conclusion follows. □

REFERENCES

[1] P.L. Butzer and H. Berens, "Semigroups of Operators and Approximation," Springer-Verlag, Berlin, 1967.

[2] P. Cannarsa and G. Da Prato, *On a functional analysis approach to parabolic equations in infinite dimensions* J. Funct. Anal., 118 (1993), 22–42.

[3] G. Da Prato and J. Zabczyk, "Stochastic Equations in Infinite Dimensions," Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1992.

[4] L. Daleckijju and S. Fomin, "Measures and Differential Equations in Infinite Dimensional Spaces," Kluwer, 1983.

[5] N. Dunford and J.T. Schwartz, "Linear Operators," Vol. II, 1956.

[6] L. Gross, *Potential theory in Hilbert spaces,* J. Funct. Analysis, 1 (1965), 123–189.

[7] J.M. Lasry and P.L. Lions, *A remark on regularization in Hilbert spaces,* Israel J. Math., 55 (1986), 257–266.

[8] J.L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation*, Publ. Math de l'I.H.E.S., 19 (1964), 5–68.

[9] Z.M. Ma and M. Rockner, "Introduction to the Theory of (Non Symmetric) Dirichlet Forms," Springer-Verlag, 1992.

[10] A.S. Nemirovski and S.M. Semenov, *The polynomial approximation of functions in Hilbert spaces*, Mat. Sb. (N.S), 92 (1973), 257–281.

[11] A. Piech, *A fundamental solution of the parabolic equation on Hilbert space,* J. Funct. An., 3 (1969), 85–114.

[12] D.W. Stroock, "Logarithmic Sobolev Inequalities for Gibbs States," Corso CIME, Springer-Verlag, 1994.

[13] H. Triebel, "Interpolation Theory, Function Spaces, Differential Operators," North-Holland, Amsterdam, 1986.