Advances in Differential Equations Volume 1, Number 4, July 1996, pp. 499 – 546

NAVIER-STOKES EQUATIONS IN THREE-DIMENSIONAL THIN DOMAINS WITH VARIOUS BOUNDARY CONDITIONS

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Abstract. In this work we develop methods for studying the Navier-Stokes equations in thin domains. We consider various boundary conditions and establish the global existence of strong solutions when the initial data belong to "large sets." Our work was inspired by the recent interesting results of G. Raugel and G. Sell [22, 23, 24] which, in the periodic case, give global existence for smooth solutions of the 3D Navier-Stokes equations in thin domains for large sets of initial conditions. We extend their results in several ways, we consider numerous boundary conditions and as it will appear hereafter, the passage from one boundary condition to another one is not necessarily straightforward. The proof of our improved results is based on precise estimates of the dependence of some classical constants on the thickness ϵ of the domain, e.g. Sobolev-type constants and the regularity constant for the corresponding Stokes problem.

As an application, we study the behavior of the average of the strong solution in the thin direction when the thickness of the domain goes to zero; we prove its convergence to the strong solution of a 2D Navier-Stokes system of equations.

0. Introduction. We are concerned in this article with the Navier-Stokes equations of viscous incompressible fluids in three-dimensional thin domains. Let Ω_ϵ be the thin domain $\Omega_\epsilon = \omega \times (0, \epsilon)$, where ω is a suitable domain in $\mathbb{R}^2$ and $0 < \epsilon < 1$.

Our aim is to study the global existence in time of the strong solutions of the 3D Navier-Stokes equations (3DNSE) in the thin domains with various boundary conditions as well as the study of the behavior of the average of the strong solutions in the thin direction when the thickness ϵ goes to zero.

The study of the global existence of strong solutions of the three-dimensional Navier-Stokes equations on thin domains was initiated by G. Raugel and G. Sell ([22, 23, 24]). Their work is inspired by methods developed by J. Hale and G. Raugel ([13, 14]) for reaction-diffusion equations and damped wave equations on thin domains. They used a dilation of the domain to obtain the dilated Navier-Stokes equations defined in the fixed domain $\Omega_1 = \omega \times (0, 1)$. Then, assuming the space periodic boundary condition, they showed that the dilated Navier-Stokes equations are a regular perturbation of the 2DNS when the thickness is small. They also obtained global existence of strong solutions when the initial data belong to "large sets."

In the same spirit, using the smallness of the domain, by imposing conditions relating the first eigenvalue of the Laplacian, the viscosity ν and the size of the initial data, J. Avrin established similar global existence results for the 3DNSE with the Dirichlet boundary condition ([3]). The results obtained in [3] are based on a contraction principle argument.

Received for publication November 1995.
AMS Subject Classifications: 34C35, 35Q30, 35Q30, 76D05.

Many other studies on partial differential equations in thin domains appear in the work of Ph. Ciarlet and his collaborators; see e.g. Ph. Ciarlet ([6]), H.L. Le Dret ([17]), and the references therein. The purpose in this work is to systematically derive the equations for plates and shells by passing to the limit, $\epsilon \to 0$, in the the equations of linear or nonlinear elasticity in three-dimensional domains.

In the present article, we develop a general study of the Navier-Stokes equations on thin domains and obtain global existence (in time) of the strong solutions for initial data belonging to large sets, for which we give a very simple characterization. The study given in this article treats systematically various boundary conditions (listed below); as we will see at the end of this article, the behavior of the solutions depends strongly on the boundary conditions. A major difference between our work and that of [22, 23, 24] is that we found it useful to avoid the dilation of the domain and work instead in the *actual domain* Ω_ϵ. The price to pay for that is that all constants appearing in the various functional inequalities depend now on ϵ. The first part of our work (Section 2) is to determine precisely the dependence on the thickness ϵ of the constants appearing in the classical Sobolev-type inequalities, often used in the study of the Navier-Stokes equations, namely the Poincaré, Ladyzhenskaya and Agmon inequalities; we also determine the dependence on ϵ of the constant appearing in the Cattabriga-Solonnikov regularity inequality; we did not find all these constants available in the literature and a number of them are new to the best of our knowledge.

A useful tool in obtaining the dependence of the constants in the functional inequalities is *the average operator* in the thin direction which allows us to use the Poincaré inequality in the thin direction. The definition of the average operator for vector functions depends on the type of boundary conditions; roughly speaking, if a component does not satisfy the Dirichlet condition in the thin direction then we take its average, otherwise we set it to be zero (see Section 1 for more details).

The boundary conditions of interest to us are combinations of the usual boundary conditions, namely the Dirichlet (D), periodic (P) and free boundary (F) conditions. We will combine these boundary conditions on $\Gamma_t \cup \Gamma_b = \omega \times \{0, \epsilon\}$ and $\Gamma_l = \partial\omega \times (0,1)$, considering (FP), (FD), (FF), (PP), (DD) and (DP) (see below). We could also consider different combinations of boundary conditions on Γ_t, Γ_b and Γ_l (which is not always straightforward as it will appear in Section 4) but we refrained from doing so to avoid lengthly developments.

The size of the large sets of initial data for which we obtain the global existence in time of the strong solution depends on the boundary condition. We divide the boundary conditions above into three types (see below for the notation):

Type I. It contains the boundary conditions (FF) and (FP), i.e., the free boundary condition in the thin direction and either the periodic or the free boundary condition on the lateral boundary. In this type of boundary conditions, we obtain that whenever the initial data satisfy

$$\lim_{\epsilon \to 0} \epsilon^q \left(\left| A_\epsilon^{\frac{1}{2}} u_0 \right|_\epsilon^2 + \left| f \right|_\epsilon^2 \right) = 0, \quad \text{for some } \; q < 1,$$

then, there exists $\epsilon_0 = \epsilon_0(\nu)$ such that for $\epsilon \le \epsilon_0$, the maximal time of existence $T(\epsilon)$ of the strong solution of the 3D Navier-Stokes equations with one of these boundary conditions satisfies

$$T(\epsilon) = +\infty.$$

Type II. It contains the boundary conditions (FD): the free boundary condition in the thin direction and the Dirichlet boundary condition on the lateral boundary, and (PP): the purely periodic boundary condition. For this type of boundary conditions, we obtain that whenever the initial data satisfy (see Sections 2 and 3 for the notation):

If for some arbitrarily fixed constants $K_1 > 0$, and $K_2 > 0$,

$$|A_\epsilon^{\frac{1}{2}} \tilde{M}_\epsilon u_0^\epsilon|_\epsilon^2 + |\tilde{M}_\epsilon f^\epsilon|_\epsilon^2 \leq K_1 \epsilon \ln\big|\ln \epsilon\big|, \quad \text{and}$$
$$|A_\epsilon^{\frac{1}{2}} \tilde{N}_\epsilon u_0^\epsilon|_\epsilon^2 + |\tilde{N}_\epsilon f^\epsilon|_\epsilon^2 \leq K_2 \ln\big|\ln \epsilon\big|,$$

then the same conclusions as for Type I hold.

Type III. It contains the boundary conditions (DD) and (DP), i.e., the Dirichlet boundary condition in the thin direction and either the Dirichlet or the periodic condition on the lateral boundary. For this type of boundary conditions, we obtain the same conclusions whenever the initial data satisfy

$$\lim_{\epsilon \to 0} \epsilon^{\frac{1}{2}} (\big|A_\epsilon^{\frac{1}{2}} u_0^\epsilon\big|_\epsilon^2 + \big|f^\epsilon\big|_\epsilon^2) = 0.$$

We also give an asymptotic expansion of the solution u^ϵ for ϵ small in the case of the channel flow. The asymptotic expansion in more general cases will be given in a forthcoming article ([28]). The results obtained for the averages are described at the end of the Introduction.

This work was motivated by the study of the Navier-Stokes equations in thin spherical shells in order to justify the Navier-Stokes equations on the sphere in view of applications to geophysical flows. The results obtained in the spherical case will be given elsewhere ([29]).

The global existence results obtained in thin domains allow us to prove the existence of the attractors, to give a characterization of them and also to determine the dependence of their fractal and Hausdorff dimensions on the thickness ϵ; all these results will appear in [33].

The setting of the problem. The nondimensionalized form of the Naiver-Stokes equations (NSE) is

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \qquad \text{in } \Omega_\epsilon \times (0, \infty), \tag{0.1}$$

$$\text{div } u = 0 \qquad \text{in } \Omega_\epsilon \times (0, \infty), \tag{0.2}$$

$$u(\cdot, 0) = u_0(\cdot) \qquad \text{in } \Omega_\epsilon. \tag{0.3}$$

Here $u = (u_1, u_2, u_3)$ is the velocity vector at point x and time t, and $p(x,t)$ is the pressure.

Equations (0.1)–(0.3) are supplemented with boundary conditions. We denote the boundary of Ω_ϵ by $\partial\Omega_\epsilon = \Gamma_t \cup \Gamma_b \cup \Gamma_l$, where

$$\Gamma_t = \omega \times \{\epsilon\}, \quad \Gamma_b = \omega \times \{0\}, \quad \text{and} \quad \Gamma_l = \partial\omega \times (0, \epsilon). \tag{0.4}$$

The boundary conditions under consideration. The boundary conditions of interest to us are combinations of the usual boundary conditions, namely the Dirichlet (D), periodic (P) and free boundary (F) conditions. More precisely, we consider the following combinations:

(FP) The free boundary condition on $\Gamma_t \cup \Gamma_b$ and the periodic condition on Γ_l; i.e., here $\omega = (0, l_1) \times (0, l_2)$ and

$$u_3 = 0 \quad \text{and} \quad \frac{\partial u_\alpha}{\partial x_3} = 0, \quad \alpha = 1, 2 \quad \text{on} \quad \Gamma_t \cup \Gamma_b;$$

and u_i, $i = 1, 2, 3$, are periodic in the directions $x_1,\ x_2$ with periods $l_1,\ l_2$ respectively, and

$$\int_{\Omega_\epsilon} (u_0)_\alpha \, dx = \int_{\Omega_\epsilon} f_\alpha \, dx = 0, \quad \alpha = 1, 2.$$

(FD) The free boundary condition on $\Gamma_t \cup \Gamma_b$ and the Dirichlet boundary condition on Γ_l; i.e., here ω is a C^2- bounded domain in $\mathbb{R}^2$, and

$$u_3 = 0 \ \text{and} \ \frac{\partial u_\alpha}{\partial x_3} = 0, \quad \alpha = 1, 2 \quad \text{on} \quad \Gamma_t \cup \Gamma_b, \quad \text{and } u = 0 \quad \text{on} \quad \Gamma_l.$$

(FF) The free boundary condition on $\partial\Omega_\epsilon$, i.e.,

$$u \cdot \vec{n} = 0, \quad \text{curl } u \times \vec{n} = 0 \quad \text{on} \quad \partial\Omega_\epsilon,$$

where $\vec{n}$ is the outward unit normal vector to $\partial\Omega_\epsilon$.

(PP) The periodic boundary condition on $\partial\Omega_\epsilon$; i.e., here $\omega = (0, l_1) \times (0, l_2)$ and u are Ω_ϵ-periodic, and, for the data

$$\int_{\Omega_\epsilon} u_0 \, dx = \int_{\Omega_\epsilon} f \, dx = 0.$$

(DD) The Dirichlet boundary condition on $\partial\Omega_\epsilon$,

$$u = 0 \quad \text{on} \quad \partial\Omega_\epsilon.$$

(DP) The Dirichlet boundary condition on $\Gamma_t \cup \Gamma_b$ and the periodic condition on Γ_l.

The mathematical setting of the problem. We denote by $H^s(\Omega_\epsilon)$, $s \in \mathbb{R}$, the Sobolev space constructed on $L^2(\Omega_\epsilon)$ and $\mathbb{L}^2(\Omega_\epsilon) = (L^2(\Omega_\epsilon))^3$, $\mathbb{H}^s(\Omega_\epsilon) = (H^s(\Omega_\epsilon))^3$. We also denote by $H_0^s(\Omega_\epsilon)$ the closure in the space $H^s(\Omega_\epsilon)$ of $\mathcal{C}_0^\infty(\Omega_\epsilon)$, the space of infinitely differentiable functions with compact support in Ω_ϵ. We need also the following spaces:

$$\dot{\mathbb{H}}^m(\Omega_\epsilon) = \Big\{ u \in \mathbb{H}^m(\Omega_\epsilon); \ \int_{\Omega_\epsilon} u \, dx = 0 \Big\}, \tag{0.5}$$

and the spaces $H_{per}^m(\Omega_\epsilon)$, which are defined with the help of Fourier series; we write

$$u(x) = \sum_{k \in \mathbb{Z}^3} u_k \exp(2ik \cdot \frac{x}{L}), \tag{0.6}$$

with $\bar{u}_k = u_{-k}$ (so that u is real valued) and

$$\frac{x}{L} = (\frac{x_1}{l_1}, \frac{x_2}{l_2}, \frac{x_3}{\epsilon}); \quad k \cdot \frac{x}{L} = k_1 \frac{x_1}{l_1} + k_2 \frac{x_2}{l_2} + k_3 \frac{x_3}{\epsilon}.$$

Then, u is said to be in $L^2(\Omega_\epsilon)$ if and only if

$$|u|^2_{L^2(\Omega_\epsilon)} = \epsilon l_1 l_2 \sum_{k \in \mathbb{Z}^3} |u_k|^2 < \infty,$$

and u is in $H^s_{per}(\Omega_\epsilon),\ s \in \mathbb{R}_+$, if and only if

$$\sum_{k \in \mathbb{Z}^3} (1 + |k|^2)^s |u_k|^2 < \infty.$$

For the mathematical setting of the Navier-Stokes equations, we consider a Hilbert space H_ϵ, which is a closed subspace of $\mathbb{L}^2(\Omega_\epsilon)$ (see [25]). Depending on the boundary condition, we define the following:

$$H_{FP} = \big\{u \in \mathbb{L}^2(\Omega_\epsilon);\ \text{div } u = 0;\ \int_{\Omega_\epsilon} u_\alpha\, dx = 0,\ u_3 = 0 \text{ on } \Gamma_t \cup \Gamma_b \\ \text{and} \quad u\big|\Gamma_\alpha = u\big|\Gamma_{\alpha+3};\ \alpha = 1, 2\big\},$$

where Γ_α and $\Gamma_{\alpha+3}$ are the faces $x_\alpha = 0$ and $x_\alpha = l_\alpha$ of $\partial\Omega_\epsilon$. The condition $u_\alpha|\Gamma_\alpha = u_{\alpha|}\Gamma_{\alpha+3}$ expresses the periodicity of u_α in the direction x_α.

$$H_{FD} = \big\{u \in \mathbb{L}^2(\Omega_\epsilon);\ \text{div } u = 0;\ u \cdot \vec{n} = 0 \ \text{ on } \ \partial\Omega_\epsilon\big\}\,;$$

$$H_{PP} = \big\{u \in \mathbb{L}^2(\Omega_\epsilon);\ \text{div } u = 0;\ \int_{\Omega_\epsilon} u\, dx = 0,\ \ u_j|\Gamma_j = u_j|\Gamma_{j+3};\ j = 1, 2, 3\big\},$$

$$H_{DD} = H_{FF} = H_{FD} \quad \text{and} \quad H_{DP} = H_{FP}.$$

Another useful space is V_ϵ, a closed subspace of $\mathbb{H}^1(\Omega_\epsilon)$, which is defined depending on the boundary condition as follows:

$$\begin{aligned}
V_{FP} &= \{u \in \mathbb{H}^1(\Omega_\epsilon) \cap H_{FP};\ u|_{\Gamma_\alpha} = u|_{\Gamma_{\alpha+3}}\},\\
V_{FD} &= \{u \in \mathbb{H}^1(\Omega_\epsilon) \cap H_{FD};\ \ u = 0 \ \text{ on } \ \Gamma_l\},\\
V_{PP} &= \{u \in \dot{\mathbb{H}}^1_{per}(\Omega_\epsilon);\ \ \text{div } u = 0\},\\
V_{DD} &= \{u \in \mathbb{H}^1_0(\Omega_\epsilon);\ \ \text{div } u = 0\}, \ \ \text{and}\\
V_{FF} &= \{u \in \mathbb{H}^1(\Omega_\epsilon) \cap H_{FF};\ \ u \cdot \vec{n} = 0 \ \text{ on } \ \partial\Omega_\epsilon\}.
\end{aligned}$$

In the remainder of this paper, unless there are differences in the proofs, we will omit the reference to the boundary condition; we denote, for instance, any one of the spaces defined above by H_ϵ or V_ϵ.

The scalar product on H_ϵ is denoted by $(\cdot,\cdot)_\epsilon$, the one on V_ϵ is denoted by $((\cdot,\cdot))_\epsilon$, and the associated norms are denoted by $|\cdot|_\epsilon$ and $||\cdot||_\epsilon$ respectively. We denote by A_ϵ the Stokes operator defined as an isomorphism from V_ϵ onto the dual V'_ϵ of V_ϵ, by

$$\forall v \in V_\epsilon; \quad \langle A_\epsilon u, v\rangle_{V'_\epsilon, V_\epsilon} = ((u,v))_\epsilon. \tag{0.7}$$

The operator A_ϵ is extended to H_ϵ as a linear unbounded operator. The domain of A_ϵ in H_ϵ is denoted by $D(A_\epsilon)$. The space $D(A_\epsilon)$ can be fully characterized using the regularity theory. We refer for the study of the regularity of the Stokes operator to [4], [7, 8], [10], [25, 26] and [32]. Here we give the characterization of the domain of the Stokes operator:

$$D(A_{FP}) = \Big\{ u \in \mathbb{H}^2(\Omega_\epsilon) \cap V_{FP}; \quad \frac{\partial u_\alpha}{\partial x_3} = 0,\ \frac{\partial u_i}{\partial x_\alpha}\Big|\Gamma_\alpha = -\frac{\partial u_i}{\partial x_\alpha}\Big|\Gamma_{\alpha+3}; \quad i = 1,2,3, \quad \alpha = 1,2 \Big\},$$

$$D(A_{FD}) = \Big\{ u \in \mathbb{H}^2(\Omega_\epsilon) \cap V_{FD}; \quad \frac{\partial u_\alpha}{\partial x_3} = 0 \text{ on } \Gamma_b \cup \Gamma_t, \quad \alpha = 1,2 \Big\},$$

$$D(A_{PP}) = \dot{\mathbb{H}}^2_{per}(\Omega_\epsilon) \quad \text{and} \quad D(A_{DD}) = \mathbb{H}^2(\Omega_\epsilon) \cap \mathbb{H}^1_0(\Omega_\epsilon).$$

$$D(A_{FF}) = \left\{ u \in \mathbb{H}^2(\Omega_\epsilon) \cap V_{FF}; \quad \text{curl } u \times \vec{n} = 0 \text{ and } u \cdot \vec{n} = 0 \text{ on } \partial\Omega_\epsilon \right\}.$$

We should also recall the Leray's projector P_ϵ, which is the orthogonal projector of $\mathbb{L}^2(\Omega_\epsilon)$ onto H_ϵ. The Stokes operator can be given with the help of the Leray projector as follows:

$$A_\epsilon u = P_\epsilon(-\Delta u), \quad \text{for } u \in D(A_\epsilon). \tag{0.8}$$

Let b_ϵ be the continuous trilinear form on V_ϵ defined by:

$$b_\epsilon(u,v,w) = \sum_{i,j=1}^{3} \int_{\Omega_\epsilon} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \qquad u,v,w \in \mathbb{H}^1(\Omega_\epsilon). \tag{0.9}$$

We denote by B_ϵ the bilinear form defined for $(u,v) \in V_\epsilon \times V_\epsilon$ by

$$\langle B_\epsilon(u,v), w\rangle_{V'_\epsilon, V_\epsilon} = b_\epsilon(u,v,w), \quad \forall w \in V_\epsilon,$$

and

$$B_\epsilon(u) = B_\epsilon(u,u).$$

We assume in this work that the data ν, u_0 and f satisfy

$$\nu > 0, \quad u_0 \in H_\epsilon \text{ (or } V_\epsilon); \quad f \in L^\infty(0,+\infty; H_\epsilon). \tag{0.10}$$

The system of equations (0.1)–(0.3), with one of the boundary conditions listed above, can be written as a differential equation in V'_ϵ :

$$u' + \nu A_\epsilon u + B_\epsilon(u) = f, \quad u(0) = u_0, \tag{0.11}$$

where u' denotes the derivative (in the distribution sense) of the function u with respect to time. We recall now the classical result of existence of solutions to problem (0.11). See [5], [9], [15], [16], [20], [18], [25, 26], etc.

Theorem 0.1. *For $u_0 \in H_\epsilon$, there exists a solution (not necessarily unique) $u = u_\epsilon$ to problem* (0.11) *such that*

$$u_\epsilon \in L^2(0,T;V_\epsilon) \cap L^\infty(0,T;H_\epsilon), \quad \forall T > 0. \tag{0.12}$$

Moreover, if $u_0 \in V_\epsilon$, then there exists $T_\epsilon = T_\epsilon(\Omega_\epsilon, \nu, u_0, f) > 0$, and a unique solution u_ϵ to problem (0.11) *such that*

$$u_\epsilon \in L^2(0,T_\epsilon;D(A_\epsilon)) \cap L^\infty(0,T_\epsilon;V_\epsilon). \tag{0.13}$$

The solution u_ϵ which satisfies (0.13) is called the strong solution of (0.11). The study of the global existence (in time) of the strong solution and the uniqueness of solutions to problem (0.11) is still open in three-dimensional domains; this work will give a partial answer to this question when the three-dimensional domain is thin.

The main results. First we state the main results concerning the global existence results. Let $R(\epsilon)$ be a monotone positive function satisfying for some $q \leq 1$

$$\lim_{\epsilon \to 0} \epsilon^q R^2(\epsilon) = 0,$$

and assume that the initial data satisfy

$$|A_\epsilon^{1/2} u_0|_\epsilon^2 + |f|_\epsilon^2 \leq R^2(\epsilon).$$

The global existence results will depend on the type of the boundary condition:

- The boundary conditions (DD) and (DP): we take $q = 1$.
- The boundary conditions (FF) and (FP): we take $q < 1$.
- The boundary conditions (PP) and (FD): we assume, in this case, a stronger condition on the initial data; i.e., for arbitrary positive constants K_1, K_2 (independent of ϵ), we assume that

$$|A_\epsilon^{1/2} \tilde{M}_\epsilon u_0|_\epsilon^2 + |\tilde{M}_\epsilon f|_\epsilon^2 \leq K_1 \epsilon \ln|\ln \epsilon| \quad \text{and}$$
$$|A_\epsilon^{1/2} \tilde{N}_\epsilon u_0|_\epsilon^2 + |\tilde{N}_\epsilon f|_\epsilon^2 \leq K_2 \ln|\ln \epsilon|,$$

where $\tilde{M}_\epsilon$ is the average operator in the thin direction and $\tilde{N}_\epsilon u = u - \tilde{M}_\epsilon u$ (see Section 1 for more details on $\tilde{M}_\epsilon$ and $\tilde{N}_\epsilon$).

Theorem A1. *With the assumptions above on the initial data, there exists $\epsilon_0 > 0$, such that*

$$\forall\ \epsilon \leq \epsilon_0,\ \forall\ u_0 \in V_\epsilon,\ \forall\ f \in H_\epsilon, \quad \textit{with} \quad |A_\epsilon^{1/2} u_0|_\epsilon^2 + |f|_\epsilon^2 \leq R^2(\epsilon)$$

the maximal time T_ϵ of existence of the strong solution u_ϵ of problem (0.1)–(0.3) *satisfies*

$$T_\epsilon = \infty.$$

In order to see the improvement that this result brings to the classical small data existence result [5], [24], [25, 26], etc., we recall that if, e.g., u_0 and f are independent on x_3 (the thin direction variable) then

$$|A_\epsilon^{1/2}u_0|_\epsilon^2 + |f|_\epsilon^2 \le C_0\epsilon,$$

where C_0 is independent of ϵ, which implies that $R(\epsilon)$ goes to zero as ϵ goes to zero, while in our results $R^2(\epsilon)$ and therefore $|A_\epsilon^{1/2}u_0|_\epsilon^2 + |f|_\epsilon^2$ can be very large for ϵ small.

Now we state the results concerning the behavior of the average, in the thin direction, of the strong solution u_ϵ of (0.1)–(0.3) when ϵ approaches zero. For this purpose, we introduce the spaces

$$\tilde{H}_P = \left\{\tilde{u} \in (L^2(\omega))^2;\ \text{div}\,'\ \tilde{u} = 0;\ \int_\omega \tilde{u}\,dx = 0,\ \ \tilde{u}_\alpha|_{\tilde{\Gamma}_\alpha} = \tilde{u}_\alpha|_{\tilde{\Gamma}_{\alpha+2}};\ \alpha = 1,2\right\},$$

where $\tilde{\Gamma}_\alpha$ is the face of $\partial\omega$ in the direction x_α.

$$\tilde{H}_D = \left\{\tilde{u} \in (L^2(\omega))^2;\ \text{div}'\ \tilde{u} = 0;\ \ \tilde{u}\cdot\vec{n} = 0 \quad \text{on} \ \ \partial\omega\right\},$$

where $\vec{n}$ is outward unit normal vector to $\partial\omega$.

$$\tilde{H}_F = \tilde{H}_D.$$

We also introduce

$$\tilde{V}_P = \tilde{H}_P \cap (\dot{H}^1_{per}(\omega))^2; \quad \tilde{V}_D = (H^1_0(\omega))^2, \ \text{ and } \ \tilde{V}_F = (H^1(\omega))^2 \cap \tilde{H}_F.$$

The 2D Navier-Stokes equations on ω are given by:

$$\begin{aligned}
&\frac{\partial \tilde{v}}{\partial t} - \nu\Delta'\tilde{v} + (\tilde{v}\cdot\nabla')\tilde{v} + \nabla'\tilde{p} = \tilde{f} && \text{in } \ \omega \times [0,\infty),\\
&\text{div}'\ \tilde{v} = 0 && \text{in } \ \omega \times [0,\infty),\\
&\tilde{v}(x',0) = \tilde{v}_0(x') && \text{in } \ \omega.
\end{aligned}$$

We denote, here and henceforth, the two-dimensional operators with a prime, for example

$$\nabla' = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, 0\right) \quad \text{and} \quad x' = (x_1, x_2).$$

The 2D Navier-Stokes equations above are supplemented with one of the following boundary conditions: The periodic boundary condition $(\tilde{P})$, the Dirichlet boundary condition $(\tilde{D})$ or the free boundary condition $(\tilde{F})$. Note that the study of the existence and uniqueness of solutions to the 2D Navier-Stokes equations is complete. See [5], [9], [15], [16], [18], [25, 26], etc.

We will prove the following:

Theorem A2. *In the case of one of the boundary conditions* (FD), (FP), (FF) or (PP), *we assume that we are given a family of functions* $u_0^\epsilon \in V_\epsilon$ *and* $f^\epsilon \in H_\epsilon$ *defined on the domains* Ω_ϵ, $0 < \epsilon < 1$ *such that*

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_0^\epsilon u_0^\epsilon(\cdot, x_3)\, dx_3 = \tilde{v}_0 \quad \textit{in } \tilde{V}\textit{-weak}$$
$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_0^\epsilon f^\epsilon(\cdot, x_3)\, dx_3 = \tilde{f} \quad \textit{in } \tilde{H}\textit{-weak.}$$

Then, for all $T > 0$, *there exists* $\epsilon_0 > 0$ *such that, for* $0 < \epsilon < \epsilon_0$, *there exists a unique strong solution* u^ϵ *of* (0.1)–(0.3) *defined on* $[0, T]$ *and*

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_0^\epsilon u^\epsilon(t; x_1, x_2, x_3)\, dx_3 = \tilde{v}(t; x_1, x_2) \quad \textit{in} \quad \mathcal{C}([0, T]; \tilde{H}) \cap L^2(0, T; \tilde{V}),$$

where $\tilde{v}$ *is the "unique" strong solution of the 2D Navier-Stokes equations above, with the boundary condition* $(\tilde{P})$ *in the case of* (PP) *or* (FP); *the boundary condition* $(\tilde{D})$ *in the case of* (FD) *and the boundary condition* $(\tilde{F})$ *in the case of* (FF).

We will also show that in the case of the Dirichlet boundary condition on either the top or the bottom of Ω_ϵ, we have

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_0^\epsilon u^\epsilon(t; x_1, x_2, x_3)\, dx_3 = 0 \quad \text{in} \quad \mathcal{C}([0, T]; \tilde{H}) \cap L^2(0, T; \tilde{V}).$$

We also give an asymptotic expansion of the solution u^ϵ for a channel flow when ϵ is small. For more details, see Section 4.

This article is organized as follows: In Section 1 we define the average operator and study its properties. Section 2 is devoted to some fundamental inequalities related to the Navier-Stokes equations on thin domains. Section 3 gives the necessary a priori estimates for the average of the strong solution in the thin direction. In Section 4, we study the behavior of the maximal time of existence of strong solutions when ϵ goes to zero, and finally in Section 5, we study the behavior of the averages as the thickness goes to zero.

1. Preliminaries. For a scalar function $\varphi \in L^2(\Omega_\epsilon)$, we define its average in the thin direction as follows:

$$(M_\epsilon \varphi)(x_1, x_2) = \frac{1}{\epsilon}\int_0^\epsilon \varphi(x_1, x_2, s)\, ds, \tag{1.1}$$

and we set

$$N_\epsilon\varphi = \varphi - M_\epsilon\varphi, \quad \text{i.e.,} \quad M_\epsilon\varphi + N_\epsilon\varphi = I_{L^2(\Omega_\epsilon)}, \tag{1.2}$$

where $I_{L^2(\Omega_\epsilon)}$ is the identity operator on $L^2(\Omega_\epsilon)$.

In order to define the average operator $\tilde{M}_\epsilon$, we will need to specify the boundary condition. For $u = (u_1, u_2, u_3) \in \mathbb{L}^2(\Omega_\epsilon)$, we write

$$\tilde{M}_\epsilon u = \begin{cases} (M_\epsilon u_1, M_\epsilon u_2, 0) & \text{for (FF), (FP) and (FD)} \\ (M_\epsilon u_1, M_\epsilon u_2, M_\epsilon u_3) & \text{for (PP)} \\ 0 & \text{for (DD) and (DP),} \end{cases} \tag{1.3}$$

and we set

$$\tilde{N}_\epsilon u = u - \tilde{M}_\epsilon u; \quad \text{i.e.,} \quad \tilde{M}_\epsilon + \tilde{N}_\epsilon u = I_{\mathbb{L}^2(\Omega_\epsilon)}. \tag{1.4}$$

The reason for which we gave different definitions for the average operator lies in the fact that there is no need to take the average of a function when its boundary values are zero at either the top or the bottom of the thin domain, i.e., when a function satisfies the Poincaré inequality in the thin direction.

It is useful to observe that for the boundary conditions under consideration, each component $(\tilde{N}_\epsilon u)_i,\ i = 1, 2, 3$ of $\tilde{N}_\epsilon u$ satisfy one of the conditions:

$$(\tilde{N}_\epsilon u)_i = 0 \text{ on } \Gamma_t \text{ and } \Gamma_b, \quad \text{or} \quad \int_0^\epsilon (\tilde{N}_\epsilon u)_i(x_1, x_2, x_3, t)\, dx_3 = 0. \tag{1.5}$$

All these operators are projectors; i.e.,

$$M_\epsilon^2 = M_\epsilon, \quad N_\epsilon^2 = N_\epsilon, \ \tilde{M}_\epsilon^2 = \tilde{M}_\epsilon, \quad \tilde{N}_\epsilon^2 = \tilde{N}_\epsilon. \tag{1.6}$$

Furthermore,we have the following properties which are obvious:

(i) M_ϵ is an orthogonal projector from $L^2(\Omega_\epsilon)$ onto $L^2(\omega)$.

(ii) $M_\epsilon N_\epsilon = 0$, and $\tilde{M}_\epsilon \tilde{N}_\epsilon = 0$ (1.7)

(iii) $M_\epsilon \nabla' = \nabla' M_\epsilon$, $N_\epsilon \nabla' = \nabla' N_\epsilon$, and $\tilde{M}_\epsilon \nabla' = \nabla' \tilde{M}_\epsilon$, $\tilde{N}_\epsilon \nabla' = \nabla' \tilde{N}_\epsilon$, (1.8)

(iv) $\varphi \in H^k(\Omega_\epsilon) \Rightarrow M_\epsilon \varphi \in H^k(\omega)$ and $N_\epsilon \varphi \in H^k(\Omega_\epsilon)$, $k \geq 0$. (1.9)

(v) The boundary condition for $\tilde{M}_\epsilon$ on $\partial\omega$ is the same as the one for u on $\partial\omega \times (0, \epsilon)$; i.e., (1.10)

If u satisfies (FD) or (DD), then $\tilde{M}_\epsilon u$ is zero on $\partial\omega$.

If u satisfies (FP) or (PP), then $\tilde{M}_\epsilon u$ is periodic.

If u satisfies (FF) , then $\tilde{M}_\epsilon u$ satisfies (FF).

In the following lemma, we give the basic properties of the operators M_ϵ and $\tilde{M}_\epsilon$. These properties hold for all the boundary conditions listed above; therefore, we omit the indices of the boundary conditions.

Lemma 1.1. *For all $u,\ v \in \mathbb{H}^1(\Omega_\epsilon)$, we have*

$$\int_{\Omega_\epsilon} \nabla \tilde{N}_\epsilon u \cdot \nabla \tilde{M}_\epsilon v\, dx = 0. \tag{1.11}$$

$$|u|_\epsilon^2 = |\tilde{M}_\epsilon u|_\epsilon^2 + |\tilde{N}_\epsilon u|_\epsilon^2 \quad \textit{and} \quad \|u\|_\epsilon^2 = \|\tilde{M}_\epsilon u\|_\epsilon^2 + \|\tilde{N}_\epsilon u\|_\epsilon^2, \quad \forall u \in \mathbb{H}^1(\Omega_\epsilon). \tag{1.12}$$

$$\textit{If} \ \ v \in V_\epsilon, \quad \textit{then } \tilde{M}_\epsilon v \in V_\epsilon \ \textit{and} \ \ \tilde{N}_\epsilon v \in V_\epsilon. \tag{1.13}$$

$$b_\epsilon(u, u, \tilde{M}_\epsilon v) = b_\epsilon(\tilde{M}_\epsilon u, \tilde{M}_\epsilon u, \tilde{M}_\epsilon v) + b_\epsilon(\tilde{N}_\epsilon u, \tilde{N}_\epsilon u, \tilde{M}_\epsilon v), \quad \textit{for} \ \ u, v \in V_\epsilon. \tag{1.14}$$

$$b_\epsilon(u,u,\tilde N_\epsilon v) = b_\epsilon(\tilde N_\epsilon u, \tilde M_\epsilon u, \tilde N_\epsilon v) + b_\epsilon(\tilde M_\epsilon u, \tilde N_\epsilon u, \tilde N_\epsilon v) + b_\epsilon(\tilde N_\epsilon u, \tilde N_\epsilon u, \tilde N_\epsilon v), \quad \textit{for} \quad u,v \in V_\epsilon. \tag{1.15}$$

For all $u \in D(A_\epsilon)$, *we have*

$$\Delta \tilde N_\epsilon u = \tilde N_\epsilon \Delta u, \quad \Delta \tilde M_\epsilon u = \tilde M_\epsilon \Delta u. \tag{1.16}$$

Proof. (i) Let $u,\ v \in \mathbb{H}^1(\Omega_\epsilon)$; we have immediately

$$\begin{aligned}\int_{\Omega_\epsilon} \nabla \tilde N_\epsilon u \cdot \nabla \tilde M_\epsilon v\, dx &= \int_{\Omega_\epsilon} \nabla' \tilde N_\epsilon u \cdot \nabla' \tilde M_\epsilon v\, dx \\ &= \int_{\Omega_\epsilon} \tilde N_\epsilon(\nabla' u) \cdot \tilde M_\epsilon(\nabla' v)\, dx = 0.\end{aligned} \tag{1.17}$$

Hence, we have (1.11).

(ii) The first Pythagorean identity in (1.12) is a consequence of the fact that $\tilde M_\epsilon$ is an orthogonal projector in $L^2(\Omega_\epsilon)$, while the second one is a consequence of (1.7), (1.8) and (1.11).

(iii) Let $v \in V_\epsilon$, it is clear that $\tilde M_\epsilon v \in \mathbb{H}^1(\Omega_\epsilon)$ and satisfies the same boundary condition as v on $\partial\Omega_\epsilon$. The only point that remains to be checked is div $\tilde M_\epsilon v = 0$: In the case of the boundary conditions (DD) and (DP), we have $\tilde M_\epsilon v = 0$. Hence, we need only to consider the boundary conditions (FF), (FP), (FD)and (PP). We integrate div $v = 0$ between 0 and ϵ and obtain

$$\text{div}'\ \tilde M_\epsilon v = \int_0^\epsilon \frac{\partial v_3}{\partial x_3}\, dx_3 = 0. \tag{1.18}$$

The second term in the left-hand side of (1.18) vanishes in all the cases (FF), (FP), (FD)and (PP). Hence,

$$\text{div}'\ \tilde M_\epsilon v = 0. \tag{1.19}$$

(iv) Let $u,\ v \in V_\epsilon$; we have

$$\begin{aligned}b_\epsilon(u,u,\tilde M_\epsilon v) = &\int_{\Omega_\epsilon} (\tilde M_\epsilon u \cdot \nabla)\tilde M_\epsilon u \cdot \tilde M_\epsilon v\, dx + \int_{\Omega_\epsilon} (\tilde N_\epsilon u \cdot \nabla)\tilde M_\epsilon u \cdot \tilde M_\epsilon v\, dx \\ &+ \int_{\Omega_\epsilon} (\tilde N_\epsilon u \cdot \nabla)\tilde N_\epsilon u \cdot \tilde M_\epsilon v\, dx + \int_{\Omega_\epsilon} (\tilde M_\epsilon u \cdot \nabla)\tilde N_\epsilon u \cdot \tilde M_\epsilon v\, dx.\end{aligned} \tag{1.20}$$

Now, since $\int_0^\epsilon \tilde N_\epsilon u\, dx_3 = \epsilon \tilde M_\epsilon\big(\tilde N_\epsilon u\big) = 0$, we have

$$\int_{\Omega_\epsilon} (\tilde N_\epsilon u \cdot \nabla)\tilde M_\epsilon u \cdot \tilde M_\epsilon v\, dx = \int_\omega \left(\int_0^\epsilon \tilde N_\epsilon u\, dx_3 \cdot \nabla\right)\tilde M_\epsilon u \cdot \tilde M_\epsilon v\, dx' = 0.$$

For the integral $\int_{\Omega_\epsilon} (\tilde M_\epsilon u \cdot \nabla)\tilde N_\epsilon u \cdot \tilde M_\epsilon v\, dx$, we observe that

$$\sum_{j=1}^3 \int_{\Omega_\epsilon} M_\epsilon u_i \frac{\partial N_\epsilon u_j}{\partial x_i} M_\epsilon v_j\, dx = \sum_{j=1}^3 \int_{\Omega_\epsilon} M_\epsilon u_i M_\epsilon v_j N_\epsilon\big(\frac{\partial u_j}{\partial x_i}\big)\, dx = 0, \tag{1.21}$$

due to (1.8), for $i = 1, 2$, while for $i = 3$, the similar integral vanishes because of (1.5), hence (1.14).

(v) Similarly, for $u,\ v \in V_\epsilon$, we have

$$b_\epsilon(u, u, \tilde{N}_\epsilon v) = \int_{\Omega_\epsilon} (\tilde{M}_\epsilon u \cdot \nabla)\tilde{M}_\epsilon u \cdot \tilde{N}_\epsilon v\, dx + \int_{\Omega_\epsilon} (\tilde{M}_\epsilon u \cdot \nabla)\tilde{N}_\epsilon u \cdot \tilde{N}_\epsilon v\, dx + \int_{\Omega_\epsilon} (\tilde{N}_\epsilon u \cdot \nabla)\tilde{M}_\epsilon u \cdot \tilde{N}_\epsilon v\, dx + \int_{\Omega_\epsilon} (\tilde{N}_\epsilon u \cdot \nabla)\tilde{N}_\epsilon u \cdot \tilde{N}_\epsilon v\, dx, \quad (1.22)$$

but

$$\int_{\Omega_\epsilon} (\tilde{M}_\epsilon u \cdot \nabla)\tilde{M}_\epsilon u \cdot \tilde{N}_\epsilon v\, dx = \int_\omega \tilde{M}_\epsilon \big[(\tilde{M}_\epsilon u \cdot \nabla')\tilde{M}_\epsilon u\big] \cdot \left(\int_0^\epsilon \tilde{N}_\epsilon v\, dx_3 \right) dx' = 0. \quad (1.23)$$

Hence (1.15) is established.

(vi) In the case of the boundary condition (DP) or (DD), the identities (1.16) are obvious, since $\tilde{M}_\epsilon u = 0$ and $\tilde{N}_\epsilon u = u$. The case of the purely periodic condition (PP) is also obvious, since $\frac{\partial}{\partial x_i}\tilde{M}_\epsilon = \tilde{M}_\epsilon \frac{\partial}{\partial x_i}$, $i = 1, 2, 3$. We need to prove (1.16) when the boundary condition is (FF), (FP) or (FD). We will only treat (FD); the two other cases can be treated similarly.

Let $v \in (\mathcal{C}_0^\infty(\omega))^2$; we have by integration by parts

$$\int_\omega \Delta\tilde{M}_\epsilon u \cdot v\, dx' = \int_\omega \Delta'\tilde{M}_\epsilon u \cdot v\, dx' = -\int_\omega \nabla'\tilde{M}_\epsilon u \cdot \nabla' v\, dx' = -\int_\omega \tilde{M}_\epsilon(\nabla' u) \cdot \nabla' v\, dx' = -\int_{\Omega_\epsilon} \nabla' u \cdot \nabla' v\, dx = \int_{\Omega_\epsilon} \Delta' u \cdot v\, dx = \int_\omega \tilde{M}_\epsilon(\Delta' u) \cdot v\, dx' = \int_\omega \tilde{M}_\epsilon(\Delta u) \cdot v\, dx'. \quad (1.24)$$

The last equality is obtained, thanks to $\frac{\partial u_\alpha}{\partial x_3} = 0$ on $\Gamma_t \cup \Gamma_b$, $\alpha = 1, 2$. Hence $\Delta\tilde{M}_\epsilon u = \tilde{M}_\epsilon \Delta u$. Then, we have on the one hand

$$\Delta u = \Delta\tilde{M}_\epsilon u + \Delta\tilde{N}_\epsilon u = \tilde{M}_\epsilon(\Delta u) + \Delta\tilde{N}_\epsilon u, \quad (1.25)$$

and on the other hand

$$\Delta u = \tilde{M}_\epsilon(\Delta u) + \tilde{N}_\epsilon(\Delta u); \quad (1.26)$$

therefore, $\Delta\tilde{N}_\epsilon u = \tilde{N}_\epsilon(\Delta u)$. □

In addition to the lemma above, we need to establish the commutativity of the operators $\tilde{M}_\epsilon$ and $\tilde{N}_\epsilon$ with the Stokes operator A_ϵ. In the case of one of the boundary conditions (FF), (FP) and (PP), we have $\Delta u = A_\epsilon u$ for $u \in D(A_\epsilon)$ (see [31]) and the commutativity follows from (1.16). In the case of the boundary conditions (DP) and (DD), we have $\tilde{M}_\epsilon = 0$. Hence the commutativity follows. There remains the case of the boundary condition (FD). First we note that

$$\Delta u|_{\Gamma_t \cup \Gamma_b} = 0, \quad \text{for} \quad u \in D(A_\epsilon).$$

Indeed, since div $u = 0$, we have

$$\frac{\partial^2 u_3}{\partial x_3^2} = -\frac{\partial^2 u_1}{\partial x_3 x_1} - \frac{\partial^2 u_2}{\partial x_3 x_2} \quad \text{in } \Omega_\epsilon$$

and since $\frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = 0$ on $\Gamma_t \cup \Gamma_b$, we have

$$\frac{\partial^2 u_3}{\partial x_3^2} = 0 \quad \text{on } \Gamma_t \cup \Gamma_b.$$

Moreover, since $u_3 = 0$ on $\Gamma_t \cup \Gamma_b$, it is clear that

$$\frac{\partial^2 u_3}{\partial x_1^2} = \frac{\partial^2 u_3}{\partial x_2^2} = 0 \quad \text{on } \Gamma_t \cup \Gamma_b.$$

Now from the characterization of the Leray projector (see [5] and [25]), we have

$$A_\epsilon u = -\Delta u + \nabla p,$$

where

$$\begin{aligned} \Delta p &= 0 && \text{in } \ \Omega_\epsilon, \\ \frac{\partial p}{\partial n_\omega} &= \Delta' u \cdot n_\omega && \text{on } \ \partial\omega \times (0, \epsilon), \\ \frac{\partial p}{\partial x_3} &= 0 && \text{on } \ \omega \times \{0, \epsilon\}. \end{aligned}$$

Applying the operator $\tilde{M}_\epsilon$ and using (1.8) and (1.16), we obtain

$$\tilde{M}_\epsilon A_\epsilon u = -\Delta \tilde{M}_\epsilon u + \nabla' M p$$

and

$$\begin{aligned} \Delta' M p &= 0 && \text{in } \omega, \\ \frac{\partial M p}{\partial n_\omega} &= (\Delta' M u) \cdot n_\omega, \end{aligned}$$

which is the characterization of the 2D Leray projector for $\Delta' \tilde{M}_\epsilon u$. Therefore

$$A_\epsilon \tilde{M}_\epsilon u = \tilde{M}_\epsilon A_\epsilon u.$$

Finally, we recall an important orthogonal property related to the trilinear form. The orthogonal property reads

$$\tilde{b}(v, v, \tilde{A} v) = 0, \qquad \forall v \in D(\tilde{A}), \tag{1.27}$$

where $\tilde{b}$ is the 2D trilinear form, and $\tilde{A}$ is to the 2D Stokes operator with either the periodic boundary condition (see [5], [26, 27]) or the free boundary condition (see [31]). The identity (1.27) will be used in our work in order to obtain better estimates of the strong solution in thin three-dimensional domains, and will apply in the cases of the boundary conditions (FF) and (FP). We state also the following lemma, which is obtained by integrating by parts several times (see [31]):

Lemma 1.2. *In the case of either boundary conditions* (PP), (FF) *or* (FP), *we have for* $u \in D(A_\epsilon)$

$$b_\epsilon(\tilde{M}_\epsilon u, \tilde{M}_\epsilon u, A_\epsilon \tilde{M}_\epsilon u) = b(\tilde{M}_\epsilon u, \tilde{M}_\epsilon u, \tilde{A}\tilde{M}_\epsilon u) = 0. \tag{1.28}$$

2. Fundamental inequalities in thin domains.

2.1. Sobolev-type inequalities. One of the basic tools in the study of nonlinear partial differential equations in thin domains is the knowledge of the exact dependence, with respect to the thickness of the domain, of the constants appearing in the Sobolev and related inequalities. In the classical form of Sobolev inequalities, the dilation invariance of the constants is emphasized. However, in thin domains, the shape of the domain is important. Therefore, we break away from isotropy and derive a version of the inequalities emphasizing the dependence of the constants on the thickness ϵ of the domain.

In our work, we will prove appropriate versions of the Poincaré inequality, Agmon's inequality and Ladyzhenskaya's inequalities. The Poincaré inequality in thin domains was obtained in [13, 14] and [22, 23, 24]. The proof is classical (see [12] and [21]). Also, Ladyzhenskaya's inequalities were obtained in [22, 23, 24] when the boundary condition is purely periodic and independently in [2] for the Dirichlet boundary condition.

We should also mention the work of Solonnikov [35] in which he obtained anisotropic inequalities for functions in $W^{m,p}(\mathbb{R}^n)$, the space of distributions which are in $L^p(\mathbb{R}^n)$ along with their derivatives of order $\leq m$. He showed that

$$\text{If} \quad b = \sum_{j=1}^{n} \beta_j < 1 \quad \text{and, either } p, q, \tau > 1 \quad \text{or}$$

$$\mu = 1 - \sum_{j=1}^{n} \frac{\alpha_j}{m_j} + \Big(\frac{1}{q} - \frac{1}{p}\Big) \sum_{j=1}^{n} \frac{1}{m_j} > 0,$$

with $q \geq p \geq 1$ and $\tau \geq 1$, then there exists a positive constant c such that

$$\big|D^\alpha u\big|_{L^q(\mathbb{R}^n)} \leq c \bigg(\prod_{j=1}^{n} \Big| \frac{\partial^{m_j} u}{\partial {x_j}^{m_j}} \Big|_{L^p(\mathbb{R}^n)}^{\beta_j} \bigg) \big|u\big|_{L^\tau(\mathbb{R}^n)}^{1-\sum_{j=1}^{n}\beta_j}, \tag{*}$$

where $D^\alpha = \dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \dfrac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$.

The method that we develop below will imply the validity of the inequality (*) in bounded parallelepipeds (see the proof of Proposition 2.2).

The following result is easy and classical (see [12], [21], [13, 14]), etc.); we incorporate it here for the sake of completeness.

Proposition 2.1 (Poincaré's inequality)**.** *For* $u \in H^1(\Omega_\epsilon)$ *satisfying one of the following conditions*:

$$\begin{cases} (i) \quad u = 0 \quad \text{on} \quad \Gamma_t, \\ (ii) \quad u = 0 \quad \text{on} \quad \Gamma_b, \\ (iii) \quad \displaystyle\int_0^\epsilon u(x_1, x_2, x_3)\, dx_3 = 0 \quad \text{a.e. in } \omega, \end{cases} \tag{2.1}$$

we have

$$|u|_{L^2(\Omega_\epsilon)} \leq \epsilon \Big|\frac{\partial u}{\partial x_3}\Big|_{L^2(\Omega_\epsilon)}. \tag{2.2}$$

We recall that (2.1) (i), (ii) or (iii) is valid for each component of a function $\tilde{N}_\epsilon u$, where u satisfies any of the boundary conditions under consideration.

Proof. First note that $\{u \in \mathcal{C}^1(\bar{\Omega}_\epsilon);\ \ u = 0 \ \text{ on } \ \Gamma_t \text{ (respectively } \Gamma_b)\}$ is dense in

$$\left\{u \in H^1(\Omega_\epsilon);\ u = 0 \ \text{ on } \ \Gamma_t \text{ (respectively } \Gamma_b)\right\},$$

and

$$\left\{u \in \mathcal{C}^1(\bar{\Omega}_\epsilon);\ \ \int_0^\epsilon u(x_1, x_2, x_3)\, dx_3 = 0,\ \forall x_1, x_2\right\} \ \text{ is dense in}$$

$$\left\{u \in H^1(\Omega_\epsilon);\ \int_0^\epsilon u(x_1, x_2, x_3)\, dx_3 = 0, \text{a.e. in } \omega\right\}.$$

Thanks to a density argument, we assume that $u \in \mathcal{C}^1(\Omega_\epsilon)$. We have for any $\zeta,\ \eta$ in $[0, \epsilon]$

$$\begin{aligned} u^2(x', \zeta) + u^2(x', \eta) &= 2u(x', \zeta)u(x', \eta) + \big(u(x', \zeta) - u(x', \eta)\big)^2 \\ &= 2u(x', \zeta)u(x', \eta) + \Big(\int_\eta^\zeta \frac{\partial u}{\partial x_3}(x', s)\, ds\Big)^2, \end{aligned} \tag{2.3}$$

with $x' = (x_1, x_2)$. We fix ζ and integrate with respect to η to obtain

$$\epsilon u^2(x', \zeta) + \int_0^\epsilon u^2(x', \eta)\, d\eta = 2u(x', \zeta) \int_0^\epsilon u(x', \eta)\, d\eta + \int_0^\epsilon \Big(\int_\eta^\zeta \frac{\partial u}{\partial x_3}(x', s)\, ds\Big)^2 d\eta. \tag{2.4}$$

Now if $u = 0$ on Γ_t (respectively Γ_b,) we take $\zeta = \epsilon$ (respectively $\zeta = 0$) and obtain from (2.4)

$$\int_0^\epsilon u^2(x', \eta)\, d\eta \leq \int_0^\epsilon |\zeta - \eta| \Big(\int_0^\epsilon \Big|\frac{\partial u}{\partial x_3}(x', s)\Big|^2\Big) ds \leq (\epsilon)^2 \Big(\int_0^\epsilon \Big|\frac{\partial u}{\partial x_3}(x', s)\Big|^2\Big) ds, \tag{2.5}$$

and (2.2) follows promptly.

If $\int_0^\epsilon u(x', \eta)\, d\eta = 0$, we infer from (2.4)

$$\begin{aligned} \int_0^\epsilon u^2(x', \eta)\, d\eta &\leq \int_0^\epsilon \Big(\int_\eta^\zeta \frac{\partial u}{\partial x_3}(x', s)\, ds\Big)^2 d\eta \leq \int_0^\epsilon |\zeta - \eta| \Big(\int_0^\epsilon \Big|\frac{\partial u}{\partial x_3}(x', s)\Big|^2\Big)\, ds \\ &\leq (\epsilon)^2 \Big(\int_0^\epsilon \Big|\frac{\partial u}{\partial x_3}(x', s)\Big|^2\Big)\, ds. \end{aligned}$$

The proof is complete. □

It is easy to see that for each boundary condition listed above, each component of $\tilde{N}_\epsilon v$, where $v \in V_\epsilon$, satisfies one of the conditions (2.1). Therefore, we have the following:

Corollary 2.1. *Under any of the boundary conditions under consideration, we have for all* $v \in V_\epsilon$

$$|\tilde{N}_\epsilon v|_\epsilon \le \epsilon \Big|\frac{\partial \tilde{N}_\epsilon v}{\partial x_3}\Big|_\epsilon. \tag{2.6}$$

Proposition 2.2 (Anisotropic Agmon's inequality)**.** *Let* $\Omega_0 = (0,1)^3$; *then there exists an absolute constant* c *such that*

$$\big|u\big|_{L^\infty(\Omega_0)} \le c\big|u\big|^{\frac{1}{4}}_{L^2(\Omega_0)} \prod_{i=1}^{3}\Big(\Big|\frac{\partial^2 u}{\partial x_i^2}\Big|_{L^2(\Omega_0)} + \Big|\frac{\partial u}{\partial x_i}\Big|_{L^2(\Omega_0)} + \big|u\big|_{L^2(\Omega_0)}\Big)^{\frac{1}{4}}, \tag{2.7}$$

for all $u \in H^2(\Omega_0)$.

Proof. We prove the lemma in three steps.

Step 1. We replace Ω_0 by $\mathbb{R}^3$ and assume that $u \in \mathcal{C}_0^2(\mathbb{R}^3)$. We write, using the Sobolev inclusion $H_0^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$,

$$\big|u\big|_{L^\infty(\mathbb{R}^3)} \le c_0 \sum_{i=1}^{3}\Big|\frac{\partial^2 u}{\partial x_i^2}\Big|_{L^2(\mathbb{R}^3)} + c_0\big|u\big|_{L^2(\mathbb{R}^3)}. \tag{2.8}$$

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$, $\lambda_i > 0$, $i = 1,2,3$, and set $y_i = \lambda_i x_i$, $i = 1,2,3$. We define the function $v_\lambda \in \mathcal{C}_0^2(\mathbb{R}^3)$ as follows:

$$v_\lambda(y_1, y_2, y_3) = u\Big(\frac{y_1}{\lambda_1}, \frac{y_2}{\lambda_2}, \frac{y_3}{\lambda_3}\Big); \tag{2.9}$$

we have immediately

$$\begin{cases} \big|v_\lambda\big|_{L^\infty(\mathbb{R}^3)} = \big|u\big|_{L^\infty(\mathbb{R}^3)}, \quad \big|v_\lambda\big|_{L^2(\mathbb{R}^3)} = (\lambda_1\lambda_2\lambda_3)^{\frac{1}{2}}\big|u\big|_{L^2(\mathbb{R}^3)} \\ \Big|\dfrac{\partial^2 v_\lambda}{\partial y_i^2}\Big|_{L^2(\mathbb{R}^3)} = \dfrac{(\lambda_1\lambda_2\lambda_3)^{\frac{1}{2}}}{\lambda_i^2}\Big|\dfrac{\partial^2 u}{\partial x_i^2}\Big|_{L^2(\mathbb{R}^3)}. \end{cases} \tag{2.10}$$

Inequality (2.8) applied to v_λ yields

$$\big|u\big|_{L^\infty(\mathbb{R}^3)} = \big|v_\lambda\big|_{L^\infty(\mathbb{R}^3)} \le \frac{c_0}{4}(\lambda_1\lambda_2\lambda_3)^{\frac{1}{2}} \sum_{i=1}^{3}\frac{1}{\lambda_i^2}\Big|\frac{\partial^2 u}{\partial x_i^2}\Big|_{L^2(\mathbb{R}^3)} + \frac{c_0}{4}(\lambda_1\lambda_2\lambda_3)^{\frac{1}{2}}\big|u\big|_{L^2(\mathbb{R}^3)}, \tag{2.11}$$

and since (2.11) is valid for all choices of $\lambda_1, \lambda_2, \lambda_3$, we can take

$$\lambda_i = \frac{\big|\frac{\partial^2 u}{\partial x_i^2}\big|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}}{\big|u\big|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}}, \quad i = 1,2,3. \tag{2.12}$$

Hence,

$$\big|u\big|_{L^\infty(\mathbb{R}^3)} \le 4c_0\big|u\big|^{\frac{1}{4}}_{L^2(\mathbb{R}^3)} \prod_{i=1}^{3}\Big|\frac{\partial^2 u}{\partial x_i^2}\Big|^{\frac{1}{4}}_{L^2(\mathbb{R}^3)}.$$

Step 2. We will show that there exists an absolute constant $c_1 > 0$, such that if $u \in \mathcal{C}^2(\bar{\Omega}_0)$, then there exists $\bar{u} \in \mathcal{C}^2(\bar{\Omega}_1)$, where $\Omega_1 = (-\frac{1}{3}, \frac{4}{3})^3$ such that

$$\Big|\frac{\partial^k \bar{u}}{\partial x_i^k}\Big|_{L^2(\Omega_1)} \leq c_1 \Big|\frac{\partial^k \bar{u}}{\partial x_i^k}\Big|_{L^2(\Omega_0)}, \quad k = 0, 1, 2;\ i = 1, 2, 3. \tag{2.13}$$

We use the classical Babitch extension operators (see [1], [34], [21]); we first extend u to the domain $(-\frac{1}{3}, \frac{4}{3}) \times (0, 1)^2$ using

$$E_k^1 u(x) = \begin{cases} u(x) & \text{for } x_1 \in [0, 1], \\ \displaystyle\sum_{j=1}^{3} (-j)^k \alpha_j u(-jx_1, x_2, x_3) & \text{for } x_1 \in [-\frac{1}{3}, 0), \\ \displaystyle\sum_{j=1}^{3} (-j)^k \alpha_j u(1 - j(x_1 - 1), x_2, x_3) & \text{for } x_1 \in (1, \frac{4}{3}), \end{cases} \tag{2.14}$$

where $(\alpha_1, \alpha_2, \alpha_3)$ is the unique solution to the system

$$\sum_{j=1}^{3} (-j)^k \alpha_j = 1, \quad k = 0, 1, 2. \tag{2.15}$$

We have immediately

$$\frac{\partial^k E_0^1 u}{\partial x_i^k} = E_k^1 \frac{\partial^k u}{\partial x_i^k}, \quad k = 0, 1, 2; \quad i = 1, 2, 3, \tag{2.16}$$

and (2.15) implies that

$$E_0^1 u \in \mathcal{C}^2([-\frac{1}{3}, \frac{4}{3}] \times [0, 1]^2) \quad \text{if} \quad u \in \mathcal{C}^2(\bar{\Omega}_0).$$

Moreover,

$$\begin{aligned} &\int_{(-1/3,4/3)\times(0,1)^2} \Big|\frac{\partial^k E_0^1 u}{\partial x_i^k}\Big|^2 dx = \int_{(-\frac{1}{3},\frac{4}{3})\times(0,1)^2} \Big|E_k^1 \frac{\partial^k u}{\partial x_i^k}\Big|^2 dx \\ &\quad = \int_{\Omega_0} \Big|\frac{\partial^k u}{\partial x_i^k}\Big|^2 dx + \int_{(-\frac{1}{3},0)\times(0,1)^2} \Big|\sum_{j=1}^{3} (-j)^k \alpha_j \frac{\partial^k u}{\partial x_i^k}(-jx_1, x_2, x_3)\Big|^2 dx \\ &\quad + \int_{(1,\frac{4}{3})\times(0,1)^2} \Big|\sum_{j=1}^{3} (-j)^k \alpha_j \frac{\partial^k u}{\partial x_i^k}(1 - j(x_1 - 1), x_2, x_3)\Big|^2 dx \leq c_1^2 \int_{\Omega_0} \Big|\frac{\partial^k u}{\partial x_i^k}\Big|^2 dx. \end{aligned} \tag{2.17}$$

The extension is complete in the direction x_1.

The same argument can be applied in the directions x_2 and x_3, but instead of applying it on the domain Ω_0, we apply it to extend the function $E_0^1 u$ defined in $(-\frac{1}{3},\frac{4}{3})\times(0,1)^2$ to the domain $(-\frac{1}{3},\frac{4}{3})^2\times(0,1)$. The extended function is denoted by $E_0^2 u$ and satisfies

$$\begin{aligned}|\frac{\partial^k E_0^2 u}{\partial x_i^k}|^2_{L^2((-\frac{1}{3},\frac{4}{3})^2\times(0,1))} &\le c_1|\frac{\partial^k E_0^1 u}{\partial x_i^k}|^2_{L^2((-\frac{1}{3},\frac{4}{3})\times(0,1)^2)}\\ &\le c_1|\frac{\partial^k u}{\partial x_i^k}|^2_{L^2(\Omega_0)},\quad k=0,1,2;\ i=1,2,3.\end{aligned}\tag{2.18}$$

Finally the extension of $E_0^2 u$ in the direction x_3 yields a function $\bar{u}\in\mathcal{C}^2(\bar{\Omega}_1)$, $\Omega_1=(-\frac{1}{3},\frac{4}{3})^3$ with

$$|\frac{\partial^k \bar{u}}{\partial x_i^k}|^2_{L^2(\Omega_1)}\le c_1^3|\frac{\partial^k u}{\partial x_i^k}|^2_{L^2(\Omega_0)},\quad k=0,1,2;\quad i=1,2,3.\tag{2.19}$$

Step 3. We will show that we can localize without "mixing the directions." Let $\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$ be such that supp $\varphi\subset(-\frac{1}{3},\frac{4}{3})$, $0\le\varphi\le 1$ and $\varphi=1$ on $[0,1]$. Set

$$\Phi(x_1,x_2,x_3)=\prod_{i=1}^{3}\varphi(x_i).\tag{2.20}$$

We have immediately (with $\bar{u}$ being the extension of u defined in Step 2)

$$\Phi\bar{u}\in\mathcal{C}_0^2(\mathbb{R}^3),\quad |u|_{L^\infty(\Omega_0)}=|\Phi\bar{u}|_{L^\infty(\mathbb{R}^3)},\quad |\Phi\bar{u}|_{L^2(\mathbb{R}^3)}\le c_2|u|_{L^2(\Omega_0)},\tag{2.21}$$

where c_2 is an absolute constant. Moreover,

$$\frac{\partial^2\Phi\bar{u}}{\partial x_i^2}=\Phi\frac{\partial^2\bar{u}}{\partial x_i^2}+2\frac{\partial\Phi}{\partial x_i}\frac{\partial\bar{u}}{\partial x_i}+\bar{u}\frac{\partial^2\Phi}{\partial x_i^2}.\tag{2.22}$$

Let $K=\big[1+\sup_{\mathbb{R}}(2|\varphi'|+|\varphi''|)\big]$; we have (with $\Omega_1=(-\frac{1}{3},\frac{4}{3})^3$)

$$\begin{aligned}|\frac{\partial^2\Phi\bar{u}}{\partial x_i^2}|_{L^2(\mathbb{R}^3)} &\le K(|\frac{\partial^2\bar{u}}{\partial x_i^2}|_{L^2(\Omega_1)}+|\frac{\partial\bar{u}}{\partial x_i}|_{L^2(\Omega_1)}+|\bar{u}|_{L^2(\Omega_1)})\\ &\le K_1(|\frac{\partial^2 u}{\partial x_i^2}|_{L^2(\Omega_0)}+|\frac{\partial u}{\partial x_i}|_{L^2(\Omega_0)}+|u|_{L^2(\Omega_0)}).\end{aligned}\tag{2.23}$$

We conclude the lemma by combining (2.13), (2.21) and (2.23). $\square$

Thanks to Proposition 2.2, it is easy to obtain

Corollary 2.2 (Agmon's inequality in thin domains)**.** *There exists a positive constant $c_0(\omega)$, independent of ϵ, such that $\forall u\in H^2(\Omega_\epsilon)$*

$$\begin{aligned}|u|_{L^\infty(\Omega_\epsilon)}\le c_0|u|^{\frac{1}{4}}_{L^2(\Omega_\epsilon)}&\left(|\frac{\partial^2 u}{\partial x_3^2}|_{L^2(\Omega_\epsilon)}+\frac{1}{\epsilon}|\frac{\partial u}{\partial x_3}|_{L^2(\Omega_\epsilon)}+\frac{1}{\epsilon^2}|u|_{L^2(\Omega_\epsilon)}\right)^{\frac{1}{4}}\\ &\times\prod_{i=1}^{2}\left(\sum_{j=1}^{2}(|\frac{\partial^2 u}{\partial x_i\partial x_j}|_{L^2(\Omega_\epsilon)}+|\frac{\partial u}{\partial x_j}|_{L^2(\Omega_\epsilon)}+|u|_{L^2(\Omega_\epsilon)}\right)^{\frac{1}{4}}.\end{aligned}\tag{2.24}$$

Proof. First assume that the domain Ω_ϵ is of the form $(0,1)^2 \times (0,\epsilon)$. We use (2.9) with $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \frac{1}{\epsilon}$. This allows us to work in the domain $(0,1)^3$ and to use (2.7). Thanks to (2.10), we have

$$|u|_{L^\infty(\Omega_\epsilon)} \leq c_0 |u|_{L^2(\Omega_\epsilon)}^{\frac{1}{4}} \Big(|\frac{\partial^2 u}{\partial x_3^2}|_{L^2(\Omega_\epsilon)} + \frac{1}{\epsilon} |\frac{\partial u}{\partial x_3}|_{L^2(\Omega_\epsilon)} + \frac{1}{\epsilon^2} |u|_{L^2(\Omega_\epsilon)} \Big)^{\frac{1}{4}} \times \prod_{i=1}^{2} \Big(|\frac{\partial^2 u}{\partial x_i^2}|_{L^2(\Omega_\epsilon)} + |\frac{\partial u}{\partial x_i}|_{L^2(\Omega_\epsilon)} + |u|_{L^2(\Omega_\epsilon)} \Big)^{\frac{1}{4}}.$$

For a general domain $\Omega_\epsilon = \omega \times (0,\epsilon)$, ω a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^2$, we obtain similarly (2.24): We proceed in the same way in the x_3 direction and, in directions x_1, x_2 we classically proceed by localization using a partition of unity. The localization procedure induces a mixing of the derivatives in the directions x_1 and x_2, hence (2.24). □

For any component $(\tilde{N}_\epsilon u)_i$ of $\tilde{N}_\epsilon u$, where $u \in D(A_\epsilon)$, we observe that besides (2.2), we have by integration by parts in x_3 :

$$\int_{\Omega_\epsilon} (\tilde{N}_\epsilon u)_i \frac{\partial^2 (\tilde{N}_\epsilon u)_i}{\partial x_3^2}\, dx = - \int_{\Omega_\epsilon} (\frac{\partial (\tilde{N}_\epsilon u)_i}{\partial x_3})^2\, dx,$$

so that

$$|\frac{\partial (\tilde{N}_\epsilon u)_i}{\partial x_3}|_{L^2(\Omega_\epsilon)} \leq \epsilon |\frac{\partial^2 (\tilde{N}_\epsilon u)_i}{\partial x_3^2}|_{L^2(\Omega_\epsilon)},$$

and

$$|\frac{\partial \tilde{N}_\epsilon u}{\partial x_3}|_{L^2(\Omega_\epsilon)} \leq \epsilon |\frac{\partial^2 \tilde{N}_\epsilon u}{\partial x_3^2}|_{L^2(\Omega_\epsilon)}. \tag{2.25}$$

Hence, combining Corollaries 2.1 and 2.2, we have

Corollary 2.3. *There exists a positive constant $c_0(\omega)$, independent of ϵ, such that*

$$|\tilde{N}_\epsilon u|_{L^\infty(\Omega_\epsilon)} \leq c_0 |\tilde{N}_\epsilon u|_{L^2(\Omega_\epsilon)}^{\frac{1}{4}} \Big(\sum_{i,j=1}^{3} |\frac{\partial^2 \tilde{N}_\epsilon u}{\partial x_i \partial x_j}|_{L^2(\Omega_\epsilon)} \Big)^{\frac{3}{4}}, \quad \forall u \in D(A_\epsilon). \tag{2.26}$$

The use of $\tilde{N}_\epsilon u$ in (2.26) is necessary to obtain a constant independent of ϵ, since $\tilde{N}_\epsilon u$ satisfies (2.2) and (2.25).

Now we give the anisotropic Ladyzhenskaya inequality.

Proposition 2.3 (Anisotropic Ladyzhenskaya's inequality)**.** *Let $\Omega = \prod_{i=1}^{3}(a_i, b_i)$. There exists an absolute constant c_0 such that*

$$|u|_{L^6(\Omega)} \leq c_0 \prod_{i=1}^{3} \Big(\frac{1}{b_i - a_i} |u|_{L^2(\Omega)} + |\frac{\partial u}{\partial x_i}|_{L^2(\Omega)} \Big)^{\frac{1}{3}}, \quad \forall u \in H^1(\Omega). \tag{2.27}$$

Proof. It is enough to establish the inequality for $u \in \mathcal{C}^\infty(\bar\Omega)$, where $\bar\Omega$ denotes the closure of Ω. Without loss of generality, we can assume that u is positive.

We write, with $\omega = \prod_{i=1}^{2}(a_i, b_i)$,

$$\int_\Omega u^6(x)\,dx \le \int_{a_3}^{b_3} dx_3 \Big[\int_{a_1}^{b_1} \max_{x_2} u^3(x)\,dx_1\Big]\Big[\int_{a_2}^{b_2} \max_{x_1} u^3(x)\,dx_2\Big]. \tag{2.28}$$

Now we estimate $\int_{a_1}^{b_1} \max_{x_2} u^3(x)\,dx_1$ and $\int_{a_2}^{b_2} \max_{x_1} u^3(x)\,dx_2$. For $a_i \le x_i \le \frac{a_i+b_i}{2}$, we have by integration by parts in t (with $1 < p < \infty$)

$$\begin{aligned}\int_0^{\frac{b_i-a_i}{2}} u^p(x', x_i + \frac{b_i-a_i}{2} - t)\,dt &= \frac{b_i-a_i}{2} u^p(x) \\ &+ p\int_0^{\frac{b_i-a_i}{2}} t u^{p-1}(x', x_i + \frac{b_i-a_i}{2} - t)\frac{\partial u}{\partial x_i}(x', x_i + \frac{b_i-a_i}{2} - t)\,dt,\end{aligned} \tag{2.29}$$

but since $t \in [0, \frac{b_i-a_i}{2}]$, we have

$$\begin{aligned}\frac{b_i-a_i}{2} u^p(x) &\le \int_{a_i}^{b_i} u^p(x)\,dx_i + \frac{p}{2}(b_i-a_i)\int_{a_i}^{b_i} u^{p-1}(x)\big|\frac{\partial u}{\partial x_i}(x)\big|\,dx_i \\ &\le \int_{a_i}^{b_i} u^{p-1}(x)\big[|u(x)| + \frac{p}{2}(b_i-a_i)\big|\frac{\partial u}{\partial x_i}(x)\big|\big]\,dx_i \\ &\le \text{(Cauchy-Schwarz)} \\ &\le \Big(\int_{a_i}^{b_i} u^{2(p-1)}(x)\,dx_i\Big)^{\frac12}\Big(\int_{a_i}^{b_i}\big[|u(x)| + \frac{p}{2}(b_i-a_i)\big|\frac{\partial u}{\partial x_i}(x)\big|\big]^2 dx_i\Big)^{\frac12}.\end{aligned} \tag{2.30}$$

The inequality above is also valid for $\frac{a_i+b_i}{2} \le x_i \le b_i$. Hence, for $i, j = 1, 2,\ i \ne j$, we have (with $p = 3$)

$$\begin{aligned}&\frac{b_i-a_i}{2}\int_{a_j}^{b_j} \max_{x_i} u^3(x)\,dx_j \\ &\le \Big(\int_\omega u^4(x)\,dx'\Big)^{\frac12}\Big(\int_\omega \big[|u(x)| + \frac{3(b_i-a_i)}{2}\big|\frac{\partial u}{\partial x_i}(x)\big|\big]^2 dx'\Big)^{\frac12}.\end{aligned} \tag{2.31}$$

Therefore,

$$\begin{aligned}\int_\Omega u^6(x)\,dx &\le \int_{a_3}^{b_3} dx_3 \Big[\int_\omega u^4(x)\,dx'\Big]\Big[\prod_{i=1}^{2}\int_\omega [\frac{2}{b_i-a_i}|u(x)| + 3|\frac{\partial u}{\partial x_i}|]^2\,dx'\Big]^{\frac12} \\ &\le \Big(\max_{x_3}\int_\omega u^4(x)\,dx'\Big)\Big[\prod_{i=1}^{2}\int_\Omega [\frac{2}{b_i-a_i}|u(x)| + 3|\frac{\partial u}{\partial x_i}|]^2\,dx\Big]^{\frac12}.\end{aligned} \tag{2.32}$$

Finally, we estimate $\max_{x_3} \int_\omega u^4(x)\, dx'$; we use inequality (2.30) with $i = 3$ and $p = 4$ and obtain after integration with respect to x_1 and x_2

$$\frac{b_3 - a_3}{2} \int_\omega u^4(x)\, dx' \leq \Big(\int_\Omega u^6(x)\, dx\Big)^{\frac{1}{2}} \Big(\int_\Omega \big[|u(x)| + 2(b_3 - a_3)\big|\frac{\partial u}{\partial x_3}\big|\big]^2\, dx\Big)^{\frac{1}{2}}. \quad (2.33)$$

Hence,

$$\max_{x_3} \int_\omega u^4(x)\, dx' \leq |u|^3_{L^6(\Omega)} \Big(\int_\Omega \big[\frac{2}{b_3 - a_3}\big|u(x)\big| + 4\big|\frac{\partial u}{\partial x_3}\big|\big]^2\, dx'\Big)^{\frac{1}{2}}. \quad (2.34)$$

Combining (2.32) and (2.34), we obtain

$$\big|u\big|^6_{L^6(\Omega)} \leq 36 \big|u\big|^3_{L^6(\Omega)} \prod_{i=1}^{3} \Big(\frac{1}{b_i - a_i}\big|u\big|_{L^2(\Omega)} + \big|\frac{\partial u}{\partial x_i}\big|_{L^2(\Omega)}\Big)^{\frac{1}{3}}, \quad \forall u \in H^1(\Omega).$$

The proof is complete ($c_0 = \sqrt[3]{36}$). □

Remark 2.1. Inequality (2.27) can be extended to hold for a general domain $\Omega_\epsilon = \omega \times (0, \epsilon)$, where ω is a $\mathcal{C}^2$ bounded domain in $\mathbb{R}^2$. We have, then, as in the case of Corollary 2.3, a mixing of the derivatives in the directions x_1 and x_2; i.e., we have the following inequality:

$$|u|_{L^6(\Omega_\epsilon)} \leq c_0(\omega)\big(\frac{1}{\epsilon}|u|_{L^2(\Omega_\epsilon)} + \big|\frac{\partial u}{\partial x_3}\big|_{L^2(\Omega_\epsilon)}\big)^{\frac{1}{3}} \big(|u|_{L^2(\Omega_\epsilon)} + \big|\frac{\partial u}{\partial x_1}\big|_{L^2(\Omega_\epsilon)} + \big|\frac{\partial u}{\partial x_2}\big|_{L^2(\Omega_\epsilon)}\big)^{\frac{2}{3}}, \quad (2.35)$$

for all u in $H^1(\Omega_\epsilon)$.

In order to avoid the appearance of ϵ in (2.35), we need to use the Poincaré inequality in the thin direction; hence, we work with $\tilde{N}_\epsilon u$ and obtain

Corollary 2.4. *There exists a positive constant c_0, independent of ϵ, such that*

$$\big|\tilde{N}_\epsilon u\big|^2_{L^6(\Omega_\epsilon)} \leq c_0 \|\tilde{N}_\epsilon u\|^2_\epsilon. \quad (2.36)$$

Proof. We take $a_3 = 0,\ b_3 = \epsilon$. Then, we have by (2.2) and (2.35)

$$\big|\tilde{N}_\epsilon u\big|_{L^6(\Omega_\epsilon)} \leq c_0(\omega)\big|\frac{\partial \tilde{N}_\epsilon u}{\partial x_3}\big|^{\frac{1}{3}}_\epsilon \big[\big|\tilde{N}_\epsilon u\big|_\epsilon + \big|\nabla' \tilde{N}_\epsilon u\big|_\epsilon\big]^{\frac{2}{3}}, \quad (2.37)$$

and (2.36) follows promptly. □

Using Proposition 2.1, Corollary 2.4 and an interpolation argument, we obtain

Lemma 2.4. *For $2 \le q \le 6$, there exists a positive constant $c(q)$, independent of ϵ, such that*

$$\left|\tilde{N}_\epsilon u\right|^2_{L^q(\Omega_\epsilon)} \le c(q)\epsilon^{\frac{6-q}{q}} \|\tilde{N}_\epsilon u\|^2_\epsilon \quad \forall\, u \in V_\epsilon. \tag{2.38}$$

2.2. Inequalities related to the Stokes operator and the trilinear form. Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \ge 2$, with a C^2-boundary $\Gamma = \partial\Omega$. We denote by $\mathcal{B}$ the second fundamental form of Γ. The quadratic form $\mathcal{B}$ is defined as follows (see [30]):

$$\mathcal{B}_{P_0}(\zeta_1, \zeta_2) = -\nabla_{\zeta_1}\vec{n} \cdot \zeta_2, \tag{2.39}$$

where ζ_1 and ζ_2 are tangent vectors to Γ at the point P_0, the vector $\vec{n}$ denotes the unit outward normal vector to Γ and $\nabla_{\zeta_1}\vec{n}$ denotes the covariant derivative, with respect to ζ_1, of the vector $\vec{n}$. We recall that, in the case of a convex domain Ω, the second fundamental form is nonpositive ([11], [30]).

For a vector field v defined on Γ, we denote its normal and tangential components by

$$v_n = v \cdot \vec{n} \quad \text{and} \quad v_T = v - v_n\vec{n}. \tag{2.40}$$

Similarly, we denote the tangential gradient of a scalar function φ by

$$\nabla_T\varphi = \nabla\varphi - (\nabla\varphi \cdot \vec{n})\vec{n} = \nabla\varphi - \frac{\partial\varphi}{\partial n}\vec{n}. \tag{2.41}$$

Now we recall the following identity due to Iooss and Grisvard; see [11].

Theorem 2.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$, and let $v \in (H^1(\Omega))^n$. Then*

$$\int_\Omega \left|div\ v\right|^2 dx - \sum_{i,j=1}^n \int_\Omega \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}\, dx = -2\langle(\gamma v)_T; \nabla_T(\gamma v \cdot \vec{n})\rangle$$
$$- \int_\Gamma \{\mathcal{B}((\gamma v)_T; (\gamma v)_T) + tr\ \mathcal{B}[(\gamma v)\cdot\vec{n}]^2\}\, d\sigma. \tag{2.42}$$

Here γ is the trace operator from $(H^1(\Omega))^n$ onto $(H^{\frac{1}{2}}(\Gamma))^n$, tr $\mathcal{B}$ is the trace of the bilinear form $\mathcal{B}$ (in the "matrix-sense") and $< \cdot;\cdot >$ is the duality bracket between $(H^{\frac{1}{2}}(\Gamma))^{n-1}$ and $(H^{-\frac{1}{2}}(\Gamma))^{n-1}$.

Assume that Ω is convex. If $\varphi \in H^2(\Omega) \cap H^1_0(\Omega)$, then writing (2.42) with $v = \nabla\varphi$ yields

$$\sum_{i,j=1}^n \int_\Omega \left|\frac{\partial^2\varphi}{\partial x_i \partial x_j}\right|^2 dx = \int_\Omega \left|\Delta\varphi\right|^2 dx + \int_\Gamma \text{tr}\ \mathcal{B}[\gamma(\nabla\varphi)\cdot\vec{n}]^2\, d\sigma. \tag{2.43}$$

However, since Ω is convex, we have tr $\mathcal{B} \le 0$. Therefore,

$$\sum_{i,j=1}^n \int_\Omega \left|\frac{\partial^2\varphi}{\partial x_i \partial x_j}\right|^2 dx \le \int_\Omega \left|\Delta\varphi\right|^2 dx. \tag{2.44}$$

The inequality (2.44) is also true for any component of a vector field $u \in \mathbb{H}^2(\Omega_\epsilon)$ satisfying any of the boundary conditions under consideration in this article. For the proof we refer the reader to the Appendix in [14].

Inequality (2.44) is an important tool for the study of the H^2-regularity of second-order strongly elliptic operators defined in convex domains (see [11]). It was used in the study of reaction-diffusion equations and damped hyperbolic equations on thin domains ([13, 14]). This type of inequality was also used in the study of Navier-Stokes equations on thin domains ([22, 23, 24]), in the case of the purely periodic boundary condition; the proof was given with the help of Fourier series ([24]). Note that inequality (2.44) gives a control of the L^2-norms of "the mixed second derivatives" of a function φ in terms of the L^2-norm of its Laplacian with a constant, which is independent of the shape of the domain (i.e., in our case independent of the thickness ϵ). Our purpose now is to obtain similar inequalities for the Stokes operator under various boundary conditions.

First, we consider the purely periodic boundary condition (PP), the free boundary condition (FF) and the boundary condition (FP). For these boundary conditions, we have

$$A_\epsilon u = -\Delta u, \quad \forall u \in D(A_\epsilon). \tag{2.45}$$

We refer to [5] and [26] for the proof of (2.45) in the case of the boundary condition (PP), and to [31] in the case of the boundary conditions (FF) and (FP). Therefore, we have

Lemma 2.5. *In the case of the boundary condition* (PP), (FF) *or* (FP), *we have*

$$\sum_{i,j=1}^{3} \Big| \frac{\partial^2 u}{\partial x_i \partial x_j} \Big|^2_{L^2(\Omega_\epsilon)} \leq \big| A_\epsilon u \big|^2_{L^2(\Omega_\epsilon)}, \quad \forall u \in D(A_\epsilon). \tag{2.46}$$

For the other boundary conditions, we will use a symmetry argument to establish that the constant c_ϵ, appearing in the classical Cattabriga-Solonnikov "H^2-regularity" inequality,

$$\sum_{i,j=1}^{3} \Big| \frac{\partial^2 u}{\partial x_i \partial x_j} \Big|^2_{L^2(\Omega_\epsilon)} \leq c_\epsilon \big| A_\epsilon u \big|^2_{L^2(\Omega_\epsilon)}, \quad \forall u \in D(A_\epsilon),$$

is independent of ϵ.

Consider the following Stokes problem:

$$-\nu \Delta \hat{u} + \nabla \hat{p} = \hat{f} \quad \text{in } \ \hat{\Omega}_\epsilon = \omega \times (-\epsilon, \epsilon), \tag{2.47}$$

$$\operatorname{div} \hat{u} = 0 \qquad \text{in } \ \hat{\Omega}_\epsilon; \tag{2.48}$$

and the boundary conditions:

$$\frac{\partial \hat{u}_1}{\partial x_3} = \frac{\partial \hat{u}_2}{\partial x_3} = 0 \quad \text{and} \quad \hat{u}_3 = 0 \quad \text{on } \ \omega \times \{-\epsilon, \epsilon\}, \tag{2.49}$$

and

$$\hat{u} = 0 \quad \text{on } \ \partial\omega \times (-\epsilon, \epsilon), \tag{2.50}$$

where $\hat{f} \in \hat{H}_\epsilon = \{\hat{v} \in \mathbb{L}^2(\hat{\Omega}_\epsilon);\ \text{div}\ \hat{v} = 0,\quad \hat{v} \cdot \vec{n} = 0 \text{ on } \partial\hat{\Omega}_\epsilon\}$.

We write $\hat{f} = (\hat{f}', \hat{f}_3)$, with $\hat{f}' = (f_1, f_2)$ and analogously, we write $\hat{u} = (\hat{u}', \hat{u}_3)$. Assume that $\hat{f}'$ is an even function with respect to x_3 and that $\hat{f}_3$ is odd in x_3. Then, it is straightforward to see that $\hat{u}'$ and $\hat{p}$ are even in x_3, while $\hat{u}_3$ is odd in x_3.

Now we are ready to prove the following:

Lemma 2.6. *Let (u, p) be a solution of the following Stokes problem:*

$$\begin{cases} -\nu\Delta u + \nabla p = f & \text{in } \Omega_\epsilon, \\ \text{div } u = 0 & \text{in } \Omega_\epsilon, \\ \frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = u_3 = 0 & \text{on } \Gamma_t \cup \Gamma_b, \\ u = 0 & \text{on } \Gamma_l. \end{cases} \tag{2.51}$$

Then, there exists a positive constant c, independent of ϵ, such that

$$\sum_{i,j=1}^{3} \Big|\frac{\partial^2 u}{\partial x_i \partial x_j}\Big|^2_{\mathbb{L}^2(\Omega_\epsilon)} \leq c\big|A_\epsilon u\big|^2_{L^2(\Omega_\epsilon)}, \quad \forall u \in D(A_\epsilon) \tag{2.52}$$

where A_ϵ is the Stokes operator associated with problem (2.51).

Proof. The regularity of the Stokes operator implies the existence of a positive constant c_ϵ, dilation invariant, such that

$$\sum_{i,j=1}^{3} \Big|\frac{\partial^2 u}{\partial x_i \partial x_j}\Big|^2_{\mathbb{L}^2(\Omega_\epsilon)} \leq c_\epsilon\big|A_\epsilon u\big|^2_{L^2(\Omega_\epsilon)} \quad \forall u \in D(A_\epsilon).$$

The constant c_ϵ is chosen to be the infimum of all constants satisfying (2.52). We extend f to $\hat{\Omega}_\epsilon = \omega \times (-\epsilon, \epsilon)$ in the following way: f_1 and f_2 are extended to be even functions $\hat{f}_1$ and $\hat{f}_2$, and f_3 is extended to be an odd function $\hat{f}_3$.

According to the symmetry argument given above, if $(\hat{u}, \hat{p})$ is a solution of the Stokes problem (2.47)–(2.50), then $(\hat{u}_1, \hat{u}_2)$ is even in x_3 and $\hat{u}_3$ is odd in x_3. Hence, $\hat{u}\big|_{\Omega_\epsilon} = u$ and the regularity of the Stokes problem on $\hat{\Omega}_\epsilon$ implies the existence of a positive constant $\hat{c}_\epsilon$ such that

$$\sum_{i,j=1}^{3} \Big|\frac{\partial^2 \hat{u}}{\partial x_i \partial x_j}\Big|^2_{\mathbb{L}^2(\hat{\Omega}_\epsilon)} \leq \hat{c}_\epsilon\big|\hat{A}_\epsilon \hat{u}\big|^2_{L^2(\hat{\Omega}_\epsilon)}, \quad \forall \hat{u} \in D(\hat{A}_\epsilon), \tag{2.53}$$

where $\hat{A}_\epsilon$ is the Stokes operator defined on $\hat{\Omega}_\epsilon$.

Now according to the symmetry of $\hat{u}$ and the fact that $\hat{u}\big|_{\Omega_\epsilon} = u$, we infer from (2.53)

$$2\sum_{i,j=1}^{3} \Big|\frac{\partial^2 u}{\partial x_i \partial x_j}\Big|^2_{\mathbb{L}^2(\Omega_\epsilon)} \leq 2\hat{c}_\epsilon\big|A_\epsilon u\big|^2_{\mathbb{L}^2(\Omega_\epsilon)}, \quad \forall u \in D(A_\epsilon). \tag{2.54}$$

Therefore, since c_ϵ was chosen to be the infimum, we have $c_\epsilon \leq \hat{c}_\epsilon$. Repeating this argument k times, with $k \geq \frac{1}{\epsilon}$, we conclude the lemma using the dilation invariance of c_ϵ. $\square$

Remark 2.2. The proof of the independence on ϵ of the constant $c(\epsilon)$ appearing in the Cattabriga-Solonnikov inequality for boundary condition (DP) is similar to the one given above and is left as an exercise.

Remark 2.3. The proof of the independence on ϵ of the constant $c(\epsilon)$ for the boundary condition (DD) is technical and long and will be given in a separate article ([32]).

Remark 2.4. The symmetry argument given above and the classical local regularity results for the Stokes operator imply the H^2 regularity of the Stokes operator, with the boundary condition (FF), (FD) or (FP) in the domain with corners Ω_ϵ.

According to Lemmas 2.5 and 2.6 and Remarks 2.2 and 2.3, we have

Theorem 2.2. *Under one of the boundary conditions under consideration, there exists a constant* c, *independent of* ϵ, *such that*

$$\sum_{i,j=1}^{3}\Big|\frac{\partial^2 u}{\partial x_i\partial x_j}\Big|^2_{\mathbb{L}^2(\Omega_\epsilon)}\leq c\big|A_\epsilon u\big|^2_{L^2(\Omega_\epsilon)},\quad \forall u\in D(A_\epsilon).$$

We end this section with some inequalities related to the trilinear form. Using Lemmas 2.5 and 2.6, we prove the following:

Lemma 2.7. *Let* $0<q<\frac{1}{2}$. *There exists a positive constant* $c_4(q)$, *independent of* ϵ, *such that, for any one of the boundary conditions* (FF), (FD), (FP), (PP), (DD) *or* (DP), *we have*

$$\big|b_\epsilon(\tilde{M}_\epsilon u,\tilde{N}_\epsilon u,w)\big|\leq c_4\epsilon^q\big|\big|\tilde{M}_\epsilon u\big|\big|_\epsilon\big|A_\epsilon\tilde{N}_\epsilon u\big|_\epsilon\big|w\big|_\epsilon\quad \forall u\in D(A_\epsilon),\ w\in\mathbb{L}^2(\Omega_\epsilon). \tag{2.55}$$

$$\big|b_\epsilon(\tilde{N}_\epsilon u,\tilde{M}_\epsilon u,w)\big|\leq c_4\epsilon^{\frac{1}{2}}\big|\big|\tilde{M}_\epsilon u\big|\big|_\epsilon\big|A_\epsilon\tilde{N}_\epsilon u\big|_\epsilon\big|w\big|_\epsilon\quad \forall u\in D(A_\epsilon),\ w\in\mathbb{L}^2(\Omega_\epsilon). \tag{2.56}$$

$$\begin{aligned}\big|b_\epsilon(\tilde{N}_\epsilon u,\tilde{N}_\epsilon u,w)\big| &\leq c_4\big|\big|\tilde{N}_\epsilon u\big|\big|^{\frac{3}{2}}_\epsilon\big|A_\epsilon\tilde{N}_\epsilon u\big|^{\frac{1}{2}}_\epsilon\big|w\big|_\epsilon\\ &\leq c_4\epsilon^{\frac{1}{2}}\big|\big|\tilde{N}_\epsilon u\big|\big|_\epsilon\big|A_\epsilon\tilde{N}_\epsilon u\big|_\epsilon\big|w\big|_\epsilon\quad \forall u\in D(A_\epsilon),\ w\in\mathbb{L}^2(\Omega_\epsilon).\end{aligned} \tag{2.57}$$

Proof. (i) We only treat the boundary conditions (FF), (FD) and (FP). The purely periodic condition (PP) is treated similarly (the only differences are in the indices of summation in (2.58)and (2.62)). The cases (DD) and (DP) are obvious. For $u\in D(A_\epsilon)$ and $w\in\mathbb{L}^2(\Omega_\epsilon)$, we have

$$\big|b_\epsilon(\tilde{M}_\epsilon u,\tilde{N}_\epsilon u,w)\big|\leq\sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\Omega_\epsilon}\big|M_\epsilon u_i\big|\Big|\frac{\partial N_\epsilon u_j}{\partial x_i}\Big|\big|w_j\big|\,dx$$

and, according to Hölder's inequality with exponents $p,\ p^*$ and 2, with $\frac{1}{p}+\frac{1}{p^*}=\frac{1}{2}$, and $2<p^*\leq 6$ (p and p^* will be chosen later in terms of q), we write

$$\big|b_\epsilon(\tilde{M}_\epsilon u,\tilde{N}_\epsilon u,w)\big|\leq\sum_{i=1}^{2}\sum_{j=1}^{3}\big|M_\epsilon u_i\big|_{L^p(\Omega_\epsilon)}\Big|\frac{\partial N_\epsilon u_j}{\partial x_i}\Big|_{L^{p^*}(\Omega_\epsilon)}\big|w_j\big|_{L^2(\Omega_\epsilon)}. \tag{2.58}$$

Since $\dfrac{\partial N_\epsilon u_j}{\partial x_i} = N_\epsilon\big(\dfrac{\partial u_j}{\partial x_i}\big)$, for $i = 1, 2$, we are able to apply Lemma 2.4 and obtain

$$\begin{aligned}\big|\frac{\partial N_\epsilon u_j}{\partial x_i}\big|_{L^{p^*}(\Omega_\epsilon)} &\le c(p^*)\epsilon^{\frac{6-p^*}{2p^*}}\big|\big|\frac{\partial N_\epsilon u_j}{\partial x_i}\big|\big|_\epsilon \\ &\le \ \text{(Theorem 2.2)} \\ &\le c(p^*)\epsilon^{\frac{6-p^*}{2p^*}}\big|A_\epsilon \tilde{N}_\epsilon u\big|_\epsilon.\end{aligned} \tag{2.59}$$

Moreover, since $H^1(\omega) \subset L^p(\omega)$, for $2 \le p < \infty$, we have

$$\big|M_\epsilon u_i\big|_{L^p(\Omega_\epsilon)} \le c(p)\epsilon^{\frac{1}{p}-\frac{1}{2}}\big|M_\epsilon u_i\big|_{H^1(\Omega_\epsilon)}. \tag{2.60}$$

Combining (2.58), (2.59) and (2.60), we obtain

$$\big|b_\epsilon(\tilde{M}_\epsilon u, \tilde{N}_\epsilon u, w)\big| \le c_4\epsilon^{\frac{1}{p}-\frac{1}{2}+\frac{6-p^*}{2p^*}}\big|\big|\tilde{M}_\epsilon u\big|\big|_\epsilon\big|A_\epsilon\tilde{N}_\epsilon u\big|_\epsilon\big|w\big|_\epsilon. \tag{2.61}$$

For $2 < p^* \le 4$, the exponent of ϵ in the inequality above $q = \frac{2}{p^*} - \frac{1}{2}$, satisfies $0 < q < \frac{1}{2}$. Note also that for any q, $0 < q < \frac{1}{2}$, there exists p^*, $2 < p^* < 4$, satisfying $q = \frac{2}{p^*} - \frac{1}{2}$. Hence, inequality (2.55) is concluded.

(ii) Now we prove inequality (2.56). We have

$$\begin{aligned}\big|b_\epsilon(\tilde{N}_\epsilon u, \tilde{M}_\epsilon u, w)\big| &\le \sum_{i=1}^{2}\sum_{j=1}^{2}\int_{\Omega_\epsilon}\big|N_\epsilon u_i\big|\big|\frac{\partial M_\epsilon u_j}{\partial x_i}\big|\big|w_j\big|\,dx \\ &\le \sum_{i=1}^{2}\sum_{j=1}^{2}\big|N_\epsilon u_i\big|_{L^\infty(\Omega_\epsilon)}\big|\frac{\partial M_\epsilon u_j}{\partial x_i}\big|_{L^2(\Omega_\epsilon)}\big|w_j\big|_\epsilon,\end{aligned} \tag{2.62}$$

and, according to Corollary 2.3 and Theorem 2.2, we have

$$\begin{aligned}\big|b_\epsilon(\tilde{N}_\epsilon u, \tilde{M}_\epsilon u, w)\big| &\le c_4\epsilon^{\frac{1}{2}}\sum_{i=1}^{2}\sum_{j=1}^{2}\big|A_\epsilon\tilde{N}_\epsilon u\big|_\epsilon\big|\frac{\partial M_\epsilon u_j}{\partial x_i}\big|_{L^2(\Omega_\epsilon)}\big|w_j\big|_{L^2(\Omega_\epsilon)} \\ &\le c_4\epsilon^{\frac{1}{2}}\big|\big|\tilde{M}_\epsilon u\big|\big|_\epsilon\big|A_\epsilon\tilde{N}_\epsilon u\big|_\epsilon\big|w\big|_{L^2(\Omega_\epsilon)}.\end{aligned} \tag{2.63}$$

(iii) Now we prove the inequality (2.57). We write

$$\big|b_\epsilon(\tilde{N}_\epsilon u, \tilde{N}_\epsilon u, w)\big| \le \sum_{i,j=1}^{3}\int_{\Omega_\epsilon}\big|N_\epsilon u_i\big|\big|\frac{\partial N_\epsilon u_j}{\partial x_i}\big|^{\frac{1}{2}}\big|N_\epsilon u_i\big|\big|\frac{\partial N_\epsilon u_j}{\partial x_i}\big|^{\frac{1}{2}}\big|w_j\big|_{L^2(\Omega_\epsilon)}\,dx, \tag{2.64}$$

and Hölder's inequality, with exponents $6, 4, 12$ and 2, yields

$$\begin{aligned}\big|b_\epsilon(\tilde{N}_\epsilon u, \tilde{N}_\epsilon u, w)\big| &\le \sum_{i,j=1}^{3}\big|N_\epsilon u_i\big|_{L^6(\Omega_\epsilon)}\big|\frac{\partial N_\epsilon u_j}{\partial x_i}\big|^{\frac{1}{2}}_{L^2(\Omega_\epsilon)}\big|\frac{\partial N_\epsilon u_j}{\partial x_i}\big|^{\frac{1}{2}}_{L^6(\Omega_\epsilon)}\big|w_j\big|_{L^2(\Omega_\epsilon)} \\ &\le c_0\sum_{i,j=1}^{3}\big|\frac{\partial N_\epsilon u_i}{\partial x_j}\big|_{L^2(\Omega_\epsilon)}\big|\frac{\partial N_\epsilon u_j}{\partial x_i}\big|^{\frac{1}{2}}_{L^2(\Omega_\epsilon)}\big|\frac{\partial^2 N_\epsilon u_j}{\partial x_i\partial x_j}\big|^{\frac{1}{2}}_{L^2(\Omega_\epsilon)}\big|w_j\big|_{L^2(\Omega_\epsilon)}.\end{aligned}$$

Hence

$$\big|b_\epsilon(\tilde N_\epsilon u,\tilde N_\epsilon u,w)\big|\le c_4\big|\big|\tilde N_\epsilon u\big|\big|_\epsilon^{\frac32}\big|A_\epsilon\tilde N_\epsilon u\big|_\epsilon^{\frac12}\big|w\big|_\epsilon. \tag{2.65}$$

We use (see (2.25))

$$\Big|\frac{\partial N_\epsilon u_j}{\partial x_i}\Big|_\epsilon\le\epsilon\Big|\frac{\partial^2 N_\epsilon u_j}{\partial x_i\partial x_3}\Big|_\epsilon\quad i,j=1,2,3,$$

and obtain, thanks to Corollary 2.4 and Theorem 2.2,

$$\big|b_\epsilon(\tilde N_\epsilon u,\tilde N_\epsilon u,w)\big|\le c_4\epsilon^{\frac12}\big|\big|\tilde N_\epsilon u\big|\big|_\epsilon\big|A_\epsilon\tilde N_\epsilon u\big|_\epsilon\big|w\big|_\epsilon. \tag{2.66}$$

The proof of Lemma 2.7 is now complete. □

3. A priori estimates. In this section we derive some a priori estimates for $\tilde M_\epsilon u$ and $\tilde N_\epsilon u$. These estimates constitute the main tool in the study of the maximal time of existence of the strong solutions and also in the study of the behavior of the averages when the thickness goes to zero.

First we write the equations satisfied by $\tilde M_\epsilon u$ and $\tilde N_\epsilon u$. Let $v\in V_\epsilon$ (we omit the indices for the boundary conditions unless they are necessary). By Lemma 1.1, $\tilde M_\epsilon v,\ \tilde N_\epsilon v\in V_\epsilon$ and

$$(\tilde M_\epsilon u,\tilde N_\epsilon v)_\epsilon=0\quad\text{ and }\quad((\tilde M_\epsilon u,\tilde N_\epsilon v))_\epsilon=0,$$

we obtain, thanks to Lemma 1.1, the following weak formulation for $\tilde M_\epsilon u$ and $\tilde N_\epsilon u$:

$$\begin{aligned}\frac{d}{dt}(\tilde M_\epsilon u,\tilde M_\epsilon v)_\epsilon+\nu(A_\epsilon^{\frac12}\tilde M_\epsilon u,A_\epsilon^{\frac12}\tilde M_\epsilon v)_\epsilon+b_\epsilon(\tilde M_\epsilon u,\tilde M_\epsilon u,\tilde M_\epsilon v)&\\+b_\epsilon(\tilde N_\epsilon u,\tilde N_\epsilon u,\tilde M_\epsilon v)=(\tilde M_\epsilon f,\tilde M_\epsilon v)_\epsilon&\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\frac{d}{dt}(\tilde N_\epsilon u,\tilde N_\epsilon v)_\epsilon+\nu(A_\epsilon^{\frac12}\tilde N_\epsilon u,A_\epsilon^{\frac12}\tilde N_\epsilon v)_\epsilon+b_\epsilon(\tilde M_\epsilon u,\tilde N_\epsilon u,\tilde N_\epsilon v)&\\+b_\epsilon(\tilde N_\epsilon u,\tilde M_\epsilon u,\tilde N_\epsilon v)+b_\epsilon(\tilde N_\epsilon u,\tilde N_\epsilon u,\tilde N_\epsilon v)=(\tilde N_\epsilon f,\tilde N_\epsilon v)_\epsilon.&\end{aligned} \tag{3.2}$$

Now we introduce the following notation: we set

$$a_0(\epsilon)=\big|A_\epsilon^{\frac12}\tilde M_\epsilon u_0\big|_\epsilon,\quad b_0(\epsilon)=\big|A_\epsilon^{\frac12}\tilde N_\epsilon u_0\big|_\epsilon,\quad \alpha(\epsilon)=\big|\tilde M_\epsilon f\big|_\epsilon,\quad \beta(\epsilon)=\big|\tilde N_\epsilon f\big|_\epsilon. \tag{3.3}$$

We also set

$$R_0^2(\epsilon)=a_0^2(\epsilon)+b_0^2(\epsilon)+\alpha^2(\epsilon)+\beta^2(\epsilon). \tag{3.4}$$

We have, according to (1.12), Lemma 1.1

$$\big|A_\epsilon^{\frac12}u_0\big|_\epsilon^2=a_0^2(\epsilon)+b_0^2(\epsilon),\ \ \big|f\big|_\epsilon^2=\alpha^2(\epsilon)+\beta^2(\epsilon)$$

and

$$R_0^2(\epsilon)=\big|A_\epsilon^{\frac12}u_0\big|_\epsilon^2+\big|f\big|_\epsilon^2. \tag{3.5}$$

Now, given $\sigma > 1$ we have the following classical fact, which is a consequence of Theorem 0.1 (see e.g. [25, 26, 27]):

$$\exists\, T^\sigma(\epsilon) > 0, \quad \text{such that} \quad \big|A_\epsilon^{\frac{1}{2}} u(t)\big|_\epsilon^2 \le \sigma R_0^2(\epsilon), \ \forall\ 0 \le t < T^\sigma(\epsilon). \tag{3.6}$$

Here $[0, T^\sigma(\epsilon))$ is the maximal interval on which (3.6) holds. It is clear that if $T^\sigma(\epsilon) < \infty$, then

$$\big|A_\epsilon^{\frac{1}{2}} u(T^\sigma(\epsilon))\big|_\epsilon^2 = \sigma R_0^2(\epsilon). \tag{3.7}$$

3.1. Estimates for $\tilde{N}_\epsilon u$. We take $v = A_\epsilon u$ in (3.2) and obtain with Lemma 2.7

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big|A_\epsilon^{\frac{1}{2}}\tilde{N}_\epsilon u\big|_\epsilon^2 + \nu\big|A_\epsilon\tilde{N}_\epsilon u\big|_\epsilon^2 \le \frac{1}{2\nu}\big|\tilde{N}_\epsilon f\big|_\epsilon^2 + \frac{\nu}{2}\big|A_\epsilon\tilde{N}_\epsilon u\big|_\epsilon^2 \\ + c_4\epsilon^{\frac{1}{2}}\big(\big|A_\epsilon^{\frac{1}{2}}\tilde{M}_\epsilon u\big|_\epsilon + \big|A_\epsilon^{\frac{1}{2}}\tilde{N}_\epsilon u\big|_\epsilon\big)\big|A_\epsilon\tilde{N}_\epsilon u\big|_\epsilon^2 + c_4\epsilon^q\big|A_\epsilon^{\frac{1}{2}}\tilde{M}_\epsilon u\big|_\epsilon\big|A_\epsilon\tilde{N}_\epsilon u\big|_\epsilon^2,\end{aligned} \tag{3.8}$$

and, since $0 < \epsilon < 1$ and $0 < q < \frac{1}{2}$, we have

$$\frac{d}{dt}\big|A_\epsilon^{\frac{1}{2}}\tilde{N}_\epsilon u\big|_\epsilon^2 + \big[\nu - 2c_4\epsilon^q\big(\big|A_\epsilon^{\frac{1}{2}}\tilde{M}_\epsilon u\big|_\epsilon + \big|A_\epsilon^{\frac{1}{2}}\tilde{N}_\epsilon u\big|_\epsilon\big)\big]\big|A_\epsilon\tilde{N}_\epsilon u\big|_\epsilon^2 \le \frac{\big|\tilde{N}_\epsilon f\big|_\epsilon^2}{\nu}. \tag{3.9}$$

Thanks to (3.6), we have

$$\big|A_\epsilon^{\frac{1}{2}}\tilde{M}_\epsilon u(t)\big|_\epsilon + \big|A_\epsilon^{\frac{1}{2}}\tilde{N}_\epsilon u(t)\big|_\epsilon \le 2\sqrt{\sigma}R_0(\epsilon), \quad 0 \le t < T^\sigma(\epsilon), \tag{3.10}$$

so that

$$\frac{d}{dt}\big|A_\epsilon^{\frac{1}{2}}\tilde{N}_\epsilon u\big|_\epsilon^2 + [\nu - 4\sqrt{\sigma}c_4\epsilon^q R_0(\epsilon)]\big|A_\epsilon\tilde{N}_\epsilon u\big|_\epsilon^2 \le \frac{\big|\tilde{N}_\epsilon f\big|_\epsilon^2}{\nu}, \quad 0 < 2q < 1. \tag{3.11}$$

At this stage, we need to make the following assumption:

$$\lim_{\epsilon \to 0} \epsilon^{2q} R_0^2(\epsilon) = 0, \quad 0 < 2q < 1. \tag{H_0}$$

We note that the assumption (H_0) is not restrictive, since physically, $R_0(\epsilon)$ can be assumed to go to zero when ϵ goes to zero, which is the case when f and u_0 are independent of ϵ. Thanks to (H_0), we can choose $\epsilon_1 = \epsilon_1(\nu, \omega, \sigma) > 0$, such that

$$4\sqrt{\sigma}c_4\epsilon^q R_0(\epsilon) \le \frac{\nu}{2}, \quad \forall \epsilon,\ 0 < \epsilon \le \epsilon_1. \tag{3.12}$$

From (3.11) and (3.12), we can write

$$\frac{d}{dt}\big|A_\epsilon^{\frac{1}{2}}\tilde{N}_\epsilon u\big|_\epsilon^2 + \frac{\nu}{2}\big|A_\epsilon\tilde{N}_\epsilon u\big|_\epsilon^2 \le \frac{\big|\tilde{N}_\epsilon f\big|_\epsilon^2}{\nu}, \quad 0 < \epsilon \le \epsilon_1, \quad 0 < t < T^\sigma(\epsilon), \tag{3.13}$$

and since by the Cauchy-Schwarz inequality

$$\big|A_\epsilon^{\frac12}\tilde N_\epsilon u\big|_\epsilon^2 \le \epsilon^2\big|A_\epsilon\tilde N_\epsilon u\big|_\epsilon^2, \tag{3.14}$$

we have

$$\frac{d}{dt}\big|A_\epsilon^{\frac12}\tilde N_\epsilon u\big|_\epsilon^2+\frac{\nu}{2\epsilon^2}\big|A_\epsilon^{\frac12}\tilde N_\epsilon u\big|_\epsilon^2\le\frac{\big|\tilde N_\epsilon f\big|_\epsilon^2}{\nu},\quad 0<\epsilon\le\epsilon_1,\ \ 0<t<T^\sigma(\epsilon). \tag{3.15}$$

Finally, Gronwall's lemma yields

$$\big|A_\epsilon^{\frac12}\tilde N_\epsilon u(t)\big|_\epsilon^2\le b_0^2(\epsilon)\exp(-\frac{\nu t}{2\epsilon^2})+\frac{2\epsilon^2}{\nu^2}\beta^2(\epsilon),\quad 0<\epsilon\le\epsilon_1,\ \ 0<t<T^\sigma(\epsilon). \tag{3.16}$$

We integrate (3.13) and (3.15) to obtain

$$\int_0^t\big|A_\epsilon^{\frac12}\tilde N_\epsilon u(s)\big|_\epsilon^2\,ds\le\frac{2\epsilon^2}{\nu^2}\beta^2(\epsilon)t+\frac{2\epsilon^2}{\nu}b_0^2(\epsilon),\quad 0<\epsilon\le\epsilon_1,\ \ 0<t<T^\sigma(\epsilon) \tag{3.17}$$

and

$$\int_0^t\big|A_\epsilon\tilde N_\epsilon u(s)\big|_\epsilon^2\,ds\le\frac{2}{\nu^2}\beta^2(\epsilon)t+\frac{2}{\nu}b_0^2(\epsilon),\quad 0<\epsilon\le\epsilon_1,\ \ 0<t<T^\sigma(\epsilon). \tag{3.18}$$

Now, thanks to the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\mathcal I=\int_0^t\big|A_\epsilon^{\frac12}\tilde N_\epsilon u(s)\big|_\epsilon^3\big|A_\epsilon\tilde N_\epsilon u(s)\big|_\epsilon\,ds\le\\ \le\sup_{0\le s\le t}\big|A_\epsilon^{\frac12}\tilde N_\epsilon u(s)\big|_\epsilon^2\left(\int_0^t\big|A_\epsilon^{\frac12}\tilde N_\epsilon u(s)\big|_\epsilon^2\,ds\right)^{\frac12}\left(\int_0^t\big|A_\epsilon\tilde N_\epsilon u(s)\big|_\epsilon^2\,ds\right)^{\frac12}.\end{aligned} \tag{3.19}$$

Therefore, combining (3.16), (3.17) and (3.18), we find

$$\begin{aligned}\mathcal I&\le\left(b_0^2(\epsilon)+\frac{2\epsilon^2\beta^2(\epsilon)}{\nu^2}\right)\left(\frac{2\epsilon^2\beta^2(\epsilon)t}{\nu^2}+\frac{2\epsilon^2b_0^2(\epsilon)}{\nu}\right)^{\frac12}\left(\frac{2\beta^2(\epsilon)t}{\nu^2}+\frac{2b_0^2(\epsilon)}{\nu}\right)^{\frac12}\\&\le4\epsilon\left(b_0^2(\epsilon)+\frac{\epsilon^2\beta^2(\epsilon)}{\nu^2}\right)\left(\frac{\beta^2(\epsilon)t}{\nu^2}+\frac{b_0^2(\epsilon)}{\nu}\right)\\&\le c_5(\nu)\epsilon\big(\epsilon^2\beta^2(\epsilon)+b_0^2(\epsilon)\big)\big(t\beta^2(\epsilon)+b_0^2(\epsilon)\big),\end{aligned} \tag{3.20}$$

where

$$c_5(\nu)=\big[2\max(1,\frac{1}{\nu^2},\frac{1}{\nu})\big]^2. \tag{3.21}$$

We collect the previous inequalities in the following lemma:

Lemma 3.1. *Under one of the boundary conditions under consideration, we assume that for some* $0 < q < \frac{1}{2}$,

$$\lim_{\epsilon\to 0} \epsilon^{2q}(\big|A_\epsilon^{\frac{1}{2}} u_0\big|_\epsilon^2 + \big|f\big|_\epsilon^2) = 0.$$

Then, there exists $\epsilon_1 = \epsilon_1(\nu) > 0$ *such that for all* ϵ, $0 < \epsilon \le \epsilon_1$, *there exists* $T^\sigma(\epsilon) > 0$ *and a positive constant* c_5, *independent of* ϵ, *such that for* $0 < \epsilon \le \epsilon_1$ *and* $0 < t < T^\sigma(\epsilon)$

$$(i) \quad \big|A_\epsilon^{\frac{1}{2}} \tilde{N}_\epsilon u(t)\big|_\epsilon^2 \le b_0^2(\epsilon)\exp(-\frac{\nu t}{2\epsilon^2}) + \frac{2\epsilon^2}{\nu^2}\beta^2(\epsilon),$$

$$(ii) \quad \int_0^t \big|A_\epsilon \tilde{N}_\epsilon u(s)\big|_\epsilon^2\, ds \le \frac{2}{\nu^2}\beta^2(\epsilon)t + \frac{2}{\nu} b_0^2(\epsilon),$$

$$(iii) \quad \int_0^t \big|A_\epsilon^{\frac{1}{2}} \tilde{N}_\epsilon u(s)\big|_\epsilon^3 \big|A_\epsilon \tilde{N}_\epsilon u(s)\big|_\epsilon\, ds \le c_5(\nu)\epsilon \left(\epsilon^2\beta^2(\epsilon) + b_0^2(\epsilon)\right)\left(t\beta^2(\epsilon) + b_0^2(\epsilon)\right).$$

Remark 3.1. In the case of the boundary conditions (DD) or (DP), the lemma above is still true with $q = \frac{1}{2}$.

We have $u = \tilde{N}_\epsilon u$ in the case of the boundary conditions (DD) or (DP). Hence, we only need inequality (2.57).

3.2. Estimates for $\tilde{M}_\epsilon u$. We take $v = u$ in (3.1) and obtain

$$\frac{1}{2}\frac{d}{dt}\big|\tilde{M}_\epsilon u\big|_\epsilon^2 + \nu\big|A_\epsilon^{\frac{1}{2}} \tilde{M}_\epsilon u\big|_\epsilon^2 + b_\epsilon(\tilde{N}_\epsilon u, \tilde{N}_\epsilon u, \tilde{M}_\epsilon u) = (\tilde{M}_\epsilon f, \tilde{M}_\epsilon u)_\epsilon. \tag{3.23}$$

According to Lemma 2.7, we have

$$\frac{1}{2}\frac{d}{dt}\big|\tilde{M}_\epsilon u\big|_\epsilon^2 + \nu\big|A_\epsilon^{\frac{1}{2}} \tilde{M}_\epsilon u\big|_\epsilon^2 \le \big|\tilde{M}_\epsilon f\big|_\epsilon \big|\tilde{M}_\epsilon u\big|_\epsilon + c_4\big|A_\epsilon^{\frac{1}{2}} \tilde{N}_\epsilon u\big|_\epsilon^{\frac{3}{2}} \big|A_\epsilon \tilde{N}_\epsilon u\big|_\epsilon^{\frac{1}{2}} \big|\tilde{M}_\epsilon u\big|_\epsilon, \tag{3.24}$$

and since

$$\big|\tilde{M}_\epsilon u\big|_\epsilon \le \frac{1}{\lambda_1}\big|\tilde{A}^{\frac{1}{2}} \tilde{M}_\epsilon u\big|_\epsilon, \tag{3.25}$$

where λ_1 is the first eigenvalue of the corresponding two-dimensional Stokes operator $\tilde{A}$, defined on ω, we obtain

$$\frac{d}{dt}\big|\tilde{M}_\epsilon u\big|_\epsilon^2 + \nu\big|A_\epsilon^{\frac{1}{2}} \tilde{M}_\epsilon u\big|_\epsilon^2 \le \frac{\big|\tilde{M}_\epsilon f\big|_\epsilon^2}{\nu\lambda_1} + \frac{c_4}{\nu\lambda_1}\big|A_\epsilon^{\frac{1}{2}} \tilde{N}_\epsilon u\big|_\epsilon^3 \big|A_\epsilon \tilde{N}_\epsilon u\big|_\epsilon, \tag{3.26}$$

and

$$\frac{d}{dt}\big|\tilde{M}_\epsilon u\big|_\epsilon^2 + \nu\lambda_1\big|\tilde{M}_\epsilon u\big|_\epsilon^2 \le \frac{\big|\tilde{M}_\epsilon f\big|_\epsilon^2}{\nu\lambda_1} + \frac{c_4}{\nu\lambda_1}\big|A_\epsilon^{\frac{1}{2}} \tilde{N}_\epsilon u\big|_\epsilon^3 \big|A_\epsilon \tilde{N}_\epsilon u\big|_\epsilon. \tag{3.27}$$

Thanks to Gronwall's lemma, we find

$$\big|\tilde{M}_\epsilon u(t)\big|_\epsilon^2 \le a_0^2(\epsilon)e^{-\nu\lambda_1 t} + \frac{\alpha^2(\epsilon)}{\nu\lambda_1} + \frac{c_4}{\nu\lambda_1}\int_0^t \big|A_\epsilon^{\frac{1}{2}} \tilde{N}_\epsilon u(s)\big|_\epsilon^3 \big|A_\epsilon \tilde{N}_\epsilon u(s)\big|_\epsilon\, ds, \tag{3.28}$$

and Lemma 3.1 (inequality (iii)) implies that

$$\left|\tilde{M}_\epsilon u(t)\right|_\epsilon^2 \leq a_0^2(\epsilon)e^{-\nu\lambda_1 t} + \frac{\alpha^2(\epsilon)}{\nu\lambda_1} + c_6(\nu)\epsilon R_n^4(\epsilon)(1+t), \tag{3.29}$$

where $c_6(\nu) = 2c_5(\nu)$ is independent of ϵ and

$$R_n^2(\epsilon) = \max\bigl(b_0^2(\epsilon), \beta^2(\epsilon)\bigr).$$

We infer from (3.26), after integration in t,

$$\int_0^t \left|A_\epsilon^{\frac{1}{2}}\tilde{M}_\epsilon u(s)\right|_\epsilon^2 ds \leq \frac{\alpha^2(\epsilon)t}{\nu^2\lambda_1} + \frac{a_0^2(\epsilon)}{\nu\lambda_1} + c_6(\nu)\epsilon R_n^4(\epsilon)(1+t). \tag{3.30}$$

To find the H^1-estimates of $\tilde{M}_\epsilon u$, we take $v = A_\epsilon u$ in (3.1) and obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left|A_\epsilon^{\frac{1}{2}}\tilde{M}_\epsilon u\right|_\epsilon^2 + \nu\left|A_\epsilon\tilde{M}_\epsilon u\right|_\epsilon^2 &= (\tilde{M}_\epsilon f, A_\epsilon\tilde{M}_\epsilon u)_\epsilon - b_\epsilon(\tilde{M}_\epsilon u, \tilde{M}_\epsilon u, A_\epsilon\tilde{M}_\epsilon u)\\ &\quad - b_\epsilon(\tilde{N}_\epsilon u, \tilde{N}_\epsilon u, A_\epsilon\tilde{M}_\epsilon u).\end{aligned} \tag{3.31}$$

We rewrite (3.31), using the L^2-scalar product and the L^2-norm on ω, as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u\right|_{L^2(\omega)}^2 + \nu\left|\tilde{A}\tilde{M}_\epsilon u\right|_{L^2(\omega)}^2 &= (\tilde{M}_\epsilon f, A_\epsilon\tilde{M}_\epsilon u)_{L^2(\omega)} - \tilde{b}(\tilde{M}_\epsilon u, \tilde{M}_\epsilon u, \tilde{A}\tilde{M}_\epsilon u)\\ &\quad - \frac{1}{\epsilon}b_\epsilon(\tilde{N}_\epsilon u, \tilde{N}_\epsilon u, A_\epsilon\tilde{M}_\epsilon u),\end{aligned} \tag{3.32}$$

where $\tilde{A}$ and $\tilde{b}$ are the two-dimensional Stokes operator and trilinear form respectively.

Now we distinguish the three types of boundary conditions:

Type I. It contains the boundary conditions (FF) and (FP). Note that for these boundary conditions, we have (see Lemma 1.2)

$$\tilde{b}(\tilde{M}_\epsilon u, \tilde{M}_\epsilon u, \tilde{A}\tilde{M}_\epsilon u) = 0. \tag{3.33}$$

Type II. It contains the boundary conditions (FD) and (PP). The lack of the orthogonal identity (3.33) forces us to use the following inequality:

$$\begin{aligned}\left|\tilde{b}(\tilde{M}_\epsilon u, \tilde{M}_\epsilon u, \tilde{A}\tilde{M}_\epsilon u)\right| &\leq c_7\left|\tilde{M}_\epsilon u\right|_{L^2(\omega)}^{\frac{1}{2}}\left|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u\right|_{L^2(\omega)}\left|\tilde{A}\tilde{M}_\epsilon u\right|_{L^2(\omega)}^{\frac{3}{2}}\\ &\leq \text{(with Young's inequality)}\\ &\leq \frac{\nu}{4}\left|\tilde{A}\tilde{M}_\epsilon u\right|_{L^2(\omega)}^2 + \frac{c_8}{\nu^3}\left|\tilde{M}_\epsilon u\right|_{L^2(\omega)}^2\left|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u\right|_{L^2(\omega)}^4,\end{aligned} \tag{3.34}$$

where c_7 and c_8 are independent of ϵ.

Type III. It contains the boundary conditions (DP) and (DD). Note that, according to our definition of $\tilde{M}_\epsilon u$ in these cases, there is nothing to do: $\tilde{M}_\epsilon u = 0$.

First we establish the a priori estimates in the case of the boundary conditions of Type I. We infer from (3.32), (3.33) and Young's inequality

$$\frac{d}{dt}\big|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u\big|_\epsilon^2+\nu\big|\tilde{A}\tilde{M}_\epsilon u\big|_\epsilon^2\leq \frac{2\big|\tilde{M}_\epsilon f\big|_\epsilon^2}{\nu}+c_9\big|A_\epsilon^{\frac{1}{2}}\tilde{N}_\epsilon u\big|_\epsilon^3\big|A_\epsilon\tilde{N}_\epsilon u\big|_\epsilon, \tag{3.35}$$

and according to

$$\big|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u\big|_\epsilon^2\leq \frac{1}{\lambda_1}\big|\tilde{A}\tilde{M}_\epsilon u\big|_\epsilon^2,$$

we have

$$\frac{d}{dt}\big|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u\big|_\epsilon^2+\nu\lambda_1\big|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u\big|_\epsilon^2\leq \frac{2\big|\tilde{M}_\epsilon f\big|_\epsilon^2}{\nu}+c_9\big|A_\epsilon^{\frac{1}{2}}\tilde{N}_\epsilon u\big|_\epsilon^3\big|A_\epsilon\tilde{N}_\epsilon u\big|_\epsilon. \tag{3.36}$$

Thanks to Gronwall's lemma and Lemma 3.1, we find

$$\big|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u(t)\big|_\epsilon^2\leq a_0^2(\epsilon)e^{-\nu\lambda_1 t}+\frac{2\big|\tilde{M}_\epsilon f\big|_\epsilon^2}{\nu^2\lambda_1}+c_{10}(\nu)\epsilon R_n^4(\epsilon)(1+t). \tag{3.37}$$

We collect the previous inequalities in the following lemma:

Lemma 3.2. *Under one of the boundary conditions* (FF) *or* (FP), *we assume that for some* $0<q<\frac{1}{2}$,

$$\lim_{\epsilon\to 0}\epsilon^{2q}\big(\big|A_\epsilon^{\frac{1}{2}}u_0\big|_\epsilon^2+\big|f\big|_\epsilon^2\big)=0.$$

Then, there exists $\epsilon_1=\epsilon_1(\nu)>0$ *such that for all* ϵ, $0<\epsilon\leq\epsilon_1$, *there exists* $T^\sigma(\epsilon)>0$ *and a positive constant* $c_{10}(\nu)$, *independent of* ϵ, *such that*

$$(i)\quad \big|\tilde{M}_\epsilon u(t)\big|_\epsilon^2\leq a_0^2(\epsilon)e^{-\nu\lambda_1 t}+\frac{\alpha^2(\epsilon)}{\nu\lambda_1}+c_6(\nu)\epsilon R_n^4(\epsilon)(1+t).$$

$$(ii)\quad \int_0^t\big|A_\epsilon^{\frac{1}{2}}\tilde{M}_\epsilon u(s)\big|_\epsilon^2\,ds\leq\frac{\alpha^2(\epsilon)t}{\nu^2\lambda_1}+\frac{a_0^2(\epsilon)}{\nu\lambda_1}+c_6(\nu)\epsilon R_n^4(\epsilon)(1+t).$$

$$(iii)\quad \big|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u(t)\big|_\epsilon^2\leq a_0^2(\epsilon)e^{-\nu\lambda_1 t}+\frac{2\big|\tilde{M}_\epsilon f\big|_\epsilon^2}{\nu^2\lambda_1}+c_{10}(\nu)\epsilon R_n^4(\epsilon)(1+t).$$

$$(iv)\quad \int_0^t\big|\tilde{A}\tilde{M}_\epsilon u(s)\big|_\epsilon^2\,ds\leq\frac{2a_0^2(\epsilon)t}{\nu}+c_{10}(\nu)\epsilon R_n^4(\epsilon)(1+t).$$

Remark 3.2. Note that the inequalities of Lemma 3.2 are similar to those of the 2D Navier-Stokes equations with the periodic boundary condition; we have, however, the perturbation term $c_{10}(\nu)\epsilon R_n^4(\epsilon)(1+t)$, which is small for ϵ small enough.

We turn now to the study of the case of the boundary conditions of Type II. We have, thanks to (3.32), (3.34) and Young's inequality

$$\begin{aligned}\frac{d}{dt}\big|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u\big|_{L^2(\omega)}^2+\nu\lambda_1\big|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u\big|_{L^2(\omega)}^2\leq \frac{2\big|\tilde{M}_\epsilon f\big|_{L^2(\omega)}^2}{\nu}+\frac{c_8}{\nu^3}\big|\tilde{M}_\epsilon u\big|_{L^2(\omega)}^2\big|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u\big|_{L^2(\omega)}^4\\ +\frac{c_{11}}{\epsilon}\big|A^{\frac{1}{2}}\tilde{N}_\epsilon u\big|_\epsilon^3\big|A_\epsilon\tilde{N}_\epsilon u\big|_\epsilon.\end{aligned} \tag{3.38}$$

We set

$$y(t) = \left|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u(t)\right|^2_{L^2(\omega)}, \quad h(t) = \frac{2\left|\tilde{M}_\epsilon f\right|^2_{L^2(\omega)}}{\nu} + \frac{c_{11}}{\epsilon}\left|A^{\frac{1}{2}}\tilde{N}_\epsilon u\right|^3_\epsilon \left|A_\epsilon \tilde{N}_\epsilon u\right|_\epsilon \tag{3.39}$$

and

$$g(t) = \frac{c_8}{\nu^3}\left|\tilde{M}_\epsilon u\right|^2_{L^2(\omega)}\left|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u\right|^2_{L^2(\omega)}. \tag{3.40}$$

We infer from (3.38)

$$\frac{dy}{dt} \leq gy + h, \tag{3.41}$$

and Gronwall's lemma yields

$$y(t) \leq y(0)\exp\left(\int_0^t g(\tau)\,d\tau\right) + \int_0^t h(s)\exp\left(\int_s^t g(\tau)\,d\tau\right)\,ds. \tag{3.42}$$

First we bound $\int_0^t g(\tau)\,d\tau$ using (3.28) and (3.29). We have

$$\begin{aligned}\int_0^t g(\tau)\,d\tau &\leq \frac{c_8}{\nu^3}\sup_{0\leq\tau\leq t}\left|\tilde{M}_\epsilon u(\tau)\right|^2_{L^2(\omega)}\int_0^t\left|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u(\tau)\right|^2_{L^2(\omega)}d\tau \\ &\leq c_{12}(\nu,\lambda_1)\left[\frac{a_0^2(\epsilon)+\alpha^2(\epsilon)}{\epsilon} + R_n^4(\epsilon)(1+t)\right]\left[\frac{t\alpha^2(\epsilon)+a_0^2(\epsilon)}{\epsilon} + R_n^4(\epsilon)(1+t)\right] \\ &\leq c_{12}(\nu,\lambda_1)\left(R_m^2(\epsilon)+R_n^4(\epsilon)(1+t)\right)\left[\frac{t\alpha^2(\epsilon)+a_0^2(\epsilon)}{\epsilon} + R_n^4(\epsilon)(1+t)\right]\end{aligned} \tag{3.43}$$

where

$$R_m^2(\epsilon) = \max\left(a_0^2(\epsilon), \alpha^2(\epsilon)\right). \tag{3.44}$$

Now we write

$$\begin{aligned}\int_0^t h(s)\exp\left(\int_s^t g(\tau)\,d\tau\right)\,ds &\leq \left(\int_0^t h(s)\,ds\right)\exp\left(\int_0^t g(\tau)\,d\tau\right) \\ &\leq \left(\frac{2tR_m^2(\epsilon)}{\nu} + c_6R_n^4(\epsilon)(1+t)\right)\exp\left(\int_0^t g(\tau)\,d\tau\right).\end{aligned} \tag{3.45}$$

Therefore,

$$\left|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u\right|^2_{L^2(\omega)} \leq c_13(\nu)\epsilon\left(R_m^2+R_n^4\right)\exp\left[c_{12}\left(R_m^2+R_n^4(1+t)\right)\left(R_m^2+R_n^4\right)(1+t)\right]. \tag{3.46}$$

Here we make the following assumption on the initial data (for the boundary conditions of Type II):

$$\text{For arbitrary } K_1 > 0 \text{ and } K_2 > 0; \quad R_m^2(\epsilon) \leq K_1\ln\left|\ln\epsilon\right|, \; R_n^2(\epsilon) \leq K_2\ln\left|\ln\epsilon\right|. \tag{H_1}$$

Taking into account the assumption (H_1), we collect the estimates of $\tilde{M}_\epsilon u$ in the case of the boundary conditions of Type II in the following:

Lemma 3.3. *Under one of the boundary conditions* (FD) *or* (PP), *we assume that for some* $0 < q < \frac{1}{2}$,

$$\lim_{\epsilon\to 0} \epsilon^{2q}(\left|A_\epsilon^{\frac{1}{2}} u_0\right|_\epsilon^2 + \left|f\right|_\epsilon^2) = 0.$$

And for arbitrary positive numbers K_1 *and* K_2, *we assume that*

$$\left|A_\epsilon^{\frac{1}{2}} \tilde{M}_\epsilon u_0\right|_\epsilon^2 + \left|\tilde{M}_\epsilon f\right|_\epsilon^2 \le K_1 \epsilon \ln\left|\ln \epsilon\right| \quad \text{and} \quad \left|A_\epsilon^{\frac{1}{2}} \tilde{N}_\epsilon u_0\right|_\epsilon^2 + \left|\tilde{N}_\epsilon f\right|_\epsilon^2 \le K_2 \ln\left|\ln \epsilon\right|.$$

Then, there exists $\epsilon_1 = \epsilon_1(\nu) > 0$, *such that for all* ϵ, $0 < \epsilon \le \epsilon_1$, *there exists* $T^\sigma(\epsilon) > 0$ *and a positive constant* $c_{13}(\nu)$, *independent of* ϵ, K_1, *and* K_2, *such that for all* $t > 0$

$$(i) \quad \left|\tilde{M}_\epsilon u(t)\right|_\epsilon^2 \le \frac{\epsilon K_1}{\lambda_1} e^{-\nu\lambda_1 t} + \frac{\epsilon K_1}{\nu \lambda_1} + c_{13}(\nu)\epsilon K_2^2(1+t).$$

$$(ii) \quad \int_0^t \left|A_\epsilon^{\frac{1}{2}} \tilde{M}_\epsilon u(s)\right|_\epsilon^2 ds \le \frac{\epsilon K_1}{\nu^2 \lambda_1}\left(\frac{t}{\nu}\right) + c_{13}(\nu)\epsilon K_2^2(1+t).$$

$$(iii) \left|A_\epsilon^{\frac{1}{2}} \tilde{M}_\epsilon u(t)\right|_\epsilon^2 \le c_{13}(\nu)\epsilon \ln\left|\ln\epsilon\right|(K_1 + K_2^2)(1+t) \\ \times \exp\left[c_{13} \ln\left|\ln \epsilon\right|\left(K_1 + K_2^2(1+t)\right)(K_1 + K_2^2)(1+t)\right].$$

4. Behavior of $T^\sigma(\epsilon)$. In this section we study the behavior of the maximal time of existence of the 3D Navier-Stokes equations in thin domains, when the thickness goes to zero. We divide our study in three parts depending on the type of the boundary condition.

Type I. It contains the boundary conditions (FF) and (FP), i.e., the free boundary condition in the thin direction and either the periodic or the free boundary condition on the lateral boundary.We will prove that $T^\sigma(\epsilon) = +\infty$ in two steps. First we prove the following:

Proposition 4.1. *In the case of the boundary conditions* (FF) *or* (FP), *we assume that* $\lim_{\epsilon\to 0} \epsilon^q(\left|A_\epsilon^{\frac{1}{2}} u_0\right|_\epsilon^2 + \left|f\right|_\epsilon^2) = 0$, *for some* $q < 1$. *Then, there exists* $\epsilon_1(\nu, \sigma, q)$ *such that*

$$\lim_{\substack{\epsilon\to 0 \\ \epsilon<\epsilon_1}} \epsilon^{2-2q} T^\sigma(\epsilon) = +\infty. \tag{4.1}$$

Proof. According to Lemmas 3.1 and 3.2, we have

$$\left|A_\epsilon^{\frac{1}{2}} \tilde{N}_\epsilon u(t)\right|_\epsilon^2 \le b_0^2(\epsilon) \exp\left(-\frac{\nu t}{2\epsilon^2}\right) + \frac{2\epsilon^2\beta^2}{\nu^2}, \quad 0 \le t \le T^\sigma(\epsilon) \tag{4.2}$$

and

$$\left|A_\epsilon^{\frac{1}{2}} \tilde{M}_\epsilon u(t)\right|_\epsilon^2 \le a_0^2(\epsilon) e^{-\nu\lambda_1 t} + \frac{\alpha^2}{\nu^2\lambda_1} + c_{10}(\nu)\epsilon R_n^4(\epsilon)(1+t). \tag{4.3}$$

Now we fix σ by setting

$$\sigma = 4\max\big(1, \frac{2}{\nu^2}, \frac{1}{\nu^2\lambda_1}\big). \tag{4.4}$$

We infer from (4.2) and (4.3)

$$\begin{aligned}\big|A_\epsilon^{\frac{1}{2}}u(t)\big|_\epsilon^2 \leq b_0^2(\epsilon)\exp\big(-\frac{\nu t}{2\epsilon^2}\big) + \frac{2\epsilon^2\beta^2}{\nu^2} + a_0^2(\epsilon)e^{-\nu\lambda_1 t} + \frac{\alpha^2}{\nu^2\lambda_1}\\ + c_{10}(\nu)\epsilon R_n^4(\epsilon)(1+t); \quad 0 \leq t \leq T^\sigma(\epsilon),\end{aligned} \tag{4.5}$$

and, with (4.4), we obtain

$$\big|A_\epsilon^{\frac{1}{2}}u(t)\big|_\epsilon^2 \leq \frac{\sigma}{4}R_0^2(\epsilon) + c_{10}(\nu)\big(\epsilon^q R_0^2(\epsilon)\big)\epsilon^{1-q}R_n^4(\epsilon)(1+t), \tag{4.6}$$

for $0 \leq t \leq T^\sigma(\epsilon)$. Now we suppose that $T^\sigma(\epsilon) < +\infty$. Then, we have

$$\big|A_\epsilon^{\frac{1}{2}}u(T^\sigma(\epsilon))\big|_\epsilon^2 = \sigma R_0^2(\epsilon) \tag{4.7}$$

and, with (4.6), we can write

$$\frac{3\sigma}{4}R_0^2(\epsilon) \leq c_{10}(\nu)\big(\epsilon^q R_0^2(\epsilon)\big)\epsilon^{1-q}R_0^2(\epsilon)\big(1 + T^\sigma(\epsilon)\big). \tag{4.8}$$

We make the obvious assumption $R_0(\epsilon) \neq 0$, and obtain

$$\frac{3\sigma}{4} \leq c_{10}(\nu)(\epsilon^q R_0^2(\epsilon))\epsilon^{1-q}\big(1 + T^\sigma(\epsilon)\big). \tag{4.9}$$

Now we claim that

$$\lim_{\epsilon\to 0}\epsilon^{1-q}T^\sigma(\epsilon) = +\infty. \tag{4.10}$$

If (4.10) doesn't hold, then there exists a positive constant L_1 and a sequence $(\epsilon_n)_{n\geq 1}$, with $0 < \epsilon_n < \epsilon_1(\nu)$, and $\lim_{n\to\infty}\epsilon_n = 0$, such that

$$T^\sigma(\epsilon_n) \leq \frac{L_1}{\epsilon_n^{1-q}}, \quad \forall\, n \geq 1. \tag{4.11}$$

Therefore,

$$\frac{3\sigma}{4} \leq L_2\epsilon_n^q R_0^2(\epsilon_n), \tag{4.12}$$

where $L_2 = L_2(\nu)$. The right-hand side of the inequality (4.12) goes to zero as n goes to ∞, hence $\sigma = 0$, which contradicts the choice of σ in (4.4). $\square$

Now we go to the second step and establish that the maximal time of existence of the strong solution of the 3D Navier-Stokes equations in thin domains is infinite, for large initial data. More precisely, we prove

Theorem 4.1. *In the case of the boundary conditions* (FF) *or* (FP), *we assume that*

$$\lim_{\epsilon\to 0} \epsilon^{2q}(\big|A_\epsilon^{\frac{1}{2}} u_0\big|_\epsilon^2 + \big|f\big|_\epsilon^2) = 0 \quad \text{for some} \quad 0 < 2q < 1.$$

Then, there exists $\epsilon_4 = \epsilon_4(\nu, q, \omega)$ *such that: for* $0 < \epsilon < \epsilon_4$, *the strong solution* u_ϵ *of* (0.1)–(0.3) *with the boundary condition* (FF) *or* (FP) *exists for all times; i.e.,* $T_\epsilon = +\infty$ *in Theorem* 0.1 *and*

$$u_\epsilon \in \mathcal{C}^0([0,\infty); V_\epsilon) \cap L^2(0,T; D(A_\epsilon)), \quad \forall\ T > 0. \tag{4.13}$$

Proof. The proof is done in two steps.

Step 1. We set

$$K_\epsilon^2 = \big|A_\epsilon^{\frac{1}{2}} \tilde{M}_\epsilon u_0\big|_\epsilon^2 + \frac{8}{\nu^2\lambda_1}\big|\tilde{M}_\epsilon f\big|_\epsilon + B_\epsilon^2, \tag{4.14}$$

where

$$B_\epsilon^2 = \big|A_\epsilon^{\frac{1}{2}} \tilde{N}_\epsilon u_0\big|_\epsilon^2 + \big|\tilde{N}_\epsilon f\big|_\epsilon^2. \tag{4.15}$$

Note that, since $\lim_{\epsilon\to 0} \epsilon^{q}(\big|A_\epsilon^{\frac{1}{2}} u_0\big|_\epsilon^2 + \big|f\big|_\epsilon^2) = 0$, we have

$$\lim_{\epsilon\to 0} \epsilon^q B_\epsilon^2 = \lim_{\epsilon\to 0} \epsilon^q K_\epsilon^2 = 0. \tag{4.16}$$

We choose $\epsilon_4 = \epsilon_4(\nu, \lambda_1, q)$ satisfying the following conditions:

$$(i) \qquad 0 < \epsilon_4 < \frac{1}{4}. \tag{4.17}$$

$$(ii) \qquad \epsilon^q K_\epsilon^2 \le 1, \quad \text{for} \quad 0 < \epsilon \le \epsilon_4. \tag{4.18}$$

$$(iii) \qquad \frac{2\epsilon^2}{\nu^2} \le \frac{1}{8}, \quad \exp\left(-\frac{\nu\lambda_1}{\epsilon^{2(1-q)}}\right) \le \frac{1}{4}, \quad \text{and} \quad \frac{16c_4(\nu)}{\nu}\max\big(1, \frac{1}{\nu^3}\big)\epsilon^q \le \frac{1}{4}. \tag{4.19}$$

$$(iv) \qquad \epsilon^{2-2q} T^\sigma(\epsilon) > 4, \quad \text{for} \quad 0 < \epsilon \le \epsilon_4. \tag{4.20}$$

The existence of ϵ_4 is obvious, since the left-hand sides of the inequalities in (ii) and (iii) go to zero when ϵ goes to zero; and, by Proposition 4.1, the left-hand side of (iv) goes to ∞ as ϵ goes to zero.

We recall that

$$\int_0^t \big|A_\epsilon^{\frac{1}{2}} \tilde{N}_\epsilon u(s)\big|_\epsilon^3 \big|A_\epsilon \tilde{N}_\epsilon u(s)\big|_\epsilon\, ds \le \frac{4}{\nu}\max\big(1, \frac{1}{\nu^3}\big) B_\epsilon^4 \big(1+t\big). \tag{4.21}$$

Hence,

$$\big|A_\epsilon^{\frac{1}{2}} \tilde{N}_\epsilon u(t)\big|_\epsilon^2 \le b_0^2(\epsilon)\exp(-\frac{\nu t}{2\epsilon^2}) + \frac{2\epsilon^2}{\nu^2}\beta^2(\epsilon), \quad 0 \le t < T^\sigma(\epsilon), \tag{4.22}$$

and

$$\big|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u(t)\big|_\epsilon^2 \le a_0^2(\epsilon)\exp\big(-\nu\lambda_1 t\big) + \frac{2\alpha^2(\epsilon)}{\nu^2\lambda_1} + \frac{16\epsilon c_4(\nu)}{\nu}\max\big(1,\frac{1}{\nu^3}\big)B_\epsilon^4(\epsilon)\big(1+t\big). \quad (4.23)$$

We set

$$t_\epsilon = \epsilon^{2(q-1)}, \quad \text{for} \;\; 0 < \epsilon \le \epsilon_4.$$

We have, according to (4.22) and (4.23)

$$\big|A_\epsilon^{\frac{1}{2}}\tilde{N}_\epsilon u(t)\big|_\epsilon^2 \le b_0^2(\epsilon)\exp\big(-\frac{\nu t_\epsilon}{2\epsilon^2}\big) + \frac{2\epsilon^2}{\nu^2}\beta^2(\epsilon), \quad t_\epsilon \le t < 2t_\epsilon, \quad (4.24)$$

and

$$\big|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u(t)\big|_\epsilon^2 \le a_0^2(\epsilon)e^{-\nu\lambda_1 t_\epsilon} + \frac{2\alpha^2(\epsilon)}{\nu^2\lambda_1} + c_5(\nu)\epsilon\left(\epsilon^q B_\epsilon^2(\epsilon)\right)B_\epsilon^2\epsilon^{1-q}\big(1+2t_\epsilon\big), \quad t_\epsilon \le t < 2t_\epsilon. \quad (4.25)$$

Now, thanks to (4.17)–(4.20), we have

$$\big|A_\epsilon^{\frac{1}{2}}\tilde{N}_\epsilon u(t)\big|_\epsilon^2 \le \frac{1}{4}B_\epsilon^2, \quad t_\epsilon \le t < 2t_\epsilon, \quad (4.26)$$

and

$$\big|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u(t)\big|_\epsilon^2 \le \frac{1}{4}K_\epsilon^2 + \frac{1}{4}B_\epsilon^2 \le \frac{1}{2}K_\epsilon^2, \quad t_\epsilon \le t < 2t_\epsilon. \quad (4.27)$$

Hence,

$$\big|A_\epsilon^{\frac{1}{2}}u(2t_\epsilon)\big|_\epsilon^2 \le \frac{3}{4}K_\epsilon^2. \quad (4.28)$$

Step 2. We use an induction argument. We write (4.22) and (4.23) when the initial data are given at the time t_0 and satisfy

$$\big|A_\epsilon^{\frac{1}{2}}\tilde{N}_\epsilon u(t_0)\big|_\epsilon^2 \le \frac{1}{2}B_\epsilon^2, \quad \text{and} \quad \big|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u(t_0)\big|_\epsilon^2 \le \frac{1}{2}K_\epsilon^2. \quad (4.29)$$

We obtain, as in the proof of Lemmas 3.1 and 3.2

$$\big|A_\epsilon^{\frac{1}{2}}\tilde{N}_\epsilon u(t)\big|_\epsilon^2 \le \frac{1}{2}B_\epsilon^2\exp\left(-\frac{\nu(t-t_0)}{2\epsilon^2}\right) + \frac{2\epsilon^2}{\nu^2}\beta^2(\epsilon), \quad t_0 \le t < T^\sigma(\epsilon), \quad (4.30)$$

and

$$\big|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u(t)\big|_\epsilon^2 \le \frac{1}{2}K_\epsilon^2 e^{-\nu\lambda_1(t-t_0)} + \frac{2\alpha^2(\epsilon)}{\nu^2\lambda_1} + c_6(\nu)\left(\epsilon^q B_\epsilon^2(\epsilon)\right)B_\epsilon^2\epsilon^{1-q}\big(1+t-t_0\big), \quad t_0 \le t < T^\sigma(\epsilon). \quad (4.31)$$

Thanks to (4.26) and (4.27), we take $t_0 = 2\epsilon^{2(q-1)}$ and obtain, as above,

$$\left|A_\epsilon^{\frac{1}{2}}\tilde{N}_\epsilon u(t)\right|_\epsilon^2 \leq \frac{1}{8}B_\epsilon^2 + \frac{1}{8}B_\epsilon^2 \leq \frac{1}{4}B_\epsilon^2, \quad 2\epsilon^{2(q-1)} \leq t < 3\epsilon^{2(q-1)}, \tag{4.32}$$

and

$$\left|\tilde{A}^{\frac{1}{2}}\tilde{M}_\epsilon u(t)\right|_\epsilon^2 \leq \frac{1}{8}K_\epsilon^2 + \frac{1}{8}K_\epsilon^2 + \frac{1}{4}B_\epsilon^2 \leq \frac{1}{2}K_\epsilon^2, \quad 2\epsilon^{2(q-1)} \leq t < 3\epsilon^{2(q-1)}. \tag{4.33}$$

Hence,

$$\left|A_\epsilon^{\frac{1}{2}}u(t)\right|_\epsilon^2 \leq \frac{3}{4}K_\epsilon^2, \quad 2\epsilon^{2(q-1)} \leq t < 3\epsilon^{2(q-1)}, \tag{4.34}$$

which implies that $T^\sigma(\epsilon) > 3\epsilon^{2(q-1)}$. We repeat this argument n times to show that

$$T^\sigma(\epsilon) > n\epsilon^{2(q-1)}; \quad \forall\ n \geq 1. \tag{4.35}$$

Hence, $T^\sigma(\epsilon) = +\infty$ for $0 < \epsilon \leq \epsilon_4$. This concludes the study for the Type I boundary conditions. □

Type II. This type contains the boundary conditions (FD) and (PP). The lack of the orthogonal property of the trilinear form, $\tilde{b}(\tilde{M}_\epsilon u, \tilde{M}_\epsilon u, \tilde{A}\tilde{M}_\epsilon u) = 0$, implies a weaker global existence result for the strong solution. We need to assume stronger conditions on the initial data; these conditions still allow large initial data and are similar to those imposed by G. Raugel and G. Sell in the case of the purely periodic boundary condition ([22, 23, 24]). More precisely, we prove the following:

Theorem 4.2. *In the case of the boundary conditions* (FD) *or* (PP), *we assume that, for arbitrary positive numbers* K_1 *and* K_2,

$$\begin{aligned} &\left|A_\epsilon^{\frac{1}{2}}\tilde{M}_\epsilon u_0\right|_\epsilon^2 + \left|\tilde{M}_\epsilon f\right|_\epsilon^2 \leq K_1\epsilon \ln\left|\ln\epsilon\right| \quad \textit{and} \\ &\left|A_\epsilon^{\frac{1}{2}}\tilde{N}_\epsilon u_0\right|_\epsilon^2 + \left|\tilde{N}_\epsilon f\right|_\epsilon^2 \leq K_2 \ln\left|\ln\epsilon\right|. \end{aligned} \tag{4.36}$$

Then, there exists $\epsilon_5 = \epsilon_5(\nu, K_1, K_2, \omega)$ *such that: for* $0 < \epsilon < \epsilon_5$, *the strong solution* u_ϵ *of* (0.1)–(0.3) *with the boundary condition* (FD) *or* (PP) *exists for all times; i.e.,*

$$u_\epsilon \in \mathcal{C}^0([0,\infty); V_\epsilon) \cap L^2(0,T; D(A_\epsilon)). \tag{4.37}$$

Proof. We set

$$R_0^2(\epsilon) = (K_1 + K_2)\ln\left|\ln\epsilon\right| + 1 + b_0^2(\epsilon) + \frac{2}{\nu^2}\beta^2(\epsilon). \tag{4.38}$$

According to (4.36), we have $\left|A_\epsilon^{\frac{1}{2}}u_0\right|_\epsilon^2 + \left|f\right|_\epsilon^2 \leq R_0^2(\epsilon)$, and since

$$\lim_{\epsilon\to 0} \epsilon^{\frac{1}{4}}\left[(K_1 + K_2)\ln\left|\ln\epsilon\right| + 1\right] = 0, \quad \text{and} \quad \lim_{\epsilon\to 0} \epsilon^{\frac{1}{4}}\left(b_0^2(\epsilon) + \frac{2}{\nu^2}\beta^2(\epsilon)\right) = 0, \tag{4.39}$$

the a priori estimates obtained in Section 3 still hold. Moreover, we recall that if $T^\sigma(\epsilon) < +\infty$, then

$$\left|A_\epsilon^{\frac{1}{2}} u(T^\sigma(\epsilon))\right|_\epsilon^2 = \sigma R_0^2(\epsilon).$$

Now we write, thanks to Lemmas 3.1 and 3.3,

$$\left|A_\epsilon^{\frac{1}{2}} u(t)\right|_\epsilon^2 \leq b_0^2(\epsilon)\exp\left(\frac{-\nu t}{2\epsilon^2}\right) + \frac{2\epsilon^2}{\nu^2}\beta^2(\epsilon) + c_{13}(\nu)\epsilon\ln\left|\ln\epsilon\right|(K_1+K_2^2)(1+t)\exp(\theta_\epsilon(t)), \tag{4.40}$$

where

$$\theta_\epsilon(t) = c_{13}(\nu)\epsilon\ln\left|\ln\epsilon\right|\left(K_1 + K_2^2(1+t)\right)(1+t).$$

First we prove that

$$\lim_{\epsilon\to 0} T^\sigma(\epsilon) = +\infty,$$

which implies the existence of $\epsilon_5(\nu, \lambda_1, K_1, K_2)$ such that $T^\sigma(\epsilon) > 1$, for $0 < \epsilon \leq \epsilon_5$. Then, the induction argument given in the proof of Theorem 4.1 (see also [24]) implies that $T^\sigma(\epsilon) = +\infty$, for $0 < \epsilon \leq \epsilon_5$.

Assume that there exists a positive constant L_3 (independent of ϵ) and a sequence $(\epsilon_n)_{n\geq 1}$, such that

$$0 < \epsilon_n \leq \epsilon_5; \quad \lim_{n\to\infty} \epsilon_n = 0 \quad \text{and} \quad T^\sigma(\epsilon_n) \leq L_3, \quad \forall\, n \geq 1. \tag{4.41}$$

We have

$$\left|A_{\epsilon_n}^{\frac{1}{2}} u\big(T^\sigma(\epsilon_n)\big)\right|_{\epsilon_n}^2 = \sigma R_0^2(\epsilon_n), \tag{4.42}$$

and we infer from (4.40)

$$\sigma R_0^2(\epsilon_n) \leq b_0^2(\epsilon_n) + \frac{2\epsilon_n^2}{\nu^2}\beta^2(\epsilon_n) + c_{13}(\nu)\epsilon_n\ln\left|\ln\epsilon_n\right|(K_1+K_2^2)(1+L_3)\exp(\theta_{\epsilon_n}(L_3)). \tag{4.43}$$

Hence, we have (with $\sigma > 2\max(1, \frac{2}{\nu^2})$)

$$\frac{\sigma}{2} R_0^2(\epsilon_n) \leq c_{13}(\nu)\epsilon_n\ln\left|\ln\epsilon_n\right|(K_1 + K_2^2)(1 + L_3)\exp(\theta_{\epsilon_n}(L_3)). \tag{4.44}$$

The right-hand side of inequality (4.44) goes to zero as n goes to ∞, while the left-hand side is greater than 2. Therefore,

$$\lim_{\epsilon\to 0} T^\sigma(\epsilon) = +\infty.$$

The induction argument is similar to the one given above and is left as an exercise. □

Type III. It contains the boundary conditions (DD) and (DP), i.e., the Dirichlet boundary condition in the thin direction and either the Dirichlet or the periodic condition on the lateral boundary. We prove the following:

Theorem 4.3. *In the case of the boundary conditions* (DD) *or* (DP), *we assume that*

$$\lim_{\epsilon\to 0}\epsilon(\left|A_\epsilon^{\frac{1}{2}}u_0\right|_\epsilon^2+\left|f\right|_\epsilon^2)=0.$$

Then, there exists $\epsilon_6=\epsilon_6(\nu)$ *such that: for* $0<\epsilon<\epsilon_4$, *the strong solution* u_ϵ *of* (0.1)–(0.3) *with the boundary condition* (DD) *or* (DP) *exists for all times; i.e.,*

$$u_\epsilon\in\mathcal{C}^0([0,\infty);V_\epsilon)\cap L^2(0,T;D(A_\epsilon)). \tag{4.45}$$

Proof. As stated in Section 1, in the case of the boundary condition (DD) or (DP), we have $\tilde{N}_\epsilon u=u$. Hence, the estimates of $\tilde{N}_\epsilon u$ obtained in Lemma 3.1 (see also Remarks 2.2, 2.3 and 3.1) hold for u. We infer from Lemma 3.1

$$\left|A_\epsilon^{\frac{1}{2}}u(t)\right|_\epsilon^2\le\left|A_\epsilon^{\frac{1}{2}}u_0\right|_\epsilon^2\exp(\frac{-\nu t}{2\epsilon^2})+\frac{\epsilon^2}{\nu^2}\left|f\right|_\epsilon^2. \tag{4.46}$$

Assume now that $T^\sigma(\epsilon)<\infty$. We have

$$\sigma R_0^2(\epsilon)\le\left|A_\epsilon^{\frac{1}{2}}u_0\right|_\epsilon^2+\frac{\epsilon^2}{\nu^2}\left|f\right|_\epsilon^2<\left|A_\epsilon^{\frac{1}{2}}u_0\right|_\epsilon^2+\left|f\right|_\epsilon^2,\quad\text{for }\ \epsilon<\nu, \tag{4.47}$$

which contradicts $\sigma>1$. Thus

$$T^\sigma(\epsilon)=+\infty,\quad\text{for }\ \epsilon<\min(\nu,\epsilon_1),$$

where ϵ_1 satisfies

$$2c_4\epsilon^{\frac{1}{2}}\sigma^{\frac{1}{2}}R_0(\epsilon)\le\frac{\nu}{2},\quad\text{for }\ 0<\epsilon\le\epsilon_1.$$

The proof is complete. □

Example of channel flow. Note that (4.46) implies that the H^1 norm of the solution u_ϵ converges to zero when $\epsilon\to 0$ as long as we are far from the time boundary layer $t=0$. We give now an example where we develop an asymptotic expansion of the solution. We consider for simplicity the case where the exterior force is tangential and given by

$$f=P_1e_1+P_2e_2, \tag{4.48}$$

where e_1 and e_2 are the unit vectors in the directions x_1 and x_2 respectively, and P_1 and P_2 are some constants. We write the Euclidean norm of f as $P=\sqrt{P_1^2+P_2^2}$. (The general case of forces and also the other boundary conditions will be given in a forthcoming article, [28].)

We consider the Navier-Stokes equations (0.1)–(0.3) with the boundary condition (DP). It is easy to see that when the exterior force f is given by (4.48) the functions

$$q_\epsilon=0\quad\text{ and }\quad w_\epsilon=-\frac{x_3(x_3-\epsilon)}{2\nu}(P_1e_1+P_2e_2) \tag{4.49}$$

are a solution of the Stationary Navier-Stokes equations

$$-\nu\Delta w + (w\cdot\nabla)w + \nabla q = P_1 e_1 + P_2 e_2 \qquad \text{in} \quad \Omega_\epsilon \tag{4.50}$$

$$\operatorname{div} w = 0 \qquad \text{in} \quad \Omega_\epsilon \tag{4.51}$$

$$w = 0 \;\text{ on } \omega\times\{0,\epsilon\} \;\text{ and } w \;\text{ is periodic in the directions } x_1, x_2. \tag{4.52}$$

Notice that

$$\begin{aligned} |w_\epsilon|^2_{L^2(\Omega_\epsilon)} &= \frac{\epsilon^5|\omega|}{12\nu^2}P^2, \quad |\nabla w_\epsilon|^2_{L^2(\Omega_\epsilon)} = \frac{\epsilon^3|\omega|}{12\nu^2}P^2, \\ |w_\epsilon|_{L^\infty(\Omega_\epsilon)} &= \frac{\epsilon^2}{8\nu}P, \qquad |\nabla w_\epsilon|_{L^\infty(\Omega_\epsilon)} = \frac{\epsilon}{2\nu}P. \end{aligned} \tag{4.53}$$

Now we prove that for ϵ small enough compared to P and ν the stationary problem (4.50)–(4.52) has a unique solution which is w_ϵ. Note that

$$(w_\epsilon\cdot\nabla)w_\epsilon = 0.$$

Let $V_\epsilon = v_\epsilon - w_\epsilon$. The equations satisfied by V_ϵ are

$$-\nu\Delta V_\epsilon + (V_\epsilon\cdot\nabla)V_\epsilon + (V_\epsilon\cdot\nabla)w_\epsilon + (w_\epsilon\cdot\nabla)V_\epsilon + \nabla q_\epsilon = 0 \;\text{ in }\; \Omega_\epsilon, \tag{4.54}$$

$$\operatorname{div} V_\epsilon = 0 \qquad \text{in} \quad \Omega_\epsilon, \tag{4.55}$$

$$V_\epsilon = 0 \;\text{ on } \omega\times\{0,\epsilon\} \;\text{ and } V_\epsilon \;\text{ is periodic in the directions } x_1, x_2. \tag{4.56}$$

We multiply (4.54) with V_ϵ and obtain

$$\begin{aligned} \nu|A_\epsilon^{\frac12}V_\epsilon|^2_\epsilon &= -\int_{\Omega_\epsilon}(V_\epsilon\cdot\nabla)\cdot V_\epsilon\,dx \le |\nabla w_\epsilon|^2_{L^\infty(\Omega_\epsilon)}|V_\epsilon|^2_\epsilon \le \epsilon|\nabla w_\epsilon|^2_{L^\infty(\Omega_\epsilon)}|A_\epsilon^{\frac12}V_\epsilon|^2_\epsilon \\ &\le \quad \text{with (4.53)} \le \frac{\epsilon^3P^2}{2\nu}|A_\epsilon^{\frac12}V_\epsilon|^2_\epsilon. \end{aligned}$$

Hence, if

$$\epsilon^3 < \frac{2\nu^2}{P^2}, \tag{4.57}$$

then

$$|A_\epsilon^{\frac12}V_\epsilon|^2_\epsilon = 0 \quad \text{and} \quad V_\epsilon = 0.$$

Now we go to the nonstationary problem and write $U_\epsilon = u_\epsilon - w_\epsilon$. The equations satisfied by U_ϵ are

$$\frac{\partial U_\epsilon}{\partial t} - \nu\Delta U_\epsilon + (U_\epsilon\cdot\nabla)U_\epsilon + (U_\epsilon\cdot\nabla)w_\epsilon + (w_\epsilon\cdot\nabla)U_\epsilon + \nabla q_\epsilon = 0 \text{ in } \;\Omega_\epsilon, \tag{4.58}$$

$$\operatorname{div} V_\epsilon = 0 \qquad \text{in} \quad \Omega_\epsilon, \tag{4.59}$$

$$U_\epsilon = 0 \;\text{ on } \omega\times\{0,\epsilon\} \;\text{ and } U_\epsilon \;\text{ is periodic in the directions } x_1, x_2, \tag{4.60}$$

$$U_\epsilon(t=0) = u_{0\epsilon} - w_\epsilon. \tag{4.61}$$

We multiply (4.58) with $A_\epsilon U_\epsilon$ and obtain, using Hölder inequalities, the inequalities of Section 2 and (4.53)

$$\frac{1}{2}\frac{d}{dt}|A_\epsilon^{\frac{1}{2}}U_\epsilon|_\epsilon^2 + \nu|A_\epsilon U_\epsilon|_\epsilon^2 \le c_0\frac{\epsilon^2\sqrt{|\omega|}P}{\nu}|A_\epsilon U_\epsilon|_\epsilon^2 + \frac{\epsilon^3 P}{8\nu}|A_\epsilon U_\epsilon|_\epsilon^2 + c_0\epsilon^{\frac{1}{2}}|A_\epsilon^{\frac{1}{2}}U_\epsilon|_\epsilon|A_\epsilon U_\epsilon|_\epsilon^2,$$

where c_0 is an absolute constant. Therefore, if ϵ is small enough so that

$$c_0\epsilon^2\sqrt{|\omega|} + \frac{\epsilon^3}{8} \le \frac{\nu^2}{P}, \tag{4.62}$$

then

$$\frac{1}{2}\frac{d}{dt}|A_\epsilon^{\frac{1}{2}}U_\epsilon|_\epsilon^2 + \nu|A_\epsilon U_\epsilon|_\epsilon^2 \le c_0\epsilon^{\frac{1}{2}}|A_\epsilon^{\frac{1}{2}}U_\epsilon|_\epsilon|A_\epsilon U_\epsilon|_\epsilon^2. \tag{4.63}$$

Assume that $|A_\epsilon^{\frac{1}{2}}U_\epsilon(0)|_\epsilon^2 < \dfrac{\nu^2}{4c_0\epsilon}$ and let T^ϵ be the maximal time such that

$$|A_\epsilon^{\frac{1}{2}}U_\epsilon(t)|_\epsilon^2 \le \frac{\nu^2}{4c_0\epsilon}, \quad \forall t\in[0,T^\epsilon). \tag{4.64}$$

Using (4.64), we infer from (4.63)

$$\forall t\in(0,T^\epsilon), \qquad \frac{1}{2}\frac{d}{dt}|A_\epsilon^{\frac{1}{2}}U_\epsilon|_\epsilon^2 + \nu|A_\epsilon U_\epsilon|_\epsilon^2 \le 0.$$

Hence $|A_\epsilon^{\frac{1}{2}}U_\epsilon(t)|_\epsilon^2$ is decreasing and $T^\epsilon = +\infty$. Moreover, with Poincaré's inequality, we have

$$\forall t>0, \qquad \frac{1}{2}\frac{d}{dt}|A_\epsilon^{\frac{1}{2}}U_\epsilon|_\epsilon^2(t) + \frac{\nu}{\epsilon^2}|A_\epsilon^{\frac{1}{2}}U_\epsilon|_\epsilon^2(t) \le 0$$

and

$$|A_\epsilon^{\frac{1}{2}}U_\epsilon|_\epsilon^2(t) \le |A_\epsilon^{\frac{1}{2}}U_\epsilon|_\epsilon^2(0)\exp(-\frac{\nu}{\epsilon^2}) \qquad \forall\ t>0.$$

Hence, for ϵ satisfying (4.62), we have the following asymptotic expansion for the strong solution $u = u_\epsilon$:

$$u_\epsilon(t) = -\frac{x_3(x_3-\epsilon)}{2\nu t}(Pe_1 + Pe_2) + U_\epsilon(t), \quad 0\le t<+\infty$$

with

$$||U_\epsilon(t)||_\epsilon^2 \le ||u_{0\epsilon} - w_\epsilon||_\epsilon^2\exp(-\frac{\nu t}{\epsilon^2}), \qquad \text{for all } t\in[0,\infty).$$

5. Behavior of the averages. In this section we establish the convergence of the average, in the thin direction, of the strong solution of the three-dimensional Navier-Stokes equations in thin domains to the strong solution of the two-dimensional Navier-Stokes equations, when the thickness of the domain goes to zero.

Consider the following two-dimensional Navier-Stokes equations:

$$\frac{\partial \tilde{v}}{\partial t} - \nu\Delta'\tilde{v} + (\tilde{v}\cdot\nabla')\tilde{v} + \nabla'\tilde{p} = \tilde{f} \quad \text{in} \quad \omega\times[0,\infty), \tag{5.1}$$

$$\text{div}'\ \tilde{v} = 0 \quad \text{in} \quad \omega\times[0,\infty), \tag{5.2}$$

$$\tilde{v}(x',0) = \tilde{v}_0(x') \quad \text{in} \quad \omega. \tag{5.3}$$

Equations (5.1)–(5.3) are supplemented with one of the following boundary conditions:

(i) The Dirichlet boundary condition: $\tilde{v} = 0$ on $\partial\omega$.

(ii) The periodic boundary condition: In this case $\omega = (0,l_1)\times(0,l_2)$ and

$$\tilde{v} \ \text{ is } \omega\text{-periodic and } \int_\omega \tilde{f}\,dx' = 0.$$

(iii) The free boundary condition:

$$\tilde{v}\cdot\vec{\tilde{n}} = 0 \quad \text{and curl } \tilde{v} = 0 \ \text{ on } \ \partial\omega,$$

where

$$\text{curl } \tilde{v} = \frac{\partial \tilde{v}_2}{\partial x_1} - \frac{\partial \tilde{v}_1}{\partial x_2}.$$

We refer to the introduction for the mathematical setting of (5.1)–(5.3). Here again, we omit the references to the boundary conditions. We single out only the case when we have the Dirichlet boundary condition in the thin direction; i.e., the boundary conditions (DD) and (DP). Note that in this case $\tilde{N}_\epsilon u = u$, and Lemma 3.1 implies

$$\sup_{\tau\le t\le T}\left|A_\epsilon^{\frac12}u^\epsilon(t)\right|_\epsilon^2 \le b_0^2(\epsilon)\exp\Big(\frac{-\nu\tau}{\epsilon^2}\Big) + \frac{\epsilon^2}{\nu^2}\beta^2(\epsilon) = \left|A_\epsilon^{\frac12}u_0^\epsilon\right|_\epsilon^2\exp\Big(\frac{-\nu\tau}{\epsilon^2}\Big) + \frac{\epsilon^2}{\nu^2}\left|f^\epsilon\right|_\epsilon^2; \tag{5.4}$$

hence, assuming that $\left|A_\epsilon^{\frac12}u_0^\epsilon\right|_\epsilon^2$ and $\left|f^\epsilon\right|_\epsilon^2$ are bounded in ϵ, we conclude that

$$\lim_{\epsilon\to 0} M_\epsilon u^\epsilon = 0, \quad \text{uniformly in } \ \mathcal{C}\big([\tau,\infty);\tilde{V}\big), \ \ \forall\ \tau > 0, \tag{5.5}$$

and if $\lim_{\epsilon\to 0}\left|A_\epsilon^{\frac12}u_0^\epsilon\right|_\epsilon^2 = 0$, then

$$\lim_{\epsilon\to 0} M_\epsilon u = 0 \quad \text{uniformly in } \ \mathcal{C}([0,\infty);\tilde{V}). \tag{5.6}$$

We proved

Theorem 5.1. *In the case of the boundary condition* (DD) *or* (DP) *"on either* Γ_t *or* Γ_b*," we assume that*

$$\exists\ K_3 > 0, \quad \textit{such that} \ \ \left|A_\epsilon^{\frac12}u_0^\epsilon\right|_\epsilon^2 + \left|f^\epsilon\right|_\epsilon^2 \le K_3, \quad 0<\epsilon<1.$$

Then,

$$\lim_{\epsilon\to 0} M_\epsilon u^\epsilon = 0 \quad \text{in} \ \ \mathcal{C}([\tau,\infty);\tilde{V}), \quad \forall\ \tau > 0.$$

Moreover, if $\lim_{\epsilon\to 0}\big|A_\epsilon^{\frac{1}{2}} u_0^\epsilon\big|_\epsilon^2 = 0$*, then*

$$\lim_{\epsilon\to 0} M_\epsilon u^\epsilon = 0 \quad \text{in} \ \ \mathcal{C}([0,\infty);\tilde{V}).$$

Now we treat the other boundary conditions. For $f \in H_\epsilon$ or $L^\infty(0,\infty;H_\epsilon)$ and $u_0^\epsilon \in V_\epsilon$, we assume the following:

$$\lim_{\epsilon\to 0} \tilde{M}_\epsilon f^\epsilon = \tilde{f} \quad \text{in} \ \ \tilde{H}\text{-weak}. \tag{5.7}$$

$$\lim_{\epsilon\to 0} \tilde{M}_\epsilon u_0^\epsilon = \tilde{v_0} \quad \text{in} \ \ \tilde{H}\text{-weak}, \quad \big|A_\epsilon^{\frac{1}{2}} \tilde{M}_\epsilon u_0^\epsilon\big|_\epsilon \text{ is bounded.} \tag{5.8}$$

Let $T > 0$ be given and fixed. Thanks to Theorems 4.1 and 4.2, there exists $\epsilon_5(\nu, \tilde{v_0}, \tilde{f})$ such that $T < T^\sigma(\epsilon)$ for $0 < \epsilon \le \epsilon_5$. Now, Lemmas 3.2 and 3.3 imply that

$$\tilde{M}_\epsilon u^\epsilon \quad \text{is bounded in } L^\infty(0,T;\tilde{V}) \quad \text{for} \ \ 0 < \epsilon \le \epsilon_5. \tag{5.9}$$

$$\tilde{M}_\epsilon u^\epsilon \quad \text{is bounded in } L^2(0,T;\tilde{V}) \quad \text{for} \ \ 0 < \epsilon \le \epsilon_5. \tag{5.10}$$

$$\tilde{M}_\epsilon u^\epsilon \quad \text{is bounded in } L^2(0,T;\tilde{H}) \quad \text{for} \ \ 0 < \epsilon \le \epsilon_5. \tag{5.11}$$

Hence, there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}}$, with $\lim_{n\to\infty}\epsilon_n = 0$, and a function v^* such that

$$\lim_{\epsilon_n\to 0} \tilde{M}_{\epsilon_n} u^{\epsilon_n} = v^* \quad \text{in} \ \ L^2(0,T;\tilde{H}) \ \ \text{(in norm)}, \tag{5.12}$$

$$\lim_{\epsilon_n\to 0} \tilde{M}_{\epsilon_n} u^{\epsilon_n} = v^* \quad \text{in } L^2(0,T;\tilde{V})\text{-weak}. \tag{5.13}$$

We rewrite the equations satisfied by $\tilde{M}_{\epsilon_n} u^\epsilon$

$$\begin{aligned}\frac{d}{dt}(\tilde{M}_{\epsilon_n}u^{\epsilon_n}, \tilde{M}_{\epsilon_n}v)_{\epsilon_n} + \nu(A_{\epsilon_n}^{\frac{1}{2}}\tilde{M}_{\epsilon_n}u^{\epsilon_n}, A_{\epsilon_n}^{\frac{1}{2}}\tilde{M}_{\epsilon_n}v)_{\epsilon_n} + b_{\epsilon_n}(\tilde{M}_{\epsilon_n}u^{\epsilon_n}, \tilde{M}_{\epsilon_n}u^{\epsilon_n}, \tilde{M}_{\epsilon_n}v)\\ + b_{\epsilon_n}(\tilde{N}_{\epsilon_n}u^{\epsilon_n}, \tilde{N}_{\epsilon_n}u^{\epsilon_n}, \tilde{M}_{\epsilon_n}v) = (\tilde{M}_{\epsilon_n}f^{\epsilon_n}, \tilde{M}_{\epsilon_n}v)_{\epsilon_n}.\end{aligned} \tag{5.14}$$

Now we use Hölder's inequality to obtain

$$\Big|\frac{1}{\epsilon_n}b_{\epsilon_n}(\tilde{N}_{\epsilon_n}u^{\epsilon_n}, \tilde{N}_{\epsilon_n}u^{\epsilon_n}, \tilde{M}_{\epsilon_n}v)\Big| \le \frac{c_0}{\epsilon_n}\big|\tilde{N}_{\epsilon_n}u^{\epsilon_n}\big|_{\mathbb{L}^4}\big|A_{\epsilon_n}^{\frac{1}{2}}\tilde{N}_{\epsilon_n}u^{\epsilon_n}\big|_{\mathbb{L}^2}\big|\tilde{M}_{\epsilon_n}v\big|_{\mathbb{L}^4}, \tag{5.15}$$

and Lemma 2.4 yields

$$\begin{aligned}&\Big|\frac{1}{\epsilon_n}b_{\epsilon_n}(\tilde{N}_{\epsilon_n}u^{\epsilon_n}, \tilde{N}_{\epsilon_n}u^{\epsilon_n}, \tilde{M}_{\epsilon_n}v)\Big| \le \frac{c_0}{\epsilon_n}\epsilon_n^{\frac{1}{4}}\big|A_{\epsilon_n}^{\frac{1}{2}}\tilde{N}_{\epsilon_n}u^{\epsilon_n}\big|_{\epsilon_n}^2 \cdot \epsilon_n^{\frac{1}{4}}\big|A_{\epsilon_n}^{\frac{1}{2}}\tilde{M}_{\epsilon_n}u^{\epsilon_n}\big|_{\mathbb{L}^2(\omega)}\\ &\le c_0\epsilon_n^{-\frac{1}{2}}\big|A_{\epsilon_n}^{\frac{1}{2}}\tilde{N}_{\epsilon_n}u^{\epsilon_n}\big|_{\epsilon_n}^2\big|A_{\epsilon_n}^{\frac{1}{2}}\tilde{M}_{\epsilon_n}u^{\epsilon_n}\big|_{\mathbb{L}^2(\omega)} \le \text{(with Lemma 3.1)}\\ &\le c_0\epsilon_n^{-\frac{1}{2}}\left(b_0^2(\epsilon_n)\exp\big(-\frac{\nu t}{2\epsilon_n^2}\big) + \frac{2\epsilon_n^2}{\nu^2}\beta^2(\epsilon)\right)\big|A_{\epsilon_n}^{\frac{1}{2}}\tilde{M}_{\epsilon_n}v\big|_{\mathbb{L}^2(\omega)}.\end{aligned} \tag{5.16}$$

Under the assumptions (5.7) and (5.8), we write $b_0^2(\epsilon_n) \le \epsilon_n K_4$, where K_4 is a positive constant independent of n. Therefore,

$$\Big|\frac{1}{\epsilon_n} b_{\epsilon_n}(\tilde{N}_{\epsilon_n} u^{\epsilon_n}, \tilde{N}_{\epsilon_n} u^{\epsilon_n}, \tilde{M}_{\epsilon_n} v)\Big| \le c_0 \epsilon_n^{-\frac{1}{2}} \big(K_4 \exp\big(-\frac{\nu t}{2\epsilon_n^2}\big) + \frac{2\epsilon_n^2}{\nu^2}\big) \big|A_{\epsilon_n}^{\frac{1}{2}} \tilde{M}_{\epsilon_n} u^{\epsilon_n}\big|_{\mathbb{L}^2(\omega)}. \tag{5.17}$$

Hence,

$$\lim_{n\to\infty} \Big|\frac{1}{\epsilon_n} b_{\epsilon_n}(\tilde{N}_{\epsilon_n} u^{\epsilon_n}, \tilde{N}_{\epsilon_n} u^{\epsilon_n}, \tilde{M}_{\epsilon_n} v)\Big| = 0. \tag{5.18}$$

Taking into account (5.9)–(5.13) and (5.18), it is straightforward to pass to the limit in (5.14) (see Chapter III of [25]). We have, for $v = (v_1, v_2, 0) \in \tilde{V}$,

$$\begin{aligned} &\frac{d}{dt}(v^*, v)_{L^2(\omega)} + \nu(\tilde{A}^{\frac{1}{2}} v^*, \tilde{A}^{\frac{1}{2}} v)_{L^2(\omega)} + \tilde{b}(v^*, v^*, v) = (\tilde{f}, v)_{L^2(\omega)}, \\ &v^*(\cdot, 0) = \tilde{v_0}. \end{aligned} \tag{5.19}$$

Finally, the uniqueness of solutions to (5.1)–(5.3) implies that $v^* = \tilde{v}$.

Now we are ready to prove the following:

Theorem 5.2. *In the case of the boundary condition* (FF), (FP), (FD) *or* (PP), *we assume that* $\big|A_\epsilon^{\frac{1}{2}} \tilde{M}_\epsilon u_0\big|_{L^2(\omega)}$ *is bounded, and also the existence of* $\tilde{f} \in \tilde{H}$ *and* $\tilde{v_0} \in \tilde{V}$, *such that*

$$\lim_{\epsilon\to 0} \tilde{M}_\epsilon f^\epsilon = \tilde{f} \quad \textit{in } \ \tilde{H}\textit{-weak.}$$
$$\lim_{\epsilon\to 0} \tilde{M}_\epsilon u_0^\epsilon = \tilde{v_0} \quad \textit{in } \tilde{H}\textit{-weak.}$$

Then, for all $T > 0$, *there exists* $\epsilon_5 = \epsilon_5(\tilde{f}, \tilde{v_0}, \nu)$ *such that*

$$\lim_{\epsilon\to 0} \tilde{M}_\epsilon u^\epsilon = \tilde{v} \quad \textit{in } \ \mathcal{C}([0,T]; \tilde{H}) \cap L^2(0,T; \tilde{V}), \tag{5.20}$$

where $\tilde{v}$ *is the unique solution to* (5.1)–(5.3), *with the "appropriate" boundary condition.*

Proof. We first prove that

$$\lim_{n\to\infty} \tilde{M}_{\epsilon_n} u^{\epsilon_n}(t) = \tilde{v}(t) \quad \text{in } \tilde{H}\text{-weak}, \ \text{ for all } \ t \in [0,T]. \tag{5.21}$$

Integrating (5.14) between 0 and t, we obtain ($\tilde{M}_{\epsilon_n} v$ is replaced with $v \in \tilde{V}$)

$$\begin{aligned} &(\tilde{M}_\epsilon u^{\epsilon_n}(t), v)_{L^2(\omega)} \\ &\quad = (\tilde{M}_\epsilon u^{\epsilon_n}(0), v)_{L^2(\omega)} + \int_0^t \Big[(\tilde{M}_\epsilon f, v)_{L^2(\omega)} - \nu(\tilde{A}^{\frac{1}{2}} \tilde{M}_\epsilon u^{\epsilon_n}(s), \tilde{A}^{\frac{1}{2}} v)_{L^2(\omega)} \\ &\qquad - \tilde{b}(\tilde{M}_\epsilon u^{\epsilon_n}(s), \tilde{M}_\epsilon u^{\epsilon_n}(s), v) - \frac{1}{\epsilon} b_\epsilon(\tilde{N}_\epsilon u^{\epsilon_n}(s), \tilde{N}_\epsilon u^{\epsilon_n}(s), v)\Big] \, ds, \end{aligned} \tag{5.22}$$

for all $v \in \tilde{V}$ and $\forall t \in [0,T]$.

The right-hand side in (5.22) converges, as $\epsilon \to 0$ (t is fixed), to

$$(\tilde{v_0}, v)_{L^2(\omega)} + \int_0^t \Big[(\tilde{f}, v)_{L^2(\omega)} - \nu(\tilde{A}^{\frac{1}{2}} \tilde{v}(s), \tilde{A}^{\frac{1}{2}} v)_{L^2(\omega)} - \tilde{b}(\tilde{v}(s), \tilde{v}(s), v) \Big] \, ds, \tag{5.23}$$

which is equal to $(\tilde{v}(t), v)_{L^2(\omega)}$.

Now we consider the following expression:

$$e_\epsilon(t) = \frac{1}{2} \big| \tilde{M}_\epsilon u(t) - \tilde{v}(t) \big|^2_{L^2(\omega)} + \nu \int_0^t \big| \tilde{A}^{\frac{1}{2}} \tilde{M}_\epsilon u(s) - \tilde{A}^{\frac{1}{2}} \tilde{M}_\epsilon \tilde{v}(s) \big|^2_{L^2(\omega)} \, ds \tag{5.24}$$

and prove that

$$\lim_{\epsilon \to 0} e_\epsilon(t) = 0, \quad \forall\, t \in [0,T]. \tag{5.25}$$

We write

$$e_\epsilon(t) = e^1_\epsilon(t) + e^2(t) + e^3_\epsilon(t), \tag{5.26}$$

where

$$\begin{cases} e^1_\epsilon(t) = \dfrac{1}{2} \big| \tilde{M}_\epsilon u(t) \big|^2_{L^2(\omega)} + \nu \displaystyle\int_0^t \big| \tilde{A}^{\frac{1}{2}} \tilde{M}_\epsilon u(s) \big|^2_{L^2(\omega)} \, ds, \\ e^2(t) = \dfrac{1}{2} \big| \tilde{v}(t) \big|^2_{L^2(\omega)} + \nu \displaystyle\int_0^t \big| \tilde{A}^{\frac{1}{2}} \tilde{v}(s) \big|^2_{L^2(\omega)} \, ds, \\ e^1_\epsilon(t) = (\tilde{M}_\epsilon u(t), \tilde{v}(t))_{L^2(\omega)} + 2\nu \displaystyle\int_0^t (\tilde{A}^{\frac{1}{2}} \tilde{M}_\epsilon u(s), \tilde{A}^{\frac{1}{2}} \tilde{M}_\epsilon \tilde{v}(s))_{L^2(\omega)} \, ds. \end{cases} \tag{5.27}$$

We set $v = \tilde{M}_{\epsilon_n} u(t)$ in (5.14) and integrate between 0 and t to obtain

$$e^1_{\epsilon_n}(t) = \frac{1}{2} \big| \tilde{M}_{\epsilon_n} u^{\epsilon_n}(0) \big|^2_{L^2(\omega)} + \int_0^t (\tilde{M}_{\epsilon_n} f, \tilde{M}_{\epsilon_n} u^{\epsilon_n}(s))_{L^2(\omega)} \, ds \\ - \frac{1}{\epsilon_n} \int_0^t b_{\epsilon_n}(\tilde{N}_{\epsilon_n} u^{\epsilon_n}, \tilde{N}_{\epsilon_n} u^{\epsilon_n}, \tilde{M}_{\epsilon_n} u^{\epsilon_n}) \, ds. \tag{5.28}$$

Thanks to (5.17), we have

$$\begin{aligned} &\Big| \int_0^t b_{\epsilon_n}(\tilde{N}_{\epsilon_n} u^{\epsilon_n}, \tilde{N}_{\epsilon_n} u^{\epsilon_n}, \tilde{M}_{\epsilon_n} u^{\epsilon_n}) \, ds \Big| \\ &\le c_4 \epsilon_n^{\frac{1}{2}} \Big(K_4 + \frac{2\epsilon_n \beta^2(\epsilon_n)}{\nu^2}\Big) \int_0^T \big| A^{\frac{1}{2}}_{\epsilon_n} \tilde{M}_{\epsilon_n} u^{\epsilon_n}(s) \big|^2_{L^2(\omega)} \, ds. \end{aligned} \tag{5.29}$$

Hence,

$$\lim_{n \to \infty} \sup_{0 \le t \le T} \Big| \int_0^t b_{\epsilon_n}(\tilde{N}_{\epsilon_n} u^{\epsilon_n}, \tilde{N}_{\epsilon_n} u^{\epsilon_n}, \tilde{M}_{\epsilon_n} u^{\epsilon_n}) \, ds \Big| = 0, \tag{5.30}$$

and

$$\lim_{n\to\infty} e^1_{\epsilon_n}(t) = \frac{1}{2}\big|\tilde{v}(0)\big|^2_{L^2(\omega)} + \int_0^t (\tilde{f}, \tilde{v}(s))_{L^2(\omega)}\, ds. \tag{5.31}$$

Moreover, integrating (5.19), with $v = \tilde{v}$ and taking into account the fact that $v^* = \tilde{v}$, we obtain

$$e^2(t) = \frac{1}{2}\big|\tilde{v}(0)\big|^2_{L^2(\omega)} + \int_0^t (\tilde{f}, \tilde{v}(s))_{L^2(\omega)}\, ds. \tag{5.32}$$

Finally we have, thanks to (5.13), (5.14) and (5.21) (t being fixed)

$$\lim_{n\to\infty} (\tilde{M}_{\epsilon_n} u^{\epsilon_n}(t), \tilde{v})_{L^2(\omega)} = \big|\tilde{v}\big|^2_{L^2(\omega)} \tag{5.33}$$

and

$$\lim_{n\to\infty} \int_0^t (A^{\frac{1}{2}}_{\epsilon_n} \tilde{M}_{\epsilon_n} u^{\epsilon_n}(s), A^{\frac{1}{2}}_{\epsilon_n} \tilde{v}(s))_{L^2(\omega)}\, ds = \int_0^t \big|A^{\frac{1}{2}}_{\epsilon_n} \tilde{v}(s)\big|^2_{L^2(\omega)}\, ds. \tag{5.34}$$

Therefore,

$$\lim_{n\to\infty} e^3_{\epsilon_n}(t) = 2e^2(t) \tag{5.35}$$

and $\lim_{n\to\infty} e_{\epsilon_n}(t) = 0,\ \forall\, t \in [0,T]$. This concludes the proof of Theorem 5.2

Acknowledgment. This work was partially supported by the National Science Foundation under Grant NSF-DMS-9400615 and by the Research Fund of Indiana University.

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