THE SCALAR CURVATURE EQUATION ON $\mathbb{R}^n$ AND $S^n$

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(Submitted by: L.A. Peletier)

Abstract. We study the existence of positive solutions for the equation $\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0$ in $\mathbb{R}^n$ ($n \geq 3$) which decay to 0 at infinity like $|x|^{2-n}$; $K(x)$ is a function which is bounded from above and below by positive constants, and no symmetry assumption on $K$ is made. We find conditions that guarantee existence for a large class of $K$’s. As a consequence one can explicitly show that the set of coefficients for which a solution exists is dense, in $C^1$ norm, in the space of positive bounded $C^1$ functions. These conditions also allow us to display a radial $K$ such that the previous problem has nonradial solutions but no radial solution. Some new results for the corresponding problem on $S^n$ are also proved.

0. Introduction. In this paper we prove some existence results for the following problem:

$$
\begin{cases}
\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0 & \text{in } \mathbb{R}^n, \\
u > 0 & u = O(|x|^{2-n}) \quad \text{as } x \to \infty,
\end{cases}
$$

(0.1)

for any $n \geq 3$. Here and throughout the paper $K$ is a function which is bounded from above and below by positive constants.

The previous equation is called the scalar curvature equation in $\mathbb{R}^n$. The existence of a positive solution for this equation amounts to the existence of a metric $g$ on $\mathbb{R}^n$ which is conformal to the standard metric $g_0$ on $\mathbb{R}^n$ ($g = u^{\frac{4}{n-2}}g_0$) and whose scalar curvature is $K$. The decay condition at $\infty$ implies that the metric $g$ gives rise (via the stereographic projection) to a metric on $S^n \setminus \{\text{a point}\}$ which is equivalent to the standard metric. This implies that the metric $g$ is incomplete.

The scalar curvature equation on $\mathbb{R}^n$ has been studied by several authors, especially after the publication of the paper by W.M. Ni [19] (see [20] for a review), and many results regarding existence of solutions to this equation, with various asymptotic behavior at infinity, are available.

As far as the author knows all the existence results regarding problem (0.1) (see for instance [16], [17], [6], [8], [5], [14], [15], [22]) hold under some symmetry assumption on the coefficient $K$, like radial symmetry, symmetry under a subgroup of the group of rotations or periodicity in one variable. The results of this paper do not assume any symmetry on $K$.

Received for publication December 1995.
AMS Subject Classifications: 35J20, 35J60, 58G03.
Theorem 1. Let $K \in C^2(\mathbb{R}^n)$ and suppose that $K_{\text{max}} := \max_{\mathbb{R}^n} K$ is achieved in at least two isolated points $y, z$. Suppose moreover that

i) there exist two constants $c > 0$ and $\rho > n - 2$ such that

$$K(x) \geq K_{\text{max}} - c|x - y|^\rho \quad \text{for } x \text{ in a neighborhood of } y,$$

$$K(x) \geq K_{\text{max}} - c|x - z|^\rho \quad \text{for } x \text{ in a neighborhood of } z;$$

ii) if $w$ is a critical point for $K$ with $K(w) \in (2^{-\frac{3}{2n}}K_{\text{max}}, K_{\text{max}})$ then either $\Delta K(w) < 0$ and $w$ is a strict local maximum or $\Delta K(w) > 0$;

iii) $\limsup_{x \to \infty} K(x) < 2^{-\frac{3}{2n}}K_{\text{max}}$;

iv) the number of critical points $w$ of $K$ which satisfy $K(w) \geq 2^{-\frac{3}{2n}}K_{\text{max}}$ is finite.

Then there exists a solution to problem (0.1).

Some remarks about Theorem 1:

- Assumption ii) requires that, in a certain range of values, there are neither critical points where $\Delta K$ vanish nor “saddle” points with $\Delta K < 0$;
- as $n$ grows $2^{-\frac{3}{2n}}$ becomes closer to 1 and assumptions ii) and iii) become weaker;
- one can substitute assumption i) with a slightly more general one; see the remark following the proof of Lemma 2.1.

Theorem 2. Let $K \in C^2(\mathbb{R}^n)$ and let $K$ have only finitely many critical points. Suppose there exists $z \in \mathbb{R}^n$ where $K$ achieves a strict local maximum and $\Delta K(z) \neq 0$ and a continuous path $x(t)$ in $\mathbb{R}^n$ connecting $z$ to some point $y$, $y \neq z$, where $K(y) \geq K(z)$. Suppose moreover that

i) $\min_t K(x(t)) > 2^{-2/(n-2)}K_{\text{max}}$;

ii) if $w$ is a critical point for $K$ with $K(w) \in ([\min_t K(x(t)), K(z))$, then either $\Delta K(w) < 0$ and $w$ is a strict local maximum or $\Delta K(w) > 0$;

iii) $\limsup_{x \to \infty} K(x) < \min_t K(x(t))$.

Then there exists a solution to problem (0.1).

The value $d := \min_t K(x(t))$ is just the highest value such that the set $\{x : K(x) \geq K(z)\} \setminus \{z\}$ can be connected to $z$ by some arc in the region $\{K \geq d\}$.

If $K(z) = K_{\text{max}}$, then the assumption $\Delta K(z) \neq 0$ can be weakened. It suffices requiring $z$ to be an isolated maximum point.

Remark. The class of $K$‘s which satisfy the assumptions of Theorem 2 is dense, in $C^1$ norm, in the class of positive bounded $C^1$ functions.

The assumption in Theorem 1 about the asymptotic behavior of $K$ near $y$ and $z$ is automatically satisfied when $n = 3$. In [4] the author has proved the following result, which implies that when $n \geq 4$ that assumption is really needed.

Theorem A ([4]). Let $n \geq 4$. Take two distinct points $y$ and $z$ in $\mathbb{R}^n$ and two positive numbers $K_z, K_y$. There exists a $C^2$ positive function $K$ with

$$K(x) = K_y - \varepsilon|x - y|^{n-2} \quad \text{for } x \text{ in a neighborhood of } y,$$

$$K(x) = K_z - \varepsilon|x - z|^{n-2} \quad \text{for } x \text{ in a neighborhood of } z$$
(for some \( \varepsilon > 0 \) small enough) such that (0.1) has no solution. The value of \( K \) on its critical points different from \( y \) and \( z \), as well as the value of \( \lim_{x \to \infty} K(x) \), can be made arbitrarily close to 0.

This theorem extends a result, proved by the author and H. Egnell in [8], regarding the nonexistence of radial solutions to (0.1) for some radial \( K \)’s.

We point out that when \( K_y = K_z = K_{\text{max}} \) the coefficient \( K \) in Theorem A satisfies every assumption in Theorem 2 except i), showing that in that theorem assumption i) can not be removed.

Theorem 2 together with a nonexistence result proved in [6], enables us to prove the following theorem.

**Theorem 3.** There exists a \( C^2 \) radial function \( K \) such that (0.1) has nonradial solutions but no radial solution.

We will explicitly construct the function \( K \) in Section 3.

The scalar curvature problem on \( S^n \), that is the existence of a solution to

\[
\begin{align*}
\Delta u - \frac{n(n-2)}{4} u + K(x)u^{\frac{n+2}{n-2}} &= 0 \quad \text{in } S^n, \\
\quad u > 0 \quad u \in H^1(S^n),
\end{align*}
\]

is closely related to problem (0.1). Problem (0.1) is equivalent (via the stereographic projection) to this one for a coefficient \( K \) which may be discontinuous in the north pole.

When \( K \in C^2(S^n) \) problem (0.2) has been studied in several papers (see [13], [1], [12], [3], [11], [10], [15]). Some strong existence results have been proved for the problem on \( S^3 \) and, for any \( n \), when \( K \) is close to a constant. Yan-Yan Li has recently proved a generalization to \( S^n, n \geq 4 \), of some of the results proved for \( S^3 \). His result, which is based on a counting index criteria, holds under the assumption that \( K \) is very flat near all its critical points. Condition i) in Theorem 1 of this paper is coherent with his assumption.

Our techniques yield some new results also for the problem on \( S^n \). Theorems 1 and 2 (as well as Theorems A and 3) continue to hold, with the obvious change of notations, also for (0.2). If \( K \in C^2(S^n) \) one can also drop the assumption regarding \( \limsup_{x \to \infty} K \).

We will not state the corresponding theorems here and we refer the reader to section 3 (theorems 6,7,8) for the precise statements.

When \( n = 3 \), under the additional assumption that

\[
\Delta K \text{ does not vanish at any critical point of } K,
\]

stronger results can be proved for problem (0.2).

**Theorem 4.** Let \( n = 3, K \in C^2(S^3) \) and let (0.3) hold. Suppose there exists \( z \in S^3 \) where \( K \) achieves a strict local maximum and and a continuous path \( x(t) \) in \( S^3 \) connecting \( z \) to some point \( y, y \neq z \), where \( K(y) \geq K(z) \). Suppose moreover that if \( w \) is a critical point for \( K \) with \( K(w) \in [\min K(x(t)), K(z)] \) and \( \Delta K(w) < 0 \) then \( w \) is a strict local maximum for \( K \). Then there exists a solution to problem (0.2).

The difference with respect to Theorem 2 is that here it is not necessary to assume \( \min K(x(t)) > 2^{-2/(n-2)}K_{\text{max}} \). A simple consequence of this result is the following.
Theorem 5. Let $n = 3$, $K \in C^2(S^3)$ and let (0.3) hold. Let us assume that $K$ has at least two local maximum points and that among the critical points of $K$ with $\Delta K < 0$ the one where $K$ has the lowest value is a strict local maximum. Then there exists a solution to problem (0.2).

To prove the existence of a critical point for the functional $\mathcal{F}$ associated to our problem we will use a min-max method. We look for a min-max over the class of continuous paths connecting two “critical points at infinity” (see section 1 for the definition).

A key point in this study is that the Palais-Smale condition does not hold. To overcome this lack of compactness one can go back to the basic elements of the variational theory studying the deformation along the flow lines of $-\nabla \mathcal{F}$. This has been done by A. Bahri and J.M. Coron in [2] and [3]. Their study allows us to regain compactness, along gradient flow, at some values of $\mathcal{F}$. These values are related to the value of the coefficient $K$ in some of its critical points. In our context, with respect to the study in [2] and [3], we have further restricted the set of critical points of $K$ which may cause loss of compactness (see Lemma 2.3 and Proposition 3.2).

The paper is organized as follows: Section 1 contains the basic definitions and recalls some of the results concerning the loss of compactness of the functional $\mathcal{F}$; in Section 2 the min-max scheme is defined and it is studied how certain assumptions on $K$ reflect on connection properties of some of the sublevel sets of $\mathcal{F}$; Section 3 contains the proofs of the existence results.

1. Loss of compactness on flow-lines. To unify the treatment of the problem on $S^n$ and the problem on $\mathbb{R}^n$ we study the existence of a solution for problem (0.2) for a coefficient $K$ which is not necessarily continuous in every point of $S^n$. For $d \geq 0$ let

$$A_d = \{ x \in S^n : K(x) \geq d \}.$$

We will say that $K$ satisfies assumption H1(d) if

$$K \in C^2(A_d), \ A_d \text{ is closed, and } K \text{ has only finitely many critical points in } A_d.$$ (H1(d))

For the results regarding exclusively $S^3$ we will assume that

$$K \in C^2(S^3), \text{ has only a finite number of critical points and } \Delta K \text{ does not vanish at any critical point of } K.$$ (H2)

Let $Lu = \Delta u - \frac{n(n-2)}{4} u$ be the conformal Laplacian on $S^n$ and, for any $u \in H^1(S^n)$, let $\|u\|^2_{-L} = \int_{S^n} |\nabla u|^2 + \frac{n(n-2)}{4} u^2 \, dv$ (here $dv$ is the volume element of $S^n$ with respect to the standard metric) and let $\langle \cdot , \cdot \rangle_{-L}$ be the corresponding scalar product. Finally let

$$M^+ = \{ \ u \in H^1(S^n) : \|u\|_{-L} = 1 ; u \geq 0 \}.$$

We will study the functional $\mathcal{F} : M^+ \to \mathbb{R}$ defined by

$$\mathcal{F}(u) = \frac{1}{\int_{S^n} K |u|^2^* \, dv}.$$
where $2^* = 2n/(n - 2)$ is the Sobolev critical exponent. Its gradient is

$$
\nabla F(u) = F(u)^{(n-2)/4} \left( u - F(u)(-L)^{-1}(K|u|^{4/(n-2)}u) \right).
$$

Any critical point $u \in M^+$ can thus be rescaled, by multiplying it by an appropriate positive constant, so that the resulting function is a weak solution of (0.2).

In a series of papers Brezis, Coron, and Bahri introduced the notion of critical point at infinity for the functional $F$. Roughly speaking this is the “limiting point” of a Palais-Smale sequence for $F$ that converges weakly to 0 but does not converge strongly. We recall that a Palais Smale sequence for $F$ is a sequence $(u_m)$ such that $(F(u_m))$ converges and $\nabla F(u_m) \to 0$ in the dual of $H^1(S^n)$.

For $a \in S^n$ and $\lambda > 0$ let

$$
\delta_{a, \lambda}(x) = c_0 \left( \frac{\lambda}{\lambda^2 + 1 + (1 - \lambda^2) \cos d(a, x)^2} \right)^{\frac{n-2}{2}}.
$$

(1.1)

Here $c_0$ is a constant, depending only on the dimension $n$, such that $\|\delta_{a, \lambda}\|_{L^1} = 1$ and $d(\cdot, \cdot)$ is the geodesic distance in $S^n$ (with respect to the standard metric). The functions $\delta_{a, \lambda}$ are the only extremal functions for the Sobolev inequality

$$
\|v\|_{-L} - S\|v\|_{2^*} \geq 0
$$

(1.2)

(here $v \in H^1(S^n)$ and $S$ is the optimal constant) and, after multiplication by a suitable constant, they are the only positive solutions of $Lu + u^{2^* - 1} = 0$. We observe that as $\lambda$ becomes large the functions $\delta_{a, \lambda}$ concentrate around the point $a$.

For a positive integer $p$ and for $\varepsilon > 0$, a neighborhood $W(p, \varepsilon)$ of a critical point at infinity is defined as follows (see for instance [3]):

$$
W(p, \varepsilon) = \left\{ u \in M^+ : \exists \lambda_1, \ldots, \lambda_p > 0; a_1, \ldots, a_p \in S^n \text{ with} \right. \\
\left. \left\|\nabla \left( u - c \sum_{i=1}^{p} K(a_i)^{\frac{2-n}{n}} \delta_{a_i, \lambda_i} \right) \right\|_2 < \varepsilon, \text{ where } c = \left( \sum_{i=1}^{p} K(a_i)^{\frac{2-n}{n}} \right)^{-\frac{1}{2}} \right. \\
\left. \lambda_i > 1/\varepsilon \ \forall i; \ \varepsilon^{-\frac{2-n}{n}} = \lambda_i/\lambda_j + \lambda_j/\lambda_i + \lambda_i \lambda_j d(a_i, a_j)^2 > 1/\varepsilon \ \forall i \neq j \right\}.
$$

For each $u \in W(p, \varepsilon)$ we optimize the approximation of $u$ by $\sum_{i=1}^{p} \alpha_i \delta_{a_i, \lambda_i}$ introducing the minimization problem (M),

$$
u \in W(p, \varepsilon) \left. \alpha_i \in S^{n-1}, \alpha_i > 0, \lambda_i > 0 \right\} \left\|\nabla \left( u - \sum_{i=1}^{p} \alpha_i \delta_{a_i, \lambda_i} \right) \right\|_2.
$$

(1.1)

**Lemma 1.1.** For any positive integer $p$ there exists $\varepsilon_0 > 0$ such that for any $u \in W(p, \varepsilon)$ with $\varepsilon < \varepsilon_0$ the minimization problem (M) has a unique solution up to permutations.

This minimization problem has already been considered in various papers, see for instance [3] and [6], and the proof of Lemma 1.1 is available there.
Lemma 1.2. Assume \((1.1)\) has no solution in \(H^1(S^n)\) and let \((u_m)\) be a Palais-Smale sequence for \(F\). Let \(K \in C^0(S^n)\). Then there exists a positive integer \(p\), a sequence \((\epsilon_m)\) converging to 0 and an extracted subsequence of \((u_m)\), that we still denote \((u_m)\), such that \(u_m \in W(p, \epsilon_m)\). There exist \(p\) points \(a_i, i = 1, \ldots, p, \) in \(S^n\) such that for all \(i, (a_i)_m \to a_i\) and
\[
F(u_m) \to S^{2^*} \left( \sum_{i=1}^{p} \frac{1}{K(a_i)^{2\gamma}} \right)^{\frac{1}{2\gamma}}.
\] (1.3)
Here the \((a_i)_m, i = 1, \ldots, p, \) are the points associated to \(u_m\) via the minimization problem \((M)\).

Lemma 1.3. Assume \((1.1)\) has no solution in \(H^1(S^n)\) and let \((u_m)\) be a Palais-Smale sequence for \(F\). Let us assume that \(K \in C^0(A_d)\) for some \(d < S^{2^*}(\lim F(u_m))^{-1}\), that \(A_d\) is closed and that \(\lim F(u_m) < 2^{2/(n-2)} S^{2^*} K_{\max}^{-1}\). Then there exists a sequence \((\epsilon_m)\) converging to 0 and an extracted subsequence of \((u_m)\), that we still denote \((u_m)\) such that \(u_m \in W(1, \epsilon_m)\). There exist \(a_1\) in \(A_d\) such that, as \(m \to \infty\), \((a_i)_m \to a_1\) and
\[
F(u_m) \to \frac{S^{2^*}}{K(a_1)}.
\]
Here \((a_i)_m\) is the point associated to \(u_m\) via the minimization problem \((M)\).

Lemma 1.2 is by now classical; see [21] or [9]. The proof of Lemma 1.3 is similar to that of Lemma 1.2 and is contained in the appendix.

A. Bahri and J.M. Coron have gone further in the study of the loss of compactness for \(F\) studying the behavior of the flow lines for the negative gradient of \(F\) near the critical points at infinity.

Lemma 1.4. Let \(u_0 \in H^1(S^n)\) and let \(u(t)\) be the solution of the problem
\[
\frac{\partial u}{\partial t} = - \nabla F(u); \quad u(0) = u_0.
\] (1.4)
Let us suppose that \(u(t)\) stays definitively in \(W(p, \epsilon)\) for any \(\epsilon\) small enough and some \(p \geq 1\).

Let \(n = 3\) and let \(K\) satisfy (H2). Then \(p = 1\) and if \(a(t)\) denotes the point associated to \(u(t)\) via the minimization problem \((M)\), then \(a(t)\) converges to some point \(a \in S^3\) with \(\nabla K(a) = 0\) and \(\Delta K(a) < 0\).

Let \(n \geq 3\) and let \(K\) satisfy H1(d) for some \(d < S^{2^*}(\lim F(u_m))^{-1}\). Let us assume that \(\lim_{t \to \infty} F(u(t)) < S^{2^*} 2^{2/(n-2)} K_{\max}^{-1}\). Then \(p = 1\) and if \(a(t)\) denotes the point associated to \(u(t)\) via the minimization problem \((M)\), then \(a(t)\) converges to some point \(a \in S^n\) with \(\nabla K(a) = 0\) and \(\Delta K(a) \leq 0\).

Note that the negative gradient flow generated by \(F\) leaves \(M^+\) invariant (see for instance [6, Lemma 2.1]) and therefore \(u(t) \in M^+\) for each \(t\).

The assertion regarding \(S^3\) has been proven in [3]; the proof of the rest of the statement is given in the appendix.

Finally we recall another result proved in [3].
Lemma 1.5. Let \( u(t) \) be a solution to (1.4); then \( \lim_{t \to \infty} \| \nabla F(u(t)) \| = 0 \).

2. Connection of the sublevel sets of \( F \). Let \( y, z \) be two distinct point in \( S^n \).

Let \( \mathcal{P}_{y, z} \) be the set of all continuous paths \( u_s : (0, \infty) \to M^+ \) such that

\[ \text{as } s \to 0, \ u_s \in W(1, \varepsilon) \text{ definitely } \forall \varepsilon > 0 \text{ and the point } a(s) \text{ (associated to } u_s \) via problem (M) converges to } y; \]

\[ \text{as } s \to \infty, \ u_s \in W(1, \varepsilon) \text{ definitely } \forall \varepsilon > 0 \text{ and the point } a(s) \text{ converges to } z. \] (2.1)

Since \( K(y), K(z) > 0 \) it is easy to see that the set of paths \( \mathcal{P}_{y, z} \) is nonempty (see for instance the proof of Lemma 2.1 below).

As \( s \to 0 \), \( F(u_s) \) tends to \( S^2 K(y)^{-1} \) while as \( s \to \infty \), \( F(u_s) \to S^2 K(z)^{-1} \). Let us assume that

\[ \forall \gamma \in \mathcal{P}_{y, z}, \quad \max_{u \in \gamma} F(u) > \max(S^2 K(y)^{-1}, S^2 K(z)^{-1}), \] (2.2)

and let

\[ \Gamma = \inf_{\gamma \in \mathcal{P}_{y, z}} \max_{u \in \gamma} F(u). \] (2.3)

In the proofs of the next two lemmas it is convenient to transform the problem on \( \mathbb{R}^n \). Let \( s \) denote the stereographic projection from \( S^n \setminus \{ \text{North pole} \} \) to \( \mathbb{R}^n \) and let \( \mathcal{D}^{1,2}(\mathbb{R}^n) \) be the completion of \( C_0^\infty(\mathbb{R}^n) \) in the norm \( \| \nabla \cdot \|_2 = (\int_{\mathbb{R}^n} |\nabla \cdot|^2 \, dx)^{1/2} \). Corresponding to \( u \in H^1(S^n) \) is \( v(x) = u(s^{-1}(x))(2/(1 + |x|^2))^{(n-2)/2} \in \mathcal{D}^{1,2}(\mathbb{R}^n) \). The function corresponding to \( \delta_{a, \lambda} \) is \( U_{b, \mu} := c_0 \lambda^{\frac{n-2}{2}} (1 + \lambda^2 \| x - a \|^2)^{-\frac{n+2}{2}} \) for suitable parameters \( b \in \mathbb{R}^n, \mu > 0 \) (about these parameters we just observe that as \( \lambda \to \infty \) then \( \mu \to \infty \) and \( b \to s(a) \)). If \( u \in M^+ \) and \( v \in \mathcal{D}^{1,2}(\mathbb{R}^n) \) is the corresponding function then \( \| \nabla v \|_2 = 1 \) and

\[ F(u) = \frac{1}{\int_{\mathbb{R}^n} K(s^{-1}(x)) \, v^{2^*} \, dx}. \]

The functions \( U_{b, \mu} \) are the only minimizers of the Sobolev inequality

\[ \| \nabla v \|_2 - S \| v \|_{2^*} \geq 0 \] (2.4)

(here \( v \in \mathcal{D}^{1,2}(\mathbb{R}^n) \) and \( S \) is the optimal constant) and, for any \( u \in \mathcal{D}^{1,2}(\mathbb{R}^n) \), they satisfy

\[ \int_{\mathbb{R}^n} U_{a, \lambda}^{2^*-1} \, u \, dx = S^{-2^*} \int_{\mathbb{R}^n} \nabla U_{a, \lambda} \nabla u \, dx. \] (2.5)

Lemma 2.1. Let \( y, z \in S^n, \ y \neq z \) and let us assume that

\[ K(x) \geq K(y) - cd(x, y)^p \quad \text{for } x \text{ in a neighborhood of } y, \]

\[ K(x) \geq K(z) - cd(x, z)^p \quad \text{for } x \text{ in a neighborhood of } z \] (2.6)
for some positive constant $c$ and some $\rho > n - 2$; then

$$\inf_{\gamma \in \mathcal{P}_{y,z}} \max_{u \in \gamma} \mathcal{F}(u) < \left( K(y)^{-\frac{n-2}{2}} + K(z)^{-\frac{n-2}{2}} \right)^{\frac{2}{n-2}} S^{2^*}.$$  

**Proof.** We may always suppose that both $y$ and $z$ are different from the North pole. Let us transform the problem on $\mathbb{R}^n$. To simplify the notations let us keep calling $y$ the point $s(y)$, $z$ the point $s(z)$ and let us denote $K(s^{-1}(x))$ by $K(x)$.

Let $u_s = sU_{y,\lambda} + tU_{z,\lambda}$, with $0 \leq s, t$, $s^2 + t^2 = 1$ and $\lambda > 0$ large. The path $u_s$, when properly normalized and transformed back in $H^1(S^n)$ belongs to $\mathcal{P}_{y,z}$. Actually the behavior at the endpoints $U_{y,\lambda}$ and $U_{z,\lambda}$ is not completely correct, but we can continue the path with the desired property.

Bahri in [2], estimate F3 page 14, proves that as $\lambda \to \infty$,

$$\int_{\mathbb{R}^n} \nabla U_{y,\lambda} \nabla U_{z,\lambda} \, dx = A\lambda^{2-n}|y-z|^{2-n} + o(\lambda^{2-n}), \tag{2.8}$$

for some suitable positive constant $A$. Let us write

$$\int_{\mathbb{R}^n} K(x)U_{y,\lambda}^{2^*-1}U_{z,\lambda} \, dx = K(y) \int_{\mathbb{R}^n} U_{y,\lambda}^{2^*-1}U_{z,\lambda} \, dx + \int_{\mathbb{R}^n} (K(x) - K(y))U_{y,\lambda}^{2^*-1}U_{z,\lambda} \, dx.$$  

The first integral on the right side, due to (2.5), is equal to $S^{-2^*} \int \nabla U_{y,\lambda} \nabla U_{z,\lambda} \, dx$ and due to (2.8) it is equal to $S^{-2^*} A\lambda^{2-n}|y-z|^{2-n} + o(\lambda^{2-n})$. The second integral on the right side is therefore equal to $o(\lambda^{2-n})$. We can conclude that

$$\int_{\mathbb{R}^n} K(x)U_{y,\lambda}^{2^*-1}U_{z,\lambda} \, dx = S^{-2^*} AK(y)\lambda^{2-n}|y_1-y_2|^{2-n} + o(\lambda^{2-n}), \tag{2.9}$$

and a similar formula holds for $\int K(x)U_{y,\lambda}U_{z,\lambda}^{2^*-1} \, dx$.

Let us study the asymptotic behavior of $\|\nabla u_s\|_2^{2^*}$ as $\lambda \to \infty$:

$$\|\nabla u_s\|_2^{2^*} = \left( (s^2 + t^2)\|U_{y,\lambda}\|_2^2 + 2st \int_{\mathbb{R}^n} \nabla U_{y,\lambda} \nabla U_{z,\lambda} \, dx \right)^{\frac{2^*}{2}}$$

$$= 1 + 2^* st A\lambda^{2-n}|y-z|^{2-n} + o(\lambda^{2-n}),$$

where the last equality follows from (2.8). Now let us prove that

$$\int_{\mathbb{R}^n} K(x)u_s^{2^*} \, dx = \int_{\mathbb{R}^n} K(x)(s^{2^*}U_{y,\lambda}^{2^*} + t^{2^*}U_{z,\lambda}^{2^*}) \, dx +$$

$$2^* s^{2^*-1}t \int_{\mathbb{R}^n} K(x)U_{y,\lambda}^{2^*-1}U_{z,\lambda} \, dx + st^{2^*-1} \int_{\mathbb{R}^n} K(x)U_{y,\lambda}U_{z,\lambda}^{2^*-1} \, dx + R(\lambda), \tag{2.10}$$

with $|R(\lambda)| \leq o(\lambda^{2-n})$. Let us suppose $n \geq 4$, that is $2^* \in (2,4)$. For each $x, y \geq 0$,

$$|(x+y)^{2^*} - x^{2^*} - y^{2^*} - 2^*(x^{2^*-1}y + xy^{2^*-1})| \leq c_1(xy)^{\frac{2^*}{2}}$$

(for some positive constant $c_1$)
and as a consequence \(|R(\lambda)| \leq c_1(st)^{-2} \int U_{y,\lambda}^2 U_{z,\lambda}^2 dx\). To estimate the last integral we split it into three parts,

\[
\left( \int_{B(y,\epsilon)} + \int_{B(z,\epsilon)} + \int_{(B(y,\epsilon) \cup B(z,\epsilon))^c} \right) U_{y,\lambda}^2 U_{z,\lambda}^2 dx,
\]

where \(2\epsilon < |y-z|\). By \(B(x, \epsilon)\) we denote the sphere centered in \(x\) and with radius \(\epsilon\). The third integral is \(O(\lambda^{-n})\) (by the dominated convergence theorem); putting \(z = \lambda(x-y)\) the first integral becomes

\[
\frac{c_2^*}{\lambda^n} \int_{\{|z|<\lambda\epsilon\}} \left(1 + |z|^2\right)^{-n/2} \left(\lambda^{-1} + |y_1 - y_2 + z\lambda^{-1}|^2\right)^{-n/2} dx
\]

\[
\leq \left(\lambda^{-1} + (1 + \epsilon)^2 |y - z|^2\right)^{-\frac{n}{2}} \frac{c_2^*}{\lambda^n} \int_{\{|z|<\lambda\epsilon\}} \left(1 + |z|^2\right)^{-n/2} dx = O(\lambda^{-n} \log(\lambda));
\]

and the second integral can be treated in the same way. This proves (2.10) for \(n \geq 4\).

The case \(n = 3\) can be treated similarly, using the inequality \(|(x+y)^6 - x^6 - y^6 - 6(x^3y + xy^3)| \leq c_4(x^4y^2 + x^2y^4)\) instead of the previous one.

Let us now examine (2.10). We rewrite this formula in the form

\[
\int_{\mathbb{R}^n} K(x) u_s^2 dx = s^{2^*} K(y) \int_{\mathbb{R}^n} U_{y,\lambda}^2 dx + t^{2^*} K(z) \int_{\mathbb{R}^n} U_{z,\lambda}^2 dx
\]

\[
+ s^{2^*} \int_{\mathbb{R}^n} (K(x) - K(y)) U_{y,\lambda}^2 dx + t^{2^*} \int_{\mathbb{R}^n} (K(x) - K(z)) U_{z,\lambda}^2 dx
\]

\[
+ 2^*(s^{2^*-1} t \int_{\mathbb{R}^n} K(x) U_{y,\lambda}^{2^*-1} U_{z,\lambda} dx + s t^{2^*-1} \int_{\mathbb{R}^n} K(x) U_{y,\lambda} U_{z,\lambda}^{2^*-1} dx) + R(\lambda).
\]

Due to (2.5) and (2.9) we get

\[
\int_{\mathbb{R}^n} K(x) u_s^2 dx = s^{-2^*} \left(s^{2^*} K(y) + t^{2^*} K(z)\right)
\]

\[
+ \int_{\mathbb{R}^n} (K(x) - K_{12}(x)) \left(s^{2^*} U_{y,\lambda}^2 + t^{2^*} U_{z,\lambda}^2\right) dx + O(\lambda^{-n})
\]

\[
+ 2^* AS^{-2^*} \lambda^{2-n} |y - z|^{2-n} \left(s^{2^*-1} t K(y) + s t^{2^*-1} K(z)\right) + R(\lambda).
\]

Here \(K_{12}(x)\) is a function equal to \(K(y)\) in a neighborhood of \(y\) and to \(K(z)\) elsewhere.

To conclude the proof of the lemma we want to show that, for \(\lambda\) large enough,

\[
\max_s F(u_s) \leq \left(K(y)^{-\frac{n-2}{2}} + K(z)^{-\frac{n-2}{2}}\right)^{\frac{n-2}{2}} S^{2^*}.
\]

Due to the previous estimates we have

\[
\lim_{\lambda \to \infty} F(u_s) = \lim_{\lambda \to \infty} \frac{\|\nabla u_s\|_2^2}{\int K(x) u_s^{2^*} dx} = \frac{S^{2^*}}{(s^{2^*} K(y) + t^{2^*} K(z))}
\]

\[
\leq \left(K(y)^{-\frac{n-2}{2}} + K(z)^{-\frac{n-2}{2}}\right)^{\frac{n-2}{2}} S^{2^*},
\]
with equality if and only if

\[
s = s' = \frac{K(z)^{\frac{n-2}{4}}}{\sqrt{K(y)^{\frac{n-2}{4}} + K(z)^{\frac{n-2}{4}}}}, \quad t = t' = \frac{K(y)^{\frac{n-2}{4}}}{\sqrt{K(y)^{\frac{n-2}{4}} + K(z)^{\frac{n-2}{4}}}}.
\]

Thus there is a problem only when \( s = s' \) and \( t = t' \). We analyze this situation more carefully. In this case,

\[
\|
abla u_{s'}
\|_2^{2^*} = 1 + a\lambda^{2-n} + o(\lambda^{2-n}),
\]

where

\[
a = 2^*A|y-z|^{2-n} \left( \frac{K(y)K(z)^{\frac{n-2}{2}}}{K(y)^{\frac{n-2}{4}} + K(z)^{\frac{n-2}{4}}} \right).
\]

Furthermore,

\[
\left( \int K(x)u_{s'} \, dx \right)^{-1} = \left( K(y)^{-\frac{n-2}{2}} + K(z)^{-\frac{n-2}{2}} \right)^{\frac{n-2}{2}} S^{2^*} \left( 1 - 2a\lambda^{2-n} - c(\lambda) \right) + o(\lambda^{2-n}),
\]

where

\[
c(\lambda) = c_5 \int_{\mathbb{R}^n} (K(x) - K_{12}(x)) \left( K(z)^{\frac{n}{2}} u_{y,\lambda}^{2^*} + K(y)^{\frac{n}{2}} u_{z,\lambda}^{2^*} \right) \, dx,
\]

and \( c_5 \) is a positive constant. Thus, combining the above estimates yields

\[
\int \frac{\|
abla u_{s'}
\|_2^{2^*}}{K(x)u_{s'}^2} \, dx \leq \left( K(y)^{-\frac{n-2}{2}} + K(z)^{-\frac{n-2}{2}} \right)^{\frac{n-2}{2}} S^{2^*} \left( 1 - a\lambda^{2-n} - c(\lambda) \right) + o(\lambda^{2-n}).
\]

Hence the lemma follows if

\[
\lim_{\lambda \to \infty} \frac{c(\lambda)}{\lambda^{2-n}} = 0. \tag{2.11}
\]

Conditions (2.6) and (2.7) transformed on \( \mathbb{R}^n \) become

\[
K(x) \geq K(y) - c|x-y|^{\rho} \quad \text{in a neighborhood of } y, \tag{2.12}
\]

\[
K(x) \geq K(z) - c|x-z|^{\rho} \quad \text{in a neighborhood of } z. \tag{2.13}
\]

Let \( \varepsilon > 0 \) be such that \( 2\varepsilon < |y-z| \) and (2.12) and (2.13) hold respectively in \( B(y, \varepsilon) \) and \( B(z, \varepsilon) \). We split the integral in the definition of \( c(\lambda) \) as follows:

\[
\left( \int_{B(y,\varepsilon)} + \int_{B(z,\varepsilon)} + \int_{(B(y,\varepsilon) \cup B(z,\varepsilon))^c} \right) (K(x) - K_{12}(x)) \times
\]

\[
\left( K(z)^{\frac{n}{2}} u_{y,\lambda}^{2^*} + K(y)^{\frac{n}{2}} u_{z,\lambda}^{2^*} \right) \, dx.
\]

By the dominated convergence theorem the last integral is equal to \( O(\lambda^{-n}) \), and also

\[
\int_{B(y,\varepsilon)} U_{z,\lambda}^{2^*} \, dx = O(\lambda^{-n}), \quad \int_{B(z,\varepsilon)} U_{y,\lambda}^{2^*} \, dx = O(\lambda^{-n}).
\]
Therefore we have only to study the asymptotics of \( \int_{B_{y,\varepsilon}} (K(x) - K_{12}(x))U_{y,\lambda}^2 \, dx \) and of \( \int_{B_{z,\varepsilon}} (K(x) - K_{12}(x))U_{z,\lambda}^2 \, dx \). Let us consider the first one (the other one can be treated in the same way):

\[
\left| \int_{B_{y,\varepsilon}} (K(x) - K_{12}(x))U_{y,\lambda}^2 \, dx \right| \leq c \int_{B_{y,\varepsilon}} |x - y|^\rho U_{y,\lambda}^2 \, dx \\
\leq c\lambda^{-n} \int_{\{|z| < \lambda\varepsilon\}} \frac{|z|^\rho}{(1 + |z|^2)^n} \, dx,
\]

and the last integral is \( o(\lambda^{2-n}) \) when \( \rho > n - 2 \).

**Remark.** One could prove the same claim assuming

\[
K(x) \geq K(y) - cd(x, y)^{\rho_1} \quad \text{for } x \text{ in a neighborhood of } y, \quad (2.14)
\]

\[
K(x) \geq K(z) - cd(x, z)^{\rho_2} \quad \text{for } x \text{ in a neighborhood of } z, \quad (2.15)
\]

for some positive constant \( c \) and any \( \rho_1, \rho_2 \) larger than \( \frac{n(n-2)}{n+2} \) and satisfying \( \frac{1}{\rho_1} + \frac{1}{\rho_2} < \frac{2}{n+2} \). To prove this it suffices to define \( u_s \) in a slightly different way, that is \( u_s = \delta_{y,\lambda} + \delta_{z,\mu} \) (with \( \lambda \) not necessarily equal to \( \mu \), and to repeat the previous proof, studying the asymptotic behavior of \( F(u_s) \) as \( \lambda, \mu \rightarrow \infty \).

One could prove that Theorem 1 continues to hold when hypothesis i) in that theorem is substituted by this more general assumption on the asymptotic behavior of \( K \) near \( y \) and \( z \).

**Lemma 2.2.** Let \( y, z \) belong to two different connected components of \( \{ x \in S^n : K(x) = K_{\max}\} \). Then

\[
\inf_{\gamma \in \Gamma_{y,z}} \max_{u \in \gamma} F(u) > \frac{S^2}{K_{\max}}.
\]

**Proof.** Let \( \Gamma = \inf_{\gamma \in \Gamma_{y,z}} \max_{u \in \gamma} F(u) \). The inequality \( \Gamma \geq \frac{S^2}{K_{\max}} \) follows from the Sobolev inequality (2.4). Let us prove that \( \Gamma > \frac{S^2}{K_{\max}} \). If not, there would be a sequence of paths \( \gamma_n \) in \( \Gamma_{y,z} \) with \( S^2 K_{\max}^{-1} < F(u) < S^2 K_{\max}^{-1} + 1/n \) for each \( u \in \gamma_n \). For each \( n \) let \( u_n \in \gamma_n \). We will determine how to choose \( u_n \) later.

By the concentration-compactness lemma of P.L. Lions (see [18]) it follows that

- (a) either \( \{u_n\} \) converges strongly to some \( u \in D^{1,2}(\mathbb{R}^n) \) with \( u \geq 0 \), \( \|\nabla u\|_2 = 1 \), or
- (b) \( |u_n|^2 \) and \( |\nabla u_n|^2 \) converge as measures respectively to \( S^2 \delta(x - x_0) \) and \( \delta(x - x_0) \), for some \( x_0 \in S^n \) such that \( K(x_0) = K_{\max} \) (here \( \delta(x) \) is the Dirac mass concentrated in 0).

If (a) holds then \( F(u) = S^2 K_{\max}^{-1} \); this may happen only if \( K \equiv K_{\max} \) and this contradicts the fact that \( \{ K = K_{\max} \} \) is not connected. Let us now show that we can choose each \( u_n \) in \( \gamma_n \) in such a way that (b) does not hold. We will call the baricenter \( x_b \) of a function \( u \in D^{1,2}(\mathbb{R}^n) \) the point \( x_b \in \mathbb{R}^n \) such that \( \int_{\mathbb{R}^n} (x - x_b) |\nabla u|^2 \, dx = 0 \). This point depends continuously on \( u \) and on each path \( \gamma \in \Gamma_{y,z} \) the baricenter moves from \( y \) to \( z \) as \( s \) varies from 0 to \( \infty \). If (b) occurs then the baricenter of \( u_n \) converges to \( x_0 \). Therefore if we choose \( u_n \in \gamma_n \) in such a way that for each \( n \) the distance of the baricenter of \( u_n \) from the set \( \{ x \in \mathbb{R}^n : K(x) = K_{\max} \} \) is larger than a fixed positive constant then (b) does not happen. This concludes the proof.
Lemma 2.3. Let \( z \in S^n \) be a strict local maximum for \( K \) and let us suppose that \( \Delta K(z) < 0 \). Moreover let \( y \) be a point of \( S^n \) different from \( z \) such that \( K(y) \geq K(z) \). Then there exists \( \sigma > 0 \) such that
\[
\inf_{\gamma \in \mathcal{P}_{y,z}} \max_{u \in \gamma} \mathcal{F}(u) > \frac{S^2}{K(z)} + \sigma.
\]

Remark. We claim that the same result holds also assuming \( K(x) \leq K(z) - cd(x, z)^p \) for any \( x \) in some neighborhood of \( z \), some \( c > 0 \) and \( p \in (0, n) \), instead of assuming \( \Delta K(z) < 0 \).

Proof. Let \( u_s \in \mathcal{P}_{y,z} \). We recall that, by definition of \( \mathcal{P}_{y,z} \), when \( s \to \infty \), \( u_s \) stays definitively in \( W(1, \varepsilon) \) for any \( \varepsilon > 0 \). Let \( \alpha_s > 0, a_s \in S^n, \lambda_s > 0 \) be the parameters associated to \( u_s \) via the minimization problem (M); we can write
\[
u_s(x) = \alpha_s \delta_{a_s, \lambda_s}(x) + e_s(x),
\]
for some \( e_s(x) \in H^1(S^n) \). The definition of \( u_s \) implies that \( \|e_s\|_{-L} \to 0 \), \( a_s \to z \), \( \lambda_s \to \infty \), as \( s \to \infty \).

The previous decomposition has already been used in various papers. As already proved in these papers (see for instance [3]) the fact that \( a_s, a_s, \lambda_s \) are the parameters that optimize the approximation of \( u_s \) by \( a \delta_{a, \lambda} \) implies that
\[
\langle e_s, \delta_{a_s, \lambda_s} \rangle_{-L} = 0; \quad \langle e_s, \frac{\delta \lambda}{\partial a} \big|_{a_s, \lambda_s} \rangle_{-L} = 0; \quad \langle e_s, \frac{\delta \lambda}{\partial a} \big|_{a_s, \lambda_s} \rangle_{-L} = 0.
\]

The condition \( \|u_s\|_{-L} = 1 \) is thus equivalent to
\[
\alpha_s^2 + \|e_s\|_{-L}^2 = 1.
\]

Let us write an expansion of \( \mathcal{F} \) as \( s \to \infty \):
\[
\int_{S^n} K(x)u_s^2 \, dx = \alpha_s^2 \int_{S^n} K(x)\delta_{a_s, \lambda_s}^2 \, dx + 2^* \alpha_s^{2^* - 1} \int_{S^n} K(x)\delta_{a_s, \lambda_s}^{2^* - 1} e_s \, dx
\]
\[
+ \frac{2^*(2^* - 1)}{2} \alpha_s^{2^* - 2} \int_{S^n} K(x)\delta_{a_s, \lambda_s}^{2^* - 2} e_s^2 \, dx + O(\|e_s\|_{-L}^3).
\]

In Lemmas 4.1 and 4.2 it is proved that for \( a_s \) close enough to the critical point \( z \),
\[
\int_{S^n} K(x)\delta_{a_s, \lambda_s}^{2^*} \, dx = K(a_s) \int_{S^n} \delta_{a_s, \lambda_s}^{2^*} \, dx + c_1 \frac{\Delta K(a_s)}{\lambda_s} + o(\frac{1}{\lambda_s^2}),
\]
\[
\left| \int_{S^n} K(x)\delta_{a_s, \lambda_s}^{2^* - 1} e_s \, dx \right| \leq c_2 \left( \frac{\|\nabla K(a_s)\|}{\lambda_s} + o(\frac{1}{\lambda_s^2}) \right) \|e_s\|_{-L},
\]
where the \( c_i \)'s are suitable positive constants. Since \( \int \delta_{a_s, \lambda_s}^{2^*} \, dx = S^{2^*} \) and \( |\nabla K(a_s)| = o(1) \) we have
\[
\int_{S^n} K(x)\delta_{a_s, \lambda_s}^{2^*} \, dx = \frac{K(a_s)}{S^{2^*}} + c_1 \frac{\Delta K(a_s)}{\lambda_s^2} + o(\frac{1}{\lambda_s^2}),
\]
\[
\left| \int_{S^n} K(x)\delta_{a_s, \lambda_s}^{2^* - 1} e_s \, dx \right| \leq o(\frac{1}{\lambda_s}) \|e_s\|_{-L}.
\]
Regarding the third integral in the right-hand side of (2.16), due to Lemma 4.1, putting all the pieces together and writing \( a_n \) in terms of \( |\epsilon| - L \), as in (2.17) one gets

\[
\int_{s}^{s'} K(x) \rho(x) \frac{\partial K(x)}{\partial x} dx = K(a_n) \int_{s}^{s'} \rho(x) \left( \frac{\partial K(x)}{\partial x} \right)^2 dx + O(\|\epsilon\|_L^2).
\]

In [7], Lemma 1.1 (and also in [3], Lemma A.2) it is proved that for any \( \epsilon \in H(\mathbb{S}^n) \)

or, equivalently,

\[
\int_{s}^{s'} K(x) \rho(x) \frac{\partial K(x)}{\partial x} dx = K(\rho)(a_n) \int_{s}^{s'} \rho(x) \left( \frac{\partial K(x)}{\partial x} \right)^2 dx + o(\|\epsilon\|_L^2).
\]

Proof: For a fixed \( \lambda > 0 \) let \( \gamma = \delta_n \); this path is in \( \mathbb{P}_{n+1} \). Actually, the behavior at the endpoints is not exactly correct, but we can continue the path with the desired property. Let \( \epsilon > 0 \), we claim that there exists \( \lambda \) such that for any \( \lambda > \lambda \),

\[
\max_{B} |F(u)| < \min_{B} |K(x)| + \epsilon.
\]

This estimate holds uniformly with respect to the path in \( \mathbb{P}_{n+1} \). Since the error term

\[
\inf_{\gamma} \max_{B} |F(u)| \leq \min_{B} |K(x)| + \epsilon.
\]

Lemma 2.4. Let \( y \in \mathbb{S}^n \) and let us assume that (2.21) holds. Let \( x(t), t \in (0, +\infty) \), be a continuous path in \( \mathbb{S}^n \) such that \( x(t) \rightarrow y \). Then

\[
\sup_{t \in (0, +\infty)} F(u) > K(y) - \epsilon^2 - 1 - \epsilon^2.
\]

Since \( x \) is a point of strict local minimum for \( K \) and \( \Delta K(x) = K(x) - K(z) + o(1) < 0 \),

\[
\int_{s}^{s'} K(x) \rho(x) \frac{\partial K(x)}{\partial x} dx \leq \mathbb{S}^n.
\]

Expanding (2.20)

\[
\int_{s}^{s'} K(x) \rho(x) \frac{\partial K(x)}{\partial x} dx = K(y) \int_{s}^{s'} \rho(x) \left( \frac{\partial K(x)}{\partial x} \right)^2 dx + o(\|\epsilon\|_L^2).
\]

THE SCALAR CURVATURE EQUATION

\[
\frac{\partial K(x)}{\partial x} = \frac{1}{|\mathbb{S}^n|} \frac{\partial K(x)}{\partial x}.
\]
We have to study the asymptotic behavior of \( \int_{S^n} K(x) \delta_{x(t),\lambda}^2 dx \) as \( \lambda \to \infty \). Let \( \sigma > 0 \) be small enough. We split the above integral into three parts,

\[
\int_{S^n} K(x) \delta_{x(t),\lambda}^2 dx = K(x(t)) \int_{S^n} \delta_{x(t),\lambda}^2 dx + 
\left( \int_{B(x(t),\sigma)} + \int_{B(x(t),\sigma)^c} \right) (K(x) - K(x(t))) \delta_{x(t),\lambda}^2 dx
\]

\( (B(x(t), \sigma) \) is the geodesic ball centered in \( x(t) \) with radius \( \sigma \). The first integral in the right hand side is equal to \( S^{-2} K(x(t)) \); the last one is bounded (in absolute value) by \( c_1 \lambda^{-n} \sigma^{-n} \), for some positive constant \( c_1 \) that does not depend on \( x(t) \). Let us now consider the second integral. We can assume that \( |\nabla K| \) is bounded in a neighborhood of the curve \( \{x(t)\} \) of radius \( \sigma \); let \( D \) be such a bound. Then

\[
\left| \int_{B(x(t),\sigma)} (K(x) - K(x(t))) \delta_{x(t),\lambda}^2 dx \right| \leq D \int_{B(x(t),\sigma)} d(x, x(t)) \delta_{x(t),\lambda}^2 dx \leq \frac{c_2}{\lambda},
\]

for some positive constant \( c_2 \) that does not depend on \( x(t) \). As a conclusion one can write

\[
\int_{S^n} K(x) \delta_{x(t),\lambda}^2 dx \geq S^{-2} K(x(t)) - \frac{c_2}{\lambda} - \frac{c_1}{\lambda^n \sigma^n}
\]

and from this the result follows immediately.

3. Existence results. In the following flow line will mean flow line for \( -\nabla F \), that is a solution to problem (1.4) for some initial value \( u_0 \).

**Lemma 3.1.** Let us assume that \( F \) has no critical points. Let \( \mathcal{P}_{y,z} \) be as in (2.1), \( \Gamma = \inf_{y,z} \max_{u \in \mathcal{P}_{y,z}} F(u) \) and let us suppose that \( \Gamma > S^2 \max(K(y)^{-1}, K(z)^{-1}) \). Let \( \varepsilon > 0 \) and \( \gamma(s) \in \mathcal{P}_{y,z} \) be such that \( \max_{u \in \mathcal{P}_{y,z}} F(u) \leq \Gamma + \varepsilon \). Then there exists \( \tilde{u} \in \gamma \) such that if \( u(t) \) denotes the flow line with initial data \( \tilde{u} \) then \( F(u(t)) \in (\Gamma, \Gamma + \varepsilon) \), for each \( t \geq 0 \).

**Proof.** Let \( \sigma > 0 \) be such that \( \Gamma - 2\sigma > S^2 \max(K(y)^{-1}, K(z)^{-1}) \). Let \( A = \{u \in M : F(u) < \Gamma - 2\sigma\} \), \( B = \{u \in M : F(u) \geq \Gamma - \sigma\} \) and for each \( u \in A \) let \( g(u) = d(u, A)/(d(u, A) + d(u, B)) \) (here \( d(\cdot, \cdot) \) denotes the distance in \( H^1(S^n) \)). For each positive \( s \) and \( t \) let \( \gamma(s, t) \) be the solution of the problem

\[
\frac{\partial u}{\partial t} = -g(u) \nabla F(u); \quad u(0) = \gamma(s, t).
\]

Note that for each fixed \( t \), \( \gamma(\cdot, t) \in \mathcal{P}_{y,z} \). Since \( \lim_{s \to 0} F(\gamma(s)) = \lim_{s \to -\infty} F(\gamma(s)) < \Gamma - \sigma \), there exists an interval \( [s_1, s_2] \) such that \( F(\gamma(s, t)) < \Gamma - \sigma \) for each \( s \in [0, s_1) \cup (s_2, \infty) \) and for each \( t \).

Let us prove that there exists \( \tilde{t} \) such that \( F(\gamma(\tilde{t})) > \Gamma \) for each \( t > 0 \). If this is not true, for each \( s \in [s_1, s_2] \) let \( \tau(s) = \inf\{t \geq 0 : F(\gamma(s, t)) \leq \Gamma\} \).
The function \( \tau(s) \) can not be bounded in \([s_1, s_2]\), since otherwise, if \( T \) is an upper bound, \( \mathcal{F}(\gamma(s, T)) \leq \Gamma \) for each \( s \). Choose \( \delta > 0 \); for each \( s \),

\[
\mathcal{F}(\gamma(s, T + \delta)) = \mathcal{F}(\gamma(s, T)) - \int_{T}^{T+\delta} g(\gamma(s, t)) \| \nabla \mathcal{F}(\gamma(s, t)) \|^2 \, dt < \Gamma
\]

and this contradicts the definition of \( \Gamma \).

We may assume that \( \tau(s) \) is unbounded in any neighborhood of some point \( \hat{s} \in [s_1, s_2] \). Due to the continuous dependence of \( \gamma(s, t) \) on \( t \), \( \mathcal{F}(\gamma(\hat{s}, \tau(\hat{s}))) \leq \Gamma \). Since \( \mathcal{F} \) has no critical point then there exists \( \delta > 0 \) such that \( \mathcal{F}(\gamma(\hat{s}, \tau(\hat{s}) + \delta)) < \Gamma \). This inequality is true also for \( s \) in some neighborhood of \( \hat{s} \) and this contradicts the definition of \( \hat{s} \).

To conclude the proof it suffices to choose \( \hat{u} = \gamma(\hat{s}) \).

**Proposition 3.2.** Let \( \mathcal{P}_{y,z} \) and \( \Gamma \) be as in (2.1), (2.3) and let \( K \) satisfy H1(d) for some \( d < S^2 \Gamma^{-1} \). Let us assume that

\[
S^2 \max(K(y)^{-1}, K(z)^{-1}) < \Gamma < 2\pi^2 S^2 \max_1,
\]

and that there exists an \( \varepsilon > 0 \) such that if \( w \) is a critical point of \( K \) and \( K(w) \in [S^2 \Gamma^{-1}, S^2 \Gamma^{-1} + \varepsilon] \), then either \( \Delta K(w) < 0 \) and \( w \) is a strict local maximum or \( \Delta K(w) > 0 \). Then there exists a solution to problem (0.2).

**Proof.** Let \( C = \{ w \in S^n : K(w) = S^2 \Gamma^{-1}, \nabla K(w) = 0, \Delta K(w) \leq 0 \} \). By assumption each point \( w \in C \) is a strict local maximum for \( K \). By Lemma 2.3 there exists a positive constant \( \sigma_w > 0 \) such that

\[
\max_{u \in \gamma} \mathcal{F}(u) > S^2 K(w)^{-1} + \sigma_w \tag{3.1}
\]

for each \( \gamma \in \mathcal{P}_{y,w} \). Let \( \sigma = \min_{w \in C} \sigma_w \) (let \( \sigma = 1 \) if \( C = \emptyset \)).

Let \( \delta > 0 \) be such that \( \delta < \min(\sigma, \varepsilon, 2\pi^2 S^2 \max_1 - \Gamma) \), and moreover let us choose \( \delta \) in such a way that there is no critical point \( w \) of \( K \) with \( K(w) \in (S^2 \Gamma^{-1}, S^2 \Gamma^{-1} + \delta) \).

Let as argue by contradiction and let us suppose that there is no solution to problem (0.2). Let \( \gamma(s) \in \mathcal{P}_{y,z} \) be a path such that \( \mathcal{F}(\gamma(s)) < \Gamma + \delta \); Lemma 3.1 states that there exists \( \hat{s} \) such that denoting by \( u(t) \) the flow line for \( -\nabla \mathcal{F} \) with initial condition \( \gamma(\hat{s}) \), then \( \mathcal{F}(u(t)) \in (\Gamma, \Gamma + \delta) \).

Due to Lemma 1.5 each “subsequence” of \( u(t) \) is a Palais-Smale sequence. Since

\[
\lim_{t \to \infty} \mathcal{F}(u(t)) \leq \Gamma + \delta < 2\pi^2 S^2 \max^{-1}_1,
\]

Lemma 1.3 implies that \( u(t) \) stays definitively in \( W(1, \mu) \) for any positive \( \mu \).

Let \( a(t) \) be the point associated to \( u(t) \) via problem (M). Lemma 1.4 says that the point \( a(t) \) converges to some point \( a \in S^n \) with \( \nabla K(a) = 0 \) and \( \Delta K(a) \leq 0 \). Furthermore \( \lim_{t \to \infty} \mathcal{F}(u(t)) = S^2 K(a)^{-1} \). Since there is no critical point \( w \) for \( K \) with \( K(w) \in (S^2 \Gamma^{-1}, S^2 \Gamma^{-1} + \delta) \), \( S^2 K(a)^{-1} = \Gamma \); that is, \( a \in C \).

The path which is the union of \( u(t), t > 0 \) and of \( \gamma(s) \) for \( s \in (0, \hat{s}] \) belongs to \( \mathcal{P}_{y,a} \) and the value of \( \max \mathcal{F} \) on this path is below \( \Gamma + \delta \). This contradicts (3.1).

We recall that hypothesis H1(d) has been defined at the beginning of Section 1.
Theorem 6. Let $K$ satisfy $H1(d)$ with $d = 2^{-\frac{n-2}{2}} K_{\text{max}}$. Suppose there exists two points $y$ and $z$ of $S^n$ belonging to two different connected components of \( \{ x \in S^n : K(x) = K_{\text{max}} \} \). Moreover, suppose that
- there exist two constants $c > 0$ and $\rho > n-2$ such that
  \[
  K(x) \geq K_{\text{max}} - cd(x,y)^\rho \quad \text{for } x \text{ in a neighborhood of } y,
  \]
  \[
  K(x) \geq K_{\text{max}} - cd(x,z)^\rho \quad \text{for } x \text{ in a neighborhood of } z;
  \]
- if $w$ is a critical point for $K$ and $K(w) \in (2^{-\frac{n-2}{2}} K_{\text{max}}, K_{\text{max}})$, then either $\Delta K(w) < 0$ and $w$ is a strict local maximum or $\Delta K(w) > 0$.

Then there exists a solution to problem (0.2).

Proof. Let $\mathbb{P}_{y,z}$ and $\Gamma$ be as in (2.1), (2.3). Due to Lemmas 2.1 and 2.2,
\[
S^2 K_{\text{max}}^{-1} < \Gamma \leq 2^{\frac{n-2}{4}} S^2 K_{\text{max}}^{-1}.
\]

The theorem follows from Proposition 3.2.

Theorem 7. Suppose there exists $z \in S^n$ where $K$ achieves a strict local maximum and $\Delta K(z) \neq 0$ and a continuous path $x(t)$ in $S^n$ connecting $z$ to some point $y$ different from $z$, where $K(y) \geq K(z)$. Suppose moreover that
- $\min_t K(x(t)) > 2^{-2/(n-2)} K_{\text{max}}$;
- if $w$ is a critical point for $K$ and $K(w) \in [\min_t K(x(t)), K(z)]$, then either $\Delta K(w) < 0$ and $w$ is a strict local maximum or $\Delta K(w) > 0$.

Finally let $K$ satisfy $H1(d)$ for some $d < \min_t K(x(t))$. Then there exists a solution to problem (0.2).

Proof. Let $\mathbb{P}_{y,z}$ and $\Gamma$ be as in (2.1), (2.3). We may choose $d > 2^{-\frac{n-2}{2}} K_{\text{max}}$. Due to Lemmas 2.3 and 2.4,
\[
S^2 K(z)^{-1} < \Gamma < d^{-1} S^2.
\]

The theorem follows from proposition 3.2.

Proof of Theorems 1 and 2. Theorems 1 and 2 follow immediately, via a stereographic projection, from the two previous results. \( \square \)

In order to prove Theorem 3 we need to extend slightly lemma 1.4 to include those $K$’s which are symmetric under the group of rotations around the North pole-South pole axis and which have critical points besides the two poles. Lemma 1.4 does not apply to these coefficients since they have infinitely many critical points.

Let $x_1, x_2, \ldots, x_{n+1}$ be euclidean coordinates in $\mathbb{R}^{n+1}$ and let $S^n$ be embedded in $\mathbb{R}^{n+1}$ in such a way that the North pole has coordinate $(0,0,\ldots,1)$ and the South pole $(0,0,\ldots,-1)$.

Lemma 1.4’. Let $K(x) = K(x_{n+1})$, and let $u(t)$ be the solution of problem (1.4) for some initial data $u_0 \in H^1(S^n)$. Let us suppose that $u(t)$ stays definitively in $W(1,\varepsilon)$ for any $\varepsilon$ small enough. Let us assume that $K(r)$ is $C^2$, with only a finite number of
critical points, in \( \{ r \in [-1, 1] : K(r) \geq d \} \), for some \( d < S^{-2} (\lim \mathcal{F}(u(t)))^{-1} \). Then, denoting by \( a(t) \) the point associated to \( u(t) \) via the minimization problem \((M)\), \( a(t)_{n+1} \) converges to some \( r \in [-1, 1] \) with \( \frac{dK}{dr}(r) = 0 \) and \( \frac{d^2K}{dr^2}(r) \leq 0 \).

See the appendix for a proof.

**Proof of Theorem 3.** Let \( K_0(r) \) be a \( C^2 \) positive function which is equal to 1 in \([0, 1/4] \), constant in \([1/2, +\infty) \) and with \( \frac{dK_0}{dr} < 0 \) in \((1/4, 1/2) \). Let \( \varphi(r) \) be equal to \( r^\alpha(2-r)^\alpha \) (for some \( \alpha > n \)) in \([0, 2] \) and identically 0 elsewhere.

We claim that problem \((0.1)\) with

\[
K(x) = K_0(|x|) - \varepsilon_0 \varphi(8|x|) - \varepsilon_\infty \varphi\left(\frac{1}{|x|}\right)
\]

has a solution, but no radial solution, for any \( \varepsilon_0, \varepsilon_\infty > 0 \) with \( \varepsilon_0, \varepsilon_\infty \) small.

The nonexistence of any radial solution follows from Theorem 0.4 in [6]. On the other hand, let \( z \) be any point with \( |z| = 1/4 \), let \( y \) be the origin and let \( P_{y,z} \) and \( \Gamma \) be as in \((2.1), (2.3)\): \( z \) and \( y \) are points where \( K \) achieves its absolute maximum and moreover there is a continuous path connecting \( z \) and \( y \) and on this path \( \min K = 1 - \varepsilon_0 \). Therefore, due to Lemmas 2.2 and 2.4, \( \Gamma \in \{S; S(1-\varepsilon_0)^{-1}\} \). For \( \varepsilon_0 \) small, the only critical points of \( K \) with values in the range \([1-\varepsilon_0, 1) \) are those on \( \{ |x| = 1/8 \} \), and on those points \( \Delta K > 0 \). Arguing exactly as in the proof of Proposition 2.3, using Lemma 1.4' instead of Lemma 1.4, one proves the existence of a solution.

**Remark.** In view of Theorem 1 in [4] it seems worth mentioning that the coefficient \( K(r) \) in the previous theorem can be made arbitrarily close (in \( C^1 \) norm) to a radial function which is strictly decreasing in \([0, 1] \) and strictly increasing in \([1, \infty) \).

Let us now prove the results regarding exclusively \( S^3 \).

**Theorem 8.** Let \( n = 3 \) and let \( K \) satisfy \( H2 \). Let \( y \) be a point with \( K(y) = K_{\max} \) and let us suppose that there exists a point \( z \), different from \( y \), where \( K \) achieves a local maximum. Furthermore let us suppose that if \( w \) is a critical point for \( K \) with \( K(w) \in \left( \frac{K(y)K(z)}{(\sqrt{K(y)}+\sqrt{K(z)})^2}, K(z) \right) \) and \( \Delta K(w) < 0 \) then \( w \) is a strict local maximum for \( K \). Then there exists a solution to problem \((0.2)\).

**Proof.** Let \( P_{y,z} \) and \( \Gamma \) be as in \((2.1), (2.3)\). Since \( z \) is a local maximum and we have assumed that \( \Delta K \neq 0 \) in each critical point of \( K \), \( \Delta K(z) < 0 \). Lemma 2.3 applies to this situation and together with Lemma 2.1 implies that

\[
S^6 K(z)^{-1} < \Gamma < \left( K(y)^{-\frac{1}{2}} + K(z)^{-\frac{1}{2}} \right)^2 S^6.
\]

One can now conclude repeating the proof of Proposition 3.2. Note that in this case, to assert that \( u(t) \) stays in \( W(1, \mu) \) for each \( \mu > 0 \) it is not necessary to assume that \( \Gamma < 2\pi^2 S^2 K_{\max}^{-1} \).

**Proof of Theorem 4.** The proof proceeds as in Theorem 8. Here

\[
S^6 K(z)^{-1} < \Gamma < S^6 \left( \min_{x \in L(t)} K(x) \right)^{-1}
\]

However, since the function \( K \) is strictly decreasing in \([0, 1] \) and strictly increasing in \([1, \infty) \), this is not a problem.

Moreover, since \( \Gamma \) is a strict local maximum for \( K \), we have that

\[
\Delta K(w) < 0 \quad \text{for all critical points } w \text{ of } K \text{ in } \{S; S(1-\varepsilon_0)^{-1}\}.
\]
due to Lemmas 2.3 and 2.4.

**Proof of Theorem 5.** Let \( z \) be a strict local maximum point for \( K \) such that \( K(z) = \min \{ K(w) : \nabla K(w) = 0, \Delta K(w) < 0 \} \) and let \( y \) be a point where \( K(y) = K_{\max} \); with this choice of \( z \) and \( y \) Theorem 5 is just a corollary of Theorem 8.

4. Appendix.

**Proof of Lemma 1.3.** We follow [21], with the necessary modifications. Given a function \( v \) defined on \( S^n, y \in S^n \) and a constant \( r > 0 \) we define now a “rescaled” function \( v_{y,r} \) on \( S^n \) in the following way: Let \( (h_1, h_2, \ldots, h_n) \) be a geodesic coordinate system on \( S^n \), centered in \( y \); we define

\[
v_{y,r}(h_1, h_2, \ldots, h_n) = r^{n-2} v(\rho h_1, \rho h_2, \ldots, \rho h_n).
\]

Furthermore given \( y \in S^n \) we denote by \( -y \in S^n \) the antipodal point. For \( u \in H^1(S^n) \) let

\[
\mathcal{E}(u) = \frac{1}{2} \int_{S^n} \left( \| \nabla u \|^2 + \frac{n(n - 2)}{4} u^2 \right) \, dv - \frac{1}{2} \int_{S^n} K(u^+)^{2r} \, dv.
\]

If \( (u_m) \) is a Palais-Smale sequence of positive functions for the functional \( \mathcal{F} \) and, for each \( m \)

\[
v^{(1)}_m = u_m \mathcal{F}(u_m)^{\frac{n-2}{2}}
\]

then \( v^{(1)}_m \) is a Palais-Smale sequence for \( \mathcal{E} \). We can write

\[
\| v^{(1)}_m \|_{-L} = \| u_m \|_{-L} \mathcal{F}(u_m)^{\frac{n-2}{2}} = \mathcal{F}(u_m)^{\frac{n-2}{2}},
\]

\[
\nabla \mathcal{E}(v^{(1)}_m) \rightarrow 0 \quad \text{strongly in } H^{-1}(S^n).
\]

Since \( \| v^{(1)}_m \|_{-L} \) is bounded, we may assume that \( v^{(1)}_m \rightharpoonup v^0 \) weakly in \( H^1(S^n) \). To be more precise this is true only for a subsequence; we will often extract subsequences without explicitly mentioning this fact. If \( v^0 \) is not identically 0, then it is a solution of problem (0.2) and therefore we may assume \( u^0 \equiv 0 \). As proved in [21], there exist sequences \( (r^1_m, r^2_m) \) of positive constants tending to 0, \( (y^1_m, r^2_m) \) of points of \( S^n \) such that

\[
(v^{(1)}_m y^1_m, r^1_m) \rightharpoonup v^{(1)} \neq 0 \quad \text{weakly in } H^1(S^n)
\]

as \( m \rightarrow \infty \). By weak continuity, \( v^{(1)} \) is a weak solution of

\[
Lu + K^{(1)}(u^+)^{2r-2} u = 0 \quad \text{in } S^n,
\]

where \( K^{(1)} \) is a weak (and almost everywhere pointwise) limit of the sequence of functions \( (r^1_m)^{-\frac{n-2}{2}} K_{y^1_m, r^1_m} \). Note that if \( u \) is a weak solution of (4.2) then \( u \geq 0 \) and

\[
0 = \| u \|^2_{-L} - \int_{S^n} K^{(1)} u^{2r} \, dv \geq \| u \|^2_{-L} - \sup_{S^n} K^{(1)} \int_{S^n} u^{2r} \, dv
\]

\[
\geq \| u \|^2_{-L} \left( 1 - \sup_{S^n} K^{(1)} S^{-2r} \| u \|^4_{-L} \right).
\]
that is, if $u \neq 0$,
\[ \frac{S^2}{\sup_{S^n} K^{(1)}} \leq \|u\|_{-\frac{4}{\alpha}}. \] (4.3)

We may assume that $y_m^1$ converges to some point $a_1 \in S^n$. Let us now prove that $a_1 \in A_d$. If this is not the case, since $A_d$ is closed,
\[ \lim_{x \to a_1} K(x) < d. \]

Moreover, by definition of $K^{(1)}$, $\sup K^{(1)} \leq d$. Therefore, by (4.3),
\[ \|v^{(1)}\|_{-\frac{4}{\alpha-L}} \geq \frac{S^2}{d}. \] (4.4)

On the other hand, by (4.1) and since the norm $\| \cdot \|_{-L}$ is invariant with respect to rescaling, $\|(v^{(1)}_m)_{y_m^1,r_m^1}\|_{-L} = \|v^{(1)}_m\|_{-L} = \mathcal{F}(u_m)^{-\frac{4}{\alpha}}$. Combining this with (4.4) one gets
\[ \lim \mathcal{F}(u_m) = \lim \|(v^{(1)}_m)_{y_m^1,r_m^1}\|_{-\frac{4}{\alpha-L}} \geq \|v^{(1)}\|_{-\frac{4}{\alpha-L}} \geq \frac{S^2}{d}, \]
which contradicts the assumptions of this Lemma.

Therefore $K(a_1) \geq d$ and moreover, since $K$ is continuous in $a_1$, $K^{(1)} \equiv K(a_1)$. But in the case $K^{(1)} = \text{constant}$, all the positive solutions of the equation (4.2) are known and we can write
\[ v^{(1)} = S^{\frac{4}{\alpha}} K(a_1)^{-\frac{\alpha+2}{\alpha}} \delta_{b,\lambda} \]
for some $\lambda > 0$, $b \in S^n$. Let
\[ v^{(2)}_m = v^{(1)}_m - (v^{(1)}_{y^1_m,1/r^1_m}). \]

Note that
\[ \|v^{(2)}_m\|_{-\frac{4}{\alpha-L}} = \|v^{(1)}_m\|_{-\frac{4}{\alpha-L}} - \|v^{(1)}_{y^1_m,1/r^1_m}\|_{-\frac{4}{\alpha-L}} + o(1), \quad \nabla \mathcal{E}(v^{(2)}_m) \to 0 \quad \text{strongly in } H^{-1}(S^n). \] (4.5)

Let us now show that $v^{(2)}_m \to 0$ strongly in $H^1(S^n)$. Let as argue by contradiction and let us suppose that this is not the case. As observed above, since problem (0.1) has no solution, then $v^{(2)}_m \to 0$ weakly. Arguing as above there exist sequences $(r^2_m)$ of positive constants tending to 0, $(y^2_m)$ of points of $S^n$ such that, as $m \to \infty$,
\[ (v^{(2)}_m)_{y^2_m,r^2_m} \to v^{(2)} \neq 0 \quad \text{weakly in } H^1(S^n) \]
and $v^{(2)}$ is a weak solution of $Lu + K^{(2)}(u^+)^{2^* - 2} u = 0$ in $S^n$, where $K^{(2)}$ is a weak limit of the sequence of functions $\{(r^2_m)^{-\frac{4}{\alpha+2}} K_{y^2_m,r^2_m}\}$. One can argue as in the case of (4.3) to prove that
\[ \frac{S^2}{\sup_{S^n} K^{(2)}} \leq \|v^{(2)}\|_{-\frac{4}{\alpha-L}}. \] (4.6)
By (4.3), (4.5) and (4.6), one gets the chain of inequalities

\[
\lim F(u_m) \overset{\mathrm{a.s.}}{\to} = \lim \|v_m^{(1)}\|^2_L = \lim \|v_m^{(2)}\|^2_L + \|v^{(1)}\|^2_L \geq \frac{S^2}{\sup K^{(1)}(x)} \cdot \|v\|^2_L + \frac{S^2}{\sup K^{(2)}(x)} \cdot \|v\|^2_L \geq \left( \frac{S^2}{\sup K^{(1)}(x)} \right)^{\frac{a}{2}} + \left( \frac{S^2}{\sup K^{(2)}(x)} \right)^{\frac{a}{2}} \geq 2 \left( \frac{S^2}{K_{\max}} \right)^{\frac{a}{2}},
\]

which contradicts the assumptions. This concludes the proof that \(v_m^{(2)} \to 0\) in \(H^1(S^n)\), that is \(v_m^{(1)} - (v^{(1)}) - y_{m,1/r_m} \to 0\). The proof is easily concluded writing this in terms of \(u_m\) and keeping in mind that \(\|u_m\|_L = 1, \|v_{h,\lambda}\|_L = 1\). \(\square\)

Let \(u \in W(1, \varepsilon)\) for some \(\varepsilon > 0\) small enough. Let \(\alpha > 0, a \in S^n, \lambda > 0\) be the parameters associated to \(u\) via the minimization problem (M); we can write

\[ u(x) = \alpha \delta_{a,\lambda}(x) + e(x), \]

for some \(e(x) \in H^1(S^n)\). The fact that \(a, \alpha, \lambda\) are the parameters that optimize the approximation of \(u\) by \(\alpha \delta_{a,\lambda}\) implies (see [3] for a proof) that

\[
\langle e, \delta_{a,\lambda} \rangle_L = 0; \quad \langle e, \frac{\partial \delta_{a,\lambda}}{\partial a} \rangle_L = 0; \quad \langle e, \frac{\partial \delta_{a,\lambda}}{\partial \lambda} \rangle_L = 0. \quad (V_0)
\]

In what follows we always assume that \(K\) satisfies H1(d).

**Lemma 4.1.** Let \(a \in A_{d+\varepsilon}\) for some \(\varepsilon > 0\) and let \(e\) satisfy \((V_0)\). Then

\[
\left| \int K(x) \delta_{a,\lambda}^{2^*-1} e \, dv \right| \leq c_1 \left( \frac{\nabla K(a)}{\lambda} + o \left( \frac{1}{\lambda} \right) \right) \|e\|_L;
\]

\[
\left| \int K(x) \delta_{a,\lambda}^{2^*-2} e^2 \, dv \right| \leq O \left( \frac{\|e\|^2_L}{\lambda} \right);
\]

\[
\int K(x) \delta_{a,\lambda}^{2^*-1} \frac{1}{\lambda} \frac{\partial \delta_{a,\lambda}}{\partial a} \, dv = -c_2 \frac{\nabla K(a)}{\lambda} + O \left( \frac{1}{\lambda^2} \right);
\]

\[
\int K(x) \delta_{a,\lambda}^{2^*-1} \frac{1}{\lambda} \frac{\partial \delta_{a,\lambda}}{\partial \lambda} \, dv = -c_3 \frac{\Delta K(a)}{\lambda^2} + o \left( \frac{1}{\lambda^2} \right).
\]

**Lemma 4.2.** Let \(z \in A_{d+\varepsilon}\) for some \(\varepsilon > 0\) and let \(z\) be a critical point for \(K\) with \(\Delta K(z) < 0\). Let \(\rho\) be small enough so that \(B(z, \rho) \subset A_{d+\varepsilon}\) and \(-\Delta K(z) \leq -2\Delta K(x)\) for any \(x \in B(z, \rho)\). Let \(a \in B(z, \rho)\) and let e satisfy \((V_0)\). Then the following expansion holds:

\[
\int (K(x) - K(a)) \delta_{a,\lambda}^{2^*} \, dv = c_4 \frac{\Delta K(a)}{\lambda^2} + o \left( \frac{1}{\lambda^2} \right).
\]

**Proof of Lemmas 4.1 and 4.2.** Let \(\tilde{K} \in C^2(S^n)\) coincide with \(K\) in \(A_d\). Assuming \(K \in C^2(S^n)\) the previous estimates have been proved in [3] (Lemmas A.6, A.7, A.8) for \(n = 3\) and generalized to any \(n\) in [2] (the proofs given for \(n = 3\) can be repeated for
any $n$ just adapting the notations). Therefore the previous estimates with $K$ replaced by $\tilde{K}$ are true.

Since the distance from $A_{d+\varepsilon}$ to $(A_d)^c$ is positive, by the dominated convergence theorem, for any $\beta > 0$ there is a positive constant $d_1 > 0$ such that, as $\lambda \to \infty$,

$$\int |\tilde{K} - K|^\beta \delta_{a,\lambda}^{2^*} \, dv \leq d_1 \lambda^{-n}.$$ 

This proves the expansion in Lemma 4.2. Since $|\lambda^{\partial \delta_{a,\lambda}}|, |\frac{1}{\lambda} \partial \delta_{a,\lambda}| \leq d_2 \delta_{a,\lambda}$ we have also

$$\int (\tilde{K} - K) \delta_{a,\lambda}^{2^*-1} \frac{1}{\lambda} \frac{\partial \delta_{a,\lambda}}{\partial a} \, dv \leq d_3 \lambda^{-n};$$

$$\int (\tilde{K} - K) \delta_{a,\lambda}^{2^*-1} \lambda \frac{\partial \delta_{a,\lambda}}{\partial \lambda} \, dv \leq d_4 \lambda^{-n}.$$ 

Similarly, by Hölder and Sobolev inequalities,

$$\int (\tilde{K} - K) \delta_{a,\lambda}^{2^*-1} e \, dv \leq d_5 \lambda^{-\frac{n+2}{2}} \|e\|_{-L};$$

$$\int (\tilde{K} - K) \delta_{a,\lambda}^{2^*-2} e^2 \, dv \leq d_6 \lambda^{-2} \|e\|^2_{-L}.$$ 

These estimates conclude the proof.

**Proof of Lemma 1.4.** For each $t$ let $\alpha(t), \lambda(t)$ be the constants associated (together with $a(t)$) to $u(t)$ via problem (M). We write $u(t) = \alpha(t)\delta_{a(t),\lambda(t)} + e_t$ with $e_t \in H^1(S^n)$. Since $u(t) \in W(1, \varepsilon)$ definitively for any $\varepsilon > 0$, $\lambda(t) \to \infty$, $\|e_t\|_{-L} \to 0$ as $t \to \infty$.

Let us start proving that for $t$ large enough $\alpha(t) \in \{K(x) \geq d + \sigma\}$ for some $\sigma > 0$ independent on $t$. As in (2.17), $\alpha(t)^2 = 1 - \|e_t\|^2_{-L}$. Formula (2.18), valid for $\|e_t\|_{-L} \to 0$, can be rewritten as

$$\int_{S^n} K(x)u(t)^2 \, dx = \int_{S^n} K(x)\delta_{a(t),\lambda(t)}^{2^*} \, dx + O(\|e_t\|_{-L});$$

moreover (see (2.22)), as $\lambda \to \infty$,

$$\int_{S^n} K(x)\delta_{a(t),\lambda(t)}^{2^*} \, dx \geq S^{-2^*} K(a(t)) + O\left(\frac{1}{\lambda(t)}\right).$$

Therefore, as $t \to \infty$,

$$\int_{S^n} K(x)u(t)^2 \, dx \geq S^{-2^*} K(a(t)) + O\left(\frac{1}{\lambda(t)}\right) + O(\|e_t\|_{-L}).$$

Since we assumed that $d < S^{2^*} (\lim \mathcal{F}(u(t)))^{-1}$, this expansion concludes the proof that $a(t) \in \{K(x) \geq d + \sigma\}$. 

In [3] Lemma 6, for \( n = 3 \), and in [2] (see pages 291, 292), for general \( n \), a system of differential equations for \( a(t), \lambda(t) \) is derived from the condition that \( u(t) \) is a flow line.

The authors prove this under the assumption \( K \in C^2(S^n) \). The same result can be proved under our weaker assumptions on \( K \) provided that \( a(t) \in A_{d+\sigma} \).

For instance, one can follow [3] (where the proof is more concise) repeating the proof of Lemma 2 and then of Lemma 6. One has just to replace the estimates needed in those lemmas, and proved in [3] when \( K \) is regular, with those proved in Lemmas 4.1 and 4.2 of this paper. We will not write here the details.

The functions \( a(t) \) and \( \lambda(t) \) satisfy the following differential equations:

\[
\frac{1}{\lambda} \frac{d\lambda}{dt} = - d_1 (1 + o(1)) \frac{\Delta K(a(t))}{K(a(t))^{n+2} \lambda^2} + O\left( \frac{\|\partial F(u(t))\|^2 + |\nabla K(a)|}{\lambda^2} \right) + o\left( \frac{1}{\lambda^2} \right),
\]  

(4.7)

\[
\frac{da}{dt} = - d_2 (1 + o(1)) \frac{\nabla K(a)}{K(a)^{n+2} \lambda^2} + O\left( \frac{\|\partial F(u(t))\|^2}{\lambda} + \frac{1}{\lambda^3} \right),
\]  

(4.8)

for some positive constants \( d_1, d_2 \). Let us perform a stereographic projection in order to be able to work linearly. Let

\[
a'(t) = a(t) - \int_{s_0}^t O\left( \frac{\|\partial F(u(s))\|^2}{\lambda} \right) ds,
\]  

(4.9)

where \( O(\|\partial F(u(t))\|^2/\lambda) \) is the one appearing in (4.8) and \( s_0 \) will be chosen later. By the mean value theorem, (4.8) becomes

\[
\frac{da'}{dt} = - d_2 (1 + o(1)) \frac{\nabla K(a')}{K(a')^{n+2} \lambda^2} + O\left( \frac{1}{\lambda^2} \int_{s_0}^t \frac{\|\partial F(u(s))\|^2}{\lambda} ds + \frac{1}{\lambda^3} \right).
\]  

(4.10)

Multiplying (4.7) by \( \lambda^2 \) one gets

\[
\left| \frac{d\lambda}{dt} \lambda - \lambda^2 O\left( \frac{\|\partial F(u(s))\|^2}{\lambda} \right) \right| \leq d_3.
\]  

(4.11)

Since on any flow line \( F \) is bounded from above,

\[
\lim_{t \to \infty} \int_0^t \|\partial F(u(s))\|^2 ds = \lim_{t \to \infty} F(u(t)) - F(u_0)
\]

is bounded. This and (4.11) imply that \( \lambda(s) \leq d_4 (s + 1) \). Let us now introduce the new variable \( \tau(t) = \int_0^t \frac{1}{M(s)} ds \). As \( t \to \infty \), \( \tau(t) \to \infty \); (4.10) becomes

\[
\frac{da'}{d\tau} = - d_2 (1 + o(1)) \frac{\nabla K(a')}{K(a')^{n+2} \lambda^2} + O\left( \int_{s_0}^{\tau} \frac{\|\partial F(u(s))\|^2}{\lambda} ds + \frac{1}{\lambda} \right).
\]  

(4.12)

Let \( y_1, y_2, \ldots, y_m \) be the critical points of \( K \) in \( \{ K \geq d + \sigma \} \). If \( K(y_i) \leq K(y_j) \) there is no trajectory of the flow associated to \( -\nabla K \) leading from \( y_i \) to \( y_j \) and the same
is true also for some neighborhood of \( y_k \) and \( y_j \). Let \( \rho > 0 \) be such that whenever \( K(y_i) \leq K(y_j) \) there is no trajectory of the flow associated to \(-\nabla K\) leading from some point in \( B(y_i, 2\rho) \) to some point in \( B(y_j, 2\rho) \). Formula (4.12) implies that

\[
\frac{1}{2} \frac{d}{d\tau} K(a'_{1})^{2} = -d_{2}(1 + o(1)) \left( \frac{\nabla K(a')}{K(a')^{2+\varepsilon}} \right) + O \left( \int_{s_{0}}^{0(t)} \frac{\| \partial F(u(s)) \|^{2}}{\lambda} \, ds + \frac{1}{\lambda} \right).
\]

Since \( \lambda \to \infty \) when \( \tau \to \infty \) and \( K \) is bounded, the previous estimates imply that \( a' \) cannot stay definitively outside of \( \cup_{i=1}^{m} B(y_{i}, \rho) \), if \( s_{0} \) is large enough. Therefore \( a' \) has to enter some of the \( B(y_{i}, \rho) \).

We claim that, for \( \tau \) large enough, if \( a' \) leaves \( B(y_{i}, \rho) \) and enters \( B(y_{k}, \rho) \) then \( K(y_{k}) > K(y_{i}) \). This is true for a trajectory of the flow associated to \(-\nabla K\) and therefore, by continuity, also for \( a' \), when \( s_{0}, \lambda \) and \( \tau \) are large enough.

Due to this \( a' \) can leave one of the \( B(y_{i}, \rho) \) to reenter another one just a finite number of times and \( a' \), after a certain time, rests in one of these neighborhoods, let us say \( B(y_{i_{0}}, \rho) \).

Choosing \( s_{0} \) large enough, the same is true for \( a \). It remains only to prove that \( \Delta K(y_{i_{0}}) \leq 0 \). Arguing by contradiction let us suppose that \( \Delta K(y_{i_{0}}) > 0 \). When \( a(t) \) is very close to \( y_{i_{0}} \), (4.7) can be rewritten as

\[
\frac{1}{\lambda} \frac{d\lambda}{dt} = -d_{1}(1 + o(1)) \frac{\Delta K(y_{i_{0}})}{K(y_{i_{0}})^{2+\varepsilon}} + O \left( \| \partial F(u(t)) \|^{2} \right).
\]

Therefore if \( \Delta K(y_{i_{0}}) > 0 \) then \( \lambda(t) \) stays bounded, which is a contradiction.

**Proof of Lemma 1.4'.** Let \( r_{1}, r_{2}, \ldots, r_{m} \) be the critical points of \( K(r) \) in \( \{ r \in [-1, 1] : K(r) \geq d \} \). Let \( U_{i}(\rho) \) be a \( \rho \) neighborhood of \( \{ x \in S^{n} : x_{n+1} = r_{i} \} \). Let us now choose \( \rho > 0 \) such that there is no trajectory of the flow associated to \(-\nabla K\) leading from some point in \( U_{i}(2\rho) \) to some point in \( U_{j}(2\rho) \) whenever \( K(r_{i}) \leq K(r_{j}) \).

The proof works as the previous one, with this new definition of the neighborhoods of the critical points.

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