Advances in Differential Equations Volume 1, Number 6, November 1996, pp. 1099 – 1122

POSITIVE STEADY STATES FOR PREY-PREDATOR MODELS WITH CROSS-DIFFUSION

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(Submitted by: Herbert Amann)

Abstract. This paper is concerned with the existence of positive solutions for boundary value problems of nonlinear elliptic systems which arise in the study of the Lotka-Volterra prey-predator model with cross-diffusion. Making use of the theory of the fixed point index we can derive sufficient conditions for the coexistence of positive steady states. Moreover, when cross-diffusion effects are comparatively small, we can get a necessary and sufficient condition for the coexistence. The uniqueness result is also given in the special case when the spatial dimension is one.

1. Introduction. Let Ω be a bounded domain in R^N with smooth boundary $\partial\Omega$. We consider the Lotka-Volterra prey-predator model with cross-diffusion effects,

$$\begin{cases} u_t = \Delta[(1+\alpha v)u] + au(1-u-cv), & \text{in } \Omega\times(0,\infty), \\ v_t = \Delta[(1+\beta u)v] + bv(1+du-v), & \text{in } \Omega\times(0,\infty), \\ u = v = 0, & \text{on } \partial\Omega\times(0,\infty), \\ u(\cdot,0) = u_0, \qquad v(\cdot,0) = v_0, & \text{in } \Omega, \end{cases} \tag{P}$$

where α, β are nonnegative constants, a, b, c, d are positive constants and u_0, v_0 are given nonnegative functions. In (P), u and v represent the population densities of prey and predator species which are interacting and migrating in the same habitat Ω. Such a density-dependent population model was first introduced by Shigesada, Kawasaki and Teramoto [24] to investigate the habitat segregation phenomena. In our model, $1+\alpha v$ or $1+\beta u$ represents the degree of the uncomfortableness of each place where the individuals are living. Each species diffuses in Ω obeying, in addition to a random movement, a repulsive force due to the population pressure by other species. The boundary condition means that the habitat Ω is surrounded by a hostile environment. For the biological background, we refer the reader to the book of Okubo [19].

We will study nonnegative steady states for (P). The stationary problem associated with (P) is

$$\begin{cases} \Delta[(1+\alpha v)u] + au(1-u-cv) = 0, & \text{in } \Omega, \\ \Delta[(1+\beta u)v] + bv(1+du-v) = 0, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \\ u \ge 0, \quad v \ge 0, & \text{in } \Omega. \end{cases} \tag{SP}$$

Received for publication December 1995.
*Partially supported by Waseda University Grant for Special Research Projects 95A-129.
AMS Subject Classifications: 35J65, 47H15, 92D25.

Especially, we are interested in positive solutions for (SP). It is said that (u, v) is a positive solution of (SP) if $u > 0$ and $v > 0$ in Ω. The main purpose of the present paper is to derive some conditions for the existence, nonexistence and uniqueness properties of positive solutions for (SP). For the one-dimensional competition model with Neumann boundary conditions, see Kan-on [12] and Mimura [18].

When there are no cross-diffusion effects ($\alpha = \beta = 0$), the coexistence problem for (SP) has been discussed by many authors (e.g., [3], [7], [8], [9], [13], [14], [15], [16], [17], [20], [27]). Especially, we have a necessary and sufficient condition for the existence of a positive solution of (SP) (see, e.g., Li [13, Theorem 1.A] or López-Gómez and Pardo [16, Theorem 3.1]). So it is possible to determine completely the coexistence region in a parameter space (a, b)(see [16]). Several typical tools for the study of (SP) are super- and sub-solution techniques (Pao [20]), bifurcation theory (Blat and Brown [3], López-Gómez [15], Yamada [27]), degree theory (Dancer [7], [8], Li [13], [14]), and their combination (López-Gómez and Pardo [16]). Among them, topological fixed point theorems are very powerful to obtain global results of positive solutions of nonlinear elliptic equations (see Amann [1]). Along this idea, Dancer [6], [7] has successfully developed the theory of the fixed point index in positive cones to investigate positive solutions of semilinear elliptic systems.

When there are cross-diffusion effects, we introduce two unknown functions U and V by

$$U = (1 + \alpha v)u \quad \text{and} \quad V = (1 + \beta u)v.$$

Since there is a one-to-one correspondence between nonnegative (u, v) and nonnegative (U, V), (SP) is rewritten in the equivalent form

$$\begin{cases} \Delta U + aU(1 - u - cv)/(1 + \alpha v) = 0, & \text{in } \Omega. \\ \Delta V + bV(1 + du - v)/(1 + \beta u) = 0, & \text{in } \Omega. \\ U = V = 0, & \text{on } \partial\Omega, \\ U \geq 0, \quad V \geq 0, & \text{in } \Omega, \end{cases} \tag{EP}$$

where we understand that u and v are smooth functions of U and V. The strategy to studying positive solutions of (EP) is essentially the same as [7] and [13]. By the reduction of (EP) to a suitable fixed point equation, we will employ the theory of fixed point index in positive cones as in Dancer [6], [7] or Li [13].

We will prepare an index formula (Proposition 2) which is a slight improvement of Dancer's formula [6] and is suitable for the application to (EP). Making use of the index formula we can get sufficient conditions for the existence of a positive solution of (SP) (or equivalently (EP)). These conditions are exhibited in a parameter space (a, b) with use of two monotone curves S_1 and S_2. It is proved that (SP) possesses a coexistence state if (a, b) lies in a region surrounded by S_1 and S_2 (see Figure 3). Generally, it is difficult to decide the nonexistence region of positive solutions of (SP). This is due to the presence of cross-diffusion effects. Indeed, when cross-diffusion effects are comparatively small to the interaction ($\alpha \leq c$ and $\beta \leq d$), we can obtain a necessary and sufficient condition for the coexistence of positive steady states; so that we get an optimal coexistence region in (a, b) space. However, the coexistence region surrounded by S_1 and S_2 curves becomes similar to the Lotka-Volterra competition system with linear diffusion and loses its optimality accordingly as β becomes large(see [11], [16] and Figure 3).

Concerning the nonstationary problem (P), we refer the reader to Amann [2] where the local solvability is discussed for a wide class of quasilinear parabolic systems including (P). However, very little is known about the global solvability of (P). For related problems with cross-diffusion terms, see Deuring [10], Pozio and Tesei [21], Redlinger[23], Yagi [25], [26] and Yamada [28].

Finally it should be noted that our existence results of positive steady states are valid for more general systems which allow a priori estimates. See Remark 6, by which we see that a competition system of Lotka-Volterra type possesses such properties.

The contents of this paper are as follows. In §2 we will give main results on positive solutions of (SP); Theorem 1 (existence), Theorem 2 (nonexistence), Theorem 3 (necessary and sufficient conditions) and Theorem 4 (uniqueness). In §3 we will give some preliminary results on the theory of fixed point index in positive cones and derive a priori estimates of solutions for (SP) or (EP). In particular, our index formula (Proposition 2) plays a very important role in the analysis. §4 is devoted to the proof of Theorem 1. By degree theory we will show some conditions which assure coexistence results for (EP). Nonexistence results of positive solutions are given in §5. Using the existence result in §4 we can derive a necessary and sufficient condition for the coexistence of positive steady states in a special situation where cross-diffusion effects are small compared with the interspecies interaction. In §6, the uniqueness of positive solutions is discussed when the spatial dimension is one.

2. Main results. Before stating main results we need some preliminary results. For each $q \in C(\bar{\Omega})$, let $\lambda_1(q)$ be the principal eigenvalue of

$$\begin{cases} -\Delta u + q(x)u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

As is well known, a variational characterization gives

$$\lambda_1(q) = \inf_{u \in H_0^1, \|u\|=1} \{\|\nabla u\|^2 + \int_\Omega q(x)u^2 dx\},$$

where $\|\cdot\|$ denotes $L^2(\Omega)-$norm. Observe that

$$\lambda_1(q_1) > \lambda_1(q_2) \qquad \text{if} \qquad q_1 \geq q_2 \ (q_1 \neq q_2). \tag{2.1}$$

For $q \equiv 0$, we simply write $\lambda_1(0) = \lambda_1$.

We need some results on the semilinear elliptic problem

$$\begin{cases} \Delta w + aw(1-w) = 0, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

It is well known that, if $a \le \lambda_1$, then $w \equiv 0$ is the only nonnegative solution of (2.2) and that, if $a > \lambda_1$, then (2.2) has a unique positive solution ϕ_a (see, e.g., Dancer [7] or Li [13]). This solution ϕ_a is strictly monotone increasing with respect to a:

$$\phi_a > \phi_b > 0 \qquad \text{in } \Omega \qquad \text{if} \qquad a > b > \lambda_1 \tag{2.3}$$

(see Blat-Brown [3] or Yamada [27]).

We will consider (SP). Since we are concerned with nonnegative solutions, it is convenient to introduce

$$U = (1 + \alpha v)u \qquad \text{and} \qquad V = (1 + \beta u)v \tag{2.4}$$

as in §1. There is a one-to-one correspondence between $(u, v) \ge 0$ and $(U, V) \ge 0$. One can also describe their relations by

$$u = u(U, V) = \frac{1}{2\beta}[\{(1 - \beta U + \alpha V)^2 + 4\beta U\}^{\frac{1}{2}} + \beta U - \alpha V - 1], \tag{2.5}$$

$$v = v(U, V) = \frac{1}{2\alpha}[\{(1 - \alpha V + \beta U)^2 + 4\alpha V\}^{\frac{1}{2}} + \alpha V - \beta U - 1]. \tag{2.6}$$

Using (2.5) and (2.6) we can rewrite (SP) in the form of (EP), where $u = u(U, V)$ and $v = v(U, V)$ are understood to be functions of (U, V) defined by (2.5) and (2.6). By virtue of the one-to-one correspondence between (u, v) and (U, V), we get the equivalence between a quasilinear system (SP) and a semilinear system (EP).

In what follows, we will concentrate ourselves on the study of (EP) in place of (SP). Making use of the existence result for (2.2), it is possible to show that (EP) has two semitrivial solutions

$$(\phi_a, 0) \quad \text{for} \quad a > \lambda_1 \qquad \text{and} \qquad (0, \phi_b) \quad \text{for} \quad b > \lambda_1$$

in addition to the trivial solution $(0, 0)$. Moreover, it will be proved by the energy method that (EP) has no positive solutions for $a \le \lambda_1$ (see Lemma 1). Since our interest lies in positive solutions of (EP), we assume $a > \lambda_1$ in most parts of the subsequent sections.

The existence is given by the following theorem.

Theorem 1. *For $a > \lambda_1$, there exists a positive solution for* (SP) *or, equivalently,* (EP) *if one of the following conditions is satisfied*:

$$\lambda_1\left(\frac{a(c\phi_b - 1)}{1 + \alpha\phi_b}\right) < 0 \quad \textit{and} \quad \lambda_1\left(\frac{-b(d\phi_a + 1)}{1 + \beta\phi_a}\right) < 0, \tag{2.7}$$

$$\lambda_1\left(\frac{a(c\phi_b - 1)}{1 + \alpha\phi_b}\right) > 0 \quad \textit{and} \quad \lambda_1\left(\frac{-b(d\phi_a + 1)}{1 + \beta\phi_a}\right) > 0. \tag{2.8}$$

We will state the meaning of Theorem 1. Regarding a and b as parameters we define two curves S_1 and S_2 in a parameter space:

$$S_1 := \{(a, b) \in R^2 : \lambda_1\left(\frac{-b(d\phi_a + 1)}{\beta\phi_a + 1}\right) = 0 \quad \text{for } a \ge \lambda_1\}, \tag{2.9}$$

$$S_2 := \{(a,b) \in R^2 : \lambda_1(\frac{a(c\phi_b - 1)}{\alpha\phi_b + 1}) = 0 \quad \text{for } b \geq \lambda_1\}. \tag{2.10}$$

Lemma A.1 in the Appendix implies that, if $d > \beta$ (resp. $d < \beta$), then S_1 is a monotone decreasing (resp. increasing) curve starting from (λ_1, λ_1) (see Figure 1). Lemma A.2 asserts that S_2 is a monotone increasing curve which starts from (λ_1, λ_1) (see Figure 2). In Theorem 1, the first inequality of (2.7) means that (a,b) lies in the right-hand side of S_2 or below S_2. The second inequality of (2.7) means that (a,b) lies above S_1. On the other hand, the first condition of (2.8) means that (a,b) lies to the left of S_2 and the second one means that (a,b) lies below S_1. Note that (2.8) never holds when (a,b) satisfies $a > \lambda_1$ and $b \leq \lambda_1$. Combining these results we can conclude from Theorem 1 that, if (a,b) lies in a region surrounded by S_1 and S_2 as in Figure 3, then (SP) has a positive solution.

Remark 1. Conditions (2.7) and (2.8) are related with the linearized stability of $(\phi_a, 0)$ and $(0, \phi_b)$ as steady states for (P). The first (resp. second) condition of (2.7) means that $(0, \phi_b)$ (resp. $(\phi_a, 0)$) is unstable, whereas the first (resp. second) condition of (2.8) gives the stability of $(0, \phi_b)$ (resp. $(\phi_a, 0)$). See also Remark 7.

Remark 2. From the view point of the bifurcation theory, we can see that positive solutions bifurcate from $(\phi_a, 0)$ when (a,b) crosses S_1 curve (see Crandall and Rabinowitz [5]). Similarly, positive solutions also bifurcate from $(0, \phi_b)$ when (a,b) moves across S_2. See also Remark 8. Moreover, Lemma A.3 tells us that a part of S_2 lies above (resp. below) S_1 if

$$\beta - d < \frac{1}{\alpha + c} \qquad (\text{resp.} \quad \beta - d > \frac{1}{\alpha + c}).$$

Although Theorem 1 asserts the existence of positive solutions for (a,b) lying in the shaded region in Figure 3, it does not give us any information concerning whether (SP) has a positive solution for (a,b) outside this region or not. A careful study of the direction of bifurcations will be required in a neighborhood of S_1 or S_2. See also [16].

We will study nonexistence properties of positive solutions to derive an optimal coexistence region. Generally this problem is difficult because we can not get nice a priori estimates for U and V. Our nonexistence results are limited to the case where cross-diffusion effects are small in the sense

$$\alpha \leq c \qquad \text{and} \qquad \beta \leq d. \tag{2.11}$$

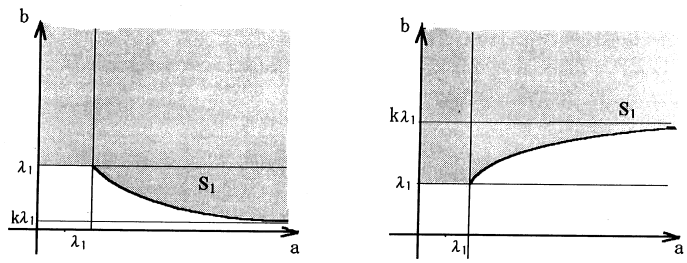
If (2.11) holds, we will show in Lemma 3 that any positive solution (U, V) of (EP) satisfies

$$\phi_a \geq U \geq u \qquad \text{and} \qquad V \geq \phi_b \qquad \text{in} \quad \Omega.$$

These a priori estimates lead us to nonexistence results. We assume that there exists a function $\psi \in C(\bar{\Omega})$ which satisfies

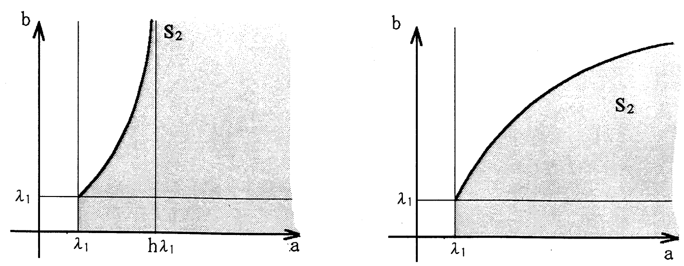
$$\frac{1 - u - cv}{1 + \alpha v} < \frac{1 - c\psi}{1 + \alpha\psi} \qquad \text{in} \quad \Omega \tag{2.12}$$

for all positive solutions (u, v) of (SP).



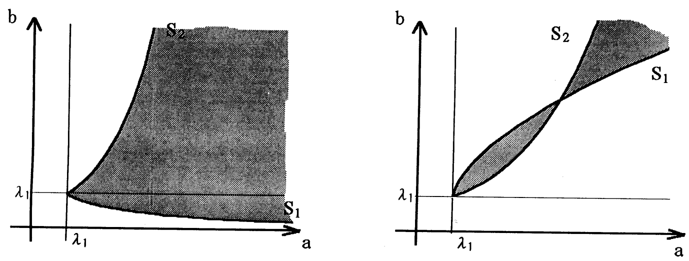
A. Case $d > \beta$ B. Case $d < \beta$

Figure 1. Profiles of S_1 curves with $k = (1+\beta)/(1+d)$. The shaded part indicates the region of (a, b) where $\lambda_1(-b(d\phi_a + 1)/(1 + \beta\phi_a)) < 0$.



A. Case $c < 1$ B. Case $c > 1$

Figure 2. Profiles of S_2 curves with $h = (1+\alpha)/(1-c)$. The shaded part indicates the region of (a, b) where $\lambda_1(a(c\phi_b - 1)/(1 + \alpha\phi_b)) < 0$.



A. Case $(\beta - d)(\alpha + c) > 1$ B. Case $(\beta - d)(\alpha + c) < 1$

Figure 3. Coexistence region described in Theorem 1.

Theorem 2. *Let α and β satisfy* (2.11). *Suppose that*

$$\lambda_1\left(\frac{a(c\psi - 1)}{1 + \alpha\psi}\right) \geq 0 \qquad \textit{for} \qquad b \geq \lambda_1, \tag{2.13}$$

where ψ *is a function satisfying* (2.12)*, or*

$$\lambda_1\Big(\frac{-b(d\phi_a+1)}{1+\beta\phi_a}\Big)\geq 0 \qquad \textit{for} \qquad b<\lambda_1. \tag{2.14}$$

Then (SP) *does not admit a positive solution.*

Condition (2.12) is rather implicit in Theorem 2. In the special case where $1+(\alpha-\alpha\beta-c\beta)\phi_b\geq 0$ in Ω, we can choose $\psi=\phi_b$ as a function satisfying (2.12). An immediate consequence of Theorem 2 is the following.

Corollary 1. *In addition to* (2.11)*, let* α *and* β *satisfy* $1+(\alpha-\alpha\beta-c\beta)\phi_b\geq 0$ *in* Ω*. Then* (SP) *does not admit a positive solution if either* (2.14) *or*

$$\lambda_1\Big(\frac{a(c\phi_b-1)}{1+\alpha\phi_b}\Big)\geq 0 \qquad \textit{for} \qquad b\geq\lambda_1 \tag{2.15}$$

holds true.

Combining Theorem 1 and Corollary 1, we obtain the following necessary and sufficient condition for the existence of positive solutions of (SP).

Theorem 3. *Assume that* $c\geq\alpha, d\geq\beta$ *and* $1+(\alpha-\alpha\beta-c\beta)\phi_b\geq 0$ *in* Ω*. Then there exists a positive solution for* (SP) *if and only if* (2.7) *is satisfied.*

Theorem 3 says that a positive solution of (SP) exists if and only if (a,b) lies in a region surrounded by S_1 and S_2 as in Figure 3A. In this sense we can get an optimal coexistence region when α and β are comparatively small.

Remark 3. When there are no cross-diffusion effects ($\alpha=\beta=0$), a necessary and sufficient condition is well known for the existence of positive solutions to (SP) (see, e.g., [8], [13] and [16]). This condition is a special case of Theorem 3.

It is a very delicate problem to determine the number of positive solutions of (SP). We can give a uniqueness result in the special case when the spatial dimension is one.

Theorem 4. *Let* $N=1$*. If* $c\geq\alpha, d\geq\beta$ *and* $\alpha-\alpha\beta-c\beta\geq 0$*, then a positive solution for* (SP) *is uniquely determined. Therefore,* (SP) *has a unique positive solution if and only if* (2.7) *is satisfied.*

Remark 4. Making use of the local bifurcation theory ([5]), we can study the direction of bifurcations of positive solutions of (SP) in parameter space (a,b) (cf. Remark 8). When α or β is large, an optimal coexistence region may be larger than the region surrounded by S_1 and S_2. In such a situation we can expect multiple coexistence states for (SP). Multiple coexistence is discussed for the Lotka-Volterra competition model with linear diffusion (see, e.g., [4] and [11]). We will discuss this subject elsewhere.

3. Preliminaries.

In this section we will collect some preliminary results which are required in the subsequent sections.

3.1. Principal eigenvalue and spectral radius.

For every $f\in C(\bar\Omega)$, let $(-\Delta+pI)^{-1}f$ denote a unique solution of

$$\begin{cases} -\Delta u+pu=f, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega. \end{cases}$$

We denote the spectral radius of an operator T in a Banach space by $r(T)$. The following proposition characterizes the principal eigenvalue of $-\Delta + q(x)$ with zero Dirichlet boundary condition in terms of the spectral radius of a suitable operator in the Banach space $C(\bar{\Omega})$.

Proposition 1. *Let $q \in C(\bar{\Omega})$ and let p be a sufficiently large number such that $p > q(x)$ for every $x \in \Omega$; then*

(i) $\lambda_1(q) > 0$ *if and only if* $r((-\Delta + pI)^{-1}(p - q(x))) < 1$,
(ii) $\lambda_1(q) < 0$ *if and only if* $r((-\Delta + pI)^{-1}(p - q(x))) > 1$,
(iii) $\lambda_1(q) = 0$ *if and only if* $r((-\Delta + pI)^{-1}(p - q(x))) = 1$.

Proof. For the proof, see Dancer [7, Proposition 1] (see also Li [13, Lemmas 2.1 and 2.3]). □

Let $a > \lambda_1$. Since ϕ_a is a positive solution of (2.2), we see from the Krein-Rutman theorem that zero is the principal eigenvalue of $-\Delta + a(\phi_a - 1)$ with zero Dirichlet condition; that is $\lambda_1(a(\phi_a - 1)) = 0$. Therefore, it follows from (2.1) that

$$\lambda_1(a(2\phi_a - 1)) > 0.$$

If we set

$$T = (-\Delta + pI)^{-1}\{p + a(1 - 2\phi_a)\} \qquad \text{for any} \quad a > \lambda_1,$$

we see from Proposition 1 that

$$r(T) < 1. \tag{3.1}$$

3.2. Index theory. We will summarize the theory of fixed point index in positive cones. The theory has been developed by Dancer [6] to study positive solutions for a certain class of nonlinear elliptic systems.

In this paper we set $E := C_0(\overline{\Omega}) \oplus C_0(\overline{\Omega})$, where $C_0(\overline{\Omega})$ denotes the space of continuous functions $u : \overline{\Omega} \to R$ such that $u|_{\partial\Omega} = 0$. Define a positive cone $W \subset E$ by $W = K \oplus K$ with $K := \{u \in C_0(\overline{\Omega}) : u \geq 0 \text{ in } \Omega\}$.

We use the notation following Dancer [6]. Let $y = (y_1, y_2)$ be an element in W; we define W_y as

$$W_y := \{x \in E : y + \gamma x \in W \quad \text{for some} \quad \gamma > 0\}.$$

Note that $\overline{W_y} = W$ for $y = (0, 0)$, $\overline{W_y} = C_0(\overline{\Omega}) \oplus K$ for $y = (y_1, 0)$ with $y_1 > 0$ and $\overline{W_y} = K \oplus C_0(\overline{\Omega})$ for $y = (0, y_2)$ with $y_2 > 0$. Especially, when both y_1 and y_2 are positive, we see $\overline{W_y} = E$. For the proof, see Dancer [7, Lemma 3].

We set

$$S_y = \overline{W_y} \cap (-\overline{W_y}). \tag{3.2}$$

Now assume that $T : E \longrightarrow E$ is a compact linear operator such that

$$T(\overline{W_y}) \subseteq \overline{W_y}. \tag{3.3}$$

If $u \in S_y$, then $Tu \in \overline{W_y}$ and $-Tu \in \overline{W_y}$ because of (3.2) and (3.3); so that $Tu \in S_y$. This fact implies that T induces a compact linear mapping $\tilde{T}$ of E/S_y into itself. We write an image of $\overline{W}_y$ under the quotient mapping $E \longrightarrow E/S_y$ as $\widetilde{W_y}$. Since $T(\overline{W}_y) \subseteq \overline{W}_y$, it follows that $\tilde{T}(\widetilde{W_y}) \subseteq \widetilde{W_y}$.

We will give a very important index formula which is essentially due to Dancer [6]. Let $A : W \longrightarrow W$ be compact and Fréchet differentiable. We denote the Fréchet derivative of A at $x \in W$ by $A'(x)$. We also assume that y is a fixed point of A and that $A'(x)$ is compact. By Lemma 1 in [6, §2], $A'(y)$ maps $\overline{W_y}$ into itself. Although the proof of Dancer's lemma is not easy, this result is an immediate consequence of the maximum principle in our application.

Combining Theorem 1 in [6, §2] and Lemma 2 in [6, §1], we can show the following proposition which is slightly different from Dancer's original result.

Proposition 2. *If* $(I - A'(y))x \neq 0$ *for every* $x \in \overline{W_y} \setminus \{0\}$*, then*

(i) $$\text{index}_W(A, y) = 0 \quad \text{if} \quad r(\widetilde{A}'(y)) > 1,$$

(ii) $$\text{index}_W(A, y) = 1 \quad \text{if} \quad r(A'(y)) < 1.$$

Remark 5. In [6] Dancer has assumed the invertibility of $A'(y)$ on the whole space E. When we apply his theory to systems such as prey-predator models and competition models, his assumption (the invertibility of $I - A'(y)$ in E) does not hold under certain circumstances. Li [13], [14] has made some changes and discussed by assuming that $(I - A'(y))x \neq 0$ for every $x \in \overline{W_y}$ and that $I - A'(y) : \overline{W_y} \longrightarrow \overline{W_y} \setminus \{0\}$ is not a surjective map.

However we find that the assumption $(I - A'(y))x \neq 0$ for every $x \in \overline{W_y} \setminus \{0\}$ is sufficient for Proposition 2 because Dancer's arguments in [6] remain valid except for the proof of (d) of Lemma 2 in [6, §1] and estimate $\|x - Tx\| \geq K\|x\|$ for $x \in \overline{W_y} \setminus \{0\}$, where $T = A'(y)$(see [6, (1)]).

The proof of (d) of Dancer's Lemma 2 is completed in the following manner. By making use of his notation, let $t \in (0, 1)$ and $w \in \overline{W_y} \setminus S_y$ satisfy $m := w - tTw \in S_y$. Suppose that there exists $x \in \overline{W_y} \setminus \{0\}$ satisfying $x - Tx = -m + (1-t)Tw$. According to Dancer's proof, there is $-w \notin \overline{W_y}$ such that $(-w) - T(-w) = -m + (1-t)Tw$. Hence we have $(x + w) - T(x + w) = 0$ with $x + w \in \overline{W_y}$. It follows from our assumption that $x = -w$. This is a contradiction.

We can show the latter inequality by contradiction. Let $\{x_n\}_{n=1}^{\infty} \subset \overline{W_y}$ be a sequence such that $x_n - Tx_n \longrightarrow 0$ as $n \to \infty$ and $\|x_n\| = 1$. Since T is compact and T maps $\overline{W_y}$ into itself, we can take a subsequence such that $Tx_{n_k} \longrightarrow v \in \overline{W_y} \setminus \{0\}$. Note that $\{x_{n_k}\}$ also converges to v. So we have $Tv = v$ with $v \in \overline{W_y} \setminus \{0\}$, which is a contradiction.

3.3. A priori estimates. We will derive some a priori estimates of positive solutions of (EP). The first lemma is concerned with the nonexistence when a is small.

Lemma 1. *If* $a \leq \lambda_1$*, then* (EP) *has no nontrivial solutions. In particular,* $(0, 0)$ *is the unique nonnegative solution of* (EP) *for* $a \leq \lambda_1$ *and* $b \leq \lambda_1$.

Proof. Let (U, V) be a positive solution of (EP). Taking $L^2(\Omega)-$inner product of the first equation of (EP) with U we get

$$\| \nabla U \|^2 = a \int_\Omega \frac{U^2}{1 + \alpha v}(1 - u - cv)dx < a \| U \|^2 .$$

Since $\| \nabla U \|^2 \geq \lambda_1 \| U \|^2$ by Poincaré's inequality, it is easy to see that $U \equiv 0$ if $a \leq \lambda_1$. □

We will give a priori estimates in the case $a > \lambda_1$.

Lemma 2. *Let* (U, V) *be a positive solution of* (EP). *Then*

$$0 \leq u(x) \leq U(x) \leq M_1 := \begin{cases} 1 & \textit{if } \alpha \leq c, \\ (c+\alpha)^2/4c\alpha & \textit{if } \alpha > c, \end{cases}$$
$$0 \leq v(x) \leq V(x) \leq M_2 := (1 + \beta M_1)(1 + dM_1),$$

for all $x \in \Omega$.

Proof. Assume $\| U \|_\infty = U(x_0) > 0$ for some $x_0 \in \Omega$. It follows from (EP) that

$$0 \leq -\Delta U(x_0) = au(x_0)(1 - u(x_0) - cv(x_0));$$

so that $1 - u(x_0) - cv(x_0) \geq 0$. Therefore,

$$u(x_0) \leq 1 \qquad \text{and} \qquad v(x_0) \leq \frac{1 - u(x_0)}{c}.$$

These estimates yield

$$\| U \|_\infty = (1 + \alpha v(x_0))u(x_0) \leq \frac{1}{c} u(x_0)(c + \alpha - \alpha u(x_0)).$$

The right-hand side is regarded as a function of $X := u(x_0)$. Taking its maximum for $0 \leq X \leq 1$, we can obtain an estimate for $\| U \|_\infty$. A bound for $\| V \|_\infty$ can be derived in the same way.

Remark 6. The above arguments are valid for the cross-diffusion system of more general type such that

$$\begin{cases} \Delta[(1 + \alpha v)u] + f(u, v) = 0, & \text{in } \Omega, \\ \Delta[(1 + \beta u)v] + g(u, v) = 0, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \\ u \geq 0, \quad v \geq 0, & \text{in } \Omega. \end{cases} \tag{3.4}$$

We will give conditions on f and g which enable us to get an a priori estimate. Using (2.4), we rewrite (3.4) as

$$\begin{cases} \Delta U + f(\dfrac{U}{1 + \alpha v}, v) = 0, & \text{in } \Omega, \\ \Delta V + g(u, \dfrac{V}{1 + \beta u}) = 0, & \text{in } \Omega, \\ U = V = 0, & \text{on } \partial\Omega, \\ U \geq 0, \quad V \geq 0, & \text{in } \Omega. \end{cases}$$

We assume the following conditions.

(A.i) There is a positive number U_0 such that, if $U \in (U_0, \infty)$ and $v \in [0, \infty)$, then $f(U/(1 + \alpha v), v) < 0$.

(A.ii) There is a positive number V_0 such that, if $V \in (V_0, \infty)$ and $u \in [0, U_0]$, then $g(u, V/(1 + \beta u)) < 0$.

Then any positive solution of (3.4) satisfies

$$0 \le u \le U \le U_0 \quad \text{and} \quad 0 \le v \le V \le V_0 \quad \text{in} \quad \Omega. \tag{3.5}$$

To show (3.5), assume $\|U\|_\infty = U(x_0) > 0$ for some $x_0 \in \Omega$. By the same argument as in the proof of Lemma 2, $f(U(x_0)/(1+\alpha v(x_0)), v(x_0)) \ge 0$. So it follows from (A.i) that $U(x_0) \le U_0$.

Note that in the case $f(u,v) = au(1-u-cv)$ and $g(u,v) = bv(1-v+du)$ we can take $U_0 = M_1$, and $V_0 = M_2$ as in Lemma 2.

The following lemma gives another a priori estimate in the special case when α and β are small.

Lemma 3. *Let (U,V) be a positive solution of* (EP). *If $c \ge \alpha$, then*

$$\phi_a \ge U \ge u \quad \text{in} \quad \Omega. \tag{3.6}$$

If $d \ge \beta$, then

$$V \ge \phi_b \quad \text{in} \quad \Omega. \tag{3.7}$$

Proof. Let (U,V) be a positive solution of (EP). Then Lemma 2 implies $U \le 1$ in case $\alpha \le c$. So

$$\begin{aligned} \Delta U + aU(1-U) &= \frac{aU}{1+\alpha v}\{(1+\alpha v)(1-U) - 1 + u + cv\} \\ &= au\{(\alpha + c)v - \frac{2\alpha v + \alpha^2 v^2}{1+\alpha v}U\} \\ &\ge auv(c + \alpha - \frac{2\alpha + \alpha^2 v}{1+\alpha v}) = \frac{auv}{1+\alpha v}(c - \alpha + c\alpha v) \ge 0. \end{aligned}$$

This fact implies that U is a subsolution of (2.2). Since (2.2) has a unique positive solution ϕ_a, the standard comparison method yields (3.6). Similarly, one can show that for $d \ge \beta$,

$$\Delta V + bV(1-V) = buv\{\beta - d - (2\beta + \beta^2 u)v\} \le 0,$$

which implies that V is a supersolution of (2.2) with a replaced by b. So (3.7) is shown by the comparison method.

4. Existence of positive solutions. In this section we will give the proof of Theorem 1. Choose a sufficiently large p such that

$$p + \frac{a}{1+\alpha v}(1-u-cv) \ge 0 \quad \text{and} \quad p + \frac{b}{1+\beta u}(1+du-v) \ge 0$$

for $0 \le u \le M_1 + 1$ and $0 \le v \le M_2 + 1$, where M_1, M_2 are positive constants in Lemma 2. Recall that M_1 and M_2 are independent of a and b. Define a mapping A in $E = C_0(\overline{\Omega}) \oplus C_0(\overline{\Omega})$ by

$$\begin{aligned} &A(U,V) \\ &:= (-\Delta + pI)^{-1}(U\{p + \frac{a}{1+\alpha v}(1-u-cv)\}, V\{p + \frac{b}{1+\beta u}(1+du-v)\}) \\ &= (-\Delta + pI)^{-1}(pU + f(u,v), pV + g(u,v)), \end{aligned} \tag{4.1}$$

where u, v are functions of U, V (see (2.5), (2.6)) and

$$f(u,v) = au(1-u-bv), \qquad g(u,v) = bv(1+du-v).$$

Clearly, (U,V) is a solution of (EP) if and only if it is a fixed point of A in W, where $W = K \oplus K$ with $K = \{u \in C_0(\overline{\Omega}) : u \geq 0 \text{ in } \Omega\}$. Taking account of Lemma 2 we set

$$D := \{(U,V) \in W : U \leq M_1 + 1 \text{ and } V \leq M_2 + 1 \quad \text{in} \ \ \Omega\};$$

then we can see from Lemma 2 that all nonnegative solutions of (EP) lie in the interior of D ($=$ int D) with respect to W. By (4.1) and the maximum principle, A maps D into W. Moreover, the regularity theory of elliptic equations tells us that A is completely continuous in E.

It is convenient to use degrees relative to W. By Lemma 2 there are no fixed points of A on the boundary of D (with respect to W). Hence it follows from the homotopy invariance of degree that $\deg_W(I - A, \text{int } D)$ is independent of a and b (see Amann [1, Theorem 11.1]).

We will complete the proof of Theorem 1 by using three lemmas. Roughly speaking, our strategy of proof is as follows. In Lemma 4 we calculate $\deg_W(I - A, \text{int } D)$. Clearly, $(0,0), (\phi_a, 0)$ and $(0, \phi_b)$ are fixed points of A in D. Lemma 5 is devoted to the calculation of their indices. In Lemma 6, we combine Lemmas 4 and 5 to conclude the existence of positive solutions by contradiction.

To calculate the index of each fixed point of A, we use Proposition 2. First of all, we will give here the expression of the Fréchet derivative of A. From (4.1) we have

$$A'(U,V)\begin{pmatrix}\hat{U}\\ \hat{V}\end{pmatrix} = (-\Delta + pI)^{-1}\left[p + \begin{pmatrix} f_u & f_v \\ g_u & g_v\end{pmatrix}\begin{pmatrix} u_U & u_V \\ v_U & v_V\end{pmatrix}\right]\begin{pmatrix}\hat{U}\\ \hat{V}\end{pmatrix}. \tag{4.2}$$

Differentiation of (2.4) yields

$$\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} = \begin{pmatrix} 1+\alpha v & \alpha u \\ \beta v & 1+\beta u\end{pmatrix}\begin{pmatrix} u_U & u_V \\ v_U & v_V\end{pmatrix}.$$

Since u, v are both nonnegative, we have

$$\begin{pmatrix} u_U & u_V \\ v_U & v_V\end{pmatrix} = \frac{1}{1+\alpha v + \beta u}\begin{pmatrix} 1+\beta u & -\alpha u \\ -\beta v & 1+\alpha v\end{pmatrix}. \tag{4.3}$$

Thus it follows from (4.2) and (4.3) that

$$\begin{aligned} A'(U,V)\begin{pmatrix}\hat{U}\\ \hat{V}\end{pmatrix} &= (-\Delta + pI)^{-1} \\ \times &\left[p + \frac{1}{1+\alpha v+\beta u}\begin{pmatrix} a(1-2u-cv) & -acu \\ bdv & b(1-2v+du)\end{pmatrix}\right. \\ \times &\left.\begin{pmatrix} 1+\beta u & -\alpha u \\ -\beta v & 1+\alpha v\end{pmatrix}\right]\begin{pmatrix}\hat{U}\\ \hat{V}\end{pmatrix}. \end{aligned} \tag{4.4}$$

Lemma 4. $\deg_W(I - A, \text{int } D) = 1$.

Proof. The proof relies on Proposition 2. First we will calculate the fixed point index of the trivial solution $(0,0)$. We have to investigate the spectrum of $A'(0,0)$; from (4.4),

$$A'(0,0)(\hat{U}, \hat{V}) = (-\Delta + pI)^{-1}((p+a)\hat{U}, (p+b)\hat{V}).$$

Clearly, $\overline{W_{(0,0)}} = W$ and, therefore, $S_{(0,0)} = \{(0,0)\}$; so that $\widetilde{A}'(0,0)$ is identical with $A'(0,0)$. Observe that

$$r(A'(0,0)) = \max\{\frac{a+p}{\lambda_1 + p}, \frac{b+p}{\lambda_1 + p}\}. \tag{4.5}$$

For $a < \lambda_1$ and $b < \lambda_1$, we can easily show that $A'(0,0)y \neq y$ on W and $r(A'(0,0)) < 1$. Using (ii) of Proposition 2 we get $\text{index}_W(A, (0,0)) = 1$. By virtue of Lemma 1, (EP) has no solutions other than $(0,0)$ for the case $a \leq \lambda_1$ and $b \leq \lambda_1$. So the excision property yields $\deg_W(I - A, \text{ int } D) = 1$ (see [1, Corollary 11.2]).

To complete the proof for the case $a \geq \lambda_1$ or $b \geq \lambda_1$, we have only to use the homotopy invariance of $\deg_W(I - A, \text{int} D)$ with respect to a and b. □

We now recall that (EP) has semitrivial solutions $(\phi_a, 0)$ (in case $a > \lambda_1$) and $(0, \phi_b)$ (in case $b > \lambda_1$). We will study their indices.

Lemma 5. (i) *For* $a > \lambda_1$ *and* $b \neq \lambda_1$, $\text{index}_W(A, (0,0)) = 0$.

(ii) *For* $a > \lambda_1, (\phi_a, 0)$ *satisfies*

$$\begin{cases} \text{index}_W(A, (\phi_a, 0)) = 0 & \text{if } \lambda_1(\frac{-b(d\phi_a + 1)}{1 + \beta\phi_a}) < 0, \\ \text{index}_W(A, (\phi_a, 0)) = 1 & \text{if } \lambda_1(\frac{-b(d\phi_a + 1)}{1 + \beta\phi_a}) > 0. \end{cases}$$

(iii) *For* $a > \lambda_1$ *and* $b > \lambda_1$, $(0, \phi_b)$ *satisfies*

$$\begin{cases} \text{index}_W(A, (0, \phi_b)) = 0 & \text{if } \lambda_1(\frac{a(c\phi_b - 1)}{1 + \alpha\phi_b}) < 0, \\ \text{index}_W(A, (0, \phi_b)) = 1 & \text{if } \lambda_1(\frac{a(c\phi_b - 1)}{1 + \alpha\phi_b}) > 0. \end{cases}$$

Proof. We will apply Proposition 2 to calculate the indices of $(0,0), (\phi_a, 0)$ and $(0, \phi_b)$.

(i) From (4.5) we see that $A'(0,0)y \neq y$ on W and $r(A'(0,0)) > 1$. Since $\widetilde{A}'(0,0) = A'(0,0)$, it follows from Proposition 2 that $\text{index}_W(A, (0,0)) = 0$.

(ii) From (4.4),

$$\begin{aligned} A'(\phi_a, 0)(\hat{U}, \hat{V}) &= (-\Delta + pI)^{-1} \\ &\times \big(\{p + a(1 - 2\phi_a)\}\hat{U} + \frac{a\phi_a\{(2\phi_a - 1)\alpha - c\}}{1 + \beta\phi_a}\hat{V}, \{p + \frac{b(1 + d\phi_a)}{1 + \beta\phi_a}\}\hat{V}\big). \end{aligned}$$

Define T_1 and T_2 by

$$\begin{aligned} T_1 &:= (-\Delta + pI)^{-1}\{p + a(1 - 2\phi_a)\}, \\ T_2 &:= (-\Delta + pI)^{-1}\{p + \frac{b(1 + d\phi_a)}{1 + \beta\phi_a}\}. \end{aligned}$$

Since $\overline{W_y} = C_0(\overline{\Omega}) \oplus K$ for $y = (\phi_a, 0)$, we see $S_y = C_0(\overline{\Omega}) \oplus \{0\}$; so that we can identify $\widetilde{A}'(\phi_a, 0)$ with T_2.

We will show by contradiction that $I - A'(\phi_a, 0) \neq 0$ on $\overline{W_y} \setminus \{(0,0)\}$ if

$$\lambda_1(q^*) \neq 0 \qquad \text{with} \qquad q^* = \frac{-b(d\phi_a + 1)}{1 + \beta\phi_a}.$$

Assume that there exists $(\xi_1, \xi_2) \in C_0(\overline{\Omega}) \oplus K$ which satisfies $A'(\phi_a, 0)(\xi_1, \xi_2) = (\xi_1, \xi_2)$ with $(\xi_1, \xi_2) \neq (0, 0)$. If $\xi_2 = 0$, then $T_1\xi_1 = \xi_1$, which implies

$$r(T_1) \geq 1. \tag{4.6}$$

This result contradicts (3.1). So we must have $T_2\xi_2 = \xi_2$ with $\xi_2 \in K \setminus \{0\}$. By the Krein-Rutman theorem, we get $r(T_2) = 1$, which is a contradiction to $\lambda_1(q^*) \neq 0$ by Proposition 1.

Since we have verified the assumption of Proposition 2, we will apply it to calculate the index of $(\phi_a, 0)$. For the case $\lambda_1(q^*) < 0$, Proposition 1 yields $r(T_2) > 1$. This fact implies $r(\widetilde{A}'(\phi_a, 0)) > 1$, so that it follows from (i) of Proposition 2 that $\text{index}_W(A, (\phi_a, 0)) = 0$.

We next consider the case $\lambda_1(q^*) > 0$, which is equivalent to

$$r(T_2) < 1 \tag{4.7}$$

by Proposition 1. We will show $r(A'(\phi_a, 0)) < 1$. Let ν be any eigenvalue of $A'(\phi_a, 0)$ and let $(\xi_1, \xi_2) \in E = C_0(\overline{\Omega}) \oplus C_0(\overline{\Omega})$ be its corresponding eigenfunction. If $\xi_2 \neq 0$, $T_2\xi_2 = \nu\xi_2$. From (4.7) it follows that $|\nu| < 1$. Whereas, if $\xi_2 = 0$, then $T_1\xi_1 = \nu\xi_1$. Since $r(T_1) < 1$ by (3.1), we see $|\nu| < 1$. Thus we have shown $r(A'(\phi_a, 0)) < 1$. Hence it is sufficient to employ (ii) of Proposition 2 to get $\text{index}_W(A, (\phi_a, 0)) = 1$.

(iii) Observe that (4.4) gives

$$\begin{aligned} A'(0, \phi_d)(\hat{U}, \hat{V}) &= (-\Delta + pI)^{-1} \\ &\times \Big(\{p + \frac{a(1 - c\phi_b)}{1 + \alpha\phi_b}\}\hat{U}, \{p + b(1 - 2\phi_b)\}\hat{V} + \frac{b\phi_b\{(2\phi_b - 1)\beta + d\}}{1 + \alpha\phi_b}\hat{U}\Big). \end{aligned}$$

Moreover, since $\overline{W_y} = K \oplus C_0(\overline{\Omega})$ for $y = (0, \phi_b)$, one can identify $\widetilde{A}'(0, \phi_b)$ with

$$(-\Delta + pI)^{-1}\{p + \frac{a(1 - c\phi_b)}{1 + \alpha\phi_b}\}.$$

Using the same arguments as (ii) we can accomplish the proof.

Remark 7. Lemma 5 together with the proof of Lemma 4 implies that the fixed point index of a trivial or semitrivial steady state is 1 if it is stable and 0 if it is unstable. Here the stability means the linearized stability of each steady state associated with (P). The latter (resp. former) inequality of (ii) implies that $(\phi_a, 0)$ is a stable (resp. unstable) steady state for (P). Similar results holds for $(0, \phi_b)$.

Indeed the linearized stability of $(\phi_a, 0)$ is studied through the eigenvalue problem

$$-\Delta\left[\begin{pmatrix}1 & \alpha\phi_a\\ 0 & 1+\beta\phi_a\end{pmatrix}\begin{pmatrix}\hat u\\ \hat v\end{pmatrix}\right]-\begin{pmatrix}a(1-2\phi_a) & -ac\phi_a\\ 0 & b(1+d\phi_a)\end{pmatrix}\begin{pmatrix}\hat u\\ \hat v\end{pmatrix}=\lambda\begin{pmatrix}\hat u\\ \hat v\end{pmatrix}\quad \text{in } \Omega$$

with $\hat u = \hat v = 0$ on $\partial\Omega$. Setting

$$\begin{pmatrix}\hat U\\ \hat V\end{pmatrix}=\begin{pmatrix}1 & \alpha\phi_a\\ 0 & 1+\beta\phi_a\end{pmatrix}\begin{pmatrix}\hat u\\ \hat v\end{pmatrix},$$

we can rewrite the above system as

$$\begin{aligned}-\Delta\begin{pmatrix}\hat U\\ \hat V\end{pmatrix}-\frac{1}{1+\beta\phi_a}\begin{pmatrix}a(1-2\phi_a) & -ac\phi_a\\ 0 & b(1+d\phi_a)\end{pmatrix}\begin{pmatrix}1+\beta\phi_a & -\alpha\phi_a\\ 0 & 1\end{pmatrix}\begin{pmatrix}\hat U\\ \hat V\end{pmatrix}\\ =\lambda\frac{1}{1+\beta\phi_a}\begin{pmatrix}1+\beta\phi_a & -\alpha\phi_a\\ 0 & 1\end{pmatrix}\begin{pmatrix}\hat U\\ \hat V\end{pmatrix}.\end{aligned}$$

By similar arguments to those used in (ii) of Lemma 5, we can see that the spectrum for the above problem consists of eigenvalues for the two problems

$$-\Delta\hat U - a(1-2\phi_a)\hat U = \lambda\hat U \quad \text{in } \Omega \quad \text{and} \quad \hat U = 0 \quad \text{on} \quad \partial\Omega, \tag{4.8}$$

$$-\Delta\hat V - \frac{b(1+d\phi_a)}{1+\beta\phi_a}\hat V = \lambda\frac{1}{1+\beta\phi_a}\hat V \quad \text{in } \Omega \quad \text{and} \quad \hat V = 0 \quad \text{on} \quad \partial\Omega. \tag{4.9}$$

From §3.1, all the eigenvalues of (4.8) are positive. The principal eigenvalue of (4.9) is positive if and only if (4.7) is true. To show this, we have only to follow the proof of (vi) of Proposition 1 in [7]. Since (4.7) is equivalent to the latter inequality of (ii), we get the stability property of $(\phi_a, 0)$.

Lemma 6. *There exists a positive solution of* (EP) *if either* (2.7) *or* (2.8) *is satisfied.*

Proof. We will complete the proof by contradiction. We begin with the case $b < \lambda_1$; then $\phi_b \equiv 0$. Note that (2.8) never holds for $b < \lambda_1$ because $\lambda_1(a(c\phi_b - 1)/(1+\alpha\phi_b)) = \lambda_1 - a < 0$. Assume that there are no nonnegative solutions of (EP) other than $(0,0)$ and $(\phi_a, 0)$; so that (EP) has no positive solutions. The excision property of fixed point index gives

$$\deg_W(I - A,\ \text{int } D) = \text{index}_W(A,(0,0)) + \text{index}_W(A,(\phi_a,0)) \tag{4.10}$$

([1, Corollary 11.2]). If (2.7) is satisfied, it follows from Lemma 5 that the right hand side of (4.10) is 0. This is a contradiction to Lemma 4; so that (EP) must have a positive solution.

When $b > \lambda_1$, we repeat the same argument as above. Assume that (EP) has no nonnegative solutions except for $(0,0)$, $(\phi_a,0)$ and $(0,\phi_b)$. Using the excision property again we have

$$\deg_W(I - A,\ \text{int } D) = \text{index}_W(A,(0,0)) + \text{index}_W(A,(\phi_a,0)) + \text{index}_W(A,(0,\phi_b)).$$

In case (2.7) is satisfied, Lemma 5 gives

$$\deg_W(I - A,\ \text{int}\ D) = 0.$$

Similarly, if (2.8) is satisfied, then

$$\deg_W(I - A,\ \text{int}\ D) = 2.$$

Both of these results contradict Lemma 4.

It remains to show the existence of a positive solution for $b = \lambda_1$ satisfying (2.7). Regard b as a parameter in (EP) with a fixed. Choose $\{b_n\}(b_n \neq \lambda_1)$ satisfying (2.7) and $\lim_{n\to\infty} b_n = \lambda_1$. Let (U_n, V_n) be a positive solution of (EP) with $b = b_n$. Since $(-\Delta)^{-1}$ is a compact operator in $C_0(\bar{\Omega})$ and $\{(U_n, V_n)\}$ is uniformly bounded by Lemma 2, there exists a subsequence, again called (U_n, V_n), such that

$$\lim_{n\to\infty} (U_n, V_n) = (U_\infty, V_\infty) \qquad \text{uniformly in} \quad \Omega.$$

We can prove in the standard manner that (U_∞, V_∞) is a solution of (EP). Moreover, define (u_n, v_n) by (2.5) and (2.6) with (U, V) replaced by (U_n, V_n) and denote the limit of (u_n, v_n) by (u_∞, v_∞). Clearly, (U_∞, V_∞) and (u_∞, v_∞) are connected by (2.4), (2.5) and (2.6). We will show by contradiction that (U_∞, V_∞) is a positive solution.

Assume $U_\infty = u_\infty = 0$ and set $\hat{U}_n = \frac{U_n}{\|U_n\|}$, $n \geq 1$, where $\|\cdot\|$ denotes the supremum norm. Then

$$-\Delta \hat{U}_n = \frac{a(1 - u_n - cv_n)}{1 + \alpha v_n}\hat{U}_n,$$

$$-\Delta V_n = b_n v_n(1 - v_n + du_n).$$

The elliptic regularity theory enables us to extract a subsequence such that $\hat{U}_n \to w \geq 0 (\neq 0)$ in $C(\Omega)$ for some w. So we have

$$-\Delta w = \frac{a(1 - cv_\infty)}{1 + \alpha v_\infty} w,$$

$$-\Delta V_\infty = \lambda_1 v_\infty(1 - v_\infty).$$

It follows from the results of §3.1 that $V_\infty = v_\infty = 0$. Since $a > \lambda_1$, we have $w = 0$, which is a contradiction to $w \neq 0$.

On the other hand, assume that $V_\infty = v_\infty = 0$ and $U_\infty = u_\infty \neq 0$. Set $\hat{V}_n = \frac{V_n}{\|V_n\|}$. Repeating the same argument as above leads to

$$-\Delta u_\infty = a(1 - u_\infty)u_\infty, \tag{4.11}$$

$$-\Delta h = \frac{\lambda_1(1 + du_\infty)}{1 + \beta u_\infty} h, \tag{4.12}$$

where $h > 0$ is a limit function of a subsequence of $\{\hat{V}_n\}$. We get $u_\infty = U_\infty = \phi_a$ from (4.11). Since $h > 0$, it follows from the Krein-Rutman theorem that (4.12) is equivalent to $\lambda_1(-b(1 + d\phi_a)/(1 + \beta\phi_a)) = 0$ with $b = \lambda_1$. This contradicts our assumption.

5. Nonexistence of positive solutions. In this section we will study nonexistence properties of positive solutions of (EP).

Proof of Theorem 2. Assume that (EP) has a positive solution (U, V) and that (2.13) is satisfied in case $b \geq \lambda_1$. Since ψ satisfies (2.12), we get

$$\Delta U + \frac{a(1-c\psi)}{1+\alpha\psi} U > 0.$$

Hence for every $p > 0$,

$$\{p + \frac{a(1-c\psi)}{1+\alpha\psi}\} U > (-\Delta + pI)U.$$

By the monotonicity of $(-\Delta + pI)^{-1}$ and the strong maximum principle (see [22]) we have

$$(-\Delta + pI)^{-1}\{p + \frac{a(1-c\psi)}{1+\alpha\psi}\} U > U. \tag{5.1}$$

Define

$$T_3 := (-\Delta + pI)^{-1}\{p + \frac{a(1-c\psi)}{1+\alpha\psi}\}.$$

As discussed by Li [13, Lemma 2.3], we see from (5.1) that $r(T_3) > 1$. This fact, together with Proposition 1, implies $\lambda_1(a(c\psi - 1)/(1+\alpha\psi)) < 0$, which is a contradiction to (2.13). Therefore, (EP) has no positive solutions when (2.13) is satisfied.

Next we consider the case $b < \lambda_1$. Let (U, V) be a positive solution for (EP). From the second equation it follows that

$$\Delta V + \frac{b(1+du)}{1+\beta u} V > 0. \tag{5.2}$$

For $\beta \leq d$, $u \mapsto (1+du)/(1+\beta u)$ is increasing. Since $u \leq \phi_a$ by (3.6), we have from (5.2) that

$$\Delta V + \frac{b(1+d\phi_a)}{1+\beta\phi_a} V > 0.$$

Therefore, making use of the same arguments as in the case $b \geq \lambda_1$ we can conclude

$$r((-\Delta + pI)^{-1}\{p + \frac{b(1+d\phi_a)}{1+\beta\phi_a}\}) > 1.$$

This fact, together with (ii) of Proposition 1, gives $\lambda_1(-b(1+d\phi_a)/(1+\beta\phi_a)) < 0$. Therefore, we see that (2.14) is also a sufficient condition for the nonexistence.

Proof of Corollary 1. We will show that $\psi = \phi_b$ satisfies (2.12) for all positive solutions (u, v) of (SP). By using $v = V/(1+\beta u)$, (2.12) can be rewritten as

$$\psi < \frac{(c+\alpha)V + u(1+\beta u)}{(\alpha + c - \alpha u)(1+\beta u)}. \tag{5.3}$$

Recall that, if $d \geq \beta$, then $V \geq \phi_b$ from (3.7). If we set

$$F(u) = \frac{(c+\alpha)\phi_b + u(1+\beta u)}{(\alpha + c - \alpha u)(1+\beta u)},$$

we have only to take ψ such that $\psi < F(u)$ for every $0 < u \leq \phi_b$; then (5.3) follows at once. It is seen that F is strictly increasing if $1 + (\alpha - \alpha\beta - c\beta)\phi_b \geq 0$ in Ω. So $F(u) > F(0) = \phi_b$ for $0 < u \leq \phi_b$. This result allows us to take $\psi = \phi_b$; so that (2.13) can be replaced by (2.15). Therefore, the assertion follows from Theorem 2.

Remark 8. It will be interesting if we can determine a nonexistence region of positive solutions of (EP) for general α and β. This problem is closely related to the direction of bifurcation of positive solutions.

We will study (EP) from the view point of the bifurcation theory (see, e.g., Crandall and Rabinowitz [5]). When we discuss the bifurcation of nontrivial solutions (U, V) of (EP) from $(\phi_a, 0)$, it is routine to carry out the linearization of (EP) with respect to U and V at $(\phi_a, 0)$ (note that u and v are dependent on U and V by (2.5) and (2.6)). As in the derivation of $A'(\phi_a, 0)$, we can show that the linearized operator associated with (EP) at $(\phi_a, 0)$ is

$$\{\Delta + a(1-2\phi_a)\}\hat{U} + \frac{a\phi_a\{(2\phi_a - 1)\alpha - c\}}{1+\beta\phi_a}\hat{V}, \tag{5.4}$$

$$\{\Delta + \frac{b(1+d\phi_a)}{1+\beta\phi_a}\}\hat{V} \qquad \text{with} \quad \hat{U} = \hat{V} = 0 \qquad \text{on} \quad \partial\Omega.$$

Let a be fixed and regard b as a bifurcation parameter. It can be seen that the spectral problem for (5.4) has zero eigenvalue when b satisfies $\lambda_1(-b(1+d\phi_a)/(1+\beta\phi_a)) = 0$ or, equivalently, (a, b) lies on S_1 curve in ab space (see (2.9)). Therefore, making use of the local bifurcation theory [5] we can prove that positive solutions of (EP) bifurcate from $(\phi_a, 0)$ when $(a, b) \in S_1$. Moreover, if $c \geq \alpha$ and $d \geq \beta$, then we can show that the bifurcation of positive solutions is supercritical; positive solutions exist for the direction $b > b(a)$.

Similarly, it is possible to show that positive solutions bifurcate from $(0, \phi_b)$ at S_2-curve by regarding a as a bifurcation parameter with b fixed. Furthermore, one can also prove that their bifurcation is supercritical when all the assumptions of Corollary 1 are satisfied. In this situation, the region surrounded by S_1 and S_2 curves is an optimal coexistence region. See also Theorem 3.

For general α and β careful analysis is required to determine the direction of bifurcation (cf., [16], [27]). We will discuss this subject elsewhere.

6. Uniqueness of the positive solution. In this section we will prove Theorem 4. For this purpose, the idea of López-Gómez and Pardo[17] is very useful. They have shown the nondegeneracy of a positive solution for prey-predator model in the special case when the spatial dimension is one.

Proof of Theorem 4. Let $\Omega = (0, 1)$ and let $(U_i, V_i), i = 1, 2$, be two positive solutions for (EP). Set $\hat{U} = U_1 - U_2$ and $\hat{V} = V_1 - V_2$. Since $U_i = (1+\alpha v_i)u_i$ and $V_i = (1+\beta u_i)v_i$

$(i = 1, 2)$,

$$\begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix} = \begin{pmatrix} 1+\alpha v_2 & \alpha u_1 \\ \beta v_2 & 1+\beta u_1 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix};$$

so that

$$\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \frac{1}{1+\alpha v_2+\beta u_1} \begin{pmatrix} 1+\beta u_1 & -\alpha u_1 \\ -\beta v_2 & 1+\alpha v_2 \end{pmatrix} \begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix}. \tag{6.1}$$

Since (U_1, V_1) and (U_2, V_2) are solutions for (EP), we can get

$$\frac{d^2}{dx^2} \begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix} + \begin{pmatrix} a(1-u_1-u_2-cv_2) & -acu_1 \\ bdv_2 & b(1-v_1-v_2+du_1) \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = 0.$$

Hence using (6.1) we have

$$\begin{cases} -\hat{U}'' + p_1\hat{U} - q_1\hat{V} = 0, \\ -\hat{V}'' + p_2\hat{V} - q_2\hat{U} = 0, \end{cases} \tag{6.2}$$

where

$$p_1 = \frac{a}{1+\alpha v_2+\beta u_1}(-1+u_1+u_2+cv_2-\beta u_1+\beta {u_1}^2+\beta u_1 u_2),$$

$$p_2 = \frac{b}{1+\alpha v_2+\beta u_1}(-1+v_1+v_2-du_1-\alpha v_2+\alpha {v_2}^2+\alpha v_1 v_2),$$

$$q_1 = \frac{au_1}{1+\alpha v_2+\beta u_1}\{(u_1+u_2)\alpha-\alpha-c\},$$

$$q_2 = \frac{bv_2}{1+\alpha v_2+\beta u_1}\{(v_1+v_2)\beta+d-\beta\}.$$

In view of (6.2) we define the operators

$$\begin{cases} L_1U := -U'' + p_1U \\ L_2V := -V'' + p_2V \end{cases} \tag{6.3}$$

with zero Dirichlet boundary conditions at $x = 0, 1$. Since U_2 and V_1 are positive, we see from (EP) that the principal eigenvalues of

$$-d^2/dx^2 + \frac{a(u_2+cv_2-1)}{1+\alpha v_2} \qquad \text{and} \qquad -d^2/dx^2 + \frac{b(v_1-du_1-1)}{1+\beta u_1}$$

are zero; that is, $\lambda_1(a(u_2+cv_2-1)/(1+\alpha v_2)) = \lambda_1(b(v_1-du_1-1)/(1+\beta u_1)) = 0$. After some calculations we can show

$$p_1 > \frac{a(u_2+cv_2-1)}{1+\alpha v_2} \qquad \text{and} \qquad p_2 > \frac{b(v_1-du_1-1)}{1+\beta u_1},$$

where we have used $\alpha - \alpha\beta - \beta c \geq 0$ and $d \geq \beta$, respectively. So it follows from (2.1) that

$$\lambda_1(p_1) > \lambda_1\left(\frac{a(u_2+cv_2-1)}{1+\alpha v_2}\right) = 0 \tag{6.4}$$

and

$$\lambda_1(p_2) > \lambda_1(\frac{b(v_1 - dv_1 - 1)}{1 + \beta u_1}) = 0. \tag{6.5}$$

Moreover, it is seen that $q_1 < 0$ and $q_2 > 0$ under the condition $\alpha \le c$ and $\beta \le d$ (note $u_i < 1$ for $i = 1, 2$). Using L_1 and L_2 defined by (6.3), we can rewrite (6.2) in the form

$$L_1\hat{U} = q_1\hat{V} \qquad \text{with} \qquad q_1 < 0, \tag{6.6}$$

$$L_2\hat{V} = q_2\hat{U} \qquad \text{with} \qquad q_2 > 0. \tag{6.7}$$

To complete the proof, we need the following lemma which is, in a sense, a generalization of the maximum principle (see Protter and Weinberger [22]).

Lemma 7 (cf. [17]). *Let $q \in C[\gamma, \delta]$ and define L by*

$$Lw = -w'' + q(x)w, \qquad \gamma < x < \delta$$

with zero Dirichlet boundary conditions at $x = \gamma, \delta$. Assume that the principal eigenvalue of L is strictly positive. If $w \in C[\gamma, \delta] \cap C^2(\gamma, \delta)$ satisfies $Lw > 0$ in (γ, δ), $w(\gamma) \ge 0$ and $w(\delta) \ge 0$, then $w > 0$ in (γ, δ).

By (6.4) the principal eigenvalue of L_1 in $(0, 1)$ is positive. Therefore, one can see from the variational characterization that the principal eigenvalue of L_1 in any subinterval of $(0, 1)$ is also positive. It follows from (6.5) that L_2 has the same property as L_1. These results imply that Lemma 7 is applicable to the restrictions of L_1 and L_2 to any subinterval of $(0, 1)$.

In the rest of the proof we follow the argument of López-Gómez and Pardo [17]. Assume that $\hat{U} \ne 0$. By the uniqueness of solutions for the Cauchy problem related to second-order differential equations, $\hat{U}$ has at most a finite number of zeros. If $\hat{U} > 0$ in $(0, 1)$, application of Lemma 7 to (6.7) implies $\hat{V} > 0$ in $(0, 1)$. Applying Lemma 7 again to (6.6), we get $\hat{U} < 0$ in $(0, 1)$. This is a contradiction. Hence, $\hat{U}$ and $\hat{V}$ must change signs in $(0, 1)$.

Assume that $\hat{U}$ vanishes at $x = x_0, x_1, x_2, \dots, x_n, x_{n+1}$ with $x_0 = 0, x_{n+1} = 1$:

$$\hat{U}(x) > 0 \quad x \in (x_{2j}, x_{2j+1}), \quad j \ge 0, 2j + 1 \le n,$$

$$\hat{U}(x) < 0 \quad x \in (x_{2j-1}, x_{2j}), \quad j \ge 1, 2j \le n.$$

By hypothesis $\hat{U}(x) > 0$ for $x \in (x_0, x_1)$ and $\hat{U}(x_0) = \hat{U}(x_1) = 0$. We will show $\hat{V}(x_1) < 0$ by contradiction. Suppose $\hat{V}(x_1) \ge 0$; then Lemma 7 with $L = L_2$ assures $\hat{V} > 0$ in (x_0, x_1) from (6.7). Applying Lemma 7 again with $L = L_1$ to (6.6) we are led to $\hat{U} < 0$ in (x_0, x_1). Since this is a contradiction, we must have $\hat{V}(x_1) < 0$. Again using the assumption that $\hat{U}(x) < 0$ for $x \in (x_1, x_2)$ and $\hat{U}(x_1) = \hat{U}(x_2) = 0$, we can apply Lemma 7 to derive $\hat{V}(x_2) > 0$. A recursive argument yields

$$\hat{V}(x_{2j}) > 0 \quad \text{and} \quad \hat{V}(x_{2j+1}) < 0 \quad \text{for} \quad x_{2j}, x_{2j+1} \in \{x_1, x_2, \dots, x_n, x_{n+1}\}.$$

This contradicts the fact $\hat{V}(x_{n+1}) = 0$. Thus $\hat{U}$ (and, therefore, $\hat{V}$) must be identically zero.

Appendix. We will give some basic properties of S_1 and S_2 defined by (2.9) and (2.10).

Lemma A.1. *If S_1 is defined by* (2.9), *it can be expressed as*

$$S_1 = \{(a, b) : b = b(a) \quad \textit{for} \quad a \geq \lambda_1\},$$

where $b(\cdot)$ is a continuous function in $[\lambda_1, \infty)$ which possesses the following properties:

(i) $b(\cdot)$ *is*

$$\begin{cases} \textit{strictly decreasing} & \textit{if } d > \beta, \\ \textit{constant } (= \lambda_1) & \textit{if } d = \beta, \\ \textit{strictly increasing} & \textit{if } d < \beta; \end{cases}$$

(ii) $b(\lambda_1) = \lambda_1$;

(iii)

$$\lim_{a \to \infty} b(a) = \frac{(1+\beta)\lambda_1}{1+d}.$$

Proof. From the variational characterization for $\lambda_1(\cdot)$, we have

$$\frac{1}{b(a)} = \sup\{\int_\Omega \frac{1+d\phi_a}{1+\beta\phi_a} w^2 dx / \int_\Omega |\nabla w|^2 \, dx : w(\neq 0) \in H_0^1(\Omega)\}. \qquad (A.1)$$

In view of this expression, it is possible to show the continuity and monotonicity of $b(\cdot)$ with use of (2.3). Moreover, since $\phi_a \to 0$ uniformly in Ω as $a \to \lambda_1$, we get (ii).

In order to show (iii) we note that for each compact set K of Ω,

$$\lim_{a \to \infty} \phi_a = 1 \qquad \text{uniformly in} \quad K \qquad (A.2)$$

(see, e.g., Dancer [7, Lemma 1]). Let ζ be an eigenfunction corresponding to $\lambda_1 = \lambda_1(0)$ with $\max_{x \in \Omega} \zeta = 1$. Then it follows from (A.1) that

$$\frac{1}{b(a)} \geq \int_\Omega \frac{1+d\phi_a}{1+\beta\phi_a} \zeta^2 \, dx / \| \nabla\zeta \|^2 = \lambda_1^{-1} \int_\Omega \frac{1+d\phi_a}{1+\beta\phi_a} \zeta^2 \, dx / \| \zeta \|^2,$$

where $\| \cdot \|$ denotes $L^2(\Omega)$ norm. Therefore, applying Lebesgue's dominated convergence theorem to the above inequality we see from (A.2) that

$$\lim_{a \to \infty} \frac{1}{b(a)} \geq \frac{1+d}{(1+\beta)\lambda_1}. \qquad (A.3)$$

Let w_a be a positive function with $\| \nabla w_a \| = 1$ such that

$$\frac{1}{b(a)} = \int_\Omega \frac{1+d\phi_a}{1+\beta\phi_a} w_a^2 \, dx. \qquad (A.4)$$

Since $\{w_a\}_{a > \lambda_1}$ is bounded in $H_0^1(\Omega)$, Rellich's theorem enables us to choose a sequence $\{w_{a_n}\}$ with $a_n \to \infty$ such that

$$\begin{aligned} & w_{a_n} \to w_\infty && \text{in } L^2(\Omega), \\ & \nabla w_{a_n} \rightharpoonup \nabla w_\infty && \text{weakly in } L^2(\Omega) \end{aligned}$$

(note $\| \nabla w_\infty \| \leq 1$). Consider the following identity:

$$\begin{aligned}&\int_\Omega \frac{1+d\phi_{a_n}}{1+\beta\phi_{a_n}} w_{a_n}^2 dx - \int_\Omega \frac{1+d}{1+\beta} w_\infty^2 dx \\ &= \left\{ \int_\Omega \frac{1+d\phi_{a_n}}{1+\beta\phi_{a_n}} (w_{a_n}^2 - w_\infty^2) dx \right\} + \left\{ \int_\Omega \left(\frac{1+d\phi_{a_n}}{1+\beta\phi_{a_n}} - \frac{1+d}{1+\beta} \right) w_\infty^2 dx \right\}.\end{aligned} \tag{A.5}$$

The first term in the right hand side is bounded from above by

$$(1+d) \int_\Omega | w_{a_n}^2 - w_\infty^2 | \, dx \leq (1+d) \| w_{a_n} - w_\infty \| \| w_{a_n} + w_\infty \|,$$

whose right hand side converges to zero as $n \to \infty$. Owing to Lebesgue's dominated convergence theorem, the second term in (A.5) also converges to zero as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \frac{1}{b(a_n)} = \frac{1+d}{1+\beta} \| w_\infty \|^2 \leq \frac{(1+d) \| w_\infty \|^2}{(1+\beta) \| \nabla w_\infty \|^2} \leq \frac{1+d}{(1+\beta)\lambda_1}, \tag{A.6}$$

where we have used $\| \nabla w_\infty \|^2 \leq 1$. Since $1/b(a)$ is monotone, it is convergent as $a \to \infty$. Hence it follows from (A.6) that

$$\lim_{a\to\infty} \frac{1}{b(a)} = \lim_{n\to\infty} \frac{1}{b(a_n)} \leq \frac{1+d}{(1+\beta)\lambda_1}. \tag{A.7}$$

Therefore, (iii) comes from (A.3) and (A.7).

Lemma A.2. *If S_2 is defined by* (2.10), *it can be expressed as*

$$S_2 = \{(a,b) : a = a(b) \quad for \quad b \geq \lambda_1\},$$

where $a(\cdot)$ is a continuous function which possesses the following properties:

(i) *$a(\cdot)$ is strictly increasing in $[\lambda_1, \infty)$;*
(ii) *$a(\lambda_1) = \lambda_1$;*
(iii)

$$\begin{cases} \lim_{b\to\infty} a(b) = \frac{(1+\alpha)\lambda_1}{1-c} & \text{if } c \leq 1, \\ \lim_{b\to\infty} a(b) = +\infty & \text{if } c > 1. \end{cases}$$

Proof. The idea of the proof is essentially the same as Lemma A.1. So observe that

$$\frac{1}{a(b)} = \sup\{ \int_\Omega \frac{1-c\phi_b}{1+\alpha\phi_b} w^2 dx / \int_\Omega | \nabla w |^2 \, dx : w(\neq 0) \in H_0^1(\Omega) \}.$$

Hence it follows that $a(\cdot)$ is strictly increasing with respect to b. For $c \leq 1$, one can prove

$$\lim_{b\to\infty} a(b) = \frac{(1+\alpha)\lambda_1}{1-c} \tag{A.8}$$

in the same way as (iii) of Lemma A.1. For $c > 1$, (A.8) is absurd; in fact we can show that $\lim_{b\to\infty} a(b) = +\infty$. □

We can give more information about S_1 and S_2 near (λ_1, λ_1).

Lemma A.3. (i) *The function* $b(\cdot)$ *defined in Lemma* A.1 *satisfies*

$$b(a) = \lambda_1 + (\beta - d)(a - \lambda_1) + o(a - \lambda_1) \qquad \text{near } \lambda_1.$$

(ii) *The function* $a(\cdot)$ *defined in Lemma* A.1 *satisfies*

$$a(b) = \lambda_1 + (\alpha + c)(b - \lambda_1) + o(b - \lambda_1) \qquad \text{near } \lambda_1.$$

Proof. The proof is based on the local bifurcation analysis and is accomplished in the same manner as [27, Lemmas 3.4 and 3.6].

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