LOCAL T–SETS AND RENORMALIZED SOLUTIONS OF DEGENERATE QUASILINEAR ELLIPTIC EQUATIONS WITH AN $L^1$-DATUM

YOUCIF ATIK
Département de Mathématiques, Ecole Normale Supérieure, 16050 Vieux–Kouba, Alger, Algérie

JEAN MICHEL RAKOTOSON
Département de Mathématiques, Université de Poitiers
40, avenue du recteur Pineau, 86022 Poitiers Cedex, France

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Abstract. In this paper we study essentially the questions of uniqueness and stability of solutions of boundary value problems associated with equations of the type:

$$-\text{div}(\hat{a}(x,u,\nabla u)) + b(x)|u|^\gamma u = \mu \in L^1(\Omega)$$

on an arbitrary open subset $\Omega$ of $\mathbb{R}^N$ with $\hat{a}(x,u,\nabla u)$ a Carathéodory nonlinear function satisfying the general conditions of Leray-Lions where the coerciveness condition is weakened to allow degeneracies and becomes

$$\hat{a}(x,u,\xi) \cdot \xi \geq a(x)|\xi|^p, \quad \forall u \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N, \quad \text{and} \quad x \text{ a.e. in } \Omega,$$

with $p > 1$ an arbitrary real number and $a$ an $L^1$-weight which might vanish or go to infinity on $S$, a closed subset of $\Omega$ whose measure is zero. Here $b$ is an $L^1$-nonnegative function with properties similar to those of the weight $a$, $\gamma$ a positive number belonging to suitable intervals. For $p > 1$ arbitrary and general weight $a$, we need new functional sets called “local T-sets” which are extensions of local Sobolev spaces. The “localization” is to handle the degeneracy. We get uniqueness and stability for $S$ satisfying a geometrical condition or $S$ and $a$ an analytic-geometrical one.

1. Introduction. Let $p > 1$ be a real number, $\Omega$ an arbitrary open subset of $\mathbb{R}^N$, and $S$ a closed subset of $\Omega$ whose $N$-dimensional Lebesgue measure is zero; in the sequel, $S$ will be called a singularity. Let $a$ and $b$ be two nonnegative functions on $\Omega$ which might degenerate (i.e., vanish or go to infinity) on $S$. We will study the questions of uniqueness and stability of renormalized solutions for boundary value problems associated to the equation

$$Au = -\text{div}(\hat{a}(x,u,\nabla u)) + b(x)|u|^\gamma u = \mu \quad \text{on} \quad \Omega$$

with a given right member $\mu$ in $L^1(\Omega)$. Here $\hat{a} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that $\hat{a}/\mu$ satisfies the general conditions of Leray-Lions ([16]) and $\gamma$ a real number with $p - 1 < \gamma < \frac{N}{N-p}(p-1)$ if $1 < p < N$ and $p - 1 < \gamma < \infty$ otherwise.

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In the nondegenerate case and $\Omega$ bounded, the problems of Leray-Lions type with measures as data, were considered, for $p > p_{c_0} = 2 - \frac{1}{N}$, by many authors; cf. [5], [22], . . . . These authors proved the existence of solutions in the framework of Sobolev spaces. However, for $1 < p \leq p_{c_0}$, this framework is too narrow to contain solutions. To overcome this difficulty, one of the authors of this paper introduced a new type of sets, the T-sets; see [23].

Here, because of the degeneracy of $\tilde{a}$, we introduce the notion of local T-sets associated to the singularity $S$ of $a$. The construction is a slight generalization of those sets introduced in [23, 24] (see also [5] for another type of such spaces). Following the same ideas as for usual Sobolev spaces, we study, in Section 3, some useful properties of these sets such as a Stampacchia-type property, Sobolev inclusions by giving some precise formulas and inequalities. In Section 4, we introduce our own notion of renormalized solution based on similar ideas as introduced by DiPerna and Lions ([12, 13]) (see also [18] and [24]). Then, we discuss the uniqueness of such a solution according to the behavior of $a$ near the singularity. For this, we consider two cases:

- First, a geometrical condition concerning directly the structure of $S$ (see Definition 5.1).
- Secondly, we impose an analytic-geometrical condition on $a$ (see Definitions 5.2 and 5.3).

Another originality of our paper is that, in Section 5, we impose only a Hölder-type condition for the mapping $u \mapsto \tilde{a}(x, u, \xi)$ (see Theorem 5.2). This is not the case in the previous paper made by various authors. In [5], the function $\tilde{a}$ does not depend on $u$, that is, $\tilde{a}(x, u, \xi) = \tilde{a}(x, \xi)$, while in [24], this map is Lipschitzian, and in [18], they treat the linear equation.

The strategy of the proof relies not only on a new choice of test function (see for instance the function $J_{\epsilon}$) but also on the use of a technique of localization of the singularity by means of some specific functions (see Lemmas 5.1–3).

At last, a novelty of this paper is the investigation of the continuity of the map $\mu \mapsto u$ (the unique renormalized solution). Since the notion of continuity is a topological one, we then make precise the right definition of convergence over a local T-sets (see Definitions 7.1 and 7.2) (our notion, with the inclusions of Section 3, implies in particular the convergence in the metric space $L^p_{\text{loc}}(\Omega)$ for some $s \in (0, 1)$.)

It is clear that as for the nondegenerate case, we must deal with critical exponents which again motivates the use of T-sets. Our results are mainly announced in [3]. Complementary details, namely on the existence theorem, can be found in [2]. Let us point out that, because of general weights considered here, we will not make use of weighted Sobolev-Poincaré inequalities (see [25], [26]). This explains the presence of the zero-order term $b(x)|u|^{\gamma - 1}u$ which will play an essential role to get the existence, uniqueness, and stability results.

2. Assumptions on $S$, $\Omega$, and $A$. We assume that $\Omega$ is an arbitrary open subset of $\mathbb{R}^N$ and $S$ is a compact subset of $\Omega$ whose Lebesgue $N$-dimensional measure $|S| = 0$. We assume that the functions $a$ and $b$ are nonnegative on $\Omega$ and can vanish or go to
infinity on $S$. Let us denote by $\mathcal{N}(S)$ the set of all neighborhoods of $S$ in $\mathbb{R}^N$ and put

$$
\mathcal{B}_0(\Omega, S) = \{ \mathcal{B} \subset \Omega \text{ bounded and open : } \exists S \in \mathcal{N}(S) \text{ such that } \mathcal{B} \subset \Omega \setminus S \}.
$$

We assume also, for all $\mathcal{B} \in \mathcal{B}_0(\Omega, S)$, and almost everywhere in $\Omega$, that

$$
0 < \inf_{\mathcal{B}} \{a, b\} \leq \sup_{\mathcal{B}} \{a, b\} < \infty, \quad \frac{a}{b} \in L^\infty(\Omega), \ b \in L^1(\Omega); \quad (ab)
$$

$$
\hat{a}(x, u, \xi) \cdot \xi \geq a(x)|\xi|^p, \ \forall u \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^N; \quad (\hat{a}1)
$$

$$
|\hat{a}(x, u, \xi)| \leq a(x)\{ |u|^{p-1} + |\xi|^{p-1} + a_0(x) \}, \ \forall u \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^N; \quad (\hat{a}2)
$$

$$
|\hat{a}(x, u, \xi) - \hat{a}(x, u, \zeta)| \cdot |\xi - \zeta| > 0, \ \forall u \in \mathbb{R}, \ \forall \xi \neq \zeta \in \mathbb{R}^N; \quad (\hat{a}3)
$$

where $a_0 \geq 0$ is a function in the weighted Lebesgue space $L^{p'}(\Omega, a)$ with $p' = p/(p-1)$. The datum $\mu$ is supposed in $L^1(\Omega)$. Finally, we denote by $\gamma$ a real number and suppose that

$$
\begin{cases}
  p - 1 < \gamma < \frac{N}{N-p} (p-1) & \text{if } 1 < p < N \\
  p - 1 < \gamma < \infty & \text{otherwise.}
\end{cases} \quad (\gamma)
$$

3. Local T-sets. For $s > 0$ and $\mathcal{O}$ an open subset of $\mathbb{R}^N$, we put

$$
L^s_{loc}(\mathcal{O}) = \{ v : \mathcal{O} \to \mathbb{R} \text{ measurable : } |v|^s \in L^1_{loc}(\mathcal{O}) \}.
$$

For $s \geq 1$, this space is locally normed in the usual way and, for $0 < s < 1$, we equip it locally with the distance

$$
\rho_\omega(u, v) = \int_\omega |u - v|^s \, dx, \quad u, v \in L^s_{loc}(\mathcal{O}), \ \omega \subset \subset \mathcal{O}.
$$

The notation $\omega \subset \subset \mathcal{O}$ means that $\omega$ is an open relatively compact subset of $\mathcal{O}$. Let $\Omega_S = \Omega \setminus S$ and

$$
Lip_p(\mathbb{R}) = \{ \Phi \in W^{1, \infty}(\mathbb{R}) : \Phi' \in L^p(\mathbb{R}), \Phi(0) = 0 \}.
$$

Notice that the functions arctan and $T_k$ ($k > 0$), the truncature at levels $-k$ and $k$, i.e., $T_k(t) = \frac{1}{2}(|t+k| - |t-k|) \ (t \in \mathbb{R})$, are in $Lip_p(\mathbb{R})$.

We define $L^{1,p}_{loc}(\Omega_S)$ to be the subset of $L^{1-p}_{loc}(\Omega_S)$ consisting of all functions with the property that

$$
\forall \Phi \in Lip_p(\mathbb{R}), \ \Phi(v) \in W^{1,p}_{loc}(\Omega_S) \text{ and }
$$

$$
\forall \omega \subset \subset \Omega_S, \ \sup_{k>0} \int_\omega \frac{|\nabla v^k|^p}{(1 + |v^k|)^{1+\delta}} < \infty, \ \forall \delta > 0,
$$

where $v^k = T_k(v)$. $L^{1,p}_{loc}(\Omega_S)$ is called a local T-set. The local T-sets share many properties with classical local Sobolev spaces. Let us give some of them on $\Omega$. 
Proposition 3.1. Let $v \in L^{1,p}_{loc}(\Omega)$ and $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ a Lipschitz function.

a) $\nabla v$ and $\nabla \Phi(v)$ exist in the usual sense almost everywhere in $\Omega$ and

$$\nabla \Phi(v)(x) = \Phi'(v(x))\nabla v(x) \quad \text{a.e. in } \Omega.$$ 

b) $v^k = T_k(v)$ satisfies almost everywhere in $\Omega$

$$\nabla v^k(x) = \begin{cases} 0 & \text{if } |v(x)| \geq k \\ \nabla v(x) & \text{if } |v(x)| < k. \end{cases}$$

c) For all subset $E$ of $\mathbb{R}$ whose measure is zero, one has

$$\nabla v(x) = 0 \quad \text{a.e. on } A = \{ x \in \Omega : v(x) \in E \}.$$

Proof. In the case when the set of the nonexistence of $\Phi'$ and the set $E$ are countable, proofs of (a), (b), and (c) may be found in [2] (see also [23]). For $E$ not necessarily countable, fixing $\omega \subset \subset \Omega$ and using Federer’s formula (see [1], [10], page 157, or [29]), one gets

$$\nabla v^k(x) = 0 \quad \text{a.e. in } \{ x \in \omega : v^k(x) \in E \}, \forall k \in \mathbb{N}^*$$

because $v^k \in W^{1,1}(\omega)$. Then (by (b)), $\nabla v(x) = 0$ almost everywhere in $A_k^\omega = A \cap \{ x \in \omega : |v(x)| < k \}$ and hence $\nabla v(x) = 0$ almost everywhere in $A_{\omega} = \bigcup_{k \in \mathbb{N}} A_k^\omega$. Therefore, using an increasing sequence $\{ \omega_j \}$ ($\omega_j \subset \subset \Omega$), converging to $\Omega$, we infer (c). To prove the chain rule of (a), we take $E = \{ t \in \mathbb{R} : \Phi'(t) \text{ does not exist} \}$ and $A$ like in (c). Since $\nabla v$ exists almost everywhere in $\Omega$, we have

$$\nabla(\Phi(v))(x) = \Phi'(v(x))\nabla v(x) \quad \text{a.e. in } \Omega \setminus A;$$

moreover, as $\Phi$ is Lipschitzian, $|E| = 0$ and (by (c)) $\nabla \Phi(v(x)) = 0$ almost everywhere in $A$. Therefore, using the usual convention that both sides are equal to zero on $A$, the chain rule is true almost everywhere in $\Omega$. \qed

The following proposition makes a link between the local T-set $L^{1,p}_{loc}(\Omega)$ and local Sobolev spaces and gives an imbedding of Sobolev’s type.

Proposition 3.2. For all $p > 1$ the inclusion $W^{1,p}_{loc}(\Omega) \subset L^{1,p}_{loc}(\Omega)$ holds and, if

a) $1 < p < N$, then $L^{1,p}_{loc}(\Omega) \subset L^{1,p}_{loc}(\Omega)$ holds and, if

$$\int_\omega |v|^p \leq C + C \int_\omega |\nabla v|^p + C \int_\omega \frac{|v|^p}{(1 + |v|)^{1+\delta}} \quad (2)$$

with $p^* = Np/(N - p)$ and $\delta = (s_c - s)(N - p)/N.$
b) \( p \geq N \), then \( L^{1,p}_{\text{loc}}(\Omega) \subset L^s_{\text{loc}}(\Omega), \ \forall s > 0, \) more precisely, for each \( \omega \subset \Omega \), \( s > 0, \) and \( q_* > \max\{1, s^{1/p}\} \), there exists a constant \( C \) such that, for all \( v \in L^{1,p}_{\text{loc}}(\Omega) \), the following inequality holds:

\[
\left( \int_\omega |v|^s \right)^{1/q_*} \leq C \int_\omega |v|^{s/q_*} + C \left[ \int_\omega \frac{|\nabla v|^p}{(1 + |v|)^{1+\delta}} \right]^{1/p}, \tag{3}
\]

with \( \delta = (1 - \frac{s}{q_*})p - 1. \)

c) \( p > N \), then \( L^{1,p}_{\text{loc}}(\Omega) = W^{1,p}_{\text{loc}}(\Omega).\)

d) \( 2 - \frac{1}{N} < p \leq N \), then \( L^{1,p}_{\text{loc}}(\Omega) \subset W^{1,q}_{\text{loc}}(\Omega), \ \forall q \in [1, \frac{N}{N-1}(p-1)]. \)

The proof of this proposition may be found in [2]; see also [23]. Using this proposition and Hölder’s inequality we can prove the following result.

**Proposition 3.3.** For each \( r \in (0, r_c) \) \( (r_c = \min\{\frac{N}{N-1}(p-1)\}) \) and each \( v \in L^{1,p}_{\text{loc}}(\Omega) \), one has \( |\nabla v| \in L^{1,p}_{\text{loc}}(\Omega) \).

### 4. Renormalized solutions for \( \mu \in L^1(\Omega) \)

We need the “almost local” T-set \( T^{1,p}_{\text{loc}}(\Omega_S,a) \) and T-subset \( \Theta^{1,p}_{\text{loc}}(\Omega_S,a) \)

\[
T^{1,p}_{\text{loc}}(\Omega_S,a) = \left\{ v \in L^{1,p}_{\text{loc}}(\Omega_S) : \forall B \in \mathcal{B}_d(\Omega, S), \ \forall \Phi \in \text{Lip}_p(\mathbb{R}), \ \Phi(v) \in W^{1,p}(B), \ \text{and} \ \sup \left\{ a^{-1} \int_{\Omega} \frac{|\nabla v|^p}{(1 + |v|)^{1+\delta}} < \infty, \forall \delta > 0 \right\} \right\}
\]

and

\[
\Theta^{1,p}_{\text{loc}}(\Omega_S,a) = \left\{ v \in T^{1,p}_{\text{loc}}(\Omega_S,a) : \lim_{m \to \infty} \int_{\{m-r \leq |v| \leq m+r\}} a|\nabla v|^p = 0, \ \forall r \geq 0 \right\}.
\]

\( \mathcal{B}_d(\Omega, S) \) was defined in the beginning of Section 2.

**Definition 4.1.** A function \( w \) is called a renormalized solution of the equation (1) if

\[
w \in \Theta^{1,p}_{\text{loc}}(\Omega_S,a) \cap L^\gamma(\Omega,b)
\]

\[
I(w, \Phi(w,\varphi)) = \int_\Omega \mu \Phi(w) \varphi, \ \forall \varphi \in L^\infty(\Omega) \cap W^{1,p}(\Omega,S), \ \forall \Phi \in W^{1,\infty}(S),
\]

where the subscript “c” stands for compact support and

\[
W^{1,p}(\Omega,S) = \{ \varphi \in W^{1,p}(\Omega) : \exists S \in \mathcal{N}(S) \text{ such that } \varphi \equiv 0 \text{ on } \Omega \cap S \}.
\]

\( \mathcal{N}(S) \) was defined in the beginning of Section 2.

Here and in the sequel we use the notation

\[
I(w, \psi) = \int_\Omega \hat{a}(x, w, \nabla w) \cdot \nabla \psi + \int_\Omega b|w|^\gamma w \psi.
\]
The hypotheses on the operator $A$, the chain rule in $L^{1,p}_{loc}(\Omega_S)$, and the definition of $T_{a_{loc}}^{1,p}(\Omega_S, a)$ permit one to see that the Definition 4.1 makes sense.

5. Uniqueness of renormalized solutions.

5.1. Conditions on $a$ and $S$ to get uniqueness. In Definitions 5.1, 5.2, 5.3, Lemmas 5.1, 5.2, and 5.3 below, we denote by $B$ an open ball of $\mathbb{R}^N$ containing the compact singularity $S$ and centered at the origin.

5.1.1. A geometrical condition.

Definition 5.1. We say that the singularity $S$ satisfies the condition $(rs)$ if

\[
\left\{
\begin{array}{l}
\text{there exists a real number } \beta \leq p \text{ such that} \\
\text{for all } B \supset S, \quad \int_{B \setminus S_n} \frac{dx}{r^p(x,S)} = O(n^\beta)(n \to \infty)
\end{array}
\right.
\]

$(rs)$

with $S_n = \{x \in \mathbb{R}^N : \rho(x,S) \leq \frac{1}{n}\}$, where $\rho$ denotes the Euclidean metric on $\mathbb{R}^N$.

5.1.2. Some analytic-geometrical conditions.

Definition 5.2. We say that the singularity $S$ and the weight $a$ satisfy the condition $(lw)$ if

\[
\left\{
\begin{array}{l}
a \text{ is Lipschitz-continuous on } \Omega, \quad a(x) = 0, \quad \forall x \in S, \\
a(x) > 0, \quad \forall x \in \Omega \setminus S, \quad \text{and there exists } \varepsilon > 0 \text{ such that} \\
\text{for all } B \supset S, \quad \int_{\Omega \cap B \setminus \Sigma_n^0} a^{1-p-\varepsilon}(x) \, dx = O(n^\varepsilon)(n \to \infty)
\end{array}
\right.
\]

$(lw)$

with $\Sigma_n^0 = \{x \in \Omega : a(x) \leq \frac{1}{n}\}$.

Definition 5.3. We say that the singularity $S$ and the weight $a$ satisfy the condition $(dw)$ if

\[
\left\{
\begin{array}{l}
a \text{ is continuous in } \Omega \setminus S, \quad a(x) > 0, \quad \forall x \in \Omega \setminus S, \\
\lim_{y \to x} a(y) = +\infty, \quad \forall x \in S, \quad \nabla a \text{ exists a.e. in } \Omega_S, \quad \text{for all } B \supset S, \\
\nabla a \in L^p(\Omega \cap B), \quad \text{and} \quad \int_{\Omega \cap B \setminus \Sigma_n^\infty} a |\nabla a|^p \, dx = O(n^p)(n \to \infty)
\end{array}
\right.
\]

$(dw)$

with $\Sigma_n^\infty = \{x \in \Omega : a(x) \geq n\} \cup S$.

Lemma 5.1. If $S$ satisfies $(rs)$ there exists a sequence $\{\ell_n\} \subset W^{1,\infty}(\mathbb{R}^N)$ such that

\[
0 \leq \ell_n \leq 1 \quad \text{in } \mathbb{R}^N, \quad \ell_n \equiv 0 \quad \text{on } S_n, \quad \lim_{n \to \infty} \ell_n(x) = 1, \quad \forall x \in \mathbb{R}^N \setminus S,
\]

\[
\lim_{n \to \infty} \nabla \ell_n(x) = 0, \quad x \text{ a.e. in } \mathbb{R}^N \setminus S.
\]

Moreover, for all $B \supset S$, there exists a constant $C > 0$ such that

\[
\int_{B \setminus S_n} |\nabla \ell_n|^p \leq C, \quad \forall n \geq 1.
\]
Lemma 5.2. If a and S satisfy (lw) there exists a sequence \( \{ \lambda_n^0 \} \subset L^\infty(\Omega) \) such that
\[
0 \leq \lambda_n^0 \leq 1 \quad \text{in } \Omega, \quad \lambda_0^0 \equiv 0 \quad \text{on } \Sigma_n^0, \quad \lim_{n \to \infty} \lambda_n^0(x) = 1, \quad \forall x \in \Omega \setminus S,
\]
\[
\lim_{n \to \infty} \nabla \lambda_n^0(x) = 0, \quad x \text{ a.e. in } \Omega \setminus S.
\]
Moreover, for all \( B \supset S \), \( \lambda_n^0 \in W^{1,p}(\Omega \cap B) \) and there exists a constant \( C > 0 \) such that
\[
\int_{\Omega \cap B \setminus \Sigma_n^0} |a| |\nabla \lambda_n^0|^p \, dx \leq C, \quad \forall n \geq 1.
\]

Lemma 5.3. If a and S satisfy (dw) there exists a sequence \( \{ \lambda_n^\infty \} \subset L^\infty(\Omega) \) such that
\[
0 \leq \lambda_n^\infty \leq 1 \quad \text{in } \Omega, \quad \lambda_0^\infty \equiv 0 \quad \text{on } \Sigma_n^\infty, \quad \lim_{n \to \infty} \lambda_n^\infty(x) = 1, \quad \forall x \in \Omega \setminus S,
\]
\[
\lim_{n \to \infty} \nabla \lambda_n^\infty(x) = 0, \quad x \text{ a.e. in } \Omega \setminus S.
\]
Moreover, for all \( B \supset S \), \( \lambda_n^\infty \in W^{1,p}(\Omega \cap B) \) and there exists a constant \( C > 0 \) such that
\[
\int_{\Omega \cap B \setminus \Sigma_n^\infty} |a| |\nabla \lambda_n^\infty|^p \, dx \leq C, \quad \forall n \geq 1.
\]

To prove Lemmas 5.1, 5.2, and 5.3, consider a function \( g \) such that
\[
g \in C^1(\mathbb{R}_+, \mathbb{R}_+), \quad 0 \leq g \leq 1, \quad g(0) = 1, \quad \text{and } g(t) = 0, \quad \forall t \geq 1
\]
and put
\[
\ell_n(x) = \begin{cases} 
g(n^{-1} \rho^{-1}(x,S)) & \text{if } x \notin S \\
0 & \text{if } x \in S,
\end{cases}
\]
\[
\lambda_n^0(x) = \begin{cases} 
g(n^{-\varepsilon/p} \rho^{-\varepsilon/p}(x)) & \text{if } x \in \overline{\Omega} \setminus S \\
0 & \text{if } x \in S,
\end{cases}
\]
\[
\lambda_n^\infty(x) = \begin{cases} 
g(a(x)/n) & \text{if } x \in \overline{\Omega} \setminus S \\
0 & \text{if } x \in S.
\end{cases}
\]

5.2. Uniqueness of renormalized solutions.

Theorem 5.1. Suppose that one of the conditions (rs), (lw), or (dw) is satisfied and \( a \in L^1(\Omega) \) (a is also supposed bounded in the case (rs)). If \( \hat{a} \) satisfies the following conditions (for all \( r, s \in \mathbb{R}, \xi \in \mathbb{R}^N \), and almost every \( x \in \Omega \))
\[
|\hat{a}(x,s,\xi)| \leq (a(x)\{s^e + |\xi|^{p-1} + a_0(x)\})^{1/p}, \quad (\hat{a}_0)
\]
\[
|\hat{a}(x,r,\xi) - \hat{a}(x,s,\xi)| \leq (a(x)|r - s|\{|r|^e + |s|^e + |\xi|^{p-1} + a_1(x)\})^{1/p}, \quad (\hat{a}_4)
\]
with \( 0 \leq e < \gamma/p' \) and \( a_0, a_1 \geq 0 \), \( a_0, a_1 \in L^{p'}(\Omega, a) \), then the equation (1) admits at most one renormalized solution in \( \Theta_{a\text{-loc}}^1(\Omega, a) \cap L^\gamma(\Omega, b) \cap L^\gamma(\Omega, a) \).

The proof of this Theorem 5.1 makes use of the important result:
**Proposition 5.1.** Under the hypotheses of Theorem 5.1, for all \( u, v \) renormalized solutions of the equation (1), we have
\[
\lim_{m \to \infty} \int_{\Omega_m} h_m(u)h_m'(v)\hat{a}(x, u, \nabla u) \cdot \nabla v = \lim_{m \to \infty} \int_{\Omega_m} h_m'(u)h_m(v)\hat{a}(x, v, \nabla v) \cdot \nabla u = 0,
\]
where \( \Omega_m = \Omega \cap \{ x \in \Omega : u^{m+1}(x) > v^{m+1}(x) \} \) and, for \( m \geq 0 \), \( h_m \) is the function:
\[
h_m(t) = \frac{1}{2} \left\{ 1 + |m + 1 - |t|| - |m - |t|| \right\} \quad (t \in \mathbb{R}).
\]

**Proof of Proposition 5.1.** Let \( u \) and \( v \) be two renormalized solutions of equation (1) belonging to \( L^\gamma(\Omega, a) \). We give the proof when \( (u,v) \) is satisfied; for the other cases the proofs are similar. Let us take in the Definition 4.1 of renormalized solution, written for \( u \),
\[
\Phi(u) = h_m(u) \quad \text{and} \quad \varphi = [h_m(u) - h_m(v)]J_\varepsilon((u^{m+1} - v^{m+1})^k)\tau_r\lambda_n^0,
\]
where \( 0 < \varepsilon < k \) are two real numbers, \( \varepsilon \) is devoted to go to zero, \( \tau_r(x) = \tau(x/r) \quad (x \in \mathbb{R}^N) \), with \( \tau \in D(\mathbb{R}^N, [0,1]) \), and \( J_\varepsilon \) are functions such that
\[
\tau(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2,
\end{cases} \quad \text{and} \quad J_\varepsilon(t) = \begin{cases} \frac{1}{\sqrt{\varepsilon}} \log \frac{1}{\varepsilon} & \text{if } \varepsilon \leq t \leq k, \\ 0 & \text{if } t \leq \varepsilon,
\end{cases}
\]
here \( K(\varepsilon) = \log \frac{\varepsilon}{2} \); \( m \) and \( r \) are two positive numbers devoted to go to infinity, \( r \) is chosen so that \( B_{2r} = B(0,2r) \supset S \), \( \{\lambda_n^0\} \) is the sequence of Lemma 5.2. We obtain
\[
(1)^{1r_m}_m + (1)^{2r}_m - (1)^{3r_m} + (1)^{4r}_m + (1)^{5r_m} + (1)^{6r}_m + (2)^{\varepsilon}_m = (3)^{\varepsilon}_m
\]
with
\[
(1)^{1r}_m = \int_{\Omega} h_m(u)[h_m(u) - h_m(v)]J_\varepsilon((u^{m+1} - v^{m+1})^k)\tau_r\lambda_n^0 \hat{a}(x, u, \nabla u) \cdot \nabla u,
\]
\[
(1)^{2r}_m = \int_{\Omega} h_m(u)h_m'(v)J_\varepsilon((u^{m+1} - v^{m+1})^k)\tau_r\lambda_n^0 \hat{a}(x, u, \nabla u) \cdot \nabla v,
\]
\[
(1)^{3r}_m = \int_{\Omega} h_m(u)h_m'(v)J_\varepsilon((u^{m+1} - v^{m+1})^k)\tau_r\lambda_n^0 \hat{a}(x, u, \nabla u) \cdot \nabla v,
\]
\[
(1)^{4r}_m = \int_{\Omega} h_m(u)[h_m(u) - h_m(v)]J_\varepsilon((u^{m+1} - v^{m+1})^k)\tau_r\lambda_n^0 \hat{a}(x, u, \nabla u)
\]
\[
\cdot \left( (\nabla (u^{m+1} - v^{m+1})^k) \right),
\]
\[
(1)^{5r}_m = \int_{\Omega} h_m(u)[h_m(u) - h_m(v)]J_\varepsilon((u^{m+1} - v^{m+1})^k)\lambda_n^0 \hat{a}(x, u, \nabla u) \cdot \nabla \tau_r,
\]
\[
(1)^{6r}_m = \int_{\Omega} h_m(u)[h_m(u) - h_m(v)]J_\varepsilon((u^{m+1} - v^{m+1})^k)\tau_r \hat{a}(x, u, \nabla u) \cdot \nabla \lambda_n^0,
\]
\[
(2)^{\varepsilon}_m = \int_{\Omega} b[u^{\gamma-1}uh_m(u)[h_m(u) - h_m(v)]J_\varepsilon((u^{m+1} - v^{m+1})^k)\tau_r\lambda_n^0,
\]
\[
(3)^{\varepsilon}_m = \int_{\Omega} \mu h_m(u)[h_m(u) - h_m(v)]J_\varepsilon((u^{m+1} - v^{m+1})^k)\tau_r\lambda_n^0.
\]
**Step I:** $n \to \infty$. Because the support of $\tau_\epsilon$ is contained in $B_{2r}$, we can consider the above integrals only on the bounded set $\Omega \cap B_{2r}$. Using properties of $\{\lambda_n^0\}$ and Lebesgue’s theorem, it is easy to pass to the limit in the integrals containing the zero-order term $b|u|^{-1}u$ and the right-hand side one $\mu$. Also, the passage to the limit in the integrals containing the leading term $\hat{a}(x,u,\nabla u)$ and free of the gradient $\nabla \lambda_n^0$ is easy. Let us denote by $f_n$ the integrand of $(1)_{mr}^{5\epsilon}$ which contains this gradient. We have $f_n \to 0$ almost everywhere in $\Omega \cap B_{2r}$ as $n \to \infty$ and, for $E \subset \Omega \cap B_{2r}$, we can write

$$
\int_E f_n \leq \left( \int_{\Omega \cap B_{2r} \setminus \Sigma^0_n} a|\nabla \lambda_n^0|^p \right)^{\frac{1}{p}} \left\{ (m+1)^{\epsilon} \left( \int_E a \right)^{\frac{1}{p}} + \left( \int_E a|\nabla u|^{m+1}|^p \right)^{\frac{1}{p}} + \left( \int_E a\lambda_0^{p\epsilon} \right)^{\frac{1}{p}} \right\}.
$$

As $\left\{ \int_{\Omega \cap B_{2r} \setminus \Sigma^0_n} a|\nabla \lambda_n^0|^p \right\}$ is bounded, this proves the equintegrability of the sequence $\{f_n\}$; hence, by Vitali’s theorem

$$
\lim_{n \to \infty} (1)_{mr}^{5\epsilon} = 0.
$$

Therefore, by passing to the limit with respect to $n$ in (4), we obtain the relation

$$
(1)^{5\epsilon}_{mr} + (1)^{2\epsilon}_{mr} - (1)^{3\epsilon}_{mr} + (1)^{4\epsilon}_{mr} + (1)^{5\epsilon}_{mr} + (2)^{\epsilon}_{mr} = (3)^{\epsilon}_{mr},
$$

where every term is equal to the corresponding term in the relation (4) without $\lambda_n^0$.

**Step II:** $r \to \infty$. In each term free of $\nabla \tau_\epsilon$, the passage to the limit may be performed using Lebesgue’s theorem and gives the corresponding term without $\tau_\epsilon$. Concerning the term containing $\nabla \tau_\epsilon$, we have

$$
|\lambda_{1\epsilon}|_{mr} \leq \int_\Omega a\{ |u|^{m+1}|^\epsilon + |\nabla u|^{m+1}|^p| \alpha \} |\nabla \tau_\epsilon| \leq \frac{1}{r} ||\nabla \tau||_{L^\infty(\mathbb{R}^N)} \left( m+1 \right)^{\epsilon} \left( \int_\Omega a \right)^{\frac{1}{p}} + \left( \int_\Omega a|\nabla u|^{m+1}|^p \right)^{\frac{1}{p}} + \left( \int_\Omega a\lambda_0^{p\epsilon} \right)^{\frac{1}{p}} \left( \int_\Omega a \right)^{\frac{1}{p}},
$$

then

$$
\lim_{r \to \infty} (1)^{5\epsilon}_{mr} = 0.
$$

Therefore, at the end of this step, we obtain the relation

$$
(1)^{1\epsilon}_{m} + (1)^{2\epsilon}_{m} - (1)^{3\epsilon}_{m} + (1)^{4\epsilon}_{m} + (2)^{\epsilon}_{m} = (3)^{\epsilon}_{m},
$$

**Step III:** $\epsilon \to 0$. Using the dominated convergence theorem and the fact that $\text{sign}(t) = \text{sign}T_k(t)$, we obtain

$$
\lim_{\epsilon \to 0} (1)^{1\epsilon}_{m} = \int_{\Omega_m} h_m'(u)[h_m(u) - h_m(v)] \hat{a}(x,u,\nabla u) \cdot \nabla u \equiv (1)^{1}_{m},
$$

$$
\lim_{\epsilon \to 0} (1)^{2\epsilon}_{m} = \int_{\Omega_m} h_m(u)h_m'(u) \hat{a}(x,u,\nabla u) \cdot \nabla u \equiv (1)^{2}_{m},
$$

$$
\lim_{\epsilon \to 0} (1)^{3\epsilon}_{m} = \int_{\Omega_m} h_m(u)h_m'(v) \hat{a}(x,u,\nabla u) \cdot \nabla v \equiv (1)^{3}_{m},
$$

$$
\lim_{\epsilon \to 0} (2)^{\epsilon}_{m} = \int_{\Omega_m} b|u|^{-1}u[h_m(u) - h_m(v)] \equiv (2)^{\epsilon}_{m},
$$

$$
\lim_{\epsilon \to 0} (3)^{\epsilon}_{m} = \int_{\Omega_m} \mu h_m(u)[h_m(u) - h_m(v)] \equiv (3)^{\epsilon}_{m},
$$

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where \( \Omega_m = \Omega \cap \{ u^{m+1} > v^{m+1} \} \). The remaining term may be written

\[
(1)^{4\varepsilon}_m = \int_{\{ x \in \Omega : |u - v| < \varepsilon \}} h_m(u)[h_m(u) - h_m(v)]J'_\varepsilon(u - v^{m+1}) \hat{a}(x, u, \nabla u) \cdot \nabla(u - v^{m+1})
\]

because \( J'_\varepsilon \) is zero on \((-\infty, \varepsilon)\]. Then, as \( h_m \) is Lipschitzian and \( h_m(v) = h_m(v^{m+1}) \),

\[
|\(1)^{4\varepsilon}_m| \leq \int_{\{ x \in u^{m+1}, v^{m+1} < k \}} \frac{u - v^{m+1}}{K(\varepsilon)(u - v^{m+1})} |\hat{a}(x, u, \nabla u)||\nabla(u - v^{m+1})|
\]

\[
\leq \frac{C}{K(\varepsilon)} \int_{\{ x \in u^{m+1}, v^{m+1} < k \}} a\{|u|^{e} + |\nabla u|^{p-1} + a_{0}\} \{|\nabla u| + |\nabla v|\}
\]

\[
+ \frac{C}{K(\varepsilon)} \int_{\{ x \in u^{m+1}, v^{m+1} > k \}} a\{|u|^{e} + |\nabla u|^{p-1} + a_{0}\}|\nabla u|.
\]

As \( u, v \in \Theta^{1, p}_{\text{aloc}}(\Omega, a) \cap L^\gamma(\Omega, a) \), each of the previous integrals has a meaning. In fact, one has, for instance, with \( A = \{ x \in \Omega : |u|, |v| \leq m + 1, \varepsilon < u - v < k \} \)

\[
\int_{A} a|\nabla u|^{p} |\nabla v| \leq \left( \int_{A} a|\nabla v| \right)^{\frac{1}{p}} \left( \int_{A} a|\nabla u| \right)^{\frac{1}{p}} \left( \int_{A} a \right)^{\frac{1}{p} - \frac{2}{p}}
\]

and

\[
\int_{A} a|\nabla u|^{p-1} |\nabla v| \leq \frac{1}{p} \int_{A} a|\nabla u|^{p} + \frac{1}{p} \int_{A} a|\nabla v|^{p};
\]

the two last integrals are finite; this follows from the definition of \( \Theta^{1, p}_{\text{aloc}}(\Omega, a) \). Then

\[
\lim_{\varepsilon \to 0} (1)^{4\varepsilon}_m = 0 \quad (\text{because } K(\varepsilon) \xrightarrow{\varepsilon \to 0} +\infty).
\]

Finally, the passage to the limit with respect to \( \varepsilon \) gives

\[
(1)^3_m = (1)_m^1 + (1)_m^2 + (2)_m - (3)_m.
\]

**Step IV:** \( m \to \infty \). We obtain

\[
\lim_{m \to \infty} \left(1\right)_m = \lim_{m \to \infty} \left(2\right)_m = 0 \quad \text{because } u \in \Theta^{1, p}_{\text{aloc}}(\Omega, a) \cap L^\gamma(\Omega, a) \quad \text{and } 0 \leq e < \frac{\gamma}{p};
\]

also, using the dominated convergence theorem

\[
\lim_{m \to \infty} (2)_m = \lim_{m \to \infty} (3)_m = 0.
\]

This proves that

\[
\lim_{m \to \infty} \int_{\Omega \cap \{ u^{m+1} > v^{m+1} \}} h_m(u)h'_m(v) \hat{a}(x, u, \nabla u) \cdot \nabla v = 0.
\]
The remaining limit of the lemma can be proved in the same way by taking

\[ \Phi(v) = h_m(v) \quad \text{and} \quad \varphi = [h_m(u) - h_m(v)]J_\varepsilon[(u^{m+1} - v^{m+1})^k] \tau_r \lambda_0^n \]

in the Definition 4.1 written for \( v \).

**Proof of Theorem 5.1.** We choose to give it in the case where the condition \( (lw) \) is satisfied; in the other cases the proofs can be carried on in the same way. Let \( u \) and \( v \) be two renormalized solutions of equation (1) in \( \Theta^{1,1}_{w,\varepsilon}(\Omega_S, a) \cap L^\gamma(\Omega, b) \cap L^\gamma(\Omega, a) \) and \( \{\lambda_0^n\} \) the localization sequence of Lemma 5.2. Taking in the Definition 4.1, written for \( u \),

\[ \Phi(u) = h_m(u) \quad \text{and} \quad \varphi = h_m(u)J_\varepsilon[(u^{m+1} - v^{m+1})^k] \tau_r \lambda_0^n, \tag{7} \]

we get a relation in which we make \( n \) and \( r \) (in this order) go to infinity to obtain (the terms containing \( \nabla \lambda_0^n \) and \( \nabla \tau_r \) disappear)

\[
\int_\Omega \hat{a}(x, u, \nabla u) \cdot \nabla \{h_m(u)h_m(v)J_\varepsilon[(u^{m+1} - v^{m+1})^k]\} \\
+ \int_\Omega b|u|^{\gamma-1}uh_m(u)h_m(v)J_\varepsilon[(u^{m+1} - v^{m+1})^k] \\
= \int_\Omega \mu h_m(u)h_m(v)J_\varepsilon[(u^{m+1} - v^{m+1})^k]. \tag{8}
\]

Let us “interchange” the roles of \( u \) and \( v \): by taking in Definition 4.1 of renormalized solution (written for \( v \))

\[ \Phi(v) = h_m(v) \quad \text{and} \quad \varphi = h_m(u)J_\varepsilon[(u^{m+1} - v^{m+1})^k] \tau_r \lambda_0^n \tag{9} \]

we obtain, after making \( n \) and \( r \) go to infinity, a relation which, by subtraction from (8), gives

\[
\int_\Omega \{\hat{a}(x, u, \nabla u) - \hat{a}(x, v, \nabla v)\} \cdot \nabla \{h_m(u)h_m(v)J_\varepsilon[(u^{m+1} - v^{m+1})^k]\} \\
+ \int_\Omega b:\{u|^{\gamma-1}u - |v|^{\gamma-1}v\}h_m(u)h_m(v)J_\varepsilon[(u^{m+1} - v^{m+1})^k] = 0.
\]

Developing this last equation we can write it as

\[
(1)_m^{1_{\varepsilon}} + (1)_m^{2_{\varepsilon}} + (1)_m^{3_{\varepsilon}} + (2)_m^{\varepsilon} = (1)_m^{4_{\varepsilon}}, \tag{10}
\]
with

\[
(1)_{m}^{\varepsilon} = \int_{\Omega} h_{m}^{\prime}(u)h_{m}(v)J_{\varepsilon}\{(u^{m+1} - v^{m+1})^{k}\}\{\hat{a}(x, u, \nabla u) - \hat{a}(x, v, \nabla v)\} \cdot \nabla u,
\]

\[
(2)_{m}^{\varepsilon} = \int_{\Omega} b\{|u|^{-1}u - |v|^{-1}v\}h_{m}(u)h_{m}(v)J_{\varepsilon}\{(u^{m+1} - v^{m+1})^{k}\}
\]

Then

\[
(1)_{m}^{\varepsilon} + (1)_{m}^{2\varepsilon} + (2)_{m}^{\varepsilon} \leq (1)_{m}^{4\varepsilon}, \tag{11}
\]

because \((1)_{m}^{3\varepsilon} \geq 0\), according to assumption \((\hat{a}.3)\). Let us now make \(\varepsilon\) go to zero. Using the dominated convergence theorem, we get

\[
\lim_{\varepsilon \downarrow 0}(1)_{m}^{\varepsilon} = \int_{\Omega} h_{m}^{\prime}(u)h_{m}(v)\{\hat{a}(x, u, \nabla u) - \hat{a}(x, v, \nabla v)\} \cdot \nabla u \equiv (1)_{m}^{1},
\]

\[
\lim_{\varepsilon \downarrow 0}(1)_{m}^{2\varepsilon} = \int_{\Omega} h_{m}(u)h_{m}(v)\{\hat{a}(x, u, \nabla u) - \hat{a}(x, v, \nabla v)\} \cdot \nabla v \equiv (1)_{m}^{2},
\]

and

\[
\lim_{\varepsilon \downarrow 0}(2)_{m}^{\varepsilon} = \int_{\Omega} b\{|u|^{-1}u - |v|^{-1}v\}h_{m}(u)h_{m}(v) \equiv (2)_{m}.
\]

Taking the assumption \((\hat{a}.4)\) into account, we can estimate the remaining term as follows:

\[
|(1)_{m}^{4\varepsilon}| = \left| \int_{\{\Omega\}^{m}} h_{m}(u)h_{m}(v)J_{\varepsilon}^{\prime}(u - v) \cdot \nabla (u - v) \right|
\]

\[
\times \{\hat{a}(x, v, \nabla v) - \hat{a}(x, u, \nabla v)\} \cdot \nabla (u - v)\right|
\]

\[
\leq \frac{1}{K(\varepsilon)} \int_{\{|u|, |v| \leq m+1\}} a\left\{|u|^{\alpha}|\nabla u| + |v|^{\beta}|\nabla u| + |\nabla v|^{p-1}|\nabla u| + a_{1}|\nabla u| + |u|^{\alpha}|\nabla v| + |v|^{\beta}|\nabla v| + |\nabla v|^{p} + a_{2}|
\]

The fact that \(u\) and \(v\) belong to \(\Theta_{a,loc}^{1}\) \(\cap L^{7}(\Omega, b)\) and the hypothesis on \(a, b\) imply that the last integral has a meaning on \(\{|u|, |v| \leq m+1\}\). Then, noticing that \(K(\varepsilon) \to 0, \infty\), we get

\[
\lim_{\varepsilon \downarrow 0}(1)_{m}^{4\varepsilon} = 0.
\]
Therefore, the passage to the limit with respect to ε in (11) gives

\[ (1)_m^1 + (1)_m^2 + (2)_m \leq 0, \quad \forall m \in \mathbb{N}^*. \] (12)

Now, making \( m \) go to infinity, since \( u, v \in \Theta^{1,p}_{\text{loc},a}(\Omega_S, a) \cap L^\gamma(\Omega, a) \), using Proposition 5.1, we get

\[ \lim_{m \to \infty} (1)_m^1 = \lim_{m \to \infty} (1)_m^2 = 0; \]

and, by the dominated convergence theorem,

\[ \lim_{m \to \infty} (2)_m = \int_{\Omega \cap \{u > v\}} b\{|u|^{\gamma-1}u - |v|^{\gamma-1}v\}. \]

Then, the relation (12) gives

\[ \int_{\Omega \cap \{u > v\}} b\{|u|^{\gamma-1}u - |v|^{\gamma-1}v\} = 0. \]

This proves that \( u \leq v \) almost everywhere in \( \Omega \). Finally, changing the roles of \( u \) and \( v \), we get \( u = v \) almost everywhere in \( \Omega \). \( \square \)

In the above uniqueness theorem we can replace the condition that \( \hat{a} \) is Lipschitz with respect to the second variable by the condition that \( \hat{a} \) is Hölder with respect to the same variable. But, we must strengthen the assumption of monotonicity of \( \hat{a} \) to get uniqueness.

**Theorem 5.2.** Suppose that \( \hat{a} \) satisfies the condition

\[ \hat{a} \text{ is Lipschitz with respect to the second variable} \]

\[ \hat{a} \text{ is Hölder with respect to the same variable. But, we must strengthen the assumption of monotonicity of \( \hat{a} \) to get uniqueness.} \]

\[ \hat{a} \text{ satisfies the condition} \]

\[ \forall \xi, \zeta \in \mathbb{R}^N \text{ and } x \text{ almost everywhere in } \Omega \text{ one has} \]

\[ [\hat{a}(x, s, \xi) - \hat{a}(x, s, \zeta)] \cdot [\xi - \zeta] \geq a(x)|\xi - \zeta|^p \quad \text{if} \quad p \geq 2 \] (13)

and

\[ [\hat{a}(x, s, \xi) - \hat{a}(x, s, \zeta)] \cdot [\xi - \zeta] \geq a(x)\frac{|\xi - \zeta|^2}{\eta + |\xi| + |\zeta|^{2-p}} \quad \text{if} \quad 1 < p < 2, \] (14)

where \( \eta \geq 0 \) is a constant. Then, under the hypotheses of Theorem 5.1 where the condition (\( \hat{a}\hat{A} \)) is replaced by

\[ \hat{a}\hat{A} \text{ there exists a nondecreasing function } d : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that} \]

\[ \int_{0+}^1 d^{-p}(s) \, ds = +\infty \quad \text{for} \quad p \geq 2 \] (15)

and

\[ \int_{0+}^1 d^{-2}(s) \, ds = +\infty \quad \text{for} \quad 1 < p < 2 \] (16)

with the property that

\[ |\hat{a}(x, r, \xi) - \hat{a}(x, s, \xi)| \leq a(x)d(|r - s|)\{|r|^r + |s|^r + |\xi|^{p-1} + a_4(x)\}, \] (17)
\[ \forall r, s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N \text{ and } x \text{ almost everywhere in } \Omega, \text{ where } 0 \leq \epsilon < \gamma/p' \text{ and } \alpha_1 \in L^{p'}(\Omega, a), \alpha_1 \geq 0, \text{ the conclusion of this Theorem 5.1 remains true.} \]

**Proof.** We argue as in the proof of Theorem 5.1, and as in this theorem, we give the proof when the condition \((l_w)\) is satisfied; in the other cases the proofs can be carried out in the same way. We write that \( u \) is a renormalized solution by taking \( \Phi(u) \) and \( \varphi \) as in (7), but here we choose, in the case \( p \geq 2 \), as \( J_\varepsilon \) the function given by

\[
J_\varepsilon(t) = \begin{cases}
\frac{1}{K(\varepsilon)} \int_0^t d^{-p'}(s) ds & \text{if } \varepsilon \leq t \leq k, \\
0 & \text{if } t \leq \varepsilon,
\end{cases}
\]

where \( K(\varepsilon) = \int_0^\varepsilon d^{-p'}(s) ds \). Arguing as in the proof of Theorem 5.1, we arrive to

\[
(1)_{m}^{1\varepsilon} + (2\varepsilon)_{m} + (3\varepsilon)_{m} + (2)_{m}^{\varepsilon} = (1)_{m}^{4\varepsilon}.
\]

(18)

Here we do not drop the positive term \( (1)_{m}^{3\varepsilon} \); it will help us to absorb a term which will come from a majoration of \( (1)_{m}^{3\varepsilon} \). In fact, using (13) and (17), we can write

\[
(1)_{m}^{1\varepsilon} + (2\varepsilon)_{m} + \int_{\{|u|,|v| \leq m^{\frac{1}{2}} \}} h_m(u)h_m(v)J'_\varepsilon(u-v)a|\nabla(u-v)|^p + (2)_{m}^{\varepsilon} \leq \int_{\{|u|,|v| \leq m^{\frac{1}{2}} \}} h_m(u)h_m(v)J'_\varepsilon(u-v)a|\nabla(u-v)|^p|u|^\varepsilon + |v|^\varepsilon + |\nabla v|^{p-1} + \alpha_1|\nabla(u-v)|.
\]

And, as in [14], we use Young’s inequality to write (with \( \beta > 0 \))

\[
d(u-v)(|u|^\varepsilon + |v|^\varepsilon + |\nabla v|^{p-1} + \alpha_1)|\nabla(u-v)| \leq \frac{\beta p}{p} |\nabla(u-v)|^p + \frac{\beta-p'}{p'} |d^{p'}(u-v)(|u|^\varepsilon + |v|^\varepsilon + |\nabla v|^{p-1} + \alpha_1)|^{p'},
\]

and, taking \( \beta = \left(\frac{2}{3}\right)^{\frac{1}{p}} \), we get

\[
(1)_{m}^{1\varepsilon} + (2)_{m}^{2\varepsilon} + (1')_{m}^{3\varepsilon} + (2)_{m}^{\varepsilon} \leq (1')_{m}^{4\varepsilon},
\]

(19)

where

\[
(1')_{m}^{3\varepsilon} = \frac{1}{2} \int_{\{|u|,|v| \leq m^{\frac{1}{2}} \}} h_m(u)h_m(v)J'_\varepsilon(u-v)a|\nabla(u-v)|^p
\]

and

\[
(1')_{m}^{4\varepsilon} = \frac{p-1}{2K(\varepsilon)} \int_{\{|u|,|v| \leq m^{\frac{1}{2}} \}} h_m(u)h_m(v)a\{|u|^\varepsilon + |v|^\varepsilon + |\nabla v|^{p-1} + \alpha_1\}^{p'}.
\]

Now we drop \( (1')_{m}^{3\varepsilon} \) and make \( \varepsilon \) go to zero in (19). As

\[
\lim_{\varepsilon \to 0} (1')_{m}^{4\varepsilon} = 0 \quad \text{because } K(\varepsilon) \to 0.
\]
we get the analog of (12). We continue as in the proof of Theorem 5.1.

In the case $1 < p < 2$ the same argument works. But, in the definition of $J_\varepsilon$, we have to replace everywhere $p'$ by 2. We make the passages to the limits with respect to $n$ and $r$ to obtain the analog of (18). We use (14) and (17) to get

\[
(1)_m^c + (1)_m^{2c} + \int_{\{|u|,|v| \leq m + 1\} \setminus \{|u| - v| < \varepsilon\}} h_m(u)h_m(v)J'_\varepsilon(u - v)a\frac{|\nabla(u - v)|^2}{(\eta + |\nabla u| + |\nabla v|)^2+p} + (2)_m^c
\]

\[
\leq \int_{\{|u|,|v| \leq m + 1\} \setminus \{|u| - v| < \varepsilon\}} h_m(u)h_m(v)J'_\varepsilon(u - v)ad(u - v)\{|u| + |v| + |\nabla v|^{p_1} + a_1\}|\nabla(u - v)|.
\]

Writing

\[
\frac{|\nabla(u - v)|}{(\eta + |\nabla u| + |\nabla v|)^2+p}d(u - v)\{|u| + |v| + |\nabla v|^{p_1} + a_1\}
\]

\[
\leq \frac{\beta^2}{2} \frac{|\nabla(u - v)|^2}{(\eta + |\nabla u| + |\nabla v|)^2+p} + \frac{\beta - 2}{2}d^2(u - v)\{|u| + |v| + |\nabla v|^{p_1} + a_1\}
\]

with $\beta > 0$ arbitrary and taking it sufficiently small, one gets after transposition and majoration

\[
(1)_m^c + (1)_m^{2c} + C(\beta) \int_{\{|u|,|v| \leq m + 1\} \setminus \{|u| - v| < \varepsilon\}} h_m(u)h_m(v)J'_\varepsilon(u - v)a\frac{|\nabla(u - v)|^2}{(\eta + |\nabla u| + |\nabla v|)^2+p} + (2)_m^c
\]

\[
\leq C(\beta) \int_{\{|u|,|v| \leq m + 1\} \setminus \{|u| - v| < \varepsilon\}} h_m(u)h_m(v)\frac{ad^2(u - v)}{K(\varepsilon)d^2(u - v)}\{|u|^{2c} + |v|^{2c}
\]

\[
+ |\nabla u|^{2(p_1-1)+a_1^2}\{|u|^{2} + |\nabla u|^{2(p_1-1)} + a_2^2\}
\]

\[
\leq C(\beta,\eta) \int_{\{|u|,|v| \leq m + 1\}} a\left\{|u|^{2c} + |v|^{2c} + |\nabla u|^{2(p_1-1)} + a_1^2 + |u|^{2c}|\nabla u|^{2-p}
\]

\[
+ |v|^{2c}|\nabla u|^{2-p} + |\nabla u|^{2(p_1-1)}|\nabla u|^{2-p} + a_2^2|\nabla u|^{2-p}
\]

\[
+ |u|^{2c}|\nabla u|^{2-p} + |v|^{2c}|\nabla u|^{2-p} + |\nabla u|^p + a_2^2|\nabla u|^{2-p}\right\}.
\]

But, for $1 < p < 2$, one has $2 - p < p$, $2(p_1-1) < p$ and $p' > 2$. This permits us to see that all the integrals of the last right-hand side make sense. We finish the proof as above.

To prove a theorem of stability we need the following result.

5.3. A priori estimates of renormalized solutions.

**Proposition 5.2.** Suppose the hypotheses of Theorem 5.1 are satisfied. Then, for every
renormalized solution \( v \) of the equation (1), the following estimates hold true:

\[
\int_{\Omega} a |\nabla \Phi(v)|^p \leq \int_{\Omega} |\mu(x)| \int_{0}^{\nu(x)} |\Phi'(s)|^p \, ds \, dx
\leq |\mu|_{L^1(\Omega)} |\Phi'|^p_{L^p(\mathbb{R})}, \quad \forall \Phi \in Lip_p(\mathbb{R}); \tag{20}
\]

\[
\int_{\Omega} a \frac{|\nabla v|^k}{(1 + |v|)^{1+\delta}} \leq \frac{1}{\delta} |\mu|_{L^1(\Omega)}, \quad \forall k > 0, \quad \forall \delta > 0; \tag{21}
\]

\[
\int_{\Omega} a \frac{|\nabla v|^p}{(1 + |v|)^{1+\delta}} \leq \frac{1}{\delta} |\mu|_{L^1(\Omega)}, \quad \delta > 0; \tag{22}
\]

\[
\int_{\Omega} b|v|^\gamma \leq \int_{\Omega} |\mu|, \tag{23}
\]

**Proof.** Let us fix \( \Phi \in Lip_p(\mathbb{R}) \) and take in the Definition 4.1 of a renormalized solution

\[\Phi^*(v) = h_m(v) \quad \text{and} \quad \varphi = \psi(v) \tau_r \lambda^0_n,\]

where \( h_m \) and \( \tau_r \) are the functions given in the Proposition 5.1 and its proof, \( r \) is chosen such that \( B(0, 2r) \supset S \). \( \psi(t) = \int_{0}^{t} |\Phi'(s)|^p \, ds \), and \( \{ \lambda^0_n \} \) is the sequence of Lemma 5.2. By making first \( n \) and second \( r \) go to infinity, we get the equality

\[
\int_{\Omega} h_m'(v) \psi'(v) \hat{a}(x, v, \nabla v) \cdot \nabla v + \int_{\Omega} h_m(v) |\Phi'(v)|^p \hat{a}(x, v, \nabla v) \cdot \nabla v
+ \int_{\Omega} b|v|^\gamma - v h_m(v) \psi(v) = \int_{\Omega} f h_m(v) \psi(v),
\]

which, by (a.1), Proposition 3.1.a, and the positiveness of the term containing \( b \), gives

\[
\int_{\Omega} h_m(v) a |\nabla \Phi(v)|^p \leq \int_{\Omega} f h_m(v) \psi(v) - \int_{\Omega} h_m'(v) \psi(v) \hat{a}(x, v, \nabla v) \cdot \nabla v.
\]

Making now \( m \) go to infinity and using the fact that \( v \in \Theta_{aloc}^{1,p}(\Omega_S, a) \cap L^\gamma(\Omega, b) \) and Fatou’s lemma, we get

\[
\int_{\Omega} a |\nabla \Phi(v)|^p \leq \int_{\Omega} |\mu(x)| \int_{0}^{\nu(x)} |\Phi'(s)|^p \, ds \, dx \leq |\mu|_{L^1(\Omega)} |\Phi'|^p_{L^p(\Omega)};
\]

this proves the estimate (20).

To prove the estimate (21), we argue as above by fixing \( k \) and \( \delta \) and taking in the Definition (4.1) of renormalized solution

\[\Phi^*(v) = h_m(v) \quad \text{and} \quad \varphi = \Phi_\delta(v^k) \tau_r \lambda^0_n, \quad \text{with} \quad \Phi_\delta(t) = \int_{0}^{t} \frac{d\sigma}{(1 + |\sigma|)^{1+\delta}},\]
To prove (22) we make $k$ go to infinity in (21). To prove (23) we take

$$\Phi(v) = h_m(v)$$

and $\varphi = T_1(\frac{v}{\varepsilon})\tau \lambda_n^0$, $\varepsilon > 0$,

with $T_1(t) = \frac{1}{2}(|t + 1| - |t - 1|)$ ($t \in \mathbb{R}$), drop the positive term

$$\frac{1}{\varepsilon} \int_\Omega h_m(v)\tau \lambda_n^0 T_1(\frac{v}{\varepsilon})\hat{a}(x, v, \nabla v) \cdot \nabla v,$$

and make $n$, $r$, and $m$ go (in this order) to infinity; the contributions of terms containing $\nabla \lambda_n^0$, $\nabla \tau$, and $h'_m(v)$ vanish. Finally, making $\varepsilon$ go to zero and using Fatou’s lemma, we get (23).

6. About the existence of solutions. The questions of existence of weak and renormalized solutions are studied in [2]. We think it is useful to recall here the definitions of these two types of solutions and some facts concerning them.

6.1. Weak solution for $\mu \in L^1(\Omega)$. We define the T-subset $\Lambda^{1,p}_{loc}(\Omega_S, a, \mu)$ as follows. It is the subset of $L^{1,p}_{loc}(\Omega_S)$ consisting of all functions $v$ with the property that, $\forall \Phi \in Lip_p(\mathbb{R})$

$$\int_\Omega a|\nabla \Phi(v)|^p \leq \int_\Omega |\mu(x)| \left| \int_0^{v(x)} |\Phi'(t)|^p dt \right| dx.$$  \hspace{1cm} (24)

Let $W_0$ be the subset of functions $u$ of $\Lambda^{1,p}_{loc}(\Omega_S, a, \mu) \cap L^\gamma(\Omega, b)$ with the property that there exists a sequence $\{u_n\} \subset W^{1,p}_0(\Omega)$ such that

$$u_n \longrightarrow u \text{ a.e. in } \Omega \text{ and } \sup_{n \geq 1} \int_\Omega a \frac{|\nabla u_n|^p}{(1 + |u_n|)^{1+\delta}} < \infty, \forall \delta > 0.$$

Definition 6.1. A function $w$ is called a weak solution of the generalized homogeneous Dirichlet problem associated with the equation $Au = \mu \in L^1(\Omega)$ on $\mathcal{D}'(\Omega_S)$ if

$$w \in W_0, \quad I(w, \varphi) = \langle \mu, \varphi \rangle, \forall \varphi \in \mathcal{D}(\Omega_S).$$  \hspace{1cm} (E)

6.2. Existence of weak solution.

Theorem 6.1. Assume conditions (ab) and \((\hat{a}.1–3)\) are satisfied. Then the problem (E) has at least one weak solution on $\mathcal{D}'(\Omega_S)$.

The proof of this theorem needs several steps: Approximation of the degenerate and irregular problem (E) by a sequence of nondegenerate and regular ones, uniform estimates of the approximate solutions and compactness of their gradients, and the passage to the limit.

We approximate the problem (E) by the sequence of problems

$$u_n \in V, \quad I_n(u_n, \varphi) = \int_\Omega \mu_n \varphi, \forall \varphi \in \mathcal{D}(\Omega_S),$$  \hspace{1cm} (E_n)
where
\[ I_n(w, \varphi) = I(w, \varphi) + \frac{1}{n} \int_{\Omega} \{ |\nabla w|^{p-2} \nabla w \cdot \nabla \varphi + |w|^{p-2} w \varphi \}, \]

\( \{ \mu_n \} \) is a sequence of \( D(\Omega) \) such that
\[ \mu_n \rightharpoonup \mu \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad |\mu_n|_{L^1(\Omega)} \leq |\mu|_{M(\Omega)}, \quad \forall n \geq 1, \]

and \( V = \{ v \in W^{1, p}_0(\Omega) : \int_{\Omega} q|\nabla v|^p < \infty \} \cap L^{r+1}(\Omega, b) \). The existence of \( u_n \) results from Theorem 2.7 of [17], page 180.

Using appropriate test functions, we can prove the following:

Lemma 6.1. The approximate solutions satisfy all the estimates of Proposition 5.2 where \( a \) is replaced by \( a + \frac{1}{n} \) and the estimate (23) by
\[ \int_{\Omega} \{ b|u_n|^\gamma + \frac{1}{n} |u_n|^{p-1} \} \leq \int_{\Omega} |\mu_n| \leq C, \quad \forall n \geq 1. \quad (25) \]

Let \( \Phi_0 = \arctan \); as the sequence \( \{ v_n \} = \{ \Phi_0(u_n) \} \) is bounded in \( L^\infty(\Omega) \), we may assume that
\[ v_n \rightharpoonup v \quad \text{in} \quad L^\infty(\Omega) \quad \text{weak-} \star. \]

Put \( u = \tan v \). Using the previous lemma, reflexivity of Sobolev spaces, the theorem of Rellich-Kondrachov, and those of Vitali, we get the following result:

Lemma 6.2. a) For all \( \omega \subset \subset \Omega_S \), the sequence \( \{ v_n \} = \{ \arctan u_n \} \) contains a subsequence \( \{ v_\nu \} = \{ \Phi_0(u_\nu) \} \) such that
\[ \Phi_0(u_\nu) \rightharpoonup \Phi_0(u) \quad \text{weakly in} \quad W^{1, q}(\omega), \quad q \in [1, p] \quad \text{and} \quad u_\nu \rightharpoonup u \quad \text{a.e. in} \ \omega. \]

In addition, \( u \in L^\gamma(\Omega, b) \) and, if \( b \in L^1(\Omega) \),
\[ \lim_{n \to \infty} \int_{\Omega} b(x)|u_n - u|^\beta = 0, \quad \forall \beta \in (0, \gamma) \quad (\text{for the whole sequence}). \]

Hence (for a subsequence denoted in the same way) \( u_n \rightharpoonup u \) almost everywhere in \( \Omega \).

b) For all \( \Phi \in Lip_p(\mathbb{R}) \), one has \( \Phi(u) \in W^{1, \hat{p}}(\Omega_S) \). In particular, \( u^k = T_k(u) \in W^{1, \hat{p}}_{loc}(\Omega_S) \), \( \forall k > 0 \).

c) For all \( \omega \subset \subset \Omega_S \), there exists \( C = C(\omega, a) > 0 \) such that
\[ \sup_{k > 0} \int_{\omega} \frac{|\nabla u^k|^p}{(1 + |u^k|)^{1+\delta}} \leq \frac{C}{\delta} |\mu|_{L^1(\Omega)}, \quad \forall \delta > 0. \]

d) For all \( k > 0 \) and all \( \varphi \in D(\Omega_S) \), there exists \( C > 0 \) such that
\[ \lim_{n \to \infty} \sup_{\varepsilon > 0} \int_{\{|u_n - u_k| \leq \varepsilon\}} \varphi \hat{u}(x, u_n, \nabla u_n) \cdot \nabla (u_n - u_k) \leq \varepsilon, \quad \forall \varepsilon > 0. \]

The compactness of gradients of approximate solutions is a consequence of the two following fundamental lemmas:
Lemma 6.3. Assume that $a \in L^1(\Omega)$. Let $\{u_n\} \subset L^{1,p}_{loc}(\Omega_S)$ such that
\[
\sup_{n \geq 1} \int_{\Omega} |u_n|^\gamma < \infty \quad \text{and} \quad \sup_{n \geq 1} \int_{\Omega} \frac{\left|\nabla u_n\right|^p}{(1 + |u_n|)^{1+\delta}} < \infty, \ \forall \delta > 0.
\]
Then
\[
\sup_{n \geq 1} \int_{\Omega} a|\nabla u_n|^{q(p-1)} < \infty, \ \forall q \in (1, \frac{\gamma}{\gamma + 1}p').
\]

Lemma 6.4. Assume that $a$ satisfies condition (ab) and (ã,1-3). Let $\{u_n\} \subset L^{1,p}_{loc}(\Omega_S)$ be a sequence having the following properties:

a) There exist $u \in L^{1,p}_{loc}(\Omega_S)$, $q \in (1, \frac{\gamma}{\gamma + 1}p')$, and a constant $C > 0$ such that
\[
u_n \longrightarrow u \quad \text{a.e. in } \Omega \quad \text{and} \quad \int_{\Omega} a|u_n|^{q(p-1)} + \int_{\Omega} a|\nabla u_n|^{q(p-1)} \leq C, \ \forall n \geq 1;
\]
b) for all $k > k_0 \geq 0$, the sequence $\{u^k_n\}$ remains in a bounded subset of $W^{1,p}_{loc}(\Omega_S)$;

c) for all $k > k_0 \geq 0$, and all $\varphi \in \mathcal{D}(\Omega_S)$, there exist two functions $\theta_k$ and $\hat{\theta}$ such that
\[
\limsup_{n \to \infty} \int_{\Omega} \varphi \hat{\theta}(x, u_n, \nabla u_n) - \nabla(u_n - u^k) \leq \theta_k(\varepsilon) + \hat{\theta}(k), \ \forall \varepsilon \in [0, \varepsilon_0],
\]
where $\varepsilon_0 > 0$, $\lim_{\varepsilon \to 0} \theta_k(\varepsilon) = 0$, $\hat{\theta}$ depending only on $k$, and $\lim_{k \to \infty} \theta(k) = 0$.

Then
\[
\int_{\Omega} a|\nabla u|^{q(p-1)} < \infty \quad \text{and} \quad \lim_{n \to \infty} \int_{\Omega} a|\nabla(u_n - u)|^{s(p-1)} = 0, \ \forall s \in (0, q),
\]
and, for a subsequence
\[
\nabla u_n' \longrightarrow \nabla u \quad \text{a.e. in } \Omega.
\]

Using the convergences of approximate solutions and their gradients, inequalities (2) and (3), the hypotheses (ab) and (ã,2), Lemmas 6.1 and 6.3, and Vitali's convergence theorem, we can pass to the limit to see that $u$ is a weak solution of the problem (E).

6.3. Renormalized solution of the homogeneous Dirichlet problem. We call $\Gamma_{ns} = \Gamma \setminus S$ the nonsingular part of the boundary $\Gamma$ of $\Omega$. For $\Gamma$ bounded this part can be empty. For $\Gamma_{ns}$ smooth, we need the almost local T-set $T^{1,p}_{aloc,\bullet}(\Omega_S, a)$ and subset $\Theta^{1,p}_{aloc,\bullet}(\Omega_S, a)$ which contain an information on the trace on this part of the boundary.

\[
T^{1,p}_{aloc,\bullet}(\Omega_S, a) = \{ v \in T^{1,p}_{aloc}(\Omega_S, a) : \forall \Phi \in Lip_p(\mathbb{R}), \Phi(v) = 0 \ \text{on } \Gamma_{ns} \}\n\]

and

\[
\Theta^{1,p}_{aloc,\bullet}(\Omega_S, a) = T^{1,p}_{aloc,\bullet}(\Omega_S, a) \cap \Theta^{1,p}_{aloc}(\Omega_S, a).
\]

Definition 6.2. A function $w$ is called a renormalized solution of the generalized homogeneous Dirichlet problem associated with the equation $\mathcal{A}u = \mu \in L^1(\Omega)$ if
\[
\begin{cases}
w \in \Theta^{1,p}_{aloc,\bullet}(\Omega_S, a) \cap L^p(\Omega, b), \quad I(w, \Phi(w)) = \int_{\Omega} \mu \Phi(w), \\
\forall \varphi \in L^p(\Omega) \cap W^{1,p}_{0}(\Omega, S), \forall \Phi \in W^{1,\infty}_c(\mathbb{R}),
\end{cases}
\]

where
\[
W^{1,p}_{0}(\Omega, S) = \{ \varphi \in W^{1,p}_{0}(\Omega) : \exists S \in \mathcal{N}(S) \text{ such that } \varphi \equiv 0 \text{ on } \Omega \cap S \}.
\]
Theorem 6.2. Under the hypotheses of Theorem 5.1 or those of Theorem 5.2 the problem \((\mathcal{E})_{\text{ren}}\) has at most one renormalized solution.

Proof. The proofs of Theorem 5.1 and Theorem 5.2 work here. It suffices to check that, for \(u\) and \(v\) in \(\Theta_{\text{aloc,s}}^{1,p}(\Omega, a) \cap L^\gamma(\Omega, b)\) the two functions \(\varphi\) chosen in (7) and (9) as test functions are as well in \(W_0^{1,p}(\Omega, S)\).


Theorem 6.3. If \(\hat{a}\) satisfies the condition (\(\hat{a}, 2\)) of Theorem 5.1, every renormalized solution of the generalized homogeneous Dirichlet problem \((\mathcal{E})_{\text{ren}}\), which is in \(W_0\), is a weak solution of the problem \((\mathcal{E})\).

Theorem 6.4. Under the hypotheses of Theorem 6.1, the problem \((\mathcal{E})_{\text{ren}}\) possesses one unique solution. More precisely, the function \(u\) built in 6.2, which is a weak solution of \((\mathcal{E})\), is also the unique solution of \((\mathcal{E})_{\text{ren}}\).

The uniqueness results have been dealt with in Theorem 6.2. To prove the existence we make use of the important result:

Proposition 6.1. For all \(k > 0\), the sequence \(\{u_n^k\}\) is strongly convergent to \(u^k\) in \(W^{1,p}_\text{loc}(\Omega_p)\) as \(n \to \infty\).

7. Continuity with respect to the second member.

Definition 7.1. A subset \(\mathcal{B}\) of \(L^{1,p}_\text{loc}(\Omega_p)\) is said to be bounded in this local T-set if

a) \(\forall \Phi \in Lip_p(\mathbb{R}), \Phi(\mathcal{B})\) is bounded in \(W^{1,p}_\text{loc}(\Omega_p)\),

b) \(\mathcal{B}\) is bounded in \(L^{1-\frac{2}{p}}_\text{loc}(\Omega_p)\), and
c) \(\forall \omega \subset \subset \Omega_p, \forall \delta > 0, \exists C = C(\omega, \delta) > 0\) such that

\[
\int_{\omega} \frac{|\nabla v|^p}{(1 + |v|)^{1+\delta}} \leq C, \forall v \in \mathcal{B}.
\]

Using essentially the Proposition 3.2, one can readily check that every subset \(\mathcal{B}\)

1. bounded in \(W^{1,p}_\text{loc}(\Omega_p)\) is also bounded in \(L^{1,p}_\text{loc}(\Omega_p)\);

2. bounded in \(L^{1-\frac{2}{p}}_\text{loc}(\Omega_p)\) is bounded in \(L^{s}_\text{loc}(\Omega_p)\), \(\forall s \in (0, \frac{N}{N-p}(p-1))\) if \(1 < p < N\) and \(\forall s > 0\) if \(p \geq N\);

3. bounded in \(L^{1,p}_\text{loc}(\Omega_p)\) is bounded in \(W^{1,q}_\text{loc}(\Omega_p)\), \(\forall q \in (0, \frac{N}{N-1}(p-1))\) provided \(2 - \frac{1}{q} < p \leq N\).

Definition 7.2. A sequence \(\{v_n\}\) of \(L^{1,p}_\text{loc}(\Omega_p)\) converges to \(v\) in this local T-set if

a) \(\{v_n\}\) is bounded in \(L^{1,p}_\text{loc}(\Omega_p)\) and

b) \(v_n \rightharpoonup v\) in \(L^{1-\frac{2}{p}}_\text{loc}(\Omega_p)\) and \(\nabla v_n \rightharpoonup \nabla v\) in \(L^{1-\frac{2}{p}}_\text{loc}(\Omega_p)^N\).

Again, using essentially the Proposition 3.2 and Vitali’s convergence theorem, one can readily check that

1. Every sequence \(\{v_n\}\) of \(W^{1,p}_\text{loc}(\Omega_p)\) converging in this space to a function \(v\) converges to \(v\) in \(L^{1,p}_\text{loc}(\Omega_p)\).\)
2. Every sequence \( \{v_n\} \) of \( L^{1,p}_{loc}(\Omega_S) \) converging to a function \( v \) in this local T-sets converges to the same function in \( L^{1,p}_{loc}(\Omega_S) \), \( \forall s \in [0, \frac{N}{N-p}(p-1)] \) if \( 1 < p < N \) and \( \forall s > 0 \) if \( p \geq N \).

3. Every sequence \( \{v_n\} \) of \( L^{1,p}_{loc}(\Omega_S) \) converging to a function \( v \) in this local T-sets converges to the same function in \( W^{1,q}_{loc}(\Omega_S) \), \( \forall q \in (0, \frac{N}{N-p}(p-1)) \) provided \( 2 - \frac{1}{N} < p \leq N \).

**Theorem 7.1.** Suppose that the equation \( (1) \) has a renormalized solution for all \( \mu \in L^1(\Omega) \). Then, under the hypotheses of Theorem 5.1, the operator \( \pi \) which associates the right-hand side of this equation with its unique renormalized solution is continuous from \( L^1(\Omega) \) (equipped with its strong topology) into \( L^{1,p}_{loc}(\Omega_S) \) (equipped with its convergence).

**Proof of Theorem 7.1.** Let \( \mu \in L^1(\Omega) \) and \( v = \pi \mu \) the renormalized solution of equation \( (1) \) with \( \mu \) in the right-hand side. Let \( \{\mu_i\} \) be a sequence of \( L^1(\Omega) \) converging to \( \mu \) in this space. We have to prove that the sequence \( \{v_i\} = \{\pi \mu_i\} \) converges to \( \mu \) in \( L^{1,p}_{loc}(\Omega_S) \) to \( \pi \mu \). We need some intermediate results.

The \( v_i \)'s satisfy all the estimates of Proposition 5.2 with the \( \mu_i \)'s in the right-hand sides. As the sequence \( \{\mu_i\} \) is bounded in \( L^1(\Omega) \), the sequence \( \{v_i\} \) is bounded in \( L^{1,p}_{loc}(\Omega_S) \) and can assume, for a subsequence (denoted in the same way), that

\[
\omega_i = \arctan v_i \rightarrow w \quad \text{in} \quad L^\infty(\Omega) \quad \text{weak-*}.
\]

Putting \( z = \tan w \), we see that the sequence \( \{v_i\} \) and the function \( z \) satisfy all the assertions of Lemma 6.2. In particular,

\[
\lim_{i \to \infty} \int_{\Omega} b(x)|v_i - z|^\beta = 0, \quad \forall \beta \in (0, \gamma) \quad \text{(for the whole sequence)}.
\]

**Lemma 7.1.** For all \( k > 0 \) and all \( \varphi \in D(\Omega_S) \), there exists \( C > 0 \) such that

\[
\limsup_{i \to \infty} \int_{\{v_i - z_k \leq \varepsilon\}} \varphi \hat{a}(x, v_i, \nabla v_i) \cdot \nabla(v_i - z_k) \leq C\varepsilon, \quad \forall \varepsilon > 0.
\]

**Proof of Lemma 7.1.** Let \( k > 0, \varphi \in D(\Omega_S), \) and \( \omega \subset \subset \Omega_S \) such that \( \text{supp} \varphi \subset \omega \). Writing the Definition 4.1 with

\[
\Phi(v_i) \doteq h_m(v_i) \quad \text{and} \quad "\varphi" \doteq \varphi T_\varepsilon(v_i^{m+1} - z_k)
\]

we get a relation in which we make \( m \) go to \( \infty \); by using Lebesgue’s dominated convergence theorem and the fact that \( v_i \in \Theta^{1,p}_{loc}(\Omega_S, a) \), we obtain

\[
\int_{\omega \cap \{v_i - z_k \leq \varepsilon\}} \varphi \hat{a}(x, v_i, \nabla v_i) \cdot \nabla(v_i - z_k) = \int_{\omega} \varphi f T_\varepsilon(v_i - z_k) - \int_{\omega} T_\varepsilon(v_i - z_k) \hat{a}(x, v_i, \nabla v_i) \cdot \nabla \varphi - \int_{\omega} \varphi \delta |v_i|^{-1} v_i T_\varepsilon(v_i - z_k).
\]
Since the $v_i$’s satisfy the estimates of Proposition 5.2 with the $\mu_i$’s in the right-hand sides, we finish the proof of our lemma by using Lemma 6.3.

As the $v_i$’s satisfy the estimates of Proposition 5.2 and the assertions of Lemma 6.2, it follows from Lemma 6.3 and 7.1 that the sequence $\{v_i\}$ and the function $z$ satisfy all the hypotheses of Lemma 6.4 with $q$ arbitrary in $(1, \frac{2}{7} + p')$. Hence,

$$\lim_{i \to \infty} \int_\Omega a|\nabla (v_i - z)|^{s(p-1)} = 0, \quad \forall s \in (0, q)$$

and, since $\{\mu_i\}$ converges strongly to $\mu$ in $L^1(\Omega)$, we can extract subsequences $\{v_j\}$ and $\{\mu_j\}$ such that

$$v_j \to z, \quad \nabla v_j \to \nabla z \quad \text{and} \quad \mu_j \to \mu \quad \text{a.e. in } \Omega. \quad (26)$$

This allows us to show that $z$ satisfies the estimates of Proposition 5.2. Therefore, $z \in L^{1, p}(\Omega_S)$ and $\{v_j\}$ converges to $z$ in this local T-set.

To prove that $z \in \Theta_{adoc}^{1, p}(\Omega_S, a)$ we write (20) for $z$ by choosing $\Phi = h_{rm}$ with

$$h_{rm}(t) = \begin{cases} 
1 & \text{if } -m + r < \sigma < m - r \\
\frac{1}{2}(m + r - \sigma) & \text{if } m - r \leq \sigma \leq m + r \\
0 & \text{if } |\sigma| > m + r \\
\frac{1}{2}(m + r + \sigma) & \text{if } -m - r \leq \sigma \leq -m + r
\end{cases}$$

and make $m$ go to $\infty$; by using the chain rule and Lebesgue’s theorem we get the result.

**End of the proof of Theorem 7.1.** Using (26) we can prove that $z$ is a renormalized solution of equation (1). In fact, let $\Phi \in W^{1, \infty}(\mathbb{R})$ and $\varphi \in L^{\infty}(\Omega) \cap W^{1, p}(\Omega, S)$. Taking $h_m(v_j)\Phi(z)\varphi_{\tau_r}$ as a “test” function in Definition 4.1, we get

$$\int_\Omega h_m(v_j)\Phi(z)\varphi_{\tau_r} \hat{a}(x, v_j, \nabla v_j) \cdot \nabla v_j + \int_\Omega h_m(v_j)\hat{a}(x, v_j, \nabla v_j) \cdot \nabla \{\Phi(z)\varphi_{\tau_r}\}$$

$$+ \int_\Omega b|v_j|^{\gamma-1} v_j h_m(v_j)\Phi(z)\varphi_{\tau_r} = \int_\Omega \mu_j h_m(v_j)\Phi(z)\varphi_{\tau_r}.$$ 

Let us make $j$ go to infinity. By virtue of the dominated convergence theorem and Vitali’s convergence theorem, we have

$$\lim_{j \to \infty} \int_\Omega \mu_j h_m(v_j)\Phi(z)\varphi_{\tau_r} = \int_\Omega \mu h_m(z)\Phi(z)\varphi_{\tau_r}$$

$$\lim_{j \to \infty} \int_\Omega b|v_j|^{\gamma-1} v_j h_m(v_j)\Phi(z)\varphi_{\tau_r} = \int_\Omega b|z|^{\gamma-1} h_m(z)\Phi(z)\varphi_{\tau_r}.$$ 

To perform the passage to the limit in the second term, we remark that

$$h_m(v_j)\hat{a}(x, v_j, \nabla v_j) \to h_m(z)\hat{a}(x, z, \nabla z) \quad \text{in} \quad L^{p'}(\Omega \cap B(0, 2r))^N \quad \text{weakly;}$$
then
\[ \lim_{j \to \infty} \int_{\Omega} h_m(v_j)\hat{a}(x, v_j, \nabla v_j) \cdot \nabla \{ \Phi(z)\varphi_r \} = \int_{\Omega} h_m(z)\hat{a}(x, z, \nabla z) \cdot \nabla \{ \Phi(z)\varphi_r \}. \]

To pass to the limit (with respect to \( j \)) in the first term we use the fact that the \( v_j \)'s are in \( \Theta^1_{\text{loc}}(\Omega_S, a) \) and the sequence \( \{\mu_j\} \) is bounded in \( L^1(\Omega) \); we get
\[ \limsup_{j \to \infty} \int_{\Omega} h'_m(v_j)\Phi(z)\varphi_r \hat{a}(x, v_j, \nabla v_j) \cdot \nabla v_j = o_r(1)(m \to \infty), \]
with \( o_r \) a function depending on \( r \). Therefore, the passage to the upper limit with respect to \( j \) gives
\[
\int_{\Omega} h_m(z)\hat{a}(x, z, \nabla z) \cdot \nabla \{ \Phi(z)\varphi_r \} + \int_{\Omega} b|z|^{\gamma-1}z h_m(z)\Phi(z)\varphi_r \\
\quad = \int_{\Omega} \mu h_m(z)\Phi(z)\varphi_r + o_r(1)(m \to \infty).
\]
Letting now \( m \) and \( r \) (in this order) go to \(+\infty\), we obtain
\[
\int_{\Omega} \hat{a}(x, z, \nabla z) \cdot \nabla \{ \Phi(z)\varphi \} + \int_{\Omega} b|z|^{\gamma-1}z \Phi(z)\varphi = \int_{\Omega} \mu \Phi(z)\varphi.
\]
This means that \( z \) is a renormalized solution of the equation (1), hence the uniqueness Theorem 5.1 implies that \( z = v = \pi \mu \). This finishes the proof of Theorem 7.1.

**Remark 7.1.** In proving the above theorem we proved the continuity of the operator \( \pi \) from \( L^1(\Omega) \) into \( L^s(\Omega, b), \forall s \in (0, \gamma) \), with respect to the strong topologies. For \( 0 < s < 1 \) the space \( L^s(\Omega, b) \) is equipped with the distance
\[
\rho(u, v) = \int_{\Omega} b(x)|u(x) - v(x)|^s \, dx, \quad u, v \in L^s(\Omega, b).
\]

**REFERENCES**


