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SYMMETRY IN EXTERIOR BOUNDARY VALUE PROBLEMS FOR QUASILINEAR ELLIPTIC EQUATIONS VIA BLOW-UP AND A PRIORI ESTIMATES

Nicola Garofalo*
Department of Mathematics, Purdue University, West Lafayette IN 47907-1935

Elena Sartori
Dipartimento di Matematica Pura e Applicata, Università di Padova
via Belzoni 7, 35131 Padova, Italy

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Abstract. Given a connected, bounded open set $\Omega_1 \subset \mathbb{R}^n$, we use a maximum principle, and compactness arguments to study the properties of the function $P(u, x)$ in (1.5) below associated to a weak solution of the exterior p-capacitary problem,

$$\operatorname{div}(|Du|^{p-2}Du) = 0 \text{ in } \Omega = \mathbb{R}^n \setminus \overline{\Omega_1}, \qquad 1 < p < n,$$

$u = 1$ on $\partial\Omega_1$, $u(x) \to 0$ as $|x| \to \infty$. As a consequence of our results we prove spherical symmetry for the solution u and for the condenser Ω_1 when the overdetermined boundary condition $|Du| = c > 0$ on $\partial\Omega_1$ is imposed. This provides a new proof of a recent result of Reichel [31].

1. Introduction. This paper is concerned with compactness methods in boundary value problems in which the ground domain is the exterior of a bounded domain. We consider the following p-capacitary problem

$$\operatorname{div}(|Du|^{p-2}Du) = 0 \qquad \text{in } \ \Omega = \mathbb{R}^n \setminus \overline{\Omega_1} \tag{1.1}$$

$$u = 1 \quad \text{on } \partial\Omega_1, \qquad u \to 0 \qquad \text{as } |x| \to \infty, \tag{1.2}$$

where $\Omega_1 \subset \mathbb{R}^n$ is a connected, bounded open set, starlike with respect to the origin (which is assumed to belong to Ω_1). When Ω_1 is a ball $B(x_o, R) =$

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$\{x \in \mathbb{R}^n \mid |x - x_o| < R\}$, then (1.1), (1.2) admits a notable spherically symmetric solution

$$u(x) \ = \ \left(\frac{R}{|x - x_o|} \right)^{\frac{n-p}{p-1}} . \tag{1.3}$$

If one considers the function

$$P(u,x) \ = \ \frac{|Du(x)|^p}{u(x)^\alpha}, \qquad \alpha > 0,$$

then it is easy to see that the choice

$$\alpha \ = \ \frac{p(n-1)}{n-p}$$

gives

$$P(u,x) \equiv \left(\frac{n-p}{p-1} \right)^p R^{-p}. \tag{1.4}$$

We note in passing that $\frac{p(n-1)}{n-p} = \frac{p^*}{n'}$, where for $1 < p < n$ the number p^* denotes the Sobolev exponent relative to p, whereas n' is the dual exponent of n.

Guided by the above considerations we now turn the attention to a weak solution of (1.1), (1.2) and to the function

$$P(u,x) \ = \ \frac{|Du(x)|^p}{u(x)^{\frac{p(n-1)}{n-p}}} \tag{1.5}$$

associated to it. Our purpose is to study the properties of $P(u,x)$. We are interested in best possible maximum principles, a priori estimates, asymptotic behavior and integral identities. As a consequence of our study we obtain a new proof of a special case of a result recently established by Reichel using the method of moving hyperplanes of Alexandrov-Serrin. Precisely, we have the following:

Theorem 1.1. *Let u be a weak solution to problem* (1.1), (1.2) *in Ω, for some $1 < p < n$, then $|Du| = c > 0$ on $\partial\Omega_1$ if and only if Ω_1 is a ball $B(x_o, R)$ with radius $R = \frac{n-p}{(p-1)c}$. The solution is then spherically symmetric about the center x_o of this ball and it is given by* (1.3).

The appropriate notion of weak solution and boundary values are explained below. We stress that we do not make any a priori requirement on the smoothness of the ground domain Ω_1. In [31] Reichel has proved symmetry in overdetermined boundary value problems for exterior domains and for various nonlinear equations which also include the degenerate elliptic one in Theorem 1.1. In this respect his result is:

Theorem 1.2. *Let Ω_1 be of class $C^{2,\alpha}$. Suppose that there exists $u \in C^{1,\alpha}(\overline{\Omega})$ such that*

$$div(|Du|^{p-2}Du) \;=\; f(u), \qquad 0 \le u < 1 \quad in \quad \Omega, \tag{1.6}$$

where $p > 1$, and $f(u)$ is assumed decreasing and Lipschitz continuous on the interval $0 \le u \le 1$. Suppose that $u = 1$ and $\frac{\partial u}{\partial \eta} = const \le 0$ on $\partial\Omega_1$, then Ω_1 is a ball and u is spherically symmetric about the center of this ball.

Theorem 1.1 corresponds to the case $f(u) \equiv 0$ in (1.6). In contrast with Theorem 1.1, where we assume no smoothness of Ω_1 and the boundary values are taken in the sense (1.9), in Theorem 1.2 the domain Ω_1 is a priori assumed to be $C^{2,\alpha}$ and the solution $u \in C^{1,\alpha}(\overline{\Omega})$. On the other hand we assume starlikeness, whereas no such hypothesis is made by Reichel.

Our approach combines several ideas based upon the maximum principle, a priori estimates, compactness arguments, integral identities. An appealing aspect of this approach is that it provides various a priori geometric and analytic information on the problem at hand which become optimal for a given known configuration. In the case of Laplace equation $p = 2$, the corresponding function in (1.5) was first introduced by Payne and Philippin [23], who also proved that such function satisfies a strong maximum principle on the capacitary potential of a bounded domain in $\mathbb{R}^n$. On a bounded domain the idea of using integral identities and a maximum principle for a suitable combination of a solution of a pde and of its gradient to deduce symmetry is originally due to H. Weinberger. In the paper [40] he gave an alternative proof for the special case $\Delta u = -1$ of Serrin's celebrated symmetry theorem [35]. In [10] John Lewis and one of us showed that Weinberger's idea could be suitably adapted to treat quasilinear operators with degenerating ellipticity such as that in (1.1) in a bounded domain on which no a priori smoothness assumption is made. In a different direction Caffarelli, Garofalo ad Segala [7] used related forms of maximum principle and compactness arguments based on *a priori* $C^{1,\alpha}$ estimates to establish pointwise bounds for the gradient and Liouville type theorems for entire solutions of quasilinear equations whose leading part is modelled on (1.1) or on the mean curvature operator. Payne and Philippin have studied extensively what they call *best possible maximum principles* for *P-functions*. These are maximum principles for some special combination of u and $|Du|$ (a P-function) which is chosen *ad hoc* depending on the problem at hand. Here u solves some elliptic semilinear or quasilinear pde, and the P-function is such that there exists a distinguished geometric configuration for which it becomes constant. Payne and Philippin used such maximum principles to prove various interesting results in potential theory (see e.g. [25], [27], [23], [24]). In [22] and [26] they obtained a symmetry result for an overdetermined boundary value problem, for the Laplacian and the conformal n-Laplacian, on a starlike bounded ring. In a related context, analogous symmetry results for the p-Laplacian, $p \neq n$, were obtained in [2], [32].

In this paper we take up the ideas in [10], [7] and combine them with those in the cited works of Payne and Philippin to study the properties of the function $P(u,x)$ in (1.5) for the exterior capacitary problem (1.1), (1.2). The three basic properties of the function P are:

(I) P satisfies the strong maximum principle.

(II) P is constant on the spherically symmetric solution to (1.1), (1.2) when Ω_1 is a ball.

(III) P is invariant under the rescaling

$$u_r(x) \; = \; r^{\frac{n-p}{p-1}} u(rx), \tag{1.7}$$

in the sense that

$$P(u_r,x) \; = \; P(u,rx).$$

Property (I) was first established in [24], but the proof there does not contain full details. For the reader's convenience we present a complete, slightly different proof in section two for the P function relative to a class of equations which includes (1.1), see Theorem 2.1. We feel that the more general situation has an independent interest, is easier to work with, and also it makes more transparent the choice of the function (1.5), see Theorem 2.2. Concerning property (II), Theorem 2.4 below proves that the spherically symmetric configuration is best possible for P, in the sense that P attains its maximum value, and thereby it is constant, if and only if the ground domain in (1.1) is a ball. To prove this theorem we exploit Serrin's results in [33], [34], the local regularity theory for uniformly elliptic equations [19], and A.D. Alexandrov's theorem on the characterization of spheres [4].

To our knowledge the important property (III) had never been observed before, and its systematic use is new even in the linear case of Laplace equation. The scale invariance of the function P expressed by (III) has remarkable implications. In Section 3, we combine it with Theorems 2.2, 2.4 and the local a priori $C^{1,\alpha}$ estimates for (1.1) to prove Theorem 3.1. The latter states that if the function P takes its supremum at infinity, then $lim_{|x|\to\infty} P(u,x)$ exists and equals

$$K(n,p)\left(\frac{n\omega_n}{cap_p\Omega_1}\right)^{\frac{p}{n-p}}, \tag{1.8}$$

where $K(n,p)$ is the constant in (3.1) and $cap_p\Omega_1$ denotes the variational p-capacity of the domain Ω_1. The explicit knowledge of this limit plays a crucial role in the proof of Theorem 1.1. The possibility of extracting a precise information at infinity from local a priori estimates ultimately relies on a delicate blow-up argument which exploits (III). In a different context, related compactness arguments based on the rescaling (1.7) were used in [18] to determine the local asymptotic behavior of a singular solution of the p-Laplacian. We thank L. Veron for kindly bringing to our attention reference [18]. In Section 4, we establish some integral identities for solutions of (1.1), (1.2) which, combined with the results previously described, allow us to infer a sharp quantitative estimate of the function P in terms of the constant (1.8), see Theorem 4.5. Finally, in Section 5, we prove Theorem 1.1.

By a weak solution in the statement of Theorem 1.1 we mean a function $u \in W^{1,p}_{loc}(\Omega)$ for which

$$\int_\Omega |Du|^{p-2} < Du, D\phi > dx \ = \ 0 \qquad \text{for every} \quad \phi \in C^\infty_o(\Omega)$$

and such that

$$u(x) \to 0 \ \text{ as } \quad |x| \to \infty.$$

The boundary conditions are assumed to hold in the following sense, see [10].

Given $\epsilon > 0$, there exists an open set $U_{1,\epsilon} \supset \partial\Omega_1$ such that

$$|1-u(x)| < \epsilon, \quad ||Du(x)| - c| < \epsilon \quad \text{for a.e.} \quad x \in U_{1,\epsilon} \cap \Omega \tag{1.9}$$

with respect to the n-dimensional Lebesgue measure. Due to the degeneracy of the p-Laplacian at points where $|Du| = 0$, a solution to (1.1) is in general only $C^{1,\alpha}_{loc}(\Omega)$ (see [9], [20], [36]). The additional assumption $|Du| = c > 0$ on $\partial\Omega_1$ makes the equation non degenerate in a neighborhood of $\partial\Omega$. This observation, needs to be carefully implemented however, since the overdetermined condition on the gradient is assumed to hold in the weak sense (1.9). We refer the reader to [10], [3], and [38] for this aspect. In particular, using the work of Vogel [38], which is based on the deep results on free boundaries by Caffarelli, Alt, and Friedman (see, e.g., [5], [6]), one proves that the boundary of Ω_1 is a C^2 surface and so by Theorem 1 in [21], the solution is in fact $C^{1,\alpha}(\overline{\Omega})$.

Before closing it is appropriate to mention that the study of symmetry in overdetermined boundary value problems is a subject which has old origins, dating back to the pioneering and insightful ideas of Lord Rayleigh [29]. Its recent history starts with two famous papers, one in geometry by A.D. Alexandrov [4], the other one in analysis due to J. Serrin [35]. From the point of view of partial differential equations Serrin's paper has had a profound impact and a lasting influence. It has stimulated an enormous amount of research and there exists nowadays a continuously growing literature on the applications of the powerful Alexandrov-Serrin method of moving hyperplanes. Among the most prominent papers based on this method we cite the celebrated works of Gidas, Ni and Nirenberg [14], [15]. Up to very recently the subject has predominantly covered the case of uniformly elliptic operators on bounded domains, or on the entire space, leaving untreated many interesting open problems which are naturally formulated for degenerate nonlinear elliptic equations or for exterior domains. In [3] Alessandrini and one of us first succeeded in adapting Serrin's ideas to prove symmetry for non-uniformly parabolic equations in a bounded cylinder modelled on the following one

$$u_t \ - \ div(|Du|^{p-2}Du) \ = \ f(u,|Du|). \tag{1.10}$$

In his interesting paper [30] Reichel extended Serrin's ideas to establish symmetry in exterior domains for semilinear uniformly elliptic equations. Subsequently, Aftalion and Busca [1] have obtained some new results for exterior domains which, in some respects, substantially improve on those in [30]. Combining the ideas in his work [30] with those in [3] Reichel has succeeded in [31] in proving symmetry theorems for overdetermined boundary value problem for various nonlinear elliptic equations in exterior domains. We have already mentioned his Theorem 1.2. Simultaneously to (and independently from) [31], in [11] the authors of the present paper have proved that Serrin's method, adapted as in [3] and [30], can be used to establish symmetry for overdetermined boundary value problems in exterior domains for degenerate parabolic equations modelled on (1.10). We explicitly note that in the right hand side of (1.10) the dependence in $|Du|$ is allowed, contrary to Reichel's result for (1.6) in Theorem 1.2. However, because of the different nature of the boundary conditions in elliptic or parabolic problems, the results in [31] and [11] should be viewed as independent from one another, in the sense that Theorem 1.2 cannot be obtained from the main result in [11]. There exist of course a large number of important papers which are based on the method of moving hyperplanes. We have only cited those which are more directly connected to the present one.

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2. A Strong Maximum Principle. In this section we consider a (weak) solution to the equation

$$\operatorname{div}(\Phi'(|Du|^2)Du) = 0 \tag{2.1}$$

in an open set $\Omega \subset \mathbb{R}^n$. We assume that $\Phi(0) = 0$ and that $\Phi \in C^3(\mathbb{R})$ satisfy the following structural assumptions.

Hypothesis. *There exist $p > 1$, $a \geq 0$ and constants $c_1, c_2 > 0$ such that for every $\sigma, \xi \in \mathbb{R}^n \setminus \{0\}$ one has*

$$c_1(a + |\sigma|)^{p-2} \leq \Phi'(|\sigma|^2) \leq c_2(a + |\sigma|)^{p-2};$$

$$c_1(a + |\sigma|)^{p-2}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(\sigma)\xi_i\xi_j \leq c_2(a + |\sigma|)^{p-2}|\xi|^2,$$

where we have let

$$a_{ij}(\sigma) = 2\Phi''(|\sigma|^2)\sigma_i\sigma_j + \Phi'(|\sigma|^2)\delta_{ij}.$$

Given a solution u to (2.1) and a function $f \in C^2(\mathbb{R})$, such that $f > 0$, we define

$$P = P(u, x) = \frac{\Psi(|Du(x)|^2)}{f(u(x))}, \tag{2.2}$$

where $\Psi(s) = 2s\Phi'(s) - \Phi(s)$. We note explicitly that the above hypothesis implies that $\Lambda(s) = \Psi'(s) = 2s\Phi''(s) + \Phi'(s) > 0$ for every $s > 0$, therefore Ψ is strictly increasing on $(0, \infty)$. In the sequel we let

$$d_{ij}(\sigma) \;=\; \frac{a_{ij}(\sigma)}{\Lambda(|\sigma|^2)}, \qquad \sigma \in \mathbb{R}^n.$$

The first result of this section is a maximum principle for the function P.

Theorem 2.1. *Let $u \in W^{1,p}_{loc}(\Omega)$ be a non constant weak solution to* (2.1) *in a connected, open set $\Omega \subset \mathbb{R}^n$. If there exists $x_o \in \Omega$ such that*

$$P(u, x_o) = \sup_{x \in \Omega} P(u, x), \tag{2.3}$$

then we must have at x_o

$$\begin{aligned} \mathcal{L}P &= \sum_{i,j=1}^{n} D_i(d_{ij}(Du) D_j P) \\ &\geq n' \frac{|Du|^2 \Psi(|Du|^2)}{f(u)^2} \left\{ \frac{f'(u)^2}{f(u)} \frac{\Psi(|Du|^2)}{2|Du|^2 \Phi'(|Du|^2)} - \frac{1}{n'} f''(u) \right\}, \end{aligned} \tag{2.4}$$

where $n' = \frac{n}{n-1}$.

Proof. We begin with the crucial remark that since $\Psi(0) = 0$ and $P(u, x) \geq 0$, if $|Du(x_o)| = 0$ we must have $P(u, x) \equiv 0$, which implies $u \equiv$ *const*, against the assumptions. Thereby, if (2.3) holds it must be $|Du(x_o)| \neq 0$. Since (2.1) is invariant with respect to the action of the orthogonal group on $\mathbb{R}^n$ we can assume without loss of generality

$$Du(x_o) = (0, ..., 0, |Du(x_o)|). \tag{2.5}$$

We let now $g(t) = f(t)^{-1}$, so that $P(u, x) = g(u(x))\Psi(|Du(x)|^2)$. In what follows we adopt the summation convention over repeated indices. Also, for a function h we use the notation $h_{,i}$, $h_{,ij}$ to indicate the partial derivatives $\frac{\partial h}{\partial x_i}$, $\frac{\partial^2 h}{\partial x_i \partial x_j}$, etc. By the regularity results in [9], [20], [36] we know that $u \in C^{1,\alpha}_{loc}(\Omega)$. In view of the fact that $|Du(x_o)| \neq 0$ it follows that $|Du| \neq 0$ in an open neighborhood V of x_o. The regularity results for quasilinear nondegenerate equations allow to infer that we actually have $u \in C^{2,\alpha}(V)$.

In such neighborhood we are thus allowed to take second derivatives of the solution u. This being said, we note that the equation (2.1) can be rewritten as

$$a_{ij}(Du)u_{,ij} \;=\; 0, \tag{2.6}$$

which, recalling the definition of d_{ij}, also gives

$$d_{ij}(Du)u_{,ij} \;=\; 0. \tag{2.7}$$

Differentiating P one obtains

$$P_{,i} = g'\Psi u_{,i} + 2g\Lambda u_{,li}\, u_{,l}\,, \qquad\qquad i = 1, ..., n.$$

In particular, from $P_{,n} = 0$ we find at x_o

$$u_{,nn} \;=\; -\,\frac{g'}{2g}\,\frac{\Psi}{\Lambda}. \tag{2.8}$$

One also has

$$u_{,n1} \;=\; ... \;=\; u_{,n(n-1)} \;=\; 0 \qquad\quad at \;\; x_o.$$

We obtain from the expression of $P_{,i}$

$$\begin{aligned}(d_{ij}(Du)P_{,i}\,)_{,j} = {}& 2gu_{,l}\,(a_{ij}(Du)(u_{,l}\,)_{,i}\,)_{,j} + 2g'a_{ij}(Du)u_{,li}\,u_{,j}\,u_{,l}\\ &+ 2ga_{ij}(Du)u_{,li}\,u_{,lj} + g'\Psi d_{ij}(Du)u_{,ij} + g'\Psi d_{ij,j}\,(Du)u_{,i}\\ &+ g''\Psi d_{ij}(Du)u_{,i}\,u_{,j} + 2g'\Lambda d_{ij}(Du)u_{,lj}\,u_l u_i.\end{aligned}$$

Differentiation of (2.6) with respect to x_l gives

$$(a_{ij}(Du)(u_{,l}\,)_{,i}\,)_{,j} \;=\; 0 \qquad\quad l = 1, ..., n. \tag{2.9}$$

To simplify the notation from now on we drop the dependence of the coefficients a_{ij}, d_{ij} from Du. Using (2.7), (2.9) we find

$$(d_{ij}P_{,i}\,)_{,j} = 2ga_{ij}u_{,li}\,u_{,lj} + 4g'a_{ij}u_{,li}\,u_{,l}\,u_{,j} + g''\Psi d_{ij}u_{,i}\,u_{,j} + g'\Psi d_{ij,j}\,u_{,i}\,. \tag{2.10}$$

Our task now is to evaluate at x_o each of the four terms in the right hand side of (2.10). Thanks to (2.5) the matrix (a_{ij}) becomes diagonal in x_o

$$(a_{ij}) = diag\,(\Phi', \Phi', ..., \Phi', \Lambda)\,. \tag{2.11}$$

This gives at x_o

$$a_{ij}u_{,li}\,u_{,lj} \;=\; \Phi' \sum_{i=1}^{n-1}\sum_{l=1}^{n} u_{,li}^{\,2} \;+\Lambda \sum_{l=1}^{n} u_{,ln}^{\,2} \;\;.$$

Schwarz inequality implies

$$\left(\sum_{i=1}^{n-1} u_{,ii}\right)^2 \;\leq\; (n-1)\sum_{i=1}^{n-1}\sum_{l=1}^{n} u_{,li}^{\,2}\,. \tag{2.12}$$

On the other hand, from (2.6) and (2.11) we find

$$\sum_{i=1}^{n-1} u_{,ii} \;=\; -\frac{\Lambda}{\Phi'}u_{,nn} \;=\; (\text{by } (2.8))\;\frac{g'}{g}\;\frac{\Psi}{2\Phi'}. \tag{2.13}$$

Using (2.13) in (2.12) one has

$$\sum_{i=1}^{n-1}\sum_{l=1}^{n} u_{,li}^{\,2} \;\geq\; \frac{1}{n-1}\left(\frac{g'}{g}\frac{\Psi}{2\Phi'}\right)^2.$$

We infer from this and (2.8)

$$a_{ij}u_{,li}\,u_{,lj} \geq \frac{1}{n-1}\Phi'\left(\frac{g'}{g}\frac{\Psi}{2\Phi'}\right)^2 + \Lambda u_{,nn}^{\,2} = \left(\frac{g'}{2g}\right)^2\left[\frac{1}{n-1}\frac{\Psi^2}{\Phi'} + \frac{\Psi^2}{\Lambda}\right].$$

In conclusion we have found

$$2g a_{ij}u_{,li}\,u_{,lj} \geq \frac{g^{'2}}{2g}\left[\frac{1}{n-1}\frac{\Psi^2}{\Phi'} + \frac{\Psi^2}{\Lambda}\right]. \tag{2.14}$$

We observe next that from the previous computations one easily obtains

$$4g' a_{ij}u_{,li}\,u_{,l}\,u_{,j} = -2\frac{g^{'2}}{g}\Psi|Du|^2, \tag{2.15}$$

$$g''\Psi d_{ij}u_{,i}\,u_{,j} = g''\Psi|Du|^2. \tag{2.16}$$

We finally look at $d_{ij},_j\, u,_i$. The computation of this term is long and tedious, but the details are straightforward, so we omit them and go directly to its final expression

$$d_{ij},_j\, u,_i = \frac{1}{\Lambda}\left[2\Phi''|Du|^2\Delta u - 2\Phi''|Du|^2 u,_{nn}\right] = \frac{2|Du|^2\Phi''}{\Lambda}\sum_{i=1}^{n-1} u,_{ii}\,.$$

We mention that the assumption $\Phi \in C^3(\mathbb{R})$ is only used in the calculation of the above term, since at some point the third derivatives of Φ are needed. However, it should be noted that the terms involving Φ''' cancel and therefore one could, working harder, weaken the hypothesis on Φ by merely assuming $\Phi \in C^2(\mathbb{R})$. From (2.13) one finally obtains

$$g'\Psi d_{ij},_j\, u,_i = \frac{g'^2}{g}\frac{\Psi^2}{\Phi'}\frac{\Phi''|Du|^2}{\Lambda}. \tag{2.17}$$

Using (2.14)-(2.17) in (2.10), and recalling that $g(t) = f(t)^{-1}$, after some simple algebraic manipulations we reach the conclusion. □

In the special case in which $\Phi'(s) = s^{p-2}$ one has $\Psi(s) = \frac{2(p-1)}{p}s^{p/2}$ and the equation (2.1) becomes

$$\mathrm{div}\,(|Du|^{p-2}Du) = 0. \tag{2.18}$$

In this situation one has

$$\frac{\Psi(s)}{2s\Phi'(s)} \equiv \frac{p-1}{p},$$

so that the choice $f(u) = u^\alpha$, $\alpha > 0$, of the function in (2.2) yields with $p' = \frac{p}{p-1}$

$$\left\{\frac{f'(u)^2}{f(u)}\frac{\Psi(|Du|^2)}{2|Du|^2\Phi'(|Du|^2)} - \frac{1}{n'}f''(u)\right\} \geq 0,$$

if and only if

$$\frac{\alpha}{p'} \geq \frac{\alpha-1}{n'}.$$

If we assume that $1 < p < n$, from the above it is then clear that the expression in the right hand side of (2.4) is nonnegative provided that

$$\alpha \leq \frac{p(n-1)}{n-p} = \frac{p^*}{n'}, \tag{2.19}$$

where $p^* = \frac{np}{n-p}$ denotes the Sobolev exponent relative to p. Theorem 2.1 thus gives the following:

Theorem 2.2. *Let u be a non constant positive weak solution to equation* (2.18) *in a connected, open set $\Omega \subset \mathbb{R}^n$. Let $1 < p < n$, then for any $\alpha > 0$ satisfying* (2.19) *the function*

$$P = P(u, x) = \frac{|Du(x)|^p}{u(x)^\alpha}$$

cannot attain a local maximum at an interior point of Ω, unless $P \equiv const$. If instead $p = n$, then one should replace the above P-function with the following one

$$P = P(u, x) = e^{-\alpha u(x)} |Du(x)|^n,$$

where $\alpha > 0$ is arbitrary.

Remark 2.3. It is important to observe that in the range $1 < p < n$ the function P in Theorem 2.2 is invariant under the rescaling

$$u_r(x) = r^\beta u(rx), \quad \beta = \frac{p}{\alpha - p},$$

in the sense that

$$P(u_r, x) = P(u, rx). \tag{2.20}$$

In particular,

$$P(u, x) = \frac{|Du(x)|^p}{u(x)^{\frac{p(n-1)}{n-p}}} \tag{2.21}$$

satisfies (2.20) with

$$u_r(x) = r^{\frac{n-p}{p-1}} u(rx). \tag{2.22}$$

As it was seen in (1.4) the P-function (1.5) is constant when evaluated on any multiple of the radial fundamental solution of (2.18), when $p \neq n$, with pole at x_o:

$$\Gamma(x) = \sigma |x - x_o|^{\frac{p-n}{p-1}}. \tag{2.23}$$

Here, σ is such that

$$\left(\frac{n-p}{p-1}\right)^{p-1} \sigma^{p-1} = \frac{1}{n\omega_n}, \tag{2.24}$$

with ω_n denoting the measure of the unit ball in $\mathbb{R}^n$. In the conformal case $p = n$ one needs to take $\alpha = n$ in the corresponding P-function in Theorem 2.2. Vice-versa, we have the following symmetry result.

Theorem 2.4. *Let $u \neq 0$ be a nonnegative solution to* $\operatorname{div}(|Du|^{p-2}Du) = 0$ *in $\Omega = \mathbb{R}^n \setminus \{0\}$, $u \to 0$ as $|x| \to \infty$, such that $P(u, \cdot)$ is constant in Ω. Then u is a multiple of the fundamental solution $\Gamma(x)$ with pole in zero.*

Proof. The Harnack inequality [33] implies $u(x) > 0$ at any $x \in \Omega$. Call P_o the constant value (> 0) of $P(u, \cdot)$ in Ω, then

$$|Du(x)|^p = P_o u(x)^{\frac{p(n-1)}{n-p}} > 0 \tag{2.25}$$

for every $x \in \Omega$. The local regularity theory [19] guarantees that $u \in C^\omega(\Omega)$. By the results in [34], either u has a removable singularity at the origin, or near this point u behaves like the fundamental solution Γ in (2.23). In the first case we would infer that u is a bounded entire solution of (2.18), which, as a consequence of Liouville theorem for the p-Laplacian, implies that $u \equiv \mathit{const}$ in $\mathbb{R}^n$. Since u vanishes at infinity we conclude that $u \equiv 0$ in $\mathbb{R}^n$, against the assumptions. Therefore, there exist positive constants c_1, c_2 and R_o such that for $0 < |x| < R_o$ one has

$$c_1 < \frac{u(x)}{\Gamma(x)} < c_2. \tag{2.26}$$

For $\epsilon > 0$ consider the level set $K_\epsilon = \{x \in \mathbb{R}^n \mid u(x) = \epsilon\}$. By the assumptions on u, K_ϵ is bounded. We claim that the origin is not a limit point for K_ϵ. If it were, there would exist a sequence $x_k \to 0$ such that $u(x_k) = \epsilon$ for each $k \in \mathbb{N}$. In view of (2.26) this is however impossible. We observe next that K_ϵ is closed. Consider, in fact, a sequence $x_k \to x_o$, with $x_k \in K_\epsilon$. Since 0 is an isolated point of K_ϵ we infer that $x_o \neq 0$, unless the sequence is constant, in which case $x_o \in K_\epsilon$. By the continuity of u in $\mathbb{R}^n \setminus \{0\}$, we conclude that $x_o \in K_\epsilon$. We have thus proved that K_ϵ is compact. We now observe that the open set $\{x \in \mathbb{R}^n \mid u(x) < \epsilon\}$ does not contain the origin. If it did, then u would have a removable singularity in zero and this is against (2.26). Thanks to (2.25), the set $K_\epsilon^* = K_\epsilon \setminus \{0\}$ is then a smooth compact hypersurface containing the origin in its interior.

On K_ϵ^* the mean curvature at x, $H_\epsilon(x)$, is given by

$$(n-1)H_\epsilon(x) = div\eta(x),$$

where $\eta(x)$ is the outer unit normal in x and *div* denotes the intrinsic divergence. Using the equation we find

$$\begin{aligned} 0 &= \operatorname{div}(|Du|^{p-2}Du) \\ &= -|Du|^{p-1}(n-1)H_\epsilon(x) + (p-1)|Du|^{p-4} < Du, Hessu(Du) > . \end{aligned}$$

This implies at every $x \in K_\epsilon^*$

$$H_\epsilon(x) = \frac{(p-1)}{(n-1)|Du|^3} < Du, Hessu(Du) > . \tag{2.27}$$

On the other hand since $P \equiv P_o$ we have on K_ϵ^*

$$\begin{aligned} 0 &= -\frac{\partial P}{\partial \eta} \\ &= < DP, \frac{Du}{|Du|} > \left[p\frac{|Du|^{p-3} < Hessu(Du), Du >}{u^{p(\frac{n-1}{n-p})}} - \frac{p(n-1)|Du|^{p+1}}{(n-p)u^{p(\frac{n-1}{n-p})+1}} \right]. \end{aligned} \tag{2.28}$$

By combining the last two equations we obtain

$$H_\epsilon(x) = \frac{(p-1)|Du|}{(n-p)u} = const.$$

Thanks to Alexandrov's theorem [4] all sets K_ϵ^* are spheres. To complete the proof of the theorem we next show that such spheres are centered at the origin, so that u can be written as $u(x) = f(r)$, with $r = |x|$. Once we know this, we observe that equation (2.18) becomes

$$\operatorname{div}(|Du|^{p-2}Du) = (p-1)|f'(r)|^{p-2} \left[f''(r) + \frac{n-1}{p-1}\frac{f'(r)}{r} \right] = 0, \tag{2.29}$$

whose solutions are of the type

$$f(r) = Ar^{\frac{p-n}{p-1}} + B,$$

where A, B are constants. Since $u(x) > 0$ and $u(x) \to 0$ as $|x| \to \infty$, we infer $B = 0$ and $A > 0$, i.e., u is a multiple of the fundamental solution Γ in (2.23).

To prove that the sets K_ϵ^* are concentric spheres we argue by contradiction and suppose that $x_o \neq 0$, where for a fixed $\epsilon > 0$ we have denoted by x_o the center of the sphere K_ϵ^*. Let $v(x) = \gamma(\epsilon)\Gamma(x - x_o)$, where $\Gamma(x - x_o)$ denotes the fundamental solution (2.23) having pole in x_o. Here, we assume to have chosen $\gamma(\epsilon)$ in such a way that $v(x) = \epsilon$ on K_ϵ^*. Since both u and v have nonvanishing gradient in Ω, for every $R > 0$ (sufficiently large) the function $w = u - v$ satisfies a linear uniformly elliptic equation in the bounded open set $\{x \in \mathbb{R}^n \mid |x| < R, u(x) < \epsilon\}$ to which the strong maximum principle in [28], or [16] can be applied, see [37]. This observation and the fact that $w(x) \to 0$ as $|x| \to \infty$ allow to infer $w \equiv 0$ in the complement of the ball whose boundary is K_ϵ^*. By the real analiticity of w in $\mathbb{R}^n \setminus \{0, x_o\}$ we conclude $u \equiv v$ in $\mathbb{R}^n \setminus \{0, x_o\}$. The smoothness of u in a neighborhood of x_o forces the conclusion $x_o = 0$. This contradiction and the arbitrariness of the choice of the level ϵ imply that all level sets are centered at the origin, thus completing the proof. □

3. A key result about the function $P(u, x)$. The purpose of this section is to establish the following basic property of the function $P(u, x)$ introduced in section two. We state it in a form which is suitable for subsequent applications. In what follows it will be convenient to adopt a short notation for a numerical quantity which will often appear in the computations. We thus set

$$K(n,p) = (n-p)\,(n\omega_n)^{\frac{p}{n-p}} \left(\frac{n-p}{p-1}\right)^{\frac{n(p-1)}{n-p}} . \tag{3.1}$$

We will denote by $cap_p(A)$ the variational p-capacity of the set A, see [17].

Theorem 3.1. *Let u be a nonnegative weak solution to* (1.1), (1.2) *in* Ω. *Let P be the function defined by* (2.21) *and set*

$$P_o = sup_{x\in\Omega} P(u,x).$$

Consider a maximizing sequence $\{x_k\}_{k\in\mathbb{N}}$ in Ω such that $P(u, x_k) \to P_o$ as $k \to \infty$. If $|x_k| \to \infty$, then $lim_{|x|\to\infty} P(u, x)$ exists and we have in fact

$$lim_{|x|\to\infty} P(u,x) = P_o = K(n,p) \left(\frac{n\omega_n}{cap_p\Omega_1}\right)^{\frac{p}{n-p}} . \tag{3.2}$$

Proof. We begin by observing that if u is a solution to (1.1), (1.2), then the comparison principle [17] implies the existence of positive constants c_1, c_2, R_o such that for $|x| \geq R_o$

$$c_1\Gamma(x) \leq u(x) \leq c_2\Gamma(x). \tag{3.3}$$

For $r > 0$ consider the rescaled function u_r defined by (2.22). Denoting by $r^{-1}\Omega_1 = \{r^{-1}x \mid x \in \Omega_1\}$, and $\Omega_r = \mathbb{R}^n \setminus \overline{r^{-1}\Omega_1}$, then u_r is a nonnegative solution to (1.1), (1.2) in Ω_r. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence as in the assumption and set $r_k = |x_k|$, so that $r_k \to \infty$ as $k \to \infty$. For every fixed $m \in \mathbb{N}$ there exists $k(m) \in \mathbb{N}$ such that for $k \geq k(m)$ the sequence of the u_{r_k} is equibounded in $K_m = \{x \in \mathbb{R}^n \mid m^{-1} < |x| < m\}$, since from (2.23) and (3.3) there exist positive constants $\tilde{c_1}$, $\tilde{c_2}$, c_1^*, c_2^*, depending on n, p, m, such that for $\xi \in K_m$ one has

$$\tilde{c_1} \leq c_1^* |\xi|^{\frac{p-n}{p-1}} \leq u_{r_k}(\xi) \leq c_2^* |\xi|^{\frac{p-n}{p-1}} \leq \tilde{c_2}.$$

From the $C^{1,\alpha}_{loc}$ estimates in [9], [20], [36] there exist positive constants M, $\alpha < 1$, $\beta < 1$, depending on $||u_{r_k}||_{L^\infty(K_m)}$, such that for every ξ_1, $\xi_2 \in K_m$

$$|u_{r_k}(\xi_1) - u_{r_k}(\xi_2)| \leq M|\xi_1 - \xi_2|^\alpha$$

$$|Du_{r_k}(\xi_1) - Du_{r_k}(\xi_2)| \leq M|\xi_1 - \xi_2|^\beta.$$

For each m we can thus extract from u_{r_k} a subsequence $u_{r_k}^{(m)}$ which converges uniformly with its first derivatives to a solution $v^{(m)}$ of (1.1) in K_m. By a diagonal process we can find a nonnegative function v which is the uniform limit in $C^{1,\alpha}$ on compact subsets of $\mathbb{R}^n \setminus \{0\}$ of a certain subsequence of u_{r_k}, which for simplicity we continue to denote by u_{r_k}, and such that

$$\operatorname{div}(|Dv|^{p-2}Dv) = 0 \quad \text{in} \mathbb{R}^n \setminus \{0\}, \qquad v(x) \to 0 \text{as} |x| \to \infty.$$

Let now $\xi_k = \frac{x_k}{r_k} \in \mathbb{S}^{n-1}$. By compactness there exists a subsequence of ξ_k, still denoted by the same symbol, such that $\xi_k \to \xi_o$ in $\mathbb{S}^{n-1}$. Clearly,

$$P(u_{r_k}, \xi_k) \to P(v, \xi_o).$$

The latter equation together with the important rescaling property (2.20) guarantee that

$$P(v, \xi_o) = P_o. \tag{3.4}$$

Since by the Harnack inequality $0 < v < 1$, this shows in particular that $P_o < \infty$. On the other hand, (2.20) implies

$$P(u_{r_k}, x) = P(u, r_k x) \leq P_o.$$

Passing to the limit as $k \to \infty$ we conclude

$$P(v, x) \leq P_o \quad \text{for every} x \in \mathbb{R}^n \setminus \{0\}. \tag{3.5}$$

From (3.4), (3.5) and Theorem 2.2 we conclude that $P(v,x) \equiv const$ in $\mathbb{R}^n \setminus \{0\}$. But then Theorem 2.4 implies the existence of a positive number γ such that

$$v(x) = \gamma\Gamma(x). \tag{3.6}$$

We prove next that (3.6) implies

$$\lim_{|x|\to\infty} \frac{u(x)}{\Gamma(x)} = (cap_p\Omega_1)^{\frac{1}{p-1}}\,, \tag{3.7}$$

$$\lim_{|x|\to\infty} |x|^{\frac{n-1}{p-1}} \left(Du - (cap_p\Omega_1)^{\frac{1}{p-1}}\, D\Gamma \right) = 0, \tag{3.8}$$

where Γ is the fundamental solution of (3.1) given in (2.23), (2.24). If we assume for a moment that (3.7), (3.8) have been proved, from them we easily obtain as $|x| \to \infty$

$$P(u,x) = (cap_p\Omega_1)^{-\frac{p}{n-p}}\, \frac{|D\Gamma(x)|^p}{\Gamma(x)^{\frac{p(n-1)}{n-p}}}\,(1+o(1))\,.$$

From this (3.2) follows using (2.10), (2.11). We are thus left with proving (3.7), (3.8).

To establish (3.7) it suffices to show that any sequence $x_k \to \infty$ admits a subsequence $x_{j_k} \to \infty$ such that

$$\frac{u(x_{j_k})}{\Gamma(x_{j_k})} \to (cap_p\Omega_1)^{\frac{1}{p-1}}\,.$$

Let $\xi_k = \frac{x_k}{r_k} \in \mathbb{S}^{n-1}$. By compactness there exist a subsequence $\{\xi_{j_k}\}_{k\in\mathbb{N}}$ of $\{\xi_k\}_{k\in\mathbb{N}}$ and $\xi_1 \in \mathbb{S}^{n-1}$ such that $\xi_{j_k} \to \xi_1$ as $k \to \infty$. We have

$$\frac{u(x_{j_k})}{\Gamma(x_{j_k})} = \frac{u_{r_{j_k}}(\xi_{j_k})}{\sigma}.$$

Reasoning as before we can find a subsequence of $u_{r_{j_k}}$, which we still denote with the same symbol, which converges to v in $C^{1,\alpha}$ of compact subsets of $\mathbb{R}^n \setminus \{0\}$. From this and from (3.4) we obtain

$$\frac{u_{r_{j_k}}(\xi_{j_k})}{\sigma} \to \frac{v(\xi_1)}{\sigma} = \gamma\frac{\Gamma(\xi_1)}{\sigma} = \gamma.$$

This proves (3.7) with the constant γ in place of $(cap_p\Omega_1)^{\frac{1}{p-1}}$. The proof of (3.8) now follows by similar considerations once we note that (3.4) gives $Dv = \gamma D\Gamma$, and that furthermore

$$r^{\frac{n-1}{p-1}} Du(r\xi) = Du_r(\xi).$$

To finish we only need to establish that $\gamma = (cap_p\Omega_1)^{\frac{1}{p-1}}$. Since u is the p-capacitary potential of Ω_1 we have

$$cap_p(\Omega_1) = -\int_{\partial B_R} |Du|^{p-2}\frac{\partial u}{\partial \eta} d\sigma,$$

where B_R is any ball containing Ω_1 and η is the normal vector pointing outside ∂B_R. From the above we obtain as $|x| \to \infty$

$$|Du|^{p-2} = \gamma^{p-2}|D\Gamma|^{p-2}(1 + o(1)),$$

$$\frac{\partial u}{\partial \eta} = \gamma\frac{\partial \Gamma}{\partial \eta}\left(1 + o(1)\right),$$

so that

$$\begin{aligned} cap_p(\Omega_0) &= \lim_{R\to\infty} \gamma^{p-1}\int_{\partial B_R} |D\Gamma|^{p-1}(1 + o(1))d\sigma \\ &= \lim_{R\to\infty} \gamma^{p-1}(1 + o(1)) = \gamma^{p-1}. \end{aligned}$$

This completes the proof of the theorem. □

4. Integral identities and their consequences. The purpose of this section is to establish some integral identities satisfied by a weak solution u to (1.1), (1.2) in $\Omega = \mathbb{R}^n\backslash\overline{\Omega_1}$ with the overdetermined condition $|Du| = c > 0$ on $\partial\Omega_1$, and, with the help of such identities, relate various geometric quantities of the ground domain Ω_1. We begin with the basic observation that thanks to the assumption $|Du| = c > 0$ on $\partial\Omega_1$, from the regularity results in [38] we infer that Ω_1 is of class C^2 and then $u \in C^{1,\alpha}_{loc}(\overline{\Omega})$, see [21]. This fact will be used without further comment in several occasions in the sequel. We note explicitly that Hopf maximum principle [28], [16] implies $\frac{\partial u}{\partial \eta} = -c$, where η is the outer unit normal to Ω_1. We start with an interesting and useful consequence of Theorem 3.1.

Lemma 4.1. *Let u be a nonnegative weak solution to* (1.1), (1.2) *in* Ω*, such that* $|Du| = c > 0$ *on* $\partial\Omega_1$*. There exist constants* $C, R_o > 0$ *such that for* $|x| > R_o$

$$u(x) \leq C|x|^{-\frac{n-p}{p-1}}, \quad |Du(x)| \leq C|x|^{-\frac{n-1}{p-1}}.$$

Proof. Consider the (extended) number P_o defined in the statement of Theorem 3.1. There are three possibilities. Either the supremum is attained at infinity, in which case Theorem 3.1 shows that $P_o < \infty$, or there exists a point $x_o \in \Omega$ such that $P(u, x_o) = P_o$, or else $P_o = \sup_{x\in\partial\Omega_1} P(u, x)$. In the second case we clearly have $P_o < \infty$. As for the third possibility, since $u = 1$ on $\partial\Omega_1$ we infer that $P_o < \infty$ as well. By the definition (2.21) of $P(u, x)$ we conclude

$$|Du(x)|^p \leq P_o u(x)^{\frac{p(n-1)}{n-p}}$$

for every $x \in \Omega$. The proof is completed by observing that (3.3) holds by the comparison theorem. □

Since u is the p-capacitary potential of Ω_1 we have [17]

$$cap_p\Omega_1 = \int_\Omega |Du|^p dx. \tag{4.1}$$

For $R > 0$ sufficiently large an integration by parts gives

$$\int_{B_R\setminus\overline{\Omega_1}} |Du|^p dx = \int_{\partial B_R} u|Du|^{p-2}\frac{\partial u}{\partial\eta}d\sigma - \int_{\partial\Omega_1} |Du|^{p-2}\frac{\partial u}{\partial\eta}d\sigma,$$

where in either boundary integral η denotes the outer unit normal to the relative domain. By Lemma 4.1 we find that

$$lim_{R\to\infty} \int_{\partial B_R} u|Du|^{p-2}\frac{\partial u}{\partial\eta}d\sigma = 0.$$

Using the overdetermined assumption we thus conclude

$$cap_p\Omega_1 = c^{p-1}|\partial\Omega_1|. \tag{4.2}$$

We consider next for $\epsilon \in (0, 1)$ the level set $K_\epsilon = \{x \in \Omega \mid u(x) > \epsilon\}$ and assume that K_ϵ is a bounded open set with $C^{1,\beta}$ boundary, for some $\beta > 0$.

For $R > 0$ large enough, so that $\overline{K_\epsilon} \subset B_R$, we let $\Omega_R = B_R \backslash \overline{K_\epsilon}$ and integrate by parts the quantity $|Du|^p < x, \eta >$ on $\partial\Omega_R$ to find

$$\int_{\partial B_R} |Du|^p < x, \eta > d\sigma - \int_{\{u=\epsilon\}} |Du|^p < x, \eta > d\sigma$$
$$= (n-p)\int_{\Omega_R} |Du|^p dx + p\int_{\partial B_R} < x, Du > \frac{\partial u}{\partial \eta}|Du|^{p-2} d\sigma$$
$$- p\int_{\{u=\epsilon\}} < x, Du > \frac{\partial u}{\partial \eta}|Du|^{p-2} d\sigma. \tag{4.3}$$

We mention that a rigorous derivation of the above identity is based on approximating u with the solutions to analogous regularized problems with boundary conditions equal to u on $\partial\Omega_R$ and then passing to the limit using the a priori $C^{1,\alpha}$ estimates up to the boundary in [21]. These arguments are by now standard and we omit them. At this point we let $R \to \infty$ in (4.3). Using Lemma 4.1 we see that the two integrals on ∂B_R converge to zero since $1 < p < n$. Observing that the outer unit normal to the exterior boundary $\{u = \epsilon\}$ of K_ϵ is $\eta = -\frac{Du}{|Du|}$, we have established the following result.

Theorem 4.2. *Let $K_\epsilon = \{x \in \Omega \mid u(x) > \epsilon\}$, $0 < \epsilon < 1$, be a level set of class $C^{1,\beta}$ for some $\beta \in (0,1)$ of a weak solution to* (1.1), (1.2). *One has*

$$(p-1)\int_{\{u=\epsilon\}} |Du|^p < x, \eta > d\sigma = (n-p)\int_{\mathbb{R}^n \backslash K_\epsilon} |Du|^p dx.$$

If we now impose also the overdetermined condition $|Du| = c > 0$ on $\partial\Omega_1$, then letting $\epsilon \to 1$ in Theorem 4.2 we find

$$n(p-1)c^p|\Omega_1| = (n-p)\int_\Omega |Du|^p dx. \tag{4.4}$$

Here, we have used the fact

$$\int_{\partial\Omega_1} < x, \eta > d\sigma = n|\Omega_1|.$$

By (4.1), (4.4) one obtains the following

Corollary 4.3. *Let* u *be a weak solution to* (1.1), (1.2) *satisfying the overdetermined assumption* $|Du| = c > 0$ *on* $\partial\Omega_1$. *The constant* c *is then prescribed by the equation*

$$cap_p\Omega_1 = \frac{n(p-1)}{n-p} c^p |\Omega_1|.$$

Coupling (4.2) with Corollary 4.3 we obtain

$$|\partial\Omega_1| = \frac{n(p-1)}{n-p} c|\Omega_1|. \tag{4.5}$$

Our next goal is to obtain some additional information on the function $P(u,x)$ on a solution to (1.1), (1.2). We return to Theorem 4.2 and consider the function $v = u/\epsilon$. By the scaling properties of the equation (1.1) we see that v is the p-capacitary potential for the bounded open set $\Omega_\epsilon = \overline{\Omega_1} \cup K_\epsilon$, so that we can restate the thesis of Theorem 4.2 in the following way

$$(p-1)\epsilon^{\frac{p(n-1)}{p-1}} \int_{\{u=\epsilon\}} P(u,x) < x, \eta > d\sigma = (n-p)\epsilon^p cap_p\Omega_\epsilon.$$

Assume now that Ω_ϵ be starlike with respect to the origin, so that $< x, \eta > \geq 0$ on $\{u = \epsilon\}$. The latter equality then gives

$$\frac{n}{n-p} \epsilon^{\frac{p(p-1)}{n-p}} |\Omega_\epsilon| max_{\{u=\epsilon\}} P(u, \cdot) \geq cap_p\Omega_\epsilon. \tag{4.6}$$

We recall next the following nonlinear version of Poincaré's theorem, see, e.g., [12].

Theorem 4.4. *Let* $A \subset \mathbb{R}^n$ *be a bounded open set and consider a ball* B_R *such that* $|B_R| = \omega_n R^n = |A|$. *One has*

$$cap_p A \geq cap_p B_R,$$

with equality if and only if A *is a ball.*

The p-capacity of a ball is known explicitly, see, e.g., [17]

$$cap_p B_R = n\omega_n \left(\frac{n-p}{p-1}\right)^{p-1} R^{n-p}. \tag{4.7}$$

Now, if B_R is such that $|B_R| = |\Omega_\epsilon|$ we obtain from Theorem 4.4 and (4.7)

$$cap_p \Omega_\epsilon \geq n\omega_n \left(\frac{n-p}{p-1}\right)^{p-1} \left(\frac{|\Omega_\epsilon|}{\omega_n}\right)^{\frac{n-p}{n}} . \tag{4.8}$$

By means of (4.8) we obtain from (4.6)

$$\max_{\{u=\epsilon\}} P(u,\cdot) \geq (n-p)\,(n\omega_n)^{\frac{p}{n-p}} \left(\frac{n-p}{p-1}\right)^{\frac{n(p-1)}{n-p}} \epsilon^{-\frac{p(p-1)}{n-p}} \,(cap_p \Omega_\epsilon)^{-\frac{p}{n-p}} . \tag{4.10}$$

It is easy to see that

$$cap_p \Omega_\epsilon = \epsilon^{1-p} cap_p \Omega_1 .$$

Using this in (4.10) we conclude the following important property of the function $P(u,x)$.

Theorem 4.5. *Let $\{u=\epsilon\}$ be a sufficiently smooth ($C^{1,\beta}$), starlike level surface of a solution u to* (1.1), (1.2), *then*

$$max_{\{u=\epsilon\}} P(u,\cdot) \geq K(n,p) \left(\frac{n\omega_n}{cap_p \Omega_1}\right)^{\frac{p}{n-p}} ,$$

where $K(n,p)$ is the constant defined by (3.1).

5. Proof of Theorem 1.1. In this section we tie up the analytic and geometric information gathered in the previous sections and finally prove Theorem 1.1. We start with a crucial lemma.

Lemma 5.1. *Let u be as in the statement of Theorem* 1.1. *The function $P(u,x)$ in* (1.5) *must attain its maximum value on $\partial\Omega_1$, unless $P(u,\cdot) \equiv$ const in Ω.*

Proof. We argue by contradiction and suppose that the maximum value P_o of $P(u,x)$ is attained either at an interior point $x_o \in \Omega$, or at infinity. In the former case we infer from Theorem 2.2 that $P \equiv const$. In the latter, we know from Theorem 3.1 that (3.2) holds. On the other hand, since Ω_1 is starlike, by the results in [8] we know that so is every level set of u, see also [13], [39] for the case $p = 2$. Furthermore, for $\epsilon \in (0,1)$ sufficiently close to 1 such level sets are smooth manifolds thanks to the overdetermined condition $|Du| = c > 0$. We can thus implement Theorem 4.5 which leads to a contradiction, unless again $P \equiv const$. This completes the proof. □

Our final step is proving Theorem 1.1.

Proof of Theorem 1.1. By Lemma 5.1 we infer that the maximum value of P occurs on $\partial\Omega_1$. We thus have $\frac{\partial P}{\partial \eta} \leq 0$, where η denotes the outer unit normal to $\partial\Omega_1$. From this information we obtain, analogously to (2.27), (2.28) in the proof of Theorem 2.4,

$$H(x) \geq \frac{p-1}{n-p} c \tag{5.1}$$

for every $x \in \partial\Omega_1$. Here, $H(x)$ represents the mean curvature in $x \in \partial\Omega_1$. Multiplying the last inequality by $< x, \eta >$ and using the formula

$$\int_{\partial\Omega_1} H(x) < x, \eta > d\sigma = |\partial\Omega_1|$$

we find

$$|\partial\Omega_1| \geq \frac{n(p-1)}{n-p} c|\Omega_1|. \tag{5.2}$$

Here, again, we have used the fact that Ω_1 be starlike so that $< x, \eta > \geq 0$. Comparing (5.2) with (4.5) we deduce that equality must hold in (5.1). By Alexandrov's theorem [4] we conclude that Ω_1 is a ball B_R centered at x_o and therefore (2.29) gives $u(x) = \left(\frac{R}{|x-x_o|}\right)^{\frac{n-p}{p-1}}$. Finally, (4.5) implies that $c = \left(\frac{n-p}{p-1}\right) \frac{1}{R}$.

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