TOPOLOGICAL DEGREE THEORIES FOR
DENSELY DEFINED MAPPINGS INVOLVING
OPERATORS OF TYPE $(S_+)$*

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Abstract. Let $X$ be a real separable reflexive Banach space with dual space $X^*$. Assume that the operator $A : X \supset \mathcal{D}(A) \to X^*$ is such that $L \subset \mathcal{D}(A)$ and $\mathcal{L} = X$, where $L$ is a subspace of $X$. It is shown that it is possible to define a topological degree for such operators $A$ that satisfy, mainly, Condition $(S_+)_{0,L}$. It is also shown that a topological degree can be defined for operators of the type $M + A$, where $M + A : X \supset \mathcal{D}(M + A) \to X^*$, $L \subset \mathcal{D}(M + A)$, and $\mathcal{L} = X$. Here, $X$ is not necessarily separable and $M$ satisfies a variant of the maximal monotonicity condition (with respect to the space $L$) as well as an approximation condition. The operator $A$ satisfies, mainly, analogues of the quasi-boundedness condition and the $(S_+)_{0,L}$ condition (with respect to the operator $M$). Properties of these degrees are studied and applications are given for nonlinear Dirichlet elliptic problems of the type

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \rho^2(u) \frac{\partial u}{\partial x_i} + a_i(x, u, \frac{\partial u}{\partial x}) \right) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(x),$$

as well as Cauchy-Dirichlet parabolic problems of the type

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, t, u, \frac{\partial u}{\partial x}) + \rho(x, t, u) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(x, t).$$

1. Introduction and preliminaries. This paper is devoted to the introduction of a topological degree for densely defined operators in reflexive

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Banach spaces and applications of this concept to nonlinear elliptic and parabolic problems. We consider two types of nonlinear densely defined operators. The first type is an operator satisfying a generalized condition \((S_{+})\), and the second is the sum of a certain generalized maximal monotone operator and an operator satisfying a generalized condition \((S_{+})\).

A degree theory for operators of type \((S_{+})\) defined on Banach spaces was developed by Skrypnik in the monograph [8] which was published in 1973. Condition \((S_{+})\) there is called Condition \(\alpha\). Extensive applications of this degree were given in Skrypnik’s monographs [9, 10].

In Section 2 we introduce the concept of a topological degree for operators \(A : X \ni D(A) \to X^{*}\), where \(X\) is a real separable and reflexive Banach space. We assume that there exists a subspace \(L\) of \(X\) such that \(L \subset D(A)\) and \(\overline{L} = X\). The main assumption on the operator \(A\) is Condition \((S_{+})_{0,L}\) (see Definition 2.1) which generalizes Condition \((S_{+})\) (with respect to the space \(L\)).

To define this degree, we show first that we can define a degree for finite-dimensional Galerkin approximations \(A_{j}\) of the operator \(A\). Then we establish the existence of the limit of the degrees of the operators \(A_{j}\) as \(j \to \infty\). This limit is defined to be the degree of the operator \(A\).

In Section 3 we introduce a topological degree for operators \(M + A : X \ni D(M + A) \to X^{*}\), where \(X\) is now a (not necessarily separable) real reflexive Banach space. We suppose that there exists a subspace \(L\) of \(X\) such that \(L \subset D(M + A)\) and \(\overline{L} = X\). We assume that the operator \(M\) satisfies a variant of the maximal monotonicity condition (Condition \(m_{2}\)) with respect to the space \(L\), as well as an approximation condition (Condition \(m_{3}\)). The operator \(A\) satisfies analogues of the quasi-boundedness condition and the \((S_{+})\) condition with respect to the operator \(M\) (Conditions \(a_{1}\), \(a_{2}\)).

The degree of the operator \(M + A\) is defined to be the degree of a certain finite-dimensional approximate mapping (Definition 3.1). We would like to bring the attention of the reader to the original construction of the degree associated with the finite intersection property. This property was employed for the first time, independently and simultaneously, in non-separable spaces by Skrypnik [8] and Browder and Hess [2].

In Section 4 we study the properties of the two degrees introduced herein. In particular, we introduce the concept of homotopic operators and prove the homotopy invariance of the degree for both classes of operators.

In Section 5 and 6 we apply the above degree theories to nonlinear differential problems. In Section 5 we study the Dirichlet problem for the elliptic
equation
\[
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \{ \rho^2(u) \frac{\partial u}{\partial x_i} + a_i(x, u, \frac{\partial u}{\partial x}) \} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(x) \quad (1.1)
\]
with the usual growth and ellipticity conditions for the coefficients \( a_i(x, u, \xi) \).
However, we make very weak growth assumptions for the function \( \rho(u) \). In particular, we may take \( \rho(u) = e^u \). The degree theory of Section 2 is applied to this differential problem.

In Section 6 we consider the Cauchy-Dirichlet problem for the nonlinear parabolic equation
\[
\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, t, u, \frac{\partial u}{\partial x}) + \rho(x, t, u) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(x, t)
\]
with the usual growth and parabolicity conditions for the coefficients \( a_i(x, t, u, \xi) \) (see, e.g., [5]). The condition on the growth of the function \( \rho(x, t, u) \), which is increasing in \( u \), involves the critical exponent. Thus, the operator corresponding to the function \( \rho(x, t, u) \) in the resulting operator equation is not compact. We reduce the parabolic problem to the operator equation \( Mu + Au = 0 \), where the operators \( M, A \) satisfy the conditions of Section 3, and establish the relevant solvability result.

We would like to note that a degree for densely defined \( A \)-proper mappings in separable Banach spaces was defined by Petryshyn in [7]. Petryshyn’s degree is a multi-valued mapping. In the case of a linear maximal monotone operator \( M \) and a nonlinear operator \( A \) defined on all of \( X \) and satisfying a monotonicity condition, a degree for the sum \( M + A \) was defined by Berkovits and Mustonen in [1].

The methods of Sections 3, 4 allow us to generalize various known results (see, e.g., [2], [4]) about the solvability of \( Mu + Au = 0 \) with a maximal monotone operator \( M \) and an operator \( A \) satisfying condition \((S)_{+}\) or a weaker monotonicity condition.

In this paper we initiate the topological approach to the study of densely defined mappings involving operators of type \((S)_{+}\). Further studies of the implications of such an approach as well as extensive applications in the field of partial differential equations will be pursued in forthcoming publications by the authors.

In what follows, the symbols \( \mathcal{R}, \mathcal{R}_{+} \) denote the real line and the set \([0, \infty)\), respectively. The symbols \( \partial D, \overline{D} \) denote the strong boundary and
the strong closure of the set $D$, respectively, in a Banach space. We use the notation $B(x, r)$ to denote the open ball around the point $x$ with radius $r > 0$. The symbols “→” (“⇒”) denote strong (weak) convergence. Unless otherwise specified, the term “continuous”, for a mapping $f$ acting between two real Banach spaces, means that $f$ is strongly continuous. Unless otherwise specified, the symbol $N$ is used to denote the set of all natural numbers.

2. Definition of degree for densely defined $(S_\dagger)$ operators. Let $X$ be a real separable reflexive Banach space with dual space $X^*$. Consider an operator $A : X \supset \mathcal{D}(A) \to X^*$ with $\mathcal{D}(A)$ dense in $X$.

We assume that there exists a subspace $L$ of the space $X$ such that

$$L \subset \mathcal{D}(A), \quad \overline{L} = X. \quad (2.1)$$

Denote by $\mathcal{F}(L)$ the set of all finite-dimensional subspaces of $L$. We can choose a sequence $\{F_j\}$, $j \in N$, such that, for each $j \in N$,

$$F_j \in \mathcal{F}(L), \quad F_j \subset F_{j+1}, \quad \dim F_j = j, \quad \overline{\bigcup_j F_j} = X. \quad (2.2)$$

We let

$$L\{F_j\} = \bigcup_{j=1}^{\infty} F_j. \quad (2.3)$$

**Definition 2.1.** We say that the operator $A$ satisfies Condition $(S_\dagger)_{0,L}$ if for every sequence $\{F_j\}$ satisfying (2.2) and every sequence $\{u_j\} \subset L$ with

$$u_j \to u_0, \quad \limsup_{j \to \infty} \langle Au_j, u_j \rangle \leq 0, \quad \lim_{j \to \infty} \langle Au_j, v \rangle = 0, \quad (2.4)$$

for some $u_0 \in X$ and any $v \in L\{F_j\}$, it follows that the sequence $u_j$ converges strongly to $u_0$, $u_0 \in \mathcal{D}(A)$ and $Au_0 = 0$.

We assume that the operator $A$ satisfies the following conditions:

$A_1)$ there exists a subspace $L$ of $X$ satisfying (2.1) and such that the operator $A$ satisfies Condition $(S_\dagger)_{0,L}$;

$A_2)$ for every $F \in \mathcal{F}(L)$, $v \in L$ the mapping $a(F, v) : F \to \mathcal{R}$, defined by $a(F, v)(u) = \langle Au, v \rangle$ is continuous;
We fix a sequence \( \{F_j\} \) satisfying the condition (2.2) and let \( \{v_j\} \subset X \) be such that \( F_j \) is the span of \( \{v_1, \ldots, v_j\} \). Define, for every \( j \in \mathbb{N} \), the finite-dimensional approximation \( A_j \) of the operator \( A \) by

\[
A_j u = \sum_{i=1}^{j} (Au,v_i)v_i, \quad \text{for } u \in F_j. \tag{2.5}
\]

We are going to define the degree of the operator \( A \) with respect to an arbitrary open subset \( D \) of the space \( X \) provided that

\[
Au \neq 0, \quad \text{for } u \in \mathcal{D}(A) \cap \partial D. \tag{2.6}
\]

**Theorem 2.1.** Let \( A : X \supset \mathcal{D}(A) \to X^* \) be an operator satisfying Conditions \( A_1 \), \( A_2 \), and let \( D \) be a bounded open set in \( X \) satisfying condition (2.6). Then there exists a number \( J \in \mathbb{N} \) such that, for \( j \geq J \), the following assertions hold:

1) the equation \( A_j u = 0 \) has no solutions belonging to \( \partial D_j \), where \( D_j = \overline{D} \cap F_j \);

2) the degree \( \deg(A_j, D_j, 0) \) of the mapping \( A_j \) on the set \( \overline{D_j} \) with respect to \( 0 \in F_j \) is well defined and independent of \( j \).

**Proof.** We shall prove our first assertion by contradiction. Assume that there exists a sequence \( u_k \in \partial D_{j(k)} \) such that \( j(k) \to \infty \) as \( k \to \infty \) and \( A_{j(k)} u_k = 0 \). We may also assume that \( u_k \) converges weakly to some \( u_0 \in X \). Then we have

\[
\langle Au_k, u_k \rangle = 0, \quad \langle Au_k, v_i \rangle = 0, \quad i = 1, \ldots, j(k), \tag{2.7}
\]

and, consequently,

\[
\lim_{k \to \infty} \langle Au_k, u_k \rangle = 0, \quad \lim_{k \to \infty} \langle Au_k, v_i \rangle = 0. \tag{2.8}
\]

The last equality is true for any \( i \). From the definition of the space \( L\{F_j\} \) we obtain

\[
\lim_{k \to \infty} \langle Au_k, v \rangle = 0, \quad \text{for } v \in L\{F_j\}. \tag{2.9}
\]

From (2.8), (2.9) and the condition \((S_+)^{0,L}\) it follows that \( u_0 \in \mathcal{D}(A) \cap \partial D \) and \( Au_0 = 0 \). This contradicts our assumption (2.6). Thus, we have the existence of a number \( J = J_1 \) for which 1) is satisfied. Consequently,

\[
A_j u \neq 0, \quad \text{for } u \in \partial D_j, \quad j \geq J_1. \tag{2.10}
\]
It follows that for each \( j \geq J_1 \) the degree \( \deg(A_j, D_j, 0) \) of the finite-dimensional mapping \( A_j \) is well defined. We prove that this degree is independent of \( j \) by using the auxiliary mapping

\[
\bar{A}_j u = \sum_{i=1}^{j-1} \langle Au, v_i \rangle v_i + \langle h_j, u \rangle v_j, \quad u \in F_j,
\]

(2.11)

where \( h_j \) is some element of \( X^* \) satisfying the conditions

\[
\langle h_j, v_i \rangle = 0 \quad \text{for } i = 1, \ldots, j-1, \quad \langle h_j, v_j \rangle = 1.
\]

(2.12)

From (2.10) we have

\[
\bar{A}_j u \neq 0, \quad \text{for } u \in \partial D_j, \ j \geq J_1 + 1,
\]

and, consequently, \( \deg(\bar{A}_j, D_j, 0) \) is well defined. By the Leray-Schauder Lemma (see [9, Chapter 2, §1]), we obtain

\[
\deg(\bar{A}_j, D_j, 0) = \deg(A_{j-1}, D_{j-1}, 0), \quad \text{for } j \geq J_1 + 1.
\]

(2.13)

In order to prove the equality

\[
\deg(\bar{A}_j, D_j, 0) = \deg(A_j, D_j, 0),
\]

(2.14)

for all large \( j \), it suffices, by virtue of the properties of the degree for finite-dimensional mappings, to establish that

\[
t_A u + (1 - t)\bar{A}_j u \neq 0, \quad \text{for } u \in \partial D_j, \ t \in [0, 1].
\]

(2.15)

We prove (2.15) by contradiction. Let us assume that there exist sequences \( u_k \in \partial D_j(k), \ \{t_k\} \in [0, 1] \) such that

\[
t_k A_{j(k)} u_k + (1 - t_k)\bar{A}_{j(k)} u_k = 0, \quad j(k) \to \infty \text{ as } k \to \infty.
\]

(2.16)

Then the following equalities hold

\[
t_k \langle Au_k, v_{j(k)} \rangle + (1 - t_k)\langle h_{j(k)}, u_k \rangle = 0,
\]

\[
\langle Au_k, v_i \rangle = 0, \quad \text{for } i = 1, \ldots, j(k) - 1.
\]

(2.17)
From (2.10) it follows that, for $j(k) \geq J_1 + 1$, we have $0 < t_k < 1$. From (2.12) and (2.17) we obtain

$$
(Au_k, u_k) = \langle h_j(k), u_k \rangle \langle Au_k, v_j(k) \rangle = \frac{1 - t_k}{t_k} \langle h_j(k), u_k \rangle^2 
$$

which, along with (2.18) and the second equality in (2.17), implies

$$
\limsup_{k \to \infty} (Au_k, u_k) \leq 0, \quad \lim_{k \to \infty} (Au_k, v) = 0
$$

for any $v \in L\{F_j\}$. Extracting a weakly convergence subsequence from $\{u_k\}$, converging to some $u_0 \in X$, we obtain by virtue of $\langle S_+ \rangle \cap D(A) \cap \partial D$ and $Au_0 = 0$, which contradicts (2.6). This establishes (2.15) and, finally, (2.14). We have from (2.13) and (2.14) that $\deg(A_j, D_j, 0)$ is independent of $j$, for all large $j$. The proof is complete. \hfill \Box

By Theorem 2.1, $\lim_{j \to \infty} \deg(A_j, D_j, 0)$ exists. We denote it by $D\{F_j\}$.

**Theorem 2.2.** Assume that conditions $A_1$, $A_2$) and (2.6) are satisfied. Then the limit

$$
D\{F_j\} = \lim_{j \to \infty} \deg(A_j, D_j, 0)
$$

does not depend on the choice of the sequence $\{F_j\}$.

**Proof.** It is necessary to show that $D\{F_j\} = D\{F_j'\}$ for any other sequence $\{F_j'\}$ having the properties (2.2). Let $\{v_j'\}$, $j = 1, 2, \ldots$, be such that $F_j'$ is the span of $\{v_1', \ldots, v_j'\}$. We may assume that for each $j$ the system $\{v_1, \ldots, v_j, v_1', \ldots, v_j'\}$ is linearly independent. Otherwise, an auxiliary system $\{\tilde{v}_j\}$ can be constructed so that both systems $\{v_1, \ldots, v_j, \tilde{v}_1, \ldots, \tilde{v}_j\}$ and $\{v_1', \ldots, v_j', \tilde{v}_1, \ldots, \tilde{v}_j\}$ are linearly independent for any $j$. The proof would then reduce to establishing the equality $D\{F_j\} = D\{\tilde{F}_j\}$, where $\tilde{F}_j$ is the span of $\{\tilde{v}_1, \ldots, \tilde{v}_j\}$.

Let $F_j''$ be the span of $\{v_1, \ldots, v_j, v_1', \ldots, v_j'\}$ and define, for $j = 1, 2, \ldots$, the finite-dimensional mappings

$$
A_{2j,i}u = \sum_{i=1}^{j} \{ \langle Au, v_i \rangle v_i + [t \langle Au, v_i' \rangle + (1 - t) \langle h_i^1, u \rangle]v_i' \}. 
$$

Here, $u \in F_{2j}'$, $t \in [0, 1]$, and $h_i^1$ are elements of the space $X^*$ satisfying the conditions

$$
\langle h_i^1, v_k \rangle = 0, \quad \langle h_i^1, v_i' \rangle = \delta_{ik}, \quad i, k = 1, \ldots, j.
$$
where $\delta_{ik} = 0$ for $i \neq k$ and 1 otherwise.

We shall verify that, for some $J_2 \in N$,

$$A_{2j,t} u \neq 0, \quad u \in \partial D'_{2j}, \quad t \in [0, 1], \quad j \geq J_2, \quad (2.23)$$

where $D''_{2j} = D \cap F''$. We prove (2.23) by contradiction, assuming that there exist sequences $\{u_k\}, \{t_k\}$ such that

$$A_{2j(k),t_k} u_k = 0, \quad u_k \in \partial D''_{2j(k)}, \quad t_k \in [0, 1], \quad j(k) \to \infty \text{ as } k \to \infty. \quad (2.24)$$

Using (2.21) we have

$$\langle Au_k, v_i \rangle = 0, \quad i = 1, \ldots, j(k),$$

$$t_k \langle Au_k, v_i' \rangle + (1 - t_k) \langle h_i^{j(k)}, u_k \rangle = 0, \quad j = 1, \ldots, j(k). \quad (2.25)$$

From (2.10) follows that $t_k > 0$ if $j(k) \geq J_1$. Since

$$u_k = u_k^{(1)} + \sum_{i=1}^{j(k)} \langle h_i^{j(k)}, u_k \rangle v_i', \quad u_k^{(1)} \in F_{j(k)},$$

we obtain, from (2.25) and (2.22),

$$\langle Au_k, u_k \rangle = -\frac{1 - t_k}{t_k} \sum_{i=1}^{j(k)} \langle h_i^{j(k)}, u_k \rangle^2. \quad (2.26)$$

From (2.26) and the first equality in (2.25) follows that

$$\lim sup_{k \to \infty} \langle Au_k, u_k \rangle \leq 0, \quad \lim_{k \to \infty} \langle Au_k, v \rangle = 0 \quad (2.27)$$

for any $v \in L\{F_j\}$. Extracting a weakly convergent subsequence of the sequence $\{u_k\}$, converging to some $u_0 \in X$, we obtain from Condition $(S_\epsilon)_{0,L}$ that $u_0 \in D(A) \cap \partial D$ and $Au_0 = 0$. We thus have the desired contradiction to the assumption (2.6) and the inequality (2.23).

In view of the properties of the degree for finite-dimensional mappings and (2.23) we obtain

$$\deg(A_{2j,0}, D''_{2j}, 0) = \deg(A_{2j,1}, D''_{2j}, 0), \quad \text{for } j \geq J_2. \quad (2.28)$$
By the Leray-Schauder lemma and the construction of the operator $A_{2j,0}$ we have

$$\deg (A_{2j,0}, D''_{2j}, 0) = \deg (A_j, D_j, 0), \quad \text{for } j \geq J_2. \quad (2.29)$$

and, consequently,

$$\deg (A_j, D_j, 0) = \deg (A_{2j,1}, D''_{2j}, 0), \quad j \geq J_2. \quad (2.30)$$

Taking into consideration the symmetric role of $v_i$ and $v'_i$ in the definition of the mapping $A_{2j,1}$ we can establish, as in the proof of (2.30), that

$$\deg (A'_j, D'_j, 0) = \deg (A_{2j,1}, D''_{2j}, 0) \quad (2.31)$$

for all large $j$. Here, $A'_j, D'_j$ are defined as $A_j, D_j$, but with $\{F'_j\}, \{v'_j\}$ in place of $\{F_j\}, \{v_j\}$, respectively.

From (2.30), (2.31) we obtain

$$\lim_{j \to \infty} \deg (A_j, D_j, 0) = \lim_{j \to \infty} \deg (A'_j, D'_j, 0)$$

and the proof of the theorem is complete. □

**Definition 2.2.** Let $A : X \supset D(A) \to X^*$ satisfy conditions $A_1, A_2$ and (2.6). Let $D$ be an open bounded subset of $X$ and let $0$ denote the zero element of $X^*$. Then the degree $\text{Deg}(A, D, 0)$ is defined by

$$\text{Deg}(A, D, 0) = \lim_{j \to \infty} \deg (A_j, D_j, 0), \quad (2.32)$$

where $\deg (A_j, D_j, 0)$ is the degree of the finite-dimensional mapping $A_j$ defined by (2.5) and $D_j = D \cap F_j$.

**3. Degree for $(S_+)$-perturbations of maximal monotone operators.** Throughout this section $X$ is a real reflexive, possibly non-separable, Banach space. Let $L$ be a subspace of $X$ and let $\mathcal{F}(L)$ be the set of all finite-dimensional subspaces of $L$.

Consider an operator $M : X \supset D(M) \to X^*$ satisfying the following conditions:

$m_1$) $M$ is monotone, i.e.,

$$\langle Mu - Mv, u - v \rangle \geq 0, \quad (3.1)$$
for every \( u, \ v \in \mathcal{D}(M) \). Moreover,

\[
L \subset \mathcal{D}(M), \quad \overline{L} = X; \quad (3.2)
\]

\( m_2 \) for every \( (u_0, h_0) \in X \times X^* \) with

\[
\langle Mu - h_0, u - u_0 \rangle \geq 0, \quad \text{for } u \in L, \quad (3.3)
\]

we have \( u_0 \in \mathcal{D}(M) \) and \( Mu_0 = h_0 \);

\( m_3 \) for any \( u_0 \in \mathcal{D}(M) \) we have

\[
\inf \{ \langle Tv - Mu_0, v - u_0 \rangle : v \in L \} = 0; \quad (3.4)
\]

\( m_4 \) for every \( F \in \mathcal{F}(L), \ v \in L \) the mapping \( m(F, v) : F \to \mathcal{R} \), defined by \( m(F, v)u = \langle Mu, v \rangle \) is continuous.

Note that the conditions \( m_2 \), \( m_3 \) are satisfied for a maximal monotone operator \( M \) if \( \mathcal{D}(M) \) is a subspace of \( X \). In this case we may set \( L = \mathcal{D}(M) \).

We also consider a second operator \( A : X \supset \mathcal{D}(A) \to X^* \) satisfying the following conditions:

\( a_1 \)

\[
L \subset \mathcal{D}(A) \quad (3.5)
\]

and \( A \) is quasi-bounded with respect to \( M \), i.e., for every number \( S > 0 \) there exists a number \( K(S) > 0 \) such that from the inequalities

\[
\langle Au + Mu, u \rangle \leq S, \quad \| u \| \leq S, \quad u \in L, \quad (3.6)
\]

we have \( \| Au \| \leq K(S) \);

\( a_2 \) the operator \( A \) satisfies the following generalized \( (S_+) \) condition with respect to \( M \) : for every sequence \( \{ u_j \} \subset L \) such that \( u_j \to u_0, \ Au_j \to h_0 \) and

\[
\limsup_{j \to \infty} \langle Au_j, u_j - u_0 \rangle \leq 0, \quad \langle Mu_j + Au_j, u_j \rangle \leq 0, \quad (3.7)
\]

for some \( u_0 \in X, \ h_0 \in X^* \), we have \( u_j \to u_0, \ u_0 \in \mathcal{D}(A) \) and \( Au_0 = h_0 \);

\( a_3 \) for every \( F \in \mathcal{F}(L), \ v \in L \) the mapping \( a(F, v) : F \to \mathcal{R} \), defined by \( a(F, v)(u) = \langle Au, v \rangle \), is continuous.
Lemma 3.1. Assume that $X$ is a reflexive Banach space and $M : X \supset \mathcal{D}(M) \to X^*$ satisfies $m_1 - m_3$, while $A : X \supset \mathcal{D}(A) \to X^*$ satisfies $a_1, a_2$). Let $D$ be a bounded open set in $X$ such that

$$Mu + Au \neq 0, \quad u \in \partial D \cap \mathcal{D}(M + A),$$  \quad (3.8)

where $\mathcal{D}(M + A) = \mathcal{D}(M) \cap \mathcal{D}(A)$. Then there exists a space $F_0 \in \mathcal{F}(L)$ such that for every space $F \in \mathcal{F}(L)$ such that $F_0 \subset F$ we have

$$Z(F_0, F) \equiv \{ u \in \partial \mathcal{D}_F \cap \mathcal{D}(M + A) : \langle Mu + Au, u \rangle \leq 0, \langle Mu + Au, v \rangle = 0, v \in F_0 \} = \emptyset,$$  \quad (3.8')

where $\mathcal{D}_F = \mathcal{D} \cap F$.

**Proof.** Assume that the contrary is true: for every $F \in \mathcal{F}(L)$ there is some $\tilde{F} \in \mathcal{F}(L)$ such that $F \subset \tilde{F}$ and $Z(F, \tilde{F}) \neq \emptyset$. Given $F \in \mathcal{F}(L)$, define the set

$$G(F) = \bigcup_{\tilde{F}} \{ (u, Au) : u \in Z(F, \tilde{F}) \} \subset X \times X^*,$$  \quad (3.9)

where the union ranges over all $\tilde{F} \in \mathcal{F}(L)$ such that $F \subset \tilde{F}$.

Since $D$ is bounded, the monotonicity of $M$ and the quasi-boundedness of $A$ yield the following inclusion:

$$G(F) \subset \overline{B(0, R)} \times \overline{B^*(0, R^*)} \subset X \times X^*,$$  \quad (3.10)

for some positive constant $R > 0$ and $R^* = K(R)$.

Denote by $\overline{G_W(F)}$ the weak closure of the set $G(F)$. Then the family

$$\{ \overline{G_W(F)} : F \in \mathcal{F}(L) \}$$  \quad (3.11)

has the finite intersection property: for every finite family $\{ F_1, \ldots, F_I \}, F_i \in \mathcal{F}(L), i = 1, \ldots, I$, we have

$$\bigcap_{1 \leq i \leq I} \overline{G_W(F_i)} \neq \emptyset.$$  \quad (3.12)

To see this, we note that our assumption at the beginning of the proof says that we can find spaces $F', F'' \in \mathcal{F}(L)$ such that

$$F_i \subset F', \quad i = 1, \ldots, I, \quad F' \subset F'', \quad Z(F', F'') \neq \emptyset.$$
Thus, by the definition of the sets $Z(F_0, F)$, $G(F)$, we have

$$
\bigcap_{1 \leq i \leq I} \overline{G_W(F_i)} \supset \bigcap_{1 \leq i \leq I} G(F_i) \supset \bigcap_{1 \leq i \leq I} \{(u, Au) : u \in Z(F_i, F'')\} \supset \{(u, Au) : u \in Z(F', F'')\} \neq \emptyset,
$$

which shows the finite intersection property for the family (3.11).

From the reflexivity of the spaces $X$, $X^*$ we have the weak compactness of the closed balls $B(0, R)$, $B^*(0, R^*)$ in $X$, $X^*$, respectively. In view of the finite intersection property for the family (3.11), we obtain the existence of a pair $(u_0, h_0)$ such that

$$
(u_0, h_0) \in \bigcap_{F \in \mathcal{F}(L)} \overline{G_W(F)}, \quad u_0 \in \overline{B(0, R)}, \quad h_0 \in \overline{B^*(0, R^*)}. \tag{3.13}
$$

We shall prove that $u_0 \in \partial D \cap D(M + A)$, $Mu_0 + Au_0 = 0$. This will contradict with the assumption (3.8).

We consider two possibilities:

a) there exists $v_0 \in L$ such that

$$
\langle Mv_0 + h_0, v_0 - u_0 \rangle \leq 0, \tag{3.14}
$$

where $u_0$, $h_0$ satisfy (3.13);

b) for every $v \in L$ we have

$$
\langle Mv + h_0, v - u_0 \rangle > 0. \tag{3.15}
$$

Let $w \in L$ be given. In Case a) we choose as $F'$ the span $\{w, v_0\}$, where $v_0 \in L$ satisfies (3.14). Then we let the sequences $\{F_j\} \subset \mathcal{F}(L)$, $\{u_j\} \subset Z(F', F_j)$ be such that

$$
\langle Mv + u_j - v \rangle \rightarrow 0, \quad Au_j \rightarrow h_0. \tag{3.16}
$$

From (3.8) we have

$$
\langle Mu_j + Au_j, u_j \rangle \leq 0, \quad \langle Mu_j + Au_j, v \rangle = 0, \quad \text{for } v \in F', \tag{3.17}
$$

and by the monotonicity of the operator $M$ we arrive at

$$
\langle Au_j, u_j - v \rangle \leq -\langle Mu_j, u_j - v \rangle = -\langle Mv, u_j - v \rangle + \langle Mv - Mu_j, u_j - v \rangle \leq \langle Mv, v - u_j \rangle \tag{3.18}
$$
for every \( v \in F' \).

Passing to the limit in (3.18) as \( j \to \infty \), we get

\[
\limsup_{j \to \infty} \langle Au_j, u_j \rangle \leq \langle h_0, v \rangle + \langle Mv, v - u_0 \rangle, \quad v \in F'.
\]

Letting \( v = v_0 \) above and using (3.14) we obtain

\[
\limsup_{j \to \infty} \langle Au_j, u_j \rangle \leq \langle h_0, u_0 \rangle,
\] (3.19)

which implies

\[
\limsup_{j \to \infty} \langle Au_j, u_j - u_0 \rangle \leq 0.
\] (3.20)

Using the property \( a_2 \) of the operator \( A \) we obtain from (3.16), (3.17) and (3.20) \( u_j \to u_0, \ u_0 \in D(A) \) and \( Au_0 = h_0 \).

Passing to the limit in (3.18) with \( v = w \) we obtain

\[
\langle Mw + h_0, w - u_0 \rangle \geq 0.
\] (3.21)

Since \( w \) is an arbitrary element of the space \( L \) we have, by Condition \( m_2 \), \( u_0 \in D(M) \) and \( Mu_0 = -h_0 \). Therefore, \( u_0 \in D(M + A), \ u_0 \in \partial D \) and \( Mu_0 + Au_0 = 0 \). This contradicts Condition (3.8) and finishes the proof of Case a).

We shall now consider Case b). By Condition \( m_2 \) we have

\[
Mu_0 = -h_0, \quad u_0 \in D(M),
\] (3.22)

and, by Condition \( m_3 \), there exists a sequence \( \{v_j\} \subset L \) such that

\[
0 \leq \langle Mv_j + h_0, v_j - u_0 \rangle < \frac{1}{j}, \quad j = 1, 2, \ldots
\] (3.23)

Let \( w \in L \) and denote by \( F_j' \) the span of \( \{w, v_1, \ldots, v_j\} \). From (3.13) follows that for each fixed \( j \) we can find sequences \( \{F_i^{(j)}\} \subset \mathcal{F}(L), \ \{u_i^{(j)}\} \subset Z(F_j', F_i^{(j)}), \ i = 1, 2, \ldots \), such that

\[
u_i^{(j)} \to u_0, \quad Au_i^{(j)} \to h_0 \quad \text{as} \ i \to \infty.
\] (3.24)
Let $X^{(1)}$ be the closure of the subspace of $X$ generated by the span of the union of all the subspaces $F^{(j)}_i$, $i, j = 1, 2, \ldots$. Since $X^{(1)}$ is separable and reflexive, being a closed subspace of the reflexive space $X$, $[X^{(1)}]^*$ is separable. We can now choose a sequence $\{h_k\} \subset [X^{(1)}]^*$ which is dense in the strong topology of $[X^{(1)}]^*$. Then a necessary and sufficient condition for a bounded sequence $\{u'_j\} \subset X^{(1)}$ to converge weakly in $X$ to $u' \in X^{(1)}$ is
\[
\langle h_k, u'_j \rangle \to \langle h_k, u' \rangle \quad \text{as} \quad j \to \infty,
\]
for every $k = 1, 2, \ldots$.

Let $Y$ be the separable closed subspace of the space $X^*$ generated by the span of $Au^{(j)}_i$ for $i, j = 1, 2, \ldots$. As above, the space $Y^*$ is separable and we can choose a sequence $\{g_l\} \subset Y^*$ which is dense in the strong topology of $Y^*$. Then a bounded sequence $\{h'_j\} \subset Y$ converges weakly in $X^*$ to $h' \in Y$ if and only if
\[
\langle g_l, h'_j \rangle \to \langle g_l, h' \rangle \quad \text{as} \quad j \to \infty,
\]
for each $l = 1, 2, \ldots$.

Define now the sequence $u_j = u^{(j)}_{i(j)}$, where $i(j)$ is chosen by the conditions
\[
|\langle h_k, u^{(j)}_{i(j)} \rangle - u_0| < \frac{1}{j}, \quad k = 1, \ldots, j,
\]
\[
|\langle g_l, Au^{(j)}_{i(j)} \rangle - h_0| < \frac{1}{j}, \quad l = 1, \ldots, j.
\]

The possibility of such a choice follows from (3.24). Then the convergence properties (3.25) and (3.26) hold for the sequences $u'_j = u_j, \ h'_j = Au_j$. Now, (3.24) and the boundedness of the sequences $u_j, \ Au_j$, which follows from (3.10), imply $u_0 \in X^{(1)}$ and $h_0 \in Y$. Consequently, we have established the following limits:
\[
u_j \to u_0, \quad Au_j \to h_0.
\]

From $u_j \equiv u^{(j)}_{i(j)} \in Z(F'_j, F^{(j)}_{i(j)})$ we have
\[
\langle Mu_j + Au_j, u_j \rangle \leq 0, \\
\langle Mu_j + Au_j, v \rangle = 0, \quad v \in F'_m, \ m \leq j.
\]

As in the case of (3.18) we obtain from (3.29)
\[
\langle Au_j, u_j \rangle \leq \langle Au_j, v \rangle + \langle Mv, v - u_j \rangle, \quad v \in F'_m, \ m \leq j.
\]
Passing to the limit in (3.20), for a fixed $m$ and $j \to \infty$, we get
\[
\limsup_{j \to \infty} \langle Au_j, u_j \rangle \leq \langle h_0, v \rangle + \langle Mv, v - u_0 \rangle.
\] (3.31)

We choose $v = v_m$ above and use (3.23) to obtain
\[
\limsup_{j \to \infty} \langle Au_j, u_j \rangle < \langle h_0, u_0 \rangle + \frac{1}{m}.
\] (3.32)

Taking into consideration the fact that $m$ can be an arbitrary natural number, we arrive at
\[
\limsup_{j \to \infty} \langle Au_j, u_j - u_0 \rangle \leq 0.
\] (3.33)

By the property $a_2$ of $A$, we conclude from (3.28), (3.33) and (3.20) that $u_j \to u_0$, $u_0 \in \mathcal{D}(A)$ and $Au_0 = h_0$. Using (3.22) we have $u_0 \in \mathcal{D}(M + A) \cap \partial D$ and $Mu_0 + Au_0 = 0$, which contradict the condition (3.8). The proof of the lemma is complete. \(\square\)

Let $F \in \mathcal{F}(L)$ and let $v_1, \ldots, v_k$ be a basis for $F$. We define a finite-dimensional mapping $(M + A)_F : F \to F$ by
\[
(M + A)_F(u) = \sum_{i=1}^{k} \langle Mu + Au, v_i \rangle v_i.
\] (3.34)

**Theorem 3.1.** Assume that $X$ is a real reflexive Banach space and $M : X \supset \mathcal{D}(M) \to X^*$ satisfies $m_1 - m_4$, while $A : X \supset \mathcal{D}(A) \to X^*$ satisfies $a_1 - a_3$. Let $D$ be an open set in $X$ such that
\[
Mu + Au \neq 0, \quad u \in \partial D \cap \mathcal{D}(M + A).
\] (3.35)

Let $F_0 \in \mathcal{F}(L)$ be the space defined in Lemma 3.1. Then for every space $F \in \mathcal{F}(L)$ with $F_0 \subset F$ we have the following relation:
\[
\deg((M + A)_F, D_F, 0) = \deg((M + A)_{F_0}, D_{F_0}, 0),
\] (3.36)

where $(M + A)_F$ is the finite-dimensional mapping defined by (3.34) and $D_F = D \cap F$.

**Proof.** From Lemma 3.1 we obtain
\[
(M + A)_F(u) \neq 0, \quad u \in \partial D_F,
\] (3.37)
where $F$ is any space in $\mathcal{F}(L)$ with $F_0 \subset F$. Consequently, the degrees of the finite-dimensional mappings on both sides of (3.36) are well defined.

Let $F \in \mathcal{F}(L)$ be such that $F_0 \subset F$ and choose a basis in $F$ of the form $v_1, \ldots, v_k$, where $v_1, \ldots, v_{k_0}$, $k_0 < k$, is a basis of $F_0$. We consider on $D_F$ the mapping

$$(M + A)^f_P (u) = \sum_{i=1}^{k_0} \langle Mu + Au, v_i \rangle v_i + \sum_{i=1}^{k-k_0} \langle f_i, u \rangle v_{i+k_0},$$

where $f_i$ is an element in $X^*$ satisfying the conditions:

$$\langle f_i, v_j \rangle = 0, \quad \text{for } j \neq i + k_0, \quad \langle f_i, v_{i+k_0} \rangle = 1,$$

for $i = 1, \ldots, k-k_0$, $j = 1, \ldots, k$. By the Leray-Schauder lemma, we have

$$\deg ((M + A)^f_P, D_F, 0) = \deg ((M + A)_{F_0}, D_{F_0}, 0).$$

(3.39)

Therefore, in order to prove the theorem it suffices to verify that

$$t(M + A)^f_P (u) + (1 - t)(M + A)^f_P (u) \neq 0, \quad u \in \partial D_F, \quad t \in [0, 1].$$

(3.40)

If (3.40) is not true, then, for some $u_0 \in \partial D_F$, $t_0 \in [0, 1]$, we have

$$\langle Mu_0 + Au_0, v_i \rangle = 0, \quad i = 1, \ldots, k_0,$$

$$t_0 \langle Mu_0 + Au_0, v_{i+k_0} \rangle + (1 - t_0) \langle f_i, u_0 \rangle = 0, \quad i = 1, \ldots, k - k_0.$$ 

(3.41)

From (3.37) follows that $t_0 > 0$. Using (3.38) we have

$$u_0 = u_0' + \sum_{i=1}^{k-k_0} \langle f_i, u_0 \rangle v_{i+k_0} \quad \text{with } u_0' \in F_0,$$

which combined with (3.41) gives

$$\langle Mu_0 + Au_0, u_0 \rangle = -\frac{1 - t_0}{t_0} \sum_{i=1}^{k-k_0} (f_i, u_0)^2 \leq 0,$$

$$\langle Mu_0 + Au_0, v \rangle = 0, \quad v \in F_0.$$

By the definition of $Z(F_0, F)$ we have $u_0 \in Z(F_0, F)$, which contradicts the assumption of Lemma 3.1 to the effect that $Z(F_0, F) = \emptyset$. It follows that (3.40) holds true and the proof is complete. □
Theorem 3.2. Assume that the space $X$, the operators $M$, $A$ and the set $D$ satisfy the conditions of Theorem 3.1. Let $F_0$, $F_0' \in \mathcal{F}(L)$ be such that for arbitrary spaces $F$, $F' \in \mathcal{F}(L)$ with $F_0 \subset F$, $F_0' \subset F'$ we have

$$Z(F_0, F) = \emptyset, \quad Z(F_0', F') = \emptyset,$$

where $Z(F_0, F)$ is defined by (3.8'). Then

$$\deg((M + A)_{F_0}, D_{F_0}, 0) = \deg((M + A)_{F_0'}, D_{F_0'}, 0),$$

where the mapping $(M + A)_{F_0}$ is defined by (3.34).

Proof. The equality (3.43) follows immediately from (3.36) with the space $F \in \mathcal{F}(L)$ such that $F_0 \subset F$, $F_0' \subset F$. \qed

Theorems 3.1 and 3.2 justify the introduction of the following definition of a degree function.

Definition 3.1. Assume that the space $X$, the operators $M$, $A$ and the set $D$ satisfy the conditions of Theorem 3.1. Then the degree $\text{Deg}(M + A, D, 0)$ is defined by

$$\text{Deg}(M + A, D, 0) = \deg((M + A)_{F_0}, D_{F_0}, 0),$$

where the operator $(M + A)_F$ is defined by (3.34), $D_{F_0} = D \cap F_0$ and $F_0$ is the finite-dimensional subspace of $L$ determined by Lemma 3.1.

4. Properties of the degree of densely defined operators. The degrees introduced in Sections 2 and 3 have all the desirable properties of the degree for finite-dimensional mappings. In this section we prove only the principles of homotopy invariance and nonzero degree.

At first, we establish the properties of the degree of Section 2. To this end, let $X_1$ be a real separable and reflexive Banach space and $D$ an open and bounded set in $X_1$. Let $A_t : X_1 \supset D(A_t) \to X_1^*$, $t \in [0, 1]$, be a one-parameter family of nonlinear operators. We assume that there exists a subspace $L$ of $X_1$ and a sequence $\{F_j\}$ satisfying the condition (2.2) and such that

$$L = X_1, \quad L\{F_j\} \subset D(A_t), \quad t \in [0, 1],$$

where the space $L\{F_j\}$ is defined by (2.3).
Definition 4.1. We say that the family \( \{A_t\} \) satisfies Condition \((S^\dagger)\) if whenever \( \{u_j\} \subset L\{F_j\}, \{t_j\} \subset [0,1] \) are such that \( u_j \to u_0, t_j \to t_0 \) and

\[
\lim_{j \to \infty} \langle A_{t_j} u_j, u_j \rangle = 0, \quad \lim_{j \to \infty} \langle A_{t_j} u_j, v \rangle = 0,
\]

for some \( u_0 \in X_1 \) and any \( v \in L\{F_j\} \), it follows that the sequence \( \{u_j\} \) converges strongly to \( u_0, u_0 \in D(A_{t_0}) \) and \( A_{t_0} u_0 = 0 \).

Definition 4.2. Let \( A^{(i)} : X_1 \supset D(A^{(i)}) \to X_1^\dagger, \ i = 0,1, \) satisfy the conditions \( A_1), A_2) \) of Section 2 with a common space \( L \). The operators \( A^{(0)}, A^{(1)} \) are called “homotopic” with respect to the bounded open set \( D \subset X_1 \) if there exists a one-parameter family of operators \( A_t : X_1 \supset D(A_t) \to X_1, \ t \in [0,1], \) such that

1) \( A^{(i)} = A_t, \ i = 0,1; \ A_t(u) \neq 0 \) for \( t \in [0,1], \ u \in D(A_t) \cap \partial D; \)

2) the family \( \{A_t\} \) satisfies Condition \((S^\dagger)\) with respect to the space \( L; \)

3) for every space \( F \subset L\{F_j\} \) and every \( v \in L\{F_j\} \) the mapping \( \tilde{a}(F,v) : F \times [0,1] \to \mathcal{R} \) defined by \( \tilde{a}(F,v)(u,t) = \langle A_t u, v \rangle \) is continuous.

Theorem 4.1. Let \( A^{(i)} : X_1 \supset D(A^{(i)}) \to X_1^\dagger, \ i = 0,1, \) satisfy Conditions \( A_1), A_2) \) of Section 2 with a common space \( L \), and assume that the operators \( A^{(0)}, A^{(1)} \) are homotopic with respect to the open and bounded set \( D \subset X_1 \).

Then

\[
\text{Deg}(A^{(0)},D,0) = \text{Deg}(A^{(1)},D,0),
\]

where the degree \( \text{Deg} \) is defined by Definition 2.2.

Proof. We choose a sequence \( \{v_j\} \) corresponding to the sequence \( \{F_j\} \) as in Section 2. Define a family of finite-dimensional mappings

\[
A_{t,j}(u) = \sum_{i=1}^{j} \langle A_t u, v_i \rangle v_i, \quad u \in F_j, \ t \in [0,1],
\]

where \( \{A_t\} \) is a family of operators which establishes a homotopy between \( A^{(0)} \) and \( A^{(1)} \) in the sense of Definition 4.2.

We show that

\[
A_{t,j}(u) \neq 0, \quad u \in \partial D_j, \ t \in [0,1], \ j \geq J_3,
\]
for a sufficiently large number $J_3$, where $D_j = D \cap F_j$. Assume that the contrary is true. Then there exist sequences $u_k \in \partial D_j(k)$, \{$t_k$\} $\subset [0, 1]$, such that

$$A_{t_k,j(k)}(u_k) = 0, \quad j(k) \to \infty \text{ as } k \to \infty. \quad (4.6)$$

We may suppose that $t_k \to t_0$, $u_k \to u_0$.

From (4.6) we have

$$\langle A_{t_k} u_k, u_k \rangle = 0, \quad \langle A_{t_k} u_k, v_i \rangle = 0, \quad i \leq j(k).$$

From this and Condition $(S_+)_{b,L}^{(t)}$ for the operators $A_{t}$ we obtain, an in the proof of Assertion 1) of Theorem 2.1, that $u_0 \in D(A_{t_0}) \cap \partial D$ and $A_{t_0} u_0 = 0$. This contradicts the property 1) in Definition 4.2 concerning homotopic operators.

By (4.5) and the properties of the degree for finite-dimensional mappings, $\deg(A_{t,j}, D_j, 0)$ does not depend on $t$ for $j \geq J_3$. We obtain

$$\deg(A_{0,j}, D_j, 0) = \deg(A_{1,j}, D_j, 0), \quad j \geq J_3. \quad (4.7)$$

Taking into consideration the fact that $A^{(0)} = A_0$, $A^{(1)} = A_1$ and passing to the limit in (4.7) as $j \to \infty$, we establish the assertion of Theorem 4.1. \qed

**Theorem 4.2.** Let $A : X_1 \supset \mathcal{D}(A) \to X_1^*$ satisfy Conditions $A_1)$, $A_2)$ of Section 2, and let $D$ be a bounded open set in $X_1$ such that (2.6) holds. Assume that

$$\text{Deg}(A, D, 0) \neq 0. \quad (4.8)$$

Then the equation $Au = 0$ has at least one solution in $\mathcal{D}(A) \cap D$.

**Proof.** Consider the finite-dimensional approximation $A_j$ of the operator $A$ defined by (2.5). From (4.8) and Definition 2.2 we conclude that

$$\deg(A_j, D, 0) \neq 0, \quad j \geq J_4, \quad (4.9)$$

for some natural number $J_4$. By the properties of the finite-dimensional degree we obtain the existence of $u_j$ such that

$$A_j u_j = 0, \quad u_j \in D_j, \quad j \geq J_4. \quad (4.10)$$

Choosing, if necessary, a weakly convergent subsequence of the sequence \{$u_j$\}, converging to some $u_0 \in X_1$, we can obtain, as in the proof of Assertion
1) of Theorem 2.1, that \( u_0 \in \overline{D} \cap D(A), \) \( Au_0 = 0. \) By (2.6), \( u_0 \not\in \partial D \) and, consequently, \( u_0 \) is a solution of the equation \( Au = 0 \) in \( D(A) \cap D. \) This is the end of the proof of Theorem 4.2. \( \square \)

We are now going to prove analogues of Theorems 4.1, 4.2 for the degree developed in Section 3. Let \( X_2 \) be a real reflexive Banach space, and let \( L \) be a subspace of \( X_2 \) such that \( L = X_2. \)

Consider the one-parameter family of operators \( M_t : X_2 \supset D(M_t) \rightarrow X_2^*, \ t \in [0, 1], \) satisfying the following conditions:

\( m_t^{(1)} \) for every \( t \in [0, 1] \) the operator \( M_t \) satisfies Conditions \( m_1) - m_3) \) of Section 3 with the space \( L \) independent of \( t; \)

\( m_t^{(2)} \) for every \( v \in L \) the mapping \( \mu(v) : [0, 1] \rightarrow X_2^*, \) defined by \( \mu(v)(t) = M_t(v) \) is continuous;

\( m_t^{(3)} \) for every \( F \in \mathcal{F}(L), \ v \in L \) the mapping \( \tilde{m}(F, v) : F \times [0, 1] \rightarrow \mathcal{R}, \) defined by \( \tilde{m}(F, v)(u, t) = \langle M_t u, v \rangle, \) is continuous.

Let \( A_t : X_2 \supset D(A_t) \rightarrow X_2^*, \ t \in [0, 1], \) be a second one-parameter family of operators satisfying the following conditions:

\( a_t^{(1)} \) for every \( t \in [0, 1], \) let \( L \subset D(A_t) \) be as in Conditions \( m_t^{(1)} - m_t^{(3)}, \) and let the family \( \{A_t\} \) be uniformly quasi-bounded, i.e., for every \( S > 0 \) there exists \( K(S) > 0 \) such that

\[
\langle M_t u + A_t u, u \rangle \leq S, \quad \|u\| \leq S, \quad u \in L, \ t \in [0, 1],
\]

implies the estimate \( \|Au\|_S \leq K(S), \) where \( \| \cdot \|_S \) is the norm of \( X_2^*; \)

\( a_t^{(2)} \) for every pair of sequences \( \{t_j\} \subset [0, 1], \ \{u_j\} \subset L \) such that \( u_j \rightarrow u_0, \ A_t u_j \rightarrow h_0, \ t_j \rightarrow t_0 \) and

\[
\lim_{j \rightarrow \infty} \sup \langle A_{t_j} u_j, u_j - u_0 \rangle \leq 0, \quad \langle M_{t_j} u_j + A_{t_j} u_j, u_j \rangle \leq 0,
\]

for some \( t_0 \in [0, 1], \ u_0 \in X_2, \ h_0 \in X_2^*, \) we have \( u_j \rightarrow u_0, \ u_0 \in D(A) \) and \( A_{t_0} u_0 = h_0; \)

\( a_t^{(3)} \) for every \( F \in \mathcal{F}(L), \ v \in L \) the mapping \( \tilde{a}(F, v) : F \times [0, 1] \rightarrow \mathcal{R}, \) defined by \( \tilde{a}(F, v)(u, t) = \langle A_t u, v \rangle \) is continuous.

**Definition 4.3.** Let \( M^{(i)} : X_2 \supset D(M^{(i)}) \rightarrow X_2^*, \ A^{(i)} : X_2 \supset D(A^{(i)}) \rightarrow X_2^*, \ i = 0, 1, \) satisfy conditions \( m_1) - m_4) \) and \( a_1) - a_3) \) of Sections 3, respectively, with a common space \( L. \) We say that the operators \( A^{(0)} + M^{(0)}, \ A^{(1)} + M^{(1)} \) are homotopic with respect to the open bounded set \( D \subset \)
$X_2$ if there exist one-parameter families of operators $M_t : X_2 \supset D(M_t) \rightarrow X_2^*$, $A_t : X_2 \supset D(A_t) \rightarrow X_2^*$ satisfying Conditions $m_t^{(1)} - m_t^{(3)}$ and $a_t^{(1)} - a_t^{(3)}$, respectively, and such that

$$M^{(i)} = M_i, \quad A^{(i)} = A_i, \quad i = 0, 1,$$  \hspace{1cm} (4.13)

and

$$M_t u + A_t u \neq 0, \quad u \in \partial D \cap D(M_t + A_t), \quad t \in [0, 1].$$ \hspace{1cm} (4.14)

**Theorem 4.3.** Assume that the operators $M^{(i)}$, $A^{(i)}$, $i = 0, 1$, satisfy Conditions $(m_1) - (m_4)$ and $(a_1) - (a_3)$ of Section 3, respectively. Assume that the operators $M^{(0)} + A^{(0)}$, $M^{(1)} + A^{(1)}$ are homotopic with respect to the bounded open set $D \subset X_2$. Then

$$\text{Deg}(M^{(0)} + A^{(0)}, D, 0) = \text{Deg}(M^{(1)} + A^{(1)}, D, 0),$$ \hspace{1cm} (4.15)

where the degree is defined in Definition 3.1.

**Proof.** For $t \in [0, 1]$ and finite-dimensional spaces $F$, $F' \in \mathcal{F}(L)$ with $F' \subset F$, we introduce the set

$$Z(F', F, t) = \{ u \in \partial D_F \cap D(M_t + A_t) : \langle M_t u + A_t u, u \rangle \leq 0, \langle M_t u + A_t u, v \rangle = 0, \quad v \in F' \},$$ \hspace{1cm} (4.16)

where $D_F = D \cap F$.

We will prove the existence of $F_0 \in \mathcal{F}(L)$ such that

$$Z(F_0, F, t) = \emptyset, \quad \text{for } F_0 \subset F, \quad t \in [0, 1].$$ \hspace{1cm} (4.17)

Assume that the contrary is true, i.e., assume that for every $F \in \mathcal{F}(L)$ there exist $t \in [0, 1]$, $\tilde{F} \in \mathcal{F}(L)$ and $F \subset \tilde{F}$ such that $Z(F, \tilde{F}, t) \neq \emptyset$. For every $F \in \mathcal{F}(L)$, define the set

$$H(F) = \bigcup_{t} \bigcup_{\tilde{F}} \{ (t, u, Au) : u \in Z(F, \tilde{F}, t), \quad F \subset \tilde{F} \in \mathcal{F}(L), \quad t \in [0, 1] \} \subset \mathcal{R} \times X_2 \times X_2^*.$$ \hspace{1cm} (4.18)

Since $D$ is bounded, the monotonicity of $M_t$ and the uniform quasi-boundedness of the family $\{A_t\}$ imply the following inclusion:

$$H(F) \subset [0, 1] \times \overline{B}(0, R) \times \overline{B^*}(0, R^*) \subset \mathcal{R} \times X_2 \times X_2^*,$$ \hspace{1cm} (4.19)
for some $R > 0$, $R^* = \mathcal{K}(R \cdot \max_{0 \leq t \leq 1} \|M_t(0)\|_*)$. Denote by $\overline{H}_W(F)$ the weak closure of the set $H(F)$. Then the family
\[
\{ \overline{H}_W(F) : F \in \mathcal{F}(L) \}
\]
has the finite intersection property. The proof of this fact follows the steps of the proof of the analogous statement for the family (3.11). It is therefore omitted. As in (3.13), we obtain the existence of $(t_0, u_0, h_0) \in [0, 1] \times B(0, R) \times B^*(0, R^*)$ such that
\[
(t_0, u_0, h_0) \in \bigcap_{F \in \mathcal{F}(L)} \overline{H}_W(F).
\]
We consider two possibilities:
\begin{enumerate}
  \item there exists $v_0 \in L$ such that
  \[
  \langle M_{t_0} v_0 - h_0, v_0 - u_0 \rangle \leq 0,
  \]
  \item for all $v \in L$ we have
  \[
  \langle M_{t_0} v - h_0, v - u_0 \rangle > 0.
  \]
\end{enumerate}

Let $w \in L$ be given. In Case a) we choose $F'$ to be the span of the set $\{w, v_0\}$, where $v_0 \in L$ satisfies (4.22). Then we have sequences $\{t_j\} \subset [0, 1]$, $\{F_j\} \subset \mathcal{F}(L)$ and $u_j \in Z(F', F_j, t_j)$ such that
\[
t_j \to t_0, \quad u_j \to u_0, \quad A_{t_j} u_j \to h_0.
\]
From (4.16) we have
\[
\langle M_{t_j} u_j + A_{t_j} u_j, u_j \rangle \leq 0, \quad \langle M_{t_j} u_j + A_{t_j} u_j, v \rangle = 0, \quad v \in F',
\]
and, by the monotonocity of the operators $M_{t_j}$,
\[
\langle A_{t_j} u_j, u_j - v \rangle \leq \langle M_{t_j} v, v - u_j \rangle, \quad v \in F'.
\]
Using the property $m_i^{(2)}$ for the family $\{M_t\}$ we can pass to the limit in (4.26), as $j \to \infty$, in order to obtain
\[
\limsup_{j \to \infty} \langle A_{t_j} u_j, u_j \rangle \leq \langle h_0, v \rangle + \langle M_{t_0} v, v - u_0 \rangle, \quad v \in F.
\]
Using (4.26), (4.27) and the properties of the operators $M_t$, $A_t$, we can prove, as in proof of the corresponding part of Lemma 3.1, that $u_0 \in \partial D \cap D(M_0 + A_0)$ and $M_0 u_0 + A_0 u_0 = 0$. This contradicts Condition (4.14) and establishes (4.17) in Case a).

Now, we consider Case b). From (4.23) for all $v \in L$ and Condition $m_2$) for the operator $M_0$, we obtain

$$M_0 u_0 = -h_0, \quad u_0 \in D(M_0). \quad (4.28)$$

By Condition $m_3$) for the operator $M_0$, there exists a sequence $\{v_j\} \subset L$ such that

$$0 \leq \langle M_0 v_j + h_0, v_j - u_0 \rangle < \frac{1}{j}, \quad j = 1, 2, \ldots. \quad (4.29)$$

Let $w \in L$ be given. Denote by $F'_j$ the span of the set $\{w, v_1, \ldots, v_j\}$. From (4.21) we obtain the existence of sequences $\{t^{(j)}_i\} \subset [0, 1]$, $\{F^{(j)}_i\} \subset F(L)$ and $\{u^{(j)}_i\} \subset Z(F'_j, F^{(j)}_i, t^{(j)}_i)$, $i = 1, 2, \ldots$, such that

$$t^{(j)}_i \to t_0, \quad u^{(j)}_i \to u_0, \quad A^{(j)}_i u^{(j)}_i \to h_0 \quad \text{as } i \to \infty. \quad (4.30)$$

As in the proof of the corresponding part of Lemma 3.1 (after (3.24)), we can introduce separable closed subspaces $X^{(1)} \subset X_2$, $Y \subset X_2^*$ such that $F^{(j)}_i \subset X^{(1)}$, $A^{(j)}_i u^{(j)}_i \subset Y$, $i, j = 1, 2, \ldots$, and then let $\{h_k\} \subset X^{(1)}$, $\{g_l\} \subset Y$, $k, l = 1, 2, \ldots$, be sequences with the same meaning as in the proof of Lemma 3.1.

Define now sequences $t_j = t^{(j)}_{i^{(j)}}$, $u_j = u^{(j)}_{i^{(j)}}$, where the number $i^{(j)}$ is chosen by the conditions

$$|t^{(j)}_{i^{(j)}} - t_0| < \frac{1}{j}, \quad |(h_k, u^{(j)}_{i^{(j)}} - u_0)| < \frac{1}{j}, \quad k = 1, \ldots, j, \quad (4.31)$$

$$|(g_l, A^{(j)}_{i^{(j)}} u^{(j)}_{i^{(j)}} - h_0)| < \frac{1}{j}, \quad l = 1, \ldots, j.$$

The boundedness of the sequences $\{u_j\}$, $\{Au_j\}$ follows from (4.19). Thus, (4.31) yields

$$t_j \to t_0, \quad u_j \to u_0, \quad A_{i^{(j)}} u_j \to h_0. \quad (4.32)$$

From $u_j \in Z(F'_j, F^{(j)}_i, t^{(j)}_i, t_j)$ we have

$$\langle M_t u_j + A_{i^{(j)}} u_j, u_j \rangle \leq 0,$$

$$\langle M_t u_j + A_{i^{(j)}} u_j, v \rangle = 0, \quad v \in F'_m, \quad m \leq j. \quad (4.33)$$
As in the proof of (3.22) we obtain

\[
\limsup_{j \to \infty} \langle A_t, u_j, u_j \rangle < \langle h_0, u_0 \rangle + \frac{1}{m}
\]

and, as in the proof of Lemma 3.1, \( u_j \to u_0, \ u_0 \in D(A_t) \) and \( A_t u_0 = h_0 \). Using (4.28) we obtain \( u_0 \in D(M_{t_0} + A_{t_0}) \cap \partial D, \ M_{t_0} u_0 + A_{t_0} u_0 = 0 \), which contradict the assumption (4.14). The proof of (4.17) is complete.

Let \( v_1, \ldots, v_{k_0} \) be a basis for the space \( F_0 \in \mathcal{F}(L) \) such that (4.17) holds. We define a finite-dimensional mapping \((M_t + A_t)_{F_0} : F_0 \to F_0\) by

\[
(M_t + A_t)_{F_0}(u) = \sum_{i=1}^{k_0} \langle M_t u + A_t u, v_i \rangle v_i.
\]

(4.34)

From (4.17) we obtain

\[
(M_t + A_t)_{F_0}(u) \neq 0, \quad u \in \partial D_{F_0}, \quad t \in [0, 1],
\]

(4.35)

and, by the properties of the degree for finite-dimensional mappings,

\[
\deg((M_0 + A_0)_{F_0}, D_{F_0}, 0) = \deg((M_1 + A_1)_{F_0}, D_{F_0}, 0).
\]

(4.36)

Taking into consideration (4.13) and Definition 3.1, we obtain from (4.36) the equality (4.15) and the proof is complete. □

**Theorem 4.4.** Assume that \( X_2 \) is a real reflexive Banach space and let \( M : X_2 \supset D(M) \to X_2 \), \( A : X_2 \supset D(A) \to X_2 \) satisfy Conditions \( m_1 \)-\( m_4 \) and \( a_1 \)-\( a_3 \), respectively. Let \( D \) be a bounded open set in \( X_2 \) such that (3.35) holds, and assume that

\[
\text{Deg}(M + A, D, 0) \neq 0,
\]

(4.37)

where the degree function is given in Definition 3.1. Then the equation

\[
Mu + Au = 0
\]

(4.38)

has at least one solution in \( D(M + A) \cap D \).

**Proof.** Assume that

\[
Mu + Au \neq 0, \quad u \in D \cap D(M + A).
\]

(4.39)
Taking into account (3.8) we have that (4.39) holds for \( u \in \overline{D} \cap \mathcal{D}(M + A) \).

As in the proof of Lemma 3.1, it is possible to prove the existence of a space \( \tilde{F}_0 \in \mathcal{F}(L) \) such that for every space \( F \in \mathcal{F}(L) \) such that \( \tilde{F}_0 \subset F \) the set

\[
\tilde{Z}(\tilde{F}_0, F) = \{ u \in \overline{D_F} \cap \mathcal{D}(M + A) : \langle Mu + Au, u \rangle \leq 0, \langle Mu + Au, v \rangle = 0, v \in \tilde{F}_0 \}
\]

is empty, where \( D_F = D \cap F \). Then for the finite-dimensional mapping \((M + A)_{\tilde{F}_0} : \tilde{F}_0 \rightarrow \tilde{F}_0\), defined by (3.34), we have

\[
(M + A)_{\tilde{F}_0}(u) \neq 0, \quad u \in \overline{D_{\tilde{F}_0}}, \tag{4.41}
\]

and, by the properties of the degree for finite-dimensional mappings, we have from (4.41)

\[
\deg ((M + A)_{\tilde{F}_0}, D_{\tilde{F}_0}, 0) = 0. \tag{4.42}
\]

From Theorem 3.2, Definition 3.1 and (4.42) we obtain

\[
\text{Deg} (M + A, D, 0) = 0,
\]

which contradicts the assumption (4.37). This finishes the proof. \( \square \)

5. Boundary value problems for densely defined nonlinear elliptic operators. Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \). We consider the boundary value problem

\[
\sum_{i=1}^n \frac{\partial}{\partial x_i} \left\{ \rho^2(u) \frac{\partial u}{\partial x_i} + a_i(x, u, \frac{\partial u}{\partial x}) \right\} = \sum_{i=1}^n \frac{\partial}{\partial x_i} f_i(x), \quad x \in \Omega, \tag{5.1}
\]

\[u(x) = 0, \quad x \in \partial \Omega, \tag{5.2}
\]

where the functions \( a_i(x, u, \xi), \ i = 1, \ldots, n\), \( \rho(u) \) satisfy the following conditions:

\(\alpha_1\) the real valued functions \( a_i(x, u, \xi), \ i = 1, 2, \ldots, n\), are defined on \( \Omega \times \mathbb{R} \times \mathbb{R}^n \) and are continuous w.r.t. \( u, \xi \) and measurable w.r.t. \( x \);

\(\alpha_2\) there exist positive constants \( \nu_1, \nu_2 \) such that for all values of \( x, u, \xi, \eta \) we have

\[
\sum_{i=1}^n \left[ a_i(x, u, \xi) - a_i(x, u, \eta) \right] (\xi_i - \eta_i) \geq \nu_1 |\xi - \eta|^m \tag{5.3}
\]
and
\[ |a_i(x, u, \xi)| \leq \nu_2(|u|^{m_1} + |\xi|^{m-1} + f(x), \]
where \( f \in L_m^\infty(\Omega), \ m_1 < \frac{n}{n-m}, \ 2 \leq m < n; \)
\( \rho_1 \) the real valued function \( \rho(u) \) is defined and continuous on \( \mathcal{R}; \)
\( \rho_2 \) there exists a positive number \( \mu \) such that for every \( u \in \mathcal{R} \) we have
\[ 0 \leq \rho(u) \leq \mu \left\{ \left| \int_0^u \rho(s)ds \right| + 1 \right\}^r, \]
where \( r \) is a constant with \( 0 \leq r < \frac{n}{n-2}. \)

We define a nonlinear operator \( A : W_0^{1,m}(\Omega) \rightarrow [W_0^{1,m}(\Omega)]^* \) by
\[ \langle Au, \phi \rangle = \sum_{i=1}^n \int_\Omega \left\{ \rho^2(u) \frac{\partial u}{\partial x_i} + a_i(x, u, \frac{\partial u}{\partial x}) \right\} \frac{\partial \phi(x)}{\partial x_i} dx, \]
for all \( \phi \in W_0^{1,m}(\Omega), \ u \in \mathcal{D}(A), \) where
\[ \mathcal{D}(A) = \{ u \in W_0^{1,m}(\Omega) : \rho^2(u) \frac{\partial u}{\partial x_i} \in L_m^\infty(\Omega) \}. \]

Set \( L = C_0^\infty(\Omega). \) Then \( L \) is a subspace of \( W_0^{1,m}(\Omega) \) such that
\[ L \subset \mathcal{D}(A), \quad \mathcal{T} = W_0^{1,m}(\Omega), \]
and the condition (2.1) for the subspace \( L \) is satisfied.

**Theorem 5.1.** Assume that Conditions \( \alpha_1), \ (\alpha_2), \ \rho_1), \ \rho_2) \) are satisfied. Then the operator \( A, \) defined by (5.6), satisfies condition \( (S_+)_0,L \) with respect to the space \( L = C_0^\infty(\Omega). \)

**Proof.** Let \( \{F_i\} \) be a sequence satisfying Condition (2.2) and let \( L\{F_i\} \) be the subspace of \( L \) defined by (2.3). We consider a sequence \( \{u_j\} \subset L, \ j = 1,2, \ldots, \) satisfying the condition (2.4). For every function \( v \in F_k \) we have, by (5.3) and the definition of the operator \( A, \)
\[ \nu_1 \int_\Omega \left| \frac{\partial(u_j - v)}{\partial x} \right|^m dx \]
\[ \leq \sum_{i=1}^n \int_\Omega \left[ a_i(x, u_j, \frac{\partial u_j}{\partial x}) - a_i(x, u_j, \frac{\partial v}{\partial x}) \right] \left( \frac{\partial u_j}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) dx \]
\[ = \langle Au_j, u_j - v \rangle - \sum_{i=1}^n \int_\Omega \left[ \rho^2(u_j) \frac{\partial u_j}{\partial x} + a_i(x, u_j, \frac{\partial v}{\partial x}) \right] \left( \frac{\partial u_j}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) dx. \]
In particular, for \( v(x) \equiv 0 \), we obtain from (2.4), (5.9) and (5.4) the inequality

\[
\limsup_{j \to \infty} \int_{\Omega} \rho^2(u_j) \left| \frac{\partial u_j}{\partial x} \right|^2 \, dx \leq K_1,
\]

for some positive constant \( K_1 \). We introduce the sequence

\[
\tilde{u}_j(x) = \tilde{\rho}(u_j(x)),
\]

where

\[
\tilde{\rho}(u) = \int_0^u \rho(s) \, ds.
\]

From (5.10) follows the boundedness of the sequence \( \{\tilde{u}_j\} \) in \( W^{1,2}_0(\Omega) \) and we can assume that \( \tilde{u}_j \) converges weakly in \( W^{1,2}_0(\Omega) \) and strongly in \( L_p(\Omega) \), \( p < \frac{2n}{n-2} \), to some function \( \tilde{u}_0 \). From this we obtain that \( \tilde{u}_j \) converges in measure to \( \tilde{u}_0 \) and \( \tilde{\rho}(u_0) \). Consequently, we have

\[
\tilde{u}_0(x) = \tilde{\rho}(u_0(x)).
\]

Taking into account (5.10) we can assume that

\[
\lim_{j \to \infty} \int_{\Omega} \rho^2(u_j) \left| \frac{\partial u_j}{\partial x} \right|^2 \, dx = R,
\]

where \( R \) is some number. From (5.13), (5.14) and \( \tilde{u}_j \rightharpoonup \tilde{u}_0 \) on \( W^{1,2}_0(\Omega) \) we have

\[
\int_{\Omega} \rho^2(u_0) \left| \frac{\partial u_0}{\partial x} \right|^2 \, dx \leq R.
\]

The sequence \( \rho(u_j) \) is bounded in \( L_\bar{q}(\Omega) \), \( \bar{q} = \frac{2n}{n-2} \cdot \frac{1}{r} > 2 \). This follows from (5.5) and the boundedness of the sequence \( \tilde{u}_j \) in \( L_{\frac{2n}{n-2}}(\Omega) \). Since \( \rho(u_j) \) converges in measure, we obtain the strong convergence of \( \rho(u_j) \) to \( \rho(u_0) \) in \( L_2(\Omega) \). Now we can pass to the limit in (5.9), for a fixed function \( v \in F_k \), and then recall (5.13), (5.14) and (5.15) to obtain

\[
\nu_1 \int_{\Omega} \left| \frac{\partial (u_0 - v)}{\partial x} \right|^m \, dx \leq -R + \sum_{i=1}^n \int_{\Omega} \rho^2(u_0) \frac{\partial u_0}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx
\]

\[
- \sum_{i=1}^n \int_{\Omega} a_i(x, u_0, \frac{\partial v}{\partial x}) \frac{\partial}{\partial x_i} (u_0 - v) \, dx
\]

\[
- \sum_{i=1}^n \int_{\Omega} \left[ \rho^2(u_0) \frac{\partial u_0}{\partial x_i} + a_i(x, u_0, \frac{\partial v}{\partial x}) \right] \frac{\partial}{\partial x_i} (u_0 - v) \, dx.
\]
This inequality is valid for \( v \in F_k \) and arbitrary \( k \). Thus, it is valid for \( v \in L\{F_j\} \). It is also true for an arbitrary function \( v \in W_0^{1,q}(\Omega) \), where \( q = 2[1 - \frac{n-2}{n}]^{-1} \), with \( r \) defined in Condition \( \rho_2 \). This follows by approximation of such a function by functions from \( L\{F_j\} \), the inequality (5.4) and the estimate

\[
\int_\Omega (\rho^2(u_0) |\frac{\partial u_0}{\partial x}|)^{q'} \, dx \leq \left\{ \int_\Omega \rho^2(u_0) |\frac{\partial u_0}{\partial x}|^2 \, dx \right\}^{\frac{q'}{2}} \left\{ \int_\Omega |\rho(u_0)|^q \, dx \right\}^{\frac{q'}{q}}, \tag{5.17}
\]

which follows from Hölder’s inequality since

\[
\frac{q'}{2} + \frac{q'}{q} = 1, \quad q' = \frac{q}{q-1}, \quad q = \frac{2n}{n-2} \cdot \frac{1}{r}.
\]

Now, we consider the functional \( l \in [W_0^{1,q}(\Omega)]^* \) defined by

\[
l(\phi) = \sum_{i=1}^n \int_\Omega \rho^2(u_0) \frac{\partial u_0(x)}{\partial x_i} \cdot \frac{\partial \phi(x)}{\partial x_i} \, dx.
\]  

From (5.16) we get the estimate

\[
-l(\phi) \leq - \int_\Omega \rho^2(u_0) |\frac{\partial u_0}{\partial x}|^2 \, dx - \sum_{i=1}^n \int_\Omega a_i(x, u_0, \frac{\partial \phi}{\partial x}) \cdot \frac{\partial (u_0 - \phi)}{\partial x_i} \, dx, \tag{5.19}
\]

which is valid for an arbitrary function \( \phi(x) \) in \( C_0^\infty(\Omega) \). We evaluate the last summand on the right-hand side on (5.19) by Hölder’s inequality and Condition (5.4) and obtain the estimate

\[
\left| \sum_{i=1}^n \int_\Omega a_i(x, u_0, \frac{\partial \phi}{\partial x}) \cdot \frac{\partial (u_0 - \phi)}{\partial x_i} \, dx \right| \leq K_2,
\]  

where \( K_2 \) is a constant independent of \( \phi \), provided that

\[
\phi \in C_0^\infty(\Omega), \quad ||\phi||_{1,m} = 1,
\]  

where \( || \cdot ||_{1,m} \) is the norm of the space \( W_0^{1,m}(\Omega) \).

Thus, for an arbitrary function \( \phi \in C_0^\infty(\Omega) \), we get from (5.19) the estimate

\[
|l(\phi)| \leq K_2 ||\phi||_{1,m}. \tag{5.22}
\]
Consequently, the functional \( l \) can be extended to a continuous linear functional on the space \( W_0^{1, m'}(\Omega) \). From Condition \( \rho_2 \) follows that it suffices to consider only the case \( q > m \). Denote this extension of \( l \) by \( \bar{l} \in \left[ W_0^{1, m}(\Omega) \right]^* \).

Recalling that the Laplace operator \( \Delta : W_0^{1, m'}(\Omega) \to \left[ W_0^{1, m}(\Omega) \right]^*, m' = \frac{m}{m-1} \), is a homeomorphism (see [6], Theorem 6.21), there exists a function \( u' \in W_0^{1, m'}(\Omega) \) such that

\[
\bar{l}(\phi) = \sum_{i=1}^{n} \int_{\Omega} \frac{\partial u'(x)}{\partial x_i} \cdot \frac{\partial \phi(x)}{\partial x_i} dx
\]  

(5.23)

for all \( \phi \in W_0^{1, m}(\Omega) \). From (5.18) and (5.23) we have

\[
\sum_{i=1}^{n} \int_{\Omega} \frac{\partial}{\partial x_i} (u''(x) - u'(x)) \frac{\partial \phi(x)}{\partial x_i} dx = 0,
\]

(5.24)

for \( \phi \in W_0^{1, q}(\Omega) \), where

\[
u''(x) = \int_{0}^{u(x)} \rho^2(s) ds.
\]

(5.25)

Recalling that the Laplacian \( \Delta : W_0^{1, q'}(\Omega) \to \left[ W_0^{1, q}(\Omega) \right]^* \) is one-to-one, we obtain from (5.24) that \( u''(x) = u'(x) \) and, consequently,

\[
\rho^2(u_0) \frac{\partial u_0(\cdot)}{\partial x_i} = \frac{\partial u'(\cdot)}{\partial x_i} \in L_{m'}(\Omega).
\]

(5.26)

We have shown that the function \( u_0 \) belongs to \( \mathcal{D}(A) \).

From (5.26) and the approximation of an arbitrary function \( v \in W_0^{1, m}(\Omega) \) by functions in \( L\{F_j\} \) follows that (5.16) is true for \( v \in W_0^{1, m}(\Omega) \). We let now in (5.16) \( v(x) = u_0(x) + tw(x) \), where \( t > 0 \) and \( w \) is an arbitrary function from \( W_0^{1, m}(\Omega) \). Dividing the resulting inequality by \( t \) and passing to the limit as \( t \to 0^+ \), we obtain in a standard way that \( Au_0 = 0 \).

We can now prove the strong convergence of the sequence \( \{u_j\} \) to \( u_0 \in W_0^{1, m}(\Omega) \). We may assume that

\[
a_i(x, u_j(x), \frac{\partial u_j(x)}{\partial x}) \to h_i(x) \quad \text{in} \quad L_{m'}(\Omega),
\]

(5.27)
with some functions \( h_i(x) \). Let \( u_j^{(0)} \in L_F \} \) be a sequence which strongly converges to \( u_0 \) in \( W_0^{1,m}(\Omega) \).

Using (5.3) we have

\[
\nu \int_\Omega \left| \frac{\partial}{\partial x} (u_j - u_j^{(0)}) \right|^m dx \\
\leq \sum_{i=1}^n \int_\Omega \left[ a_i(x, u_j, \frac{\partial u_j}{\partial x}) - a_i(x, u_j^{(0)}, \frac{\partial u_j^{(0)}}{\partial x}) \right] \frac{\partial}{\partial x_i} (u_j - u_j^{(0)}) dx \\
= \langle Au_j, u_j \rangle - \int_\Omega \rho^2(u_j) \left| \frac{\partial u_j(x)}{\partial x} \right|^2 dx - \sum_{i=1}^n \int_\Omega a_i(x, u_j, \frac{\partial u_j}{\partial x}) \cdot \frac{\partial u_j^{(0)}}{\partial x} dx \\
- \sum_{i=1}^n \int_\Omega a_i(x, u_j, \frac{\partial u_j^{(0)}}{\partial x}) \frac{\partial (u_j - u_j^{(0)})}{\partial x} dx. 
\]

Passing to the limit as \( j \to \infty \) and taking into account (2.4), (5.14), (5.15), (5.27) and Condition (5.4) we get

\[
\nu \limsup_{j \to \infty} \int_\Omega \left| \frac{\partial}{\partial x} (u_j - u_j^{(0)}) \right|^m dx \\
\leq - \sum_{i=1}^n \int_\Omega \left\{ \rho^2(u_0) \frac{\partial u_0}{\partial x_i} + h_i(x) \right\} \frac{\partial u_0(x)}{\partial x_i} dx. 
\]

We now show that the right-hand side of (5.29) equals zero. From

\[
\lim_{j \to \infty} \langle Au_j, v \rangle = 0,
\]

for \( v \in L_F \} \), the weak convergence of \( \bar{u}_j \) to \( \rho(u_0) \) in \( W_0^{1,2}(\Omega) \), the strong convergence of \( \rho(u_j) \) to \( \rho(u_0) \) in \( L_2(\Omega) \) and (5.27) we have

\[
\sum_{i=1}^n \int_\Omega \left\{ \rho^2(u_0) \frac{\partial u_0}{\partial x_i} + h_i(x) \right\} \frac{\partial v(x)}{\partial x_i} dx = 0,
\]

for an arbitrary function \( v \in L_F \} \).

From (5.26) and \( h_i \in L_{m'}(\Omega) \) follows that (5.30) is actually true for an arbitrary function \( v \in W_0^{1,m}(\Omega) \). Letting in (5.30) \( v(x) = u_0(x) \) we obtain that the right-hand side of (5.29) is equal to zero. Then (5.29) implies the strong convergence of the sequence \( \{u_j\} \) to \( u_0 \) in \( W_0^{1,m}(\Omega) \). This is the end of the proof. \( \square \)
Corollary 5.1. Assume that the conditions of Theorem 5.1 are satisfied and let $D$ be an open bounded subset of $W^{1,m}_0(\Omega)$ such that $Au \neq 0$, $u \in \partial D \cap D(A)$, where the operator $A$ is defined by (5.6). Then, by Definition 2.2, the degree of this operator, $\text{Deg}(A,D,0)$, is well defined.

The application of the degree theory for the operator $A$ to the solvability of the boundary value problem (5.1), (5.2) is connected with the consideration of a parametric family of such problems. We formulate below one such result.

Consider the one-parameter family of boundary value problems

$$
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ \rho^{2t}(u) \frac{\partial u}{\partial x_i} + t a_i(x, u, \frac{\partial u}{\partial x}) + (1-t) \left| \frac{\partial u(x)}{\partial x} \right|^{m-2} \frac{\partial u(x)}{\partial x} \right\} = t \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(x), \quad x \in \Omega,
$$

where $f_i \in L_m(\Omega)$, $t \in [0,1]$ and the functions $a_i(x,u,\xi)$, $\rho(u)$ satisfy the conditions $\alpha_1$, $\alpha_2$, $\rho_1$, $\rho_2$. In the usual way, we can define the solution of this problem for fixed $t$ in the space $W^{1,m}_0(\Omega)$ such that $\rho^{2t}(u) \frac{\partial u}{\partial x_i} \in L_m(\Omega)$.

Define the parametric operator family

$$
A_t : W^{1,m}_0(\Omega) \supset \mathcal{D}(A_t) \to [W^{1,m}_0(\Omega)]^* 
$$

by

$$
\langle A_t u, \phi \rangle = \sum_{i=1}^{n} \int_{\Omega} \left\{ \rho^{2t}(u) \frac{\partial u}{\partial x_i} + t a_i(x, u, \frac{\partial u}{\partial x}) - f_i(x) \right\} 
+ (1-t) \left| \frac{\partial u(x)}{\partial x} \right|^{m-2} \frac{\partial u(x)}{\partial x} \left( \frac{\partial \phi(x)}{\partial x} \right) dx
$$

for $\phi \in W^{1,m}_0(\Omega)$, $u \in \mathcal{D}(A_t)$, where

$$
\mathcal{D}(A_t) = \{ u \in W^{1,m}_0(\Omega) : \rho^{2t}(u) \frac{\partial u}{\partial x_i} \in L_m(\Omega) \}.
$$

Theorem 5.2. Assume that the conditions $\alpha_1$, $\alpha_2$, $\rho_1$, $\rho_2$ are satisfied. Then the parametric family $\{ A_t \}$, $t \in [0,1]$, defined by (5.33), satisfies the condition $(S_+)^{(l)}_{0L}$ of Definition 4.1 with $L = C^\infty_0(\Omega)$.

The proof of Theorem 5.2 is analogous to that of Theorem 5.1. It is therefore omitted.
Theorem 5.3. Assume that the conditions $\alpha_1$, $\alpha_2$, $\rho_1$, $\rho_2$ are satisfied, $f_i \in L_{m'}(\Omega)$, $i = 1, 2, \ldots, n$, and suppose that there exists a number $M$ such that for any solution $u \in W^{1,m}_0(\Omega)$ of the problem (5.31)$_t$, (5.32)$_t$, $t \in [0, 1]$, the following estimate holds

$$
\|u\|_{1,m} \leq M. \tag{5.35}
$$

Then the problem (5.1), (5.2) has at least one solution in $W^{1,m}_0(\Omega)$.

Proof. The solvability of the boundary value problem (5.31)$_t$, (5.32)$_t$ is equivalent to the solvability of the operator equation

$$
A_t u = 0. \tag{5.36}
$$

We must prove the existence of the solution of this equation for $t = 1$.

From (5.35) we have

$$
A_t u \neq 0, \quad u \in \partial B(0, M + 1) \cap D(A_t), \tag{5.37}
$$

where $B(0, M + 1) \subset W^{1,m}_0(\Omega)$. Using Theorem 4.1 and (5.2) we obtain

$$
\text{Deg}(A_1, B(0, M + 1), 0) = \text{Deg}(A_0, B(0, M + 1), 0). \tag{5.38}
$$

The degree on the right-hand side of (5.38) is the one given by [10, Chapter 2, Theorem 4.4]. Since the operator $A_0$ is defined on all of $W^{1,m}_0(\Omega)$ its degree, in the sense of Definition 2.2, coincides with the degree in [10]. Moreover, the operator $A_0$ satisfies the inequality

$$
\langle A_0 u, u \rangle > 0, \quad u \in \partial B(0, M + 1). \tag{5.39}
$$

Consequently, from (5.38) we have

$$
\text{Deg}(A_1, B(0, M + 1), 0) = 1,
$$

and the solvability of the equation $A_1 u = 0$ follows from theorem 4.2. The proof is finished. □

6. Initial-boundary value problems for densely defined nonlinear parabolic operators. Let $\Omega$ be a bounded open set in $\mathbb{R}^n$. We consider, in the cylindrical domain $Q = \Omega \times (0, T)$ the parabolic problem

$$
\frac{\partial u}{\partial t} - \sum_{i=1}^n \frac{\partial}{\partial x_i} a_i(x, t, u, \frac{\partial u}{\partial x}) + \rho(x, t, u) = \sum_{i=1}^n \frac{\partial}{\partial x_i} f_i(x, t, u), \quad (x, t) \in Q, \tag{6.1}
$$
Degree Theories

\[
  u(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T),
\]
\[
  u(x, 0) = 0 \quad x \in \Omega.
\]

We assume the following conditions:

\(\alpha'\) the real valued functions \(a_i(x, t, u, \xi), \ i = 1, 2, \ldots, n,\) are defined on \(Q \times \mathbb{R} \times \mathbb{R}^n\) and are continuous in \(u, \ \xi\) and measurable in \(x, t;\)

\(\alpha''\) there exist positive constants \(\nu_1, \ \nu_2\) such that for all values of \(x, t, u, \ \xi, \ \eta\) we have

\[
  \sum_{i=1}^{n} [a_i(x, t, u, \xi) - a_i(x, t, u, \eta)] (\xi_i - \eta_i) \geq \nu_1 |\xi - \eta|^m
\]

and

\[
  |a_i(x, t, u, \xi)| \leq \nu_2 [u|\nu_0 + |\xi|^{-m-1}] + g(x, t),
\]

where

\[
  m_0 < m - 1 + \frac{2(m - 1)}{n}, \quad g \in L_{m'}(Q), \quad 1 < m < n;
\]

\(\rho'\) the real valued function \(\rho(x, t, u)\) is defined on \(Q \times \mathbb{R}\) and is continuous in \(u\) and measurable in \(x, \ t;\)

\(\rho''\) the function \(\rho(x, t, u)\) is nondecreasing w.r.t. \(u\) and there exists a positive number \(\mu\) such that

\[
  |\rho(x, t, u)| \leq \mu(1 + |u|)^{\frac{m(n+2) - 1}{n}}.
\]

Set \(X = L_{m}(0, T, W^{1, m}_{0}(\Omega))\) and define the operators \(M : X \supset \mathcal{D}(M) \to X^*, \ A : X \supset \mathcal{D}(A) \to X^*\) as follows:

\[
  \langle Mu, \phi \rangle = \left\langle \frac{\partial u}{\partial t}, \phi \right\rangle + \int_{Q} \rho(x, t, u(x, t)) \phi(x, t) dx dt,
\]

\[
  \langle Au, \phi \rangle = \sum_{i=1}^{n} \int_{Q} a_i(x, t, u, \frac{\partial u}{\partial x}) \frac{\partial \phi(x, t)}{\partial x_i} dx dt,
\]

where \(\phi \in X,\) and

\[
  \mathcal{D}(M) = \{ u \in X : \frac{\partial u}{\partial t} \in X^*, \ u(x, 0) = 0 \},
\]
\[ \mathcal{D}(A) = \{ u \in X : u \in L_m(Q) \}, \quad \bar{m} = \frac{m_0 m}{m - 1}, \]  

(6.10)

where \( m_0 \) is the number from Condition \( \alpha'' \).

It is well known (see [10, Proposition 23.23]) that for every function \( u \in \mathcal{D}(M) \) we have the formula

\[ \langle \frac{\partial u}{\partial t}, u \rangle = \frac{1}{2} \int_{\Omega} u^2(x, T) dx. \]  

(6.11)

From (6.11) and for \( u \in \mathcal{D}(M) \) we have

\[ \sup_{0 \leq t \leq T} \int_{\Omega} u^2(x, t) dx < +\infty. \]  

(6.12)

It is also well known (see (6.53) below) that from \( u \in X \) and (6.12) we have

\[ \iint_Q |u(x, t)|^\frac{m(n+2)}{m-n} \, dx \, dt < +\infty. \]  

(6.13)

Consequently, we obtain the inclusion

\[ \mathcal{D}(M) \subset \mathcal{D}(A). \]  

(6.14)

From Hölder’s inequality and Conditions \( \alpha' \), \( \alpha'' \) follows that the operator \( A \) is well defined by (6.8) on \( \mathcal{D}(A) \). In order to verify that the operator \( M \) is also well defined by (6.7) on \( \mathcal{D}(M) \), we will prove first the following lemma.

**Lemma 6.1.** Assume that \( u \in \mathcal{D}(M) \). Then the estimate

\[ |\iint_Q \rho(x, t, u(x, t)) \phi(x, t) \, dx \, dt| \leq C_1 \| \phi \| \]  

(6.15)

holds for every \( \phi \in C^\infty(\overline{Q}) \cap X \), where the constant \( C_1 \) is independent of \( \phi \). Here, \( \| \cdot \| \) is the norm of the space \( L_m(0, T, W^{1, m}_0(\Omega)) \).

**Proof.** The integral on the left-hand side of (6.15) is finite by virtue of (6.13) and we will estimate it. By Hölder’s inequality we have

\[
\left| \int_{Q} \rho(x, t, u(x, t)) \phi(x, t) \, dx \, dt \right| \\
\leq \left\{ \int_0^T \left[ \int_{\Omega} |\rho(x, t, u(x, t))|^{m'} \, dx \right]^{\frac{m'}{m'}} \, dt \right\}^{\frac{1}{m'}} \left\{ \int_0^T \left[ \int_{\Omega} |\phi(x, t)|^{m} \, dx \right]^{\frac{m}{m}} \, dt \right\}^{\frac{1}{m}},
\]  

(6.16)
where \( m_* = \frac{nm}{n-m} \). Using the embedding \( W^{1,m}_0(\Omega) \rightarrow L_{m_*}(\Omega) \) we obtain the estimate
\[
\left\{ \int_0^T \left[ \int_{\Omega} |\phi(x,t)|^{m_*} \, dx \right] \frac{m_*}{m} \, dt \right\}^{\frac{1}{m}} \leq C_2\|\phi\|. \tag{6.17}
\]
It is necessary to verify now that the first factor on the right-hand side of (6.16) is finite.

It is easy to see that
\[
\left( \frac{m(n+2)}{n} - 1 \right) \cdot m_* = m_* \left( \frac{1}{z} + 2(1 - \frac{1}{z}) \right), \quad \text{where } z = \frac{m_* - 1}{m - 1} > 1. \tag{6.18}
\]
From (6.6) and Hölder’s inequality we obtain
\[
\left\{ \int_0^T \left[ \int_{\Omega} |\rho(x,t,u(x,t))|^{m_*} \, dx \right] \frac{m_*}{m} \, dt \right\}^{\frac{1}{m}} \leq C_3 \left\{ \int_0^T \left[ \int_{\Omega} |u(x,t)|^{m_*} \, dx \right] \frac{m_*}{m} \left( 1 - \frac{1}{z} \right) \, dt \right\}^{\frac{1}{m}}. \tag{6.19}
\]
The right-hand side of (6.19) is finite by (6.12), (6.17). This completes the proof. □

From Lemma 6.1 we obtain that for \( u \in D(M) \) the operator \( M \) is well defined by (6.7). We will prove that the operator \( M \) is maximal monotone. The proof is based on the solvability of the following equation
\[
\frac{\partial u}{\partial t} - \sum_{i=1}^n \frac{\partial}{\partial x_i} a_i'(x,t,\frac{\partial u}{\partial x}) + \rho(x,t,u) = \sum_{i=1}^n \frac{\partial}{\partial x_i} f_i(x,t), \quad (x,t) \in Q, \tag{6.20}
\]
with the boundary condition (6.2) and the initial condition (6.3). The function \( a_i'(x,t,\xi) \) in (6.20) is defined by the equality
\[
a_i'(x,t,\xi) = \left| \xi - \frac{\partial u'(x,t)}{\partial x} \right|^{m-2} \cdot \left( \xi_i - \frac{\partial u'(x,t)}{\partial x_i} \right), \tag{6.21}
\]
where \( u' \in X \) is fixed.

We shall say that a function \( u \in D(M) \) is a solution of the problem (6.1)-(6.3) if for every \( \phi \in X \) we have
\[
\langle \frac{\partial u}{\partial t}, \phi \rangle + \int_Q \left\{ \sum_{i=1}^n a_i(x,t,u,\frac{\partial u}{\partial x},\frac{\partial \phi}{\partial x}) + \rho(x,t,u)\phi \right\} \, dx \, dt = -\langle f, \phi \rangle, \tag{6.22}
\]
where
\[
\langle f, \phi \rangle = \sum_{i=1}^n \int_Q f_i(x,t) \frac{\partial \phi}{\partial x_i} \, dx \, dt.
\]
Lemma 6.2. Assume that the conditions \( \rho', \rho'' \) are satisfied. Then for every \( n \)-tuple of functions \( f_i \in L_{m'}(Q), i = 1, 2, \ldots, n \), the initial-boundary value problem (6.20), (6.2), (6.3) has at least one solution \( u \in \mathcal{D}(M) \).

Proof. We prove the lemma by the standard Galerkin method. Choose a sequence of functions \( v_i \in C_0^\infty(\Omega) \) so that

\[
\int_{\Omega} v_i(x)v_j(x)dx = \delta^j_i, \quad \overline{\bigcup_{1 \leq j < \infty} F_j} = W_0^{1,m}(\Omega),
\]

where \( \delta^j_i \) is the Kronecker \( \delta \) function, \( F_j \) is the span of the set \( \{v_1, \ldots, v_j\} \) and the bar over \( \cup F_j \) denotes closure in \( W_0^{1,m}(\Omega) \).

Define, for a fixed \( j \), the functions \( C_{kj} \in W_0^{1,m'}(0,T), k = 1, \ldots, j \), as the solutions of the Cauchy problems

\[
\frac{dC_{kj}(t)}{dt} = -\int_{\Omega} \left\{ \sum_{i=1}^n a_i'(x,t) \frac{\partial u_j}{\partial x_i} \frac{\partial v_k}{\partial x_i} + \rho(x,t,u_j)v_k \right\} dx
- \int_{\Omega} \left\{ \sum_{i=1}^n f_i(x,t) \frac{\partial v_k}{\partial x_i} dx, \quad k = 1, \ldots, j, \right.
\]

\[
C_{kj}(0) = 0, \quad k = 1, \ldots, j,
\]

respectively, where

\[
u_j(x,t) = \sum_{l=1}^j C_{lj}(t)v_l'(x).
\]

The solvability of the problem (6.24), (6.25) is possible via the Leray-Schauder method as in the proof of Lemma 4 in [3].

Multiplying (6.24) by \( C_{kj} \), summing up the resulting equalities w.r.t. \( k \) ranging from 1 to \( j \), and then integrating w.r.t. \( t \) from 0 to \( T' \leq T \), we obtain the estimate

\[
sup_{0 \leq t \leq T} \int_{\Omega} |u_j(x,t)|^2 dx + \int_{Q_T} |\frac{\partial u_j(x,t)}{\partial x}|^m dx dt \leq C_4,
\]

where \( C_4 \) is a positive constant independent of \( j \). Consequently, passing to a subsequence, if necessary, we may assume that

\[
u_j \rightharpoonup u_0 \quad \text{in} \quad L_m(0,T,W_0^{1,m}(\Omega))
\]
and

\[ a_i'(\cdot, \cdot, \frac{\partial u_j}{\partial x}) \rightarrow a_i^{(0)}(\cdot, \cdot) \in L_{m'}(Q), \quad i = 1, \ldots, n, \quad (6.29) \]

for some functions \( u_0(x, t), \ a_i^{(0)}(x, t) \), where \( u_0(x, t) \) satisfies the inequality

\[ \int \int_Q \left| \frac{\partial u_0(x, t)}{\partial x} \right|^m \ dx \ dt \leq C_4. \quad (6.30) \]

From the inequalities (6.16), (6.17), (6.19) and (6.27) we obtain that the functional \( r(u_j) \in X^* \), given by

\[ \langle r(u_j), \phi \rangle = \int \int_Q \rho(x, t, u_j)\phi(x, t) \ dx \ dt, \quad \phi \in X, \quad (6.31) \]

is well defined and the sequence \( \{r(u_j)\}, \ j = 1, 2, \ldots, \) is bounded in \( X^* \). Consequently, we may assume that

\[ r(u_j) \rightarrow r_0 \quad \text{in} \ X^*. \quad (6.32) \]

Now, we will prove that the function \( u_0(x, t) \) defined by (6.28) belongs to \( \mathcal{D}(M) \). Denote by \( L_j \) the space of all functions of the form

\[ \sum_{k=1}^j d_k(t)v_k(x), \quad d_k \in C_0^\infty(0, T). \]

Multiplying (6.24) by \( d_k \), summing up the resulting equalities w.r.t. \( k \) and then integrating w.r.t. \( t \) we obtain

\[ \int \int_Q \left\{ -u_j(x, t) \frac{\partial w}{\partial t} + \sum_{i=1}^n a_i'(x, t, \frac{\partial u_j}{\partial x}) \frac{\partial w}{\partial x_i} \right\} \ dx \ dt + \langle r(u_j), w \rangle = \langle f, w \rangle, \quad (6.33) \]

where

\[ w(x, t) = \sum_{k=1}^j d_k(t)v_k(x) \quad \text{with} \ w \in L_j. \]

Let \( w(x, t) \) be an arbitrary function in \( L_{j_0} \), for a fixed \( j_0 \). Passing to the limit in (6.33), as \( j \rightarrow \infty \), we obtain from (6.29), (6.32)

\[ \int \int_Q u_0(x, t) \frac{\partial w(x, t)}{\partial t} \ dx \ dt \]

\[ = \sum_{i=1}^n \int \int_Q a_i^{(0)}(x, t) \frac{\partial w}{\partial x_i} \ dx \ dt + \langle r_0, w \rangle - \langle f, w \rangle = \langle a_0 + r_0 - f, w \rangle, \quad (6.34) \]
where \( a_0 \in X^* \) is defined by

\[
\langle a_0, \phi \rangle = \sum_{i=1}^{n} \int_{Q} a_i^0(x,t) \frac{\partial \phi(x,t)}{\partial x_i} \, dx \, dt, \quad \phi \in X. \tag{6.35}
\]

By a suitable approximation we can check that (6.34) is actually true for an arbitrary \( w \in C^\infty_0(Q) \). This means that the derivative of \( u(x,t) \) w.r.t. \( t \), in the sense of distributions, is well defined and

\[
\frac{\partial u_0}{\partial t} = -a_0 - r_0 + f \in X^*. \tag{6.36}
\]

Using the initial conditions (6.25) it is easy to verify that \( u_0(x,0) = 0 \). We have established that \( u_0 \in \mathcal{D}(M) \).

Now, we will prove that \( u_0(x,t) \) is a solution of the problem (6.20), (6.2), (6.3). Define a functional \( a(u) \in X^* \) by

\[
\langle a(u), \phi \rangle = \sum_{i=1}^{n} \int_{Q} a_i'(x,t) \frac{\partial \phi(x,t)}{\partial x_i} \, dx \, dt, \tag{6.37}
\]

for \( u, \phi \in X \). Denote by \( \overline{L}_j \) the linear space of functions of the form

\[
\sum_{k=1}^{j} g_k(t)v_k(x), \quad \text{where} \quad g_k \in C^\infty[0,T], \quad g_k(0) = 0, \quad k = 1, \ldots, j.
\]

Using the assumed conditions on the functions \( a_i'(x,t,\xi), \rho(x,t,u) \) we have

\[
\int_{Q} \frac{\partial}{\partial t} \left( u_j - \psi \right) \cdot \left( u_j - \psi \right) dx \, dt + \langle a(u_j) - a(\psi), u_j - \psi \rangle + \langle r(u_j) - r(\psi), u_j - \psi \rangle \geq 0, \tag{6.38}
\]

for an arbitrary \( \psi \in \overline{L}_{j_0}, \quad j_0 \leq j \), with the functional \( r(\psi) \) defined analogously as in (6.31). Using the system (6.24) and the condition (6.23) for \( v_i(x) \), we rewrite the last inequality in the form

\[
\langle f, u_j - \psi \rangle - \int_{Q} \frac{\partial \psi}{\partial t} \left( u_j - \psi \right) dx \, dt - \langle a(\psi) + r(\psi), u_j - \psi \rangle \geq 0, \tag{6.39}
\]
where \( f \) is defined in (6.22).

Passing to the limit in (6.39) as \( j \to \infty \) we obtain

\[
- \int_Q \frac{\partial \psi}{\partial t} (u_0 - \psi) dx dt + \langle f - a(\psi) - r(\psi), u_0 - \psi \rangle \geq 0, \tag{6.40}
\]

for \( \psi \in \bar{L}_{j_0} \) with a fixed \( j_0 \). From (6.36) we have

\[
f = \frac{\partial u_0}{\partial t} + a_0 + r_0.
\]

Substituting this value of \( f \) in (6.40) we get

\[
- \int_Q \frac{\partial \psi}{\partial t} (u_0 - \psi) dx dt + \langle \frac{\partial u_0}{\partial t} + a_0 + r_0 - a(\psi) - r(\psi), u_0 - \psi \rangle \geq 0. \tag{6.41}
\]

This last inequality is valid for \( \psi \in \bar{L}_{j_0} \) and any number \( j_0 \in \mathbb{N} \). Taking into account that \( \cup_j \bar{L}_j \) is dense in the space

\[
\{ u \in X = L_m(0,T,W^{1,m}_0(\Omega) : \frac{\partial u}{\partial t} \in X^*, u(x,0) = 0 \},
\]

with norm

\[
\| u \| + \| \frac{\partial u}{\partial t} \|_*
\]

(see [10, Proposition 23.23]), we obtain from (6.41) the inequality

\[
\langle \frac{\partial u_0}{\partial t} - \frac{\partial \psi}{\partial t}, u_0 - \psi \rangle + \langle a_0 + r_0 - a(\psi) - r(\psi), u_0 - \psi \rangle \geq 0, \tag{6.42}
\]

for all \( \psi \in \mathcal{D}(M) \).

Letting in (6.42) \( \psi(x,t) = u_0(x,t) + \lambda \phi(x,t) \), where \( \lambda \) is an arbitrary positive number and \( \phi \in \mathcal{D}(M) \cap C^\infty(\bar{Q}) \), we get

\[
\lambda^2 \langle \frac{\partial \phi}{\partial t}, \phi \rangle - \lambda \langle a_0 + r_0 - a(u_0 + \lambda \phi) - r(u_0 + \lambda \phi), \phi \rangle \geq 0. \tag{6.43}
\]

Dividing above by \( \lambda \) and passing to the limit as \( \lambda \to 0 \), we obtain

\[
- \langle a_0 + r_0 - a(u_0) - r(u_0), \phi \rangle \geq 0.
\]
Since this inequality is valid also for the function \(-\phi(x, t)\) instead of \(\phi(x, t)\), we have
\[
\langle a_0 + r_0 - a(u_0) - r(u_0), \phi \rangle = 0,
\] (6.44)
for \(\phi \in \mathcal{D}(M) \cap C^\infty(Q)\). By a suitable approximation we obtain (6.44) for an arbitrary \(\phi \in X\).

Consequently, from (6.44) we have
\[
a_0 + r_0 = a(u_0) + r(u_0),
\] (6.45)
and from (6.36), (6.45) we conclude that \(u_0(x, t)\) is the solution to the problem (6.20), (6.2), (6.3). This is the end of the proof. \(\square\)

**Theorem 6.1.** Assume that the conditions \(\rho', \rho''\) are satisfied. Then the operator \(M\), defined by (6.7) is maximal monotone.

**Proof.** Using (6.11) and the condition \(\rho'\) we obtain for arbitrary \(u, v \in \mathcal{D}(M)\) the inequality
\[
\langle Mu - Mv, u - v \rangle \geq 0,
\] (6.46)
i.e., the monotonicity of the operator \(M\).

It is only necessary to show that for each pair \((u', f') \in X \times X^*\) such that
\[
\langle Mu - f', u - u' \rangle \geq 0,
\] (6.47)
for all \(u \in \mathcal{D}(M)\), the following assertion holds: \(u' \in \mathcal{D}(M)\) and \(Mu' = f'\). Let \(u', f'\) be a pair satisfying (6.47) and consider the initial-boundary value problem (6.20), (6.2), (6.3), where the function \(u'(x, t)\) in (6.21) is the same as the function \(u'\) above, and the functions \(f_i(x, t)\) in (6.20) are defined from the following representation of the functional \(f'\):
\[
\langle f', \phi \rangle = -\sum_{i=1}^n \int_Q f_i(x, t) \frac{\partial \phi(x, t)}{\partial x_i} dx dt, \quad \phi \in X.
\] (6.48)
By Lemma 6.2, the problem under consideration has a solution \(u''(x, t)\) for which we have
\[
\frac{\partial u''}{\partial t} = -a(u'') - r(u'') + f', \quad u'' \in \mathcal{D}(M),
\] (6.49)
where the functionals \(a(u''), r(u'')\) are defined according to (6.37), (6.31). Since
\[
Mu'' = \frac{\partial u''}{\partial t} + r(u''),
\] (6.50)
we have from (6.49) and (6.47) the following inequality
\[
\iint_Q \left| \frac{\partial}{\partial x} [u''(x, t) - u'(x, t)] \right|^m \, dx \, dt = \langle a(u''), u'' - u' \rangle
\]
\[
= -(Mu'' - f', u'' - u') \leq 0.
\]
This implies \( u'(x, t) = u''(x, t) \) and we have established that \( u' \in \mathcal{D}(M) \) by virtue of (6.49). From (6.21) and (6.37) follows that \( a(u') = 0 \), and (6.50), (6.49) imply
\[
Mu' = Mu'' = f',
\]
which ends the proof of the theorem. \( \square \)

**Theorem 6.2.** Assume that the conditions \( \alpha' \), \( \alpha'' \), \( \rho' \), \( \rho'' \) are satisfied and let \( D \) be an open bounded set in \( X = L_m(0, T; W_0^{1,m}(\Omega)) \) such that
\[
Mu + Au \neq 0, \quad u \in \partial D \cap \mathcal{D}(M + A),
\]
where the operators \( M, A \) are defined by (6.7) and (6.8), respectively. Then the degree of the operator \( M + A, \Deg(M + A, D, 0) \), is well defined by Definition 2.2.

**Proof.** It is necessary to check that the operators \( M \) and \( A \) satisfy the conditions \( m_1 \) – \( m_3 \), \( a_1 \) – \( a_3 \) of Section 3. Conditions \( m_1 \), \( m_3 \) for the operator \( M \) are satisfied by virtue of Theorem 6.1 with \( L = \mathcal{D}(M) \). The equality (3.4) is trivial now because we can choose in it \( v = u_0 \). Conditions \( m_4 \), \( a_3 \) for the operators \( M, A \) follow from our assumptions \( \alpha' \), \( \rho' \).

The inclusion (3.5) for \( A \) follows with \( L = \mathcal{D}(M) \) from (6.14). In order to check the quasi-boundedness of the operator \( A \) we note that the inequalities
\[
\langle Au + Mu, u \rangle \leq N, \quad \|u\| \leq N, \quad u \in \mathcal{D}(M),
\]
conditions \( \alpha'' \), \( \rho'' \) and the formula (6.11) imply the following estimate:
\[
\sup_{0 \leq t \leq T} \int_{\Omega} u^2(x, t) \, dx \leq C_5(N) \left\{ 1 + \iint_Q |u|^{m(1+\frac{1}{m})} \, dx \, dt \right\},
\]
for some constant \( C_5(N) \) which depends only on known parameters and \( N \). Evaluating the last integral by Hölder’s inequality and using (6.17) we obtain
\[
\iint_Q |u(x, t)|^{m + \frac{2m}{m^*}} \, dx \, dt
\]
\[
\leq \int_0^T \left\{ \int_{\Omega} |u(x, t)|^{m} \, dx \right\}^{m^{*'}} \cdot \left\{ \int_{\Omega} |u(x, t)|^{2} \, dx \right\}^{1 - \frac{m}{m^*}} \, dt
\]
\[
\leq C_6(N) \sup_{0 \leq t \leq T} \left\{ \int_{\Omega} |u(x, t)|^{2} \, dx \right\}^{1 - \frac{m}{m^*}}
\]
with \( m_* = \frac{nm}{n-m} \). From (6.52), (6.53) we have

\[
\sup_{0 < t < T} \int_\Omega u^2(x,t)dx \leq C_\gamma(N), \quad \int_Q |u(x,t)|^{m+\frac{2m}{n}}dxdt \leq C_\gamma(N). \tag{6.54}
\]

Now, the inequality \( \| Au \|_* \leq K(N) \) follows from (6.5) and Hölder’s inequality and the quasi-boundedness of the operator \( A \) is established.

We verify that the operator \( A \) satisfies the generalized condition \((S_+)\) w.r.t. the operator \( M \). Let \( \{ u_j \} \subset \mathcal{D}(M) \) be a sequence with the properties

\[
\begin{align*}
  u_j & \rightharpoonup u_0, \quad Au_j \rightharpoonup h_0, \\
  \limsup_{j \to \infty} \langle Au_j, u_j - u_0 \rangle & \leq 0, \quad \langle Mu_j + Au_j, u_j \rangle \leq 0. \tag{6.55}
\end{align*}
\]

Then, as in the proof of the estimates (6.54), we obtain

\[
\sup_{0 < t < T} \int_\Omega u_j^2(x,t)dx \leq C_8, \quad \int_Q |u_j(x,t)|^{m+\frac{2m}{n}}dxdt \leq C_8, \tag{6.56}
\]

for some constant \( C_8 \) independent of \( j \).

Using the formula (6.11), the second inequality in (6.55) and Conditions \( \alpha'', \rho'' \) we get the estimate

\[
\frac{1}{2} \int_\Omega \left[ u_j^2(x,t_1) - u_j^2(x,t_2) \right]dx \\
\leq \int_{t_1}^{t_2} \int_\Omega \left\{ \rho(x,t,0)u_j + \sum_{i=1}^{n} a_i(x,t,u_j,0) \frac{\partial u_j}{\partial x_i} \right\}dxdt, \tag{6.57}
\]

for any points \( t_1, \ t_2 \in [0,T] \). Estimating the right-hand side of (6.57) by Hölder’s inequality and using (6.5) and (6.54) we establish the estimate

\[
\int_\Omega \left[ u_j^2(x,t_1) - u_j^2(x,t_2) \right]dx \leq C_9|t_1 - t_2|^{\delta}, \tag{6.58}
\]

for some positive number \( \delta \) independent of \( j \). From the boundedness of the sequence \( \{ u_j \} \) in \( X \), its weak convergence to \( u_0 \) and the inequalities (6.56), (6.58) we conclude that \( u_j \) converges strongly to \( u_0 \) in \( L_p(Q) \) with \( p < m + \frac{2m}{n} \).
The strong convergence of \( u_j \) to \( u_0 \) in \( X \) follows from (6.4), (6.5), (6.55) and

\[
\sum_{i=1}^{n} \int_{Q} [a_i(x, t, u_j, \frac{\partial u_j}{\partial x}) - a_i(x, t, u_j, \frac{\partial u_0}{\partial x})] \cdot \frac{\partial (u_j - u_0)}{\partial x_i} dx dt
\]

\[
= \langle Au_j, u_j - u_0 \rangle - \sum_{i=1}^{n} \int_{Q} a_i(x, t, u_j, \frac{\partial u_0}{\partial x}) \cdot \frac{\partial (u_j - u_0)}{\partial x_i} dx dt,
\]

(6.59)

because the limit of last integral equals zero.

The equality \( Au_0 = h_0 \) follows immediately from the established strong convergence of \( u_j \) in \( L_p(Q) \), \( p < m + \frac{2m}{n} \) and in \( X \). This completes the proof of Theorem 6.2. □

We formulate below one application of degree theory to the solvability of the initial-boundary value problem (6.1)-(6.3). Consider a one-parameter family of problems

\[
\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i,\tau(x, t, u, \frac{\partial u}{\partial x}) + \tau \rho(x, t, u) = \tau \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(x, t),
\]

(6.60)

with the boundary condition (6.2) and the initial condition (6.3), where \( 0 \leq \tau \leq 1 \),

\[
a_{i,\tau}(x, t, u, \xi) = a_i(x, t, \tau u, \xi) - (1 - \tau) a_i(x, t, 0, 0)
\]

(6.61),

and the functions \( a_i(x, t, u, \xi) \), \( \rho(x, t, u) \) satisfying the conditions \( \alpha' \), \( \alpha'' \), \( \rho' \), \( \rho'' \).

Define the parametric families of operators \( M_\tau : X \supset D(M) \to X^* \), \( A_\tau : X \supset D(A) \to X^* \) by

\[
\langle M_\tau u, \phi \rangle = \langle \frac{\partial u}{\partial t}, \phi \rangle + \tau \int_{Q} \rho(x, t, u(x, t)) \phi(x, t) dx dt,
\]

(6.62)

\[
\langle A_\tau u, \phi \rangle = \sum_{i=1}^{n} \int_{Q} [a_{i,\tau}(x, t, u, \frac{\partial u}{\partial x}) + \tau f_i(x, t)] \cdot \frac{\partial \phi(x, t)}{\partial x_i} dx dt,
\]

(6.63)

where \( X = L_m(0, T, W_0^{1,m}(\Omega)) \) and \( D(M), D(A) \) are defined by (6.9), (6.10).
Then we have
therefore
Theorem 6.3. Assume that the conditions $\alpha', \alpha'', \rho', \rho''$ are satisfied. Then the parametric families $M_\tau, A_\tau, \tau \in [0, 1]$, defined by (6.62) and (6.63), satisfy the conditions $m^{(1)}_\tau - m^{(3)}_\tau, a^{(1)}_\tau - a^{(3)}_\tau$ of Section 4.

The proof of Theorem 6.3 is analogous to the proof of Theorem 6.2. It is therefore omitted.

**Theorem 6.4.** Assume that the conditions $\alpha', \alpha'', \rho', \rho''$ are satisfied, $f_i \in L^m(Q), i = 1, \ldots, n$, and suppose that there exists a constant $M$ such that for an arbitrary solution $u \in D(M)$ of the problem (6.61)$_\tau$, (6.2), (6.3) we have $\|u\| \leq M$, where $\| \cdot \|$ is the norm of the space $L^m(0,T,W^{1,m}_0(\Omega))$. Then the problem (6.1), (6.2), (6.3) has at least one solution in $D(M)$.

The proof of Theorem 6.4 is analogous to the proof Theorem 5.3, and is based on Theorems 4.3 and 4.4. It is necessary to note only that for an arbitrary function $u \in D(M)$ we have $(M_0u + A_0u, u) \geq 0$.

**REFERENCES**


