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EXISTENCE AND STABILITY OF SOLUTIONS TO PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DELAY

Wolfgang M. Ruess
Fachbereich Mathematik, Universität Essen, D-45117 Essen, Germany

(Submitted by: M.G. Crandall)

Abstract. Results on (a) the existence and (b) asymptotic stability of mild and of strong solutions to the nonlinear partial functional differential equation with delay (FDE) $\dot{u}(t) + Bu(t) \ni F(u_t),\ t \geq 0,\ u_0 = \varphi \in E$, are presented. The 'partial differential expression' B will be a, generally multivalued, accretive operator, and the history-responsive operator F will be allowed to be (defined and) Lipschitz continuous on 'thin' subsets of the initial-history space E of functions from an interval $I \subset (-\infty, 0]$ to the state Banach space X. As one of the main results, it is shown that the well-established solution theory on strong, mild and integral solutions to the underlayed counterpart to (FDE) of the nonlinear initial-value problem (CP) $\dot{u}(t) + Bu(t) \ni f(t),\ t \geq 0,\ u(0) = u_0 \in X$, can fully be extended to the more general initial-history problem (FDE). The results are based on the relation of the solutions to (FDE) to those of an associated nonlinear Cauchy problem in the initial-history space E. Applications to models from population dynamics and biology are presented.

1. Introduction. Our objective is to study existence and asymptotic stability of mild and strong solutions to the following partial functional differential equation with delay:

$$\text{(FDE)} \quad \begin{cases} \dot{x}(t) + Bx(t) \ni F(x_t), & t \geq 0 \\ x_{|I} = \varphi \in \hat{E}. \end{cases}$$

$B \subset X \times X$ is a (generally) nonlinear and multivalued operator in a Banach (state) space X, with $(B + \alpha I)$ accretive, some $\alpha \in \mathbb{R}$, and, for given $I = [-r, 0]$, $r > 0$ (finite delay), or $I = \mathbb{R}^-$ (infinite delay), and $t \geq 0$, $x_t : I \to X$

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is the history of x up to t : $x_t(s) = x(t+s)$, $s \in I$, and $\varphi : I \to X$ is a given initial history out of a space E of functions from I to X. Moreover, F is a given history-responsive operator with domain $\hat{E} \subset E$ and range in X, Lipschitz on the (possibly 'thin') subset $\hat{E}$ of E with Lipschitz constant $M > 0$.

The existing discussion of (FDE) can mainly be structured into two parts:
(I) Existence of local solutions to (FDE) for the case of (merely) continuous history-responsive operators $F : D(F) \subset E \to X$.
(II) Existence and smoothness properties of global solutions to (FDE) under global or 'local' (in a wide sense) Lipschitz conditions on F.

Two particular techniques of proof have turned out essential and most common:
(A) Discrete scheme limit techniques – familiar from the nonlinear Cauchy problem ($F \equiv 0$ in (FDE)), or further direct methods on the level of the state space X; cf. [37, 38, 39, 40, 67] and references therein.
(B) The evolution operator approach via an associated evolution problem in the initial-history space E; cf. [2, 6, 17, 19, 20, 25, 35, 42, 43, 44, 53, 56, 59, 60, 63, 64, 70, 72, 71].

For a more exhaustive list of references on the work on (FDE), the reader is referred to [10, 36, 62].

The object of this paper is to take up the case (II) of history-responsive operators F that are Lipschitz continuous on suitably chosen subsets $\hat{E}$ of their domain, and to use the evolution operator approach (B) for the analysis of (FDE). This technique of proof consists of the following program:
Associate with (FDE) the operator A in E defined by

$$\begin{cases} D(A) = \{\varphi \in \hat{E} \mid \ \varphi' \in E, \varphi(0) \in D(B), \varphi'(0) \in F(\varphi) - B\varphi(0)\} \\ A\varphi := -\varphi' \ , \varphi \in D(A), \end{cases} \tag{1.1}$$

and consider the following statements:
(**S1**) $-A$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on $cl\, D(A) \subset \hat{E}$ of type γ, with $\gamma = \max\{0, M+\alpha\}$: $\|S(t)\varphi - S(t)\psi\| \leq e^{\gamma t}\|\varphi - \psi\|$, $t \geq 0$, $\varphi, \psi \in cl\, D(A)$.
(**S2**) If $\varphi \in cl\, D(A)$, and $x_\varphi : I \cup \mathbb{R}^+ \to X$ is defined by

$$x_\varphi(t) = \begin{cases} \varphi(t) & t \in I \\ (S(t)\varphi)(0) & t \geq 0, \end{cases} \tag{1.2}$$

then $S(t)\varphi = (x_\varphi)_t$ (i.e., $(S(t))_{t\geq 0}$ acts as a translation).
(**S3**) For $\varphi \in cl\, D(A)$, the function x_φ from (**S2**) solves (FDE).

This program has been initiated in [57, 68, 70, 72, 71] for F globally (defined and) Lipschitz continuous, and in [6, 7] for F globally or just locally (i.e., on norm-balls in E) Lipschitz continuous, together with separate results on statement (**S2**) in [25, 56], and corresponding results for the nonautonomous case in [17, 19, 53]. (For B continuous single-valued, cf. [30, 33, 34] and the references therein, and for B and F linear, see [2]). Essentially, (**S3**) is shown to hold for B single-valued, $(B+\alpha I)$ $m-$accretive, F globally Lipschitz, the state space X having a uniformly convex dual X^*, and the initial history $\varphi \in \hat{D}(A)$ (= the generalized domain of A). The question, however, in which 'weak' sense the function x_φ of (1.2) is a solution to (FDE) in general, remained open.

In view of concrete examples from biology and population dynamics (cf. [59, 63, 64, 65], and see section 5 below), these results are subject to restrictions not easily met in applications. While the operator B ought to be allowed to be multivalued and not necessarily $\alpha - m-$ accretive, the operator F in general not only fails to be globally Lipschitz, but may not even be defined globally or on norm-balls, but defined and Lipschitz continuous only on 'thin' subsets of the initial-history space E. Finally, the assumption on X to have a uniformly convex dual (for (**S3**) to hold) is unduly restrictive and is imposed by the search for strong solutions to (FDE).

In the following, we take up the 'local approach to global solutions' as developed in [59, 60, 63, 64, 65] avoiding these restrictions, and establish (**S3**) in full generality. The following are the main results.

1. For all $\varphi \in cl\, D(A)$, the function x_φ from (1.2) in (**S2**) above is a global mild solution to (FDE). It is a strong solution provided $(B + \alpha I)$ is $m-$accretive, X has the Radon-Nikodym property, and $\varphi \in \hat{D}(A)$ (Section 2). This shows that the well-established solution theory on strong, mild and integral solutions to the nonlinear Cauchy problem ($F \equiv 0$, i.e., no influence of the history) fully extends to the more general initial-history problem (FDE).

2. The solutions to (FDE) are asymptotically stable provided $(M+\alpha) < 0$ (i.e., if the damping $-\alpha I$ dominates the influence of the history M = Lipschitz constant of F on $\hat{E}$) (Section 3).

These results will be derived both for initial-history spaces of continuous functions (Sections 2 and 3) and of $L^1(\mathbb{R}^-, \nu; X)-$functions (Section 4). Parts of the results of Sections 2 and 3 have been announced in [61].

Notation and terminology. Given a subset D of a Banach space Y, $cl\, D$ will denote its (norm-) closure in Y. Recall [13] that a subset $C \subset$

$Y \times Y$ is said to be *accretive* in Y if for each $\lambda > 0$ and each pair $[x_i, y_i] \in C$, $i \in \{1, 2\}$, we have

$$\|(x_1 + \lambda y_1) - (x_2 + \lambda y_2)\| \geq \|x_1 - x_2\|,$$

maximal accretive if there exists no accretive subset of $Y \times Y$ strictly larger than C, and *m-accretive* in Y if, in addition, $R(I + \lambda C) = Y$ for all $\lambda > 0$. Moreover, for $\lambda > 0$, $J_\lambda^C = (I + \lambda C)^{-1}$, and $C_\lambda = \lambda^{-1}(I - J_\lambda^C)$. Recall ([11]) that if $R(I + \lambda C) \supset cl\, D(C)$ for all $\lambda > 0$ sufficiently small, the generalized domain $\hat{D}(C)$ of C is given by $\hat{D}(C) = \{y \in cl\, D(C) \mid \lim_{\lambda \to 0^+} \|C_\lambda y\| < \infty\}$.

Finally, we shall need the following item of notation concerning duality maps: for $x, y \in Y$, $\langle y, x \rangle_s = \sup\{\langle y, x^* \rangle : x^* \in J(x)\}$, where, for $x \in Y$, $J(x) = \{x^* \in Y^* : \langle x, x^* \rangle = \|x\|, \|x^*\| \leq 1\}$ is the (generally, multivalued) duality map of Y.

For all these notions and the general theory of accretive sets and evolution equations, the reader is referred to [4, 12, 13, 24, 50].

2. Existence of global solutions to (FDE). Throughout this and the subsequent section, we shall work with initial-history spaces of continuous functions. In the finite delay case $I = [-r, 0]$, $r > 0$, we take the initial-history space to be $E = C([-r, 0]; X)$, endowed with the sup-norm. In the infinite delay case $I = (-\infty, 0]$, E will be chosen to be a weighted sup-norm space of the type $E_v = \{\varphi \in C(\mathbb{R}^-, X) : v\varphi \in BUC(\mathbb{R}^-, X)\}$, with norm $\|\varphi\|_v := \sup\{v(s)\|\varphi(s)\| : s \in \mathbb{R}^-\}$, where the (weight-) function $v : \mathbb{R}^- \to (0, 1]$ has the following properties:

(v1) v is continuous, nondecreasing, and $v(0) = 1$;

(v2) There exists a constant $M_v \geq 0$ such that $\left|\frac{v(s+u)}{v(s)} - 1\right| \leq M_v|u|$ for all $s, u \leq 0$.

Typical such weight functions are $v(s) \equiv 1$ (with, in this case, $E_v = BUC(\mathbb{R}^-, X)$ with sup-norm), $v(s) = e^{\mu s}$, or $v(s) = (1 + |s|)^{-\mu}$, $\mu \geq 0$ (spaces of 'fading memory type'). (The Banach spaces E_v are sometimes called UC_g- spaces, $v = 1/g$, and have been considered by various authors, cf. [1, 27, 32] and the further references listed therein.)

The following assumptions will make the 'local approach' to global solutions work, compare [63].

(**A1**) The operator $B \subset X \times X$ is such that $(B + \alpha I)$ is accretive for some $\alpha \in \mathbb{R}$. $\hat{X} \subset X$ and $\hat{E} \subset E$ are closed subsets of X and E, respectively, such that

(1) $F : \hat{E} \to X$ is continuous and bounded (i.e., transforms bounded sets of $\hat{E}$ to bounded sets in X), with the property that

(2) there exists $M > 0$ such that, for all $\varphi, \psi \in \hat{E}$ with $\|\varphi - \psi\| = \|\varphi(0) - \psi(0)\|$, $\|F(\varphi) - F(\psi)\| \le M\|\varphi - \psi\|$, and, moreover,

(3) for $x \in \hat{X}$, $\psi \in \hat{E}$, and $\lambda > 0$ with $\lambda\gamma < 1$, with $\gamma := \max\{0, M + \alpha\}$, if $\varphi_x \in E$ is the solution to

$$\varphi - \lambda\varphi' = \psi, \qquad \varphi(0) = x,$$

then $\varphi_x \in \hat{E}$.

(**A2**) If $\psi \in \hat{E}$ and $\lambda > 0$ with $\lambda\gamma < 1$, then

$$(\psi(0) + \lambda F(\varphi_x)) \in (I + \lambda B)(D(B) \cap \hat{X})$$

for each $x \in \hat{X}$.

The subsequent results on (FDE) are all subject to assumptions (**A1**) and (**A2**) being in effect.

In the following, we shall call a continuous function $u : I \cup \mathbb{R}^+ \to X$ a *strong*, *mild*, or *integral* solution of (FDE) if $u_{|I} = \varphi$, and, on $\mathbb{R}^+$, u is a solution of the respective kind to the Cauchy problem

$$\text{(CP)} \quad \begin{cases} \dot{u}(t) + Bu(t) \ni f(t)\ , & t \ge 0 \\ u(0) = u_0 \end{cases}$$

with $f(t) = F(u_t)$ and $u_0 = \varphi(0)$.

Recall that a strong solution fulfills (CP) a.e. $t > 0$, a mild solution is the limit of discrete-scheme (DS) approximate solutions, and that, with B being $\alpha-$accretive, *strong* implies *mild* implies *integral*, cf. [4, 12, 13, 24, 50]. (The precise definitions will be reported subsequently at the appropriate places.)

Our first result establishes (**S3**) in the above setting, and thus shows that the evolution operator approach via E in fact leads to the best possible result for (FDE), namely, the existence of global mild solutions in the general context of (**A1**) and (**A2**).

Theorem 2.1. *For all $\varphi \in cl\, D(A)$, the function x_φ from* (1.2) *of* (**S2**) *is a global mild solution to* (FDE) *with $(x_\varphi)_t \in \hat{E}$ and $x_\varphi(t) \in \hat{X} \cap cl\, D(B)$ for all $t \ge 0$. In particular, it is an integral solution of type α, i.e.,*

$$\begin{aligned} &\|x_\varphi(t) - x\| - \|x_\varphi(r) - x\| \\ &\quad \le \alpha \int_r^t \|x_\varphi(\tau) - x\|\, d\tau + \int_r^t \langle F((x_\varphi)_\tau) - y, x_\varphi(\tau) - x\rangle_s\, d\tau \end{aligned} \tag{2.1}$$

for all $[x,y] \in B$ *and all* $0 \le r \le t$.
Finally, given $\varphi, \psi \in cl\, D(A)$,

$$e^{-\alpha t}\|x_\varphi(t) - x_\psi(t)\| - e^{-\alpha r}\|x_\varphi(r) - x_\psi(r)\| \qquad (2.2)$$
$$\le \int_r^t e^{-\alpha\tau}\langle F(S(\tau)\varphi) - F(S(\tau)\psi), x_\varphi(\tau) - x_\psi(\tau)\rangle_s d\tau$$

for all $0 \le r \le t$.

Proof. We prove that (**S1**)–(**S3**) of Section 1 are fulfilled, with (**S3**) in the sense of mild solutions.

Step 1 : The proof of [63, Thm. 2.1] reveals that

$$(A + \gamma I) \quad \text{is accretive in} \quad E, \qquad (2.3)$$

$$R(I + \lambda A) \supset \hat{E} \quad \text{for all} \quad \lambda > 0,\ \lambda\gamma < 1, \ \text{and} \qquad (2.4)$$

$$J_\lambda^A\varphi(0) = J_\lambda^B[\varphi(0) + \lambda F(J_\lambda^A\varphi)] \quad \text{for all} \quad \varphi \in \hat{E},\ \lambda > 0,\ \lambda\gamma < 1. \qquad (2.5)$$

(Notice that for this part of the proof assumptions (**A1**) and (**A2**) are sufficient, and that the additional assumptions of [63, Thm. 2.1] on $\hat{E}$ to be convex and F to be Lipschitz on all of $\hat{E}$ are not needed.) In particular, (**S1**) holds. According to [25, 56], (**S2**) holds as well.

Step 2: Since, by (2.3) and (2.4), the operator A is $\gamma-$accretive and fulfills the 'strong range condition' $R(I + \lambda A) \supset cl\, D(A)$, given any $\varphi \in cl\, D(A)$, the function $\Phi(\cdot) := S(\cdot)\varphi$ is the unique mild solution to the Cauchy problem in E

$$\begin{cases} \dot{\Phi}(t) + A\Phi(t) = 0, & t \ge 0 \\ \Phi(0) = \varphi, \end{cases}$$

with

$$S(t)\varphi = \lim_{\lambda \to 0^+} (J_\lambda^A)^{[t/\lambda]}\varphi \qquad (2.6)$$

uniformly over bounded $t-$intervals, cf. [50, Ch. 5.3]. (For $0 < \lambda \le t$, $[t/\lambda]$ denotes the largest integer $\le (t/\lambda)$.)

For fixed $T > 0$, and $\epsilon > 0$, choose $0 < \lambda < \epsilon$, $\lambda\gamma < 1$, small enough so that, according to (**A1**), (2.6), and Lebesgue's Dominated Convergence Theorem,

$$\int_0^{T+1} \left\| F(S(\tau)\varphi) - F((J_\lambda^A)^{[\tau/\lambda]+1}\varphi) \right\| d\tau < \epsilon. \qquad (2.7)$$

Let N_λ be the smallest integer with $\lambda N_\lambda \geq T$, and let $u_0 = \varphi(0)$, $u_k = ((J_\lambda^A)^k\varphi)(0)$, $k \in \{1, ..., N_\lambda\}$. Then, since, by the definition of $D(A)$,

$$\frac{u_k - u_{k-1}}{\lambda} = (-A(J_\lambda^A)^k\varphi)(0) \in F((J_\lambda^A)^k\varphi) - Bu_k,$$

and, according to (2.7),

$$\sum_{k=1}^{N_\lambda} \int_{(k-1)\lambda}^{k\lambda} \left\| F(S(\tau)\varphi) - F((J_\lambda^A)^k\varphi) \right\| d\tau$$
$$\leq \int_0^{T+1} \left\| F(S(\tau)\varphi) - F((J_\lambda^A)^{[\tau/\lambda]+1}\varphi) \right\| d\tau < \epsilon,$$

the systems $\{u_0, ..., u_{N_\lambda}\}, \{F(J_\lambda^A\varphi), ..., F((J_\lambda^A)^{N_\lambda}\varphi)\}$ constitute an $\epsilon-$discretization of (CP) with $u_0 = \varphi(0)$ and, according to (**S2**), $f(t) = F(S(t)\varphi)$ $= F((x_\varphi)_t)$. Moreover, according to (2.6), the piecewise constant function $u_\epsilon : [0, N_\lambda\lambda] \to X$, defined by

$$u_\epsilon(t) = \begin{cases} u_0 & , \quad t = 0 \\ u_k & , \quad t \in ((k-1)\lambda, k\lambda] \end{cases}$$

converges to $x_\varphi(\cdot) = S(\cdot)\varphi(0)$ uniformly over $[0, T]$. According to the definition of mild solutions to evolution equations (cf. [4, Ch. 1.3]), and, concerning (2.2), according to the stability inequalities for mild solutions (cf. [4, Ch. 6.2]), this completes the proof of Theorem 2.1 .

Theorem 2.1 immediately leads to the following extension of the known results on the existence of strong solutions to (FDE). While referring to Banach spaces with the Radon-Nikodym property (RNP) (cf. [15]), we recall that this class includes the classes of reflexive Banach spaces and of separable dual spaces.

Theorem 2.2. *If* $(B + \alpha I)$ *is* $m-$*accretive in* X, *and* X *has the Radon-Nikodym property then, for every* $\varphi \in \hat{D}(A)$, *the function* x_φ *from* (**S2**) *is a global strong solution to* (FDE), *i.e., it is locally absolutely continuous on* $\mathbb{R}^+$ *and differentiable a.e.* $\mathbb{R}^+$ *such that* $\dot{x}_\varphi(t) + B(x_\varphi(t)) \ni F((x_\varphi)_t)$ *a.e.* $t > 0$. *Moreover,* $(x_\varphi)_t \in \hat{E}$ *and* $x_\varphi(t) \in \hat{X} \cap \hat{D}(B)$ *for all* $t \geq 0$.

Proof. If $\varphi \in \hat{D}(A)$, the motion $S(\cdot)\varphi : \mathbb{R}^+ \to E$ is Lipschitz continuous on bounded $t-$intervals, and thus so is $x_{\varphi_{|\mathbb{R}^+}} : \mathbb{R}^+ \to X$. Hence, since X

has (RNP), $x_{\varphi|\mathbb{R}^+}$ is differentiable a.e. $t > 0$. Using the integral inequality (2.1) above and adapting the argument of proof of [50, Thm. 5.6] to the inhomogeneous Cauchy problem (CP) with $f(\cdot) = F(S(\cdot)\varphi)$ (and keeping in mind that $x_\varphi(t) \in \hat{D}(B) \subset cl\, D(B)$, $t \geq 0$) serves to complete the proof of Theorem 2.2.

Remarks 2.3. 1. In the context of Theorems 2.1 and 2.2, if conditions (**A1**) (1) and (2) are strengthened to F being Lipschitz continuous on $\hat{E}$, we have:
(a) The solutions to (FDE) as asserted by these theorems are unique among all mild solutions x to (FDE) with $x_t \in \hat{E}$ for all $t \geq 0$. (This can be read from the argument of proof for the special case of [63, Prop. 2.4].)
(b) $\{\varphi \in \hat{E} : \varphi(0) \in cl\,(D(B) \cap \mathcal{D}_B)\} \subset cl\, D(A) \subset \{\varphi \in \hat{E} : \varphi(0) \in cl\, D(B)\}$, with $\mathcal{D}_B := \bigcup_{\kappa>0} \left(\bigcap_{0<\lambda<\kappa} R(I+\lambda B)\right)$ ([63, Prop. 2.6]).
(c) $\hat{D}(A) = \{\varphi \in cl\, D(A) : v\varphi$ Lipschitz continuous on I and $\varphi(0) \in \hat{D}(B)\}$ ([63, Prop. 3.3]).

2. With regard to strong, mild, and integral solutions, Theorems 2.1 and 2.2 are optimal, for they put the existence results for the initial-history problem (FDE) in parallel with those for the underlayed counterpart of the initial-value problem (CP) ($F \equiv 0$).

Inequality (2.2), crucial for the asymptotic stability results of section 3 below, is a particular feature of this aspect. For strong solutions, (2.2) is a consequence of the solutions fulfilling (FDE) pointwise a.e. $t > 0$; Theorem 2.1 removes this restriction by proving all functions x_φ, $\varphi \in cl\, D(A)$, to be mild solutions. (Recall that, in general, mild solutions may fail to fulfill the respective evolution equation at any $t > 0$.)

3. Assumptions (**A1**) and (**A2**) are automatically fulfilled in the global case of F being globally Lipschitz and B being $m-$accretive, with the choice of $\hat{X} = X$ and $\hat{E} = E$, respectively. The same is true for F Lipschitz on norm-balls, see [63, Prop. 2.7, (b)]. Thus, Theorems 2.1 and 2.2 extend and complete previous related works on (FDE) in [6, 7, 35, 59, 60, 63, 64, 70, 72].

4. Notice the following particular aspects of Theorem 2.1:
(a) The result includes an assertion on flow invariance: by (2.5), and $x_\varphi(t) = (S(t)\varphi)(0)$, and the fact that the semigroup $(S(t))_{t\geq 0}$ leaves $cl\, D(A)$ invariant, automatically $x_\varphi(t) \in \hat{X}$ and $(x_\varphi)_t \in \hat{E}$ for all $\varphi \in cl\, D(A)$, $t \geq 0$.
(b) The generality in assumptions (**A1**) and (**A2**) allows for a great deal of flexibility with regard to applications: while, under just local Lipschitz continuity assumptions on the operator F, in general only local solutions

to (FDE) can be expected, given a concrete problem of type (FDE), conditions (**A1**) and (**A2**) provide instructions for specifying subsets $\hat{E}$ of initial histories for which, uniformly over $\hat{E}$, there do exist global solutions.

As a combination of these two aspects, conditions (**A1**) and (**A2**) may be viewed as *a priori* estimates on subsets of initial histories that allow for both global existence of solutions and flow invariance at the same time. (This feature has been used to great advantage in [63, 64] for applications to models from biology and population dynamics where, most naturally, $\hat{E}$ is a truncated cone of nonnegative initial histories; compare section 5 below.)

5. A related program, addressing the nonautonomous counterpart of (FDE) under 'local' accretiveness assumptions on the $B(t)'s$ and 'local' Lipschitz assumptions on the $F(t;\cdot)'s$, both for continuous initial histories $E = C([-R,0];X)$ and L^1- initial histories $E = L^1(-R,0;X)$, leading to the existence of local integral solutions, is presented in [22, 23, 21]. The assumptions are in the spirit of assumptions (**A1**) and (**A2**), but not really comparable to them: while the analog of (**A1**) is more restrictive in that the operators $F(t;\cdot)$ are supposed to be globally defined and Lipschitz on norm-balls, the analog of (**A2**) for this special case of $\hat{X}$ = norm-ball in X is somewhat more general. The difference with regard to the results is that the weakening towards (**A2**) in general leads to just local existence of (integral) solutions. (For details, see the cited works.)

Remarks 2.4: The linear case. Assume that $B \subset X \times X$ is linear (possibly multi-valued), and $F : \hat{E} \to X$ Lipschitz continuous.

Case 1: $-B$ (single-valued) is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ of bounded linear operators on X. Theorem 2.1 then extends the results of [14, 29, 33], [52, Chps. B-IV.3, C-IV.3] and [68, 69, 81]): *For all* $\varphi \in \hat{E}$, *the function* x_φ *from* (**S2**) *is a weak solution to* (FDE), *i.e., it is given by the variation-of-constants formula*

$$x_\varphi(t) = T(t)(\varphi(0)) + \int_0^t T(t-s)F((x_\varphi)_s)\,ds, \qquad t \geq 0.$$

It suffices to note that, for B linear, the mild solution of the inhomogeneous Cauchy problem is equal to the solution given by the variation-of-constants formula (plus the fact that, in the present case, $cl\,D(A) = \hat{E}$ [63, Prop. 2.6]).

Notice that this representation of the solution to (FDE) with B linear reduces the question of when this solution is even strong or classical to the question of the differentiability properties of the integral term – as is usual for the inhomogeneous linear Cauchy problem.

If $F \in B(E,X)$ (= bounded linear operators $E \to X$), then x_φ is the unique classical solution to (FDE) for all $\varphi \in D(A)$. However, in this respect, the following more general results hold:

Case 2: There exist $\omega \in \mathbb{R}$, and $M > 0$ such that, for all $\mu > \omega$,

$$(\mu I + B)^{-1} \in B(X) \quad \text{and} \quad \left\|(\mu I + B)^{-n}\right\| \leq \frac{M}{(\mu-\omega)^n}, \quad n \in \mathbb{N}. \tag{2.8}$$

Then, in the setting of Theorem 2.1, *we have*: *if* $\varphi \in D(A)$, $\varphi'(0) \in cl\, D(B)$, *and either* (i) $F(S(\cdot)\varphi)$ *is differentiable a.e.* $\mathbb{R}^+$, *or* (ii) X *has the Radon-Nikodym property, or* (iii) $F \in B(E,X)$, *then* x_φ *is a classical solution to* (FDE).

These results follow from Theorem 2.1, combined with the regularity results for the inhomogeneous Cauchy problem for linear (possibly multivalued) $\omega-$accretive operators, cf. [4, section 5.4].

Note. Single-valued, not necessarily densely defined linear operators B satisfying (2.8) are being considered in quite a number of recent publications and are called Hille-Yosida operators. This class is included in the general nonlinear setup: even for B multivalued as above, (2.8) is equivalent to $(B + \omega I)$ being m-accretive for an equivalent norm on X.

3. Asymptotic stability of solutions to (FDE). Throughout this section, we shall need the following strengthening of (**A1**) (2) and (3) on the Lipschitz continuity of F in $\hat{E}$.

(**A1**) (4) There exists $M > 0$ such that $\|F(\varphi) - F(\psi)\| \leq M\|\varphi - \psi\|$ for all $\varphi, \psi \in \hat{E}$.

Moreover, in the following, given $\varphi \in cl\, D(A)$, $x_\varphi : I \cup \mathbb{R}^+ \to X$ will denote the unique mild solution to (FDE) as asserted by Theorem 2.1 and Remark 2.3.1 (a) above.

As usual, solutions to (FDE) will be called *asymptotically stable* if, for any $\varphi, \psi \in cl\, D(A)$,

$$\lim_{t\to\infty} \left\|x_\varphi(t) - x_\psi(t)\right\| = 0,$$

and *exponentially asymptotically stable* if there exist $K, \delta > 0$ such that

$$\left\|x_\varphi(t) - x_\psi(t)\right\| \leq Ke^{-\delta t}\|\varphi - \psi\|, \quad t \geq 0.$$

If, for the moment, we let $\tilde{B} := (B + \alpha I)$, and consider the associated nonlinear Cauchy problem

$$\text{(CP)} \quad \begin{cases} \dot{x}(t) + \tilde{B}x(t) \ni 0\ , & t \geq 0 \\ x(0) = \varphi(0), \end{cases}$$

then (FDE) may be viewed as a perturbation of (CP) with a 'damping' represented by $-\alpha I$ and a 'forcing' represented by the influence F of the history. Since $\tilde{B}$ is accretive and thus its solution semigroup is nonexpansive, hence the solutions are stable ($\|x(t)-y(t)\| \le \|x(s)-y(s)\|$, $0 \le s \le t$, for any two solutions x, y), one is naturally led to ask the following question:

(**Q**) If, in (FDE), $-\alpha > M$, i.e., if the damping strictly dominates the influence of the history, are solutions to (FDE) asymptotically stable ?

In the finite-delay case, the answer is positive and solutions are even exponentially asymptotically stable, cf. [56]. In the infinite-delay case, however, one has to find a way in between two extreme cases depending on the particular choice of the weight v for the initial-history space $E = E_v$:

1. For $v(s) \equiv 1$, i.e., $E_v = (BUC(\mathbb{R}^-, X), sup - norm)$, $-\alpha > M$ does not imply asymptotic stability for (FDE) [64, Example 4.1.A].

2. For $v(s) = e^{\mu s}$ (or, more generally, v such that $s \to v(s)e^{-\mu s}$ is nondecreasing), $\mu > 0$, and $-\alpha > M$, solutions to (FDE) are exponentially asymptotically stable [57, Cor. 4.1], [64, Thm. 3.6].

For the infinite-delay case, in addition to the basic assumptions (v1) and (v2), the following special properties of the weight functions v will play a role in answering question (**Q**).

$$(v3) \quad \lim_{s\to-\infty} v(s) = 0; \qquad\qquad (v3^*) \quad \lim_{t\to\infty} \sup_{s\le -t} \frac{v(s)}{v(s+t)} = 0.$$

Clearly, (v3*) implies (v3). If, for $\mu > 0$, $v_1(s) = e^{\mu s}$, and $v_2(s) = (1 + |s|)^{-\mu}$, $s \le 0$, then v_1 and v_2 both fulfill (v3), v_1 fulfills (v3*), but v_2 fails to satisfy (v3*).

Theorem 3.1. *Consider* (FDE) *in the infinite-delay case with initial- history space* E_v, *with* v *satisfying* (v3), *and assume that* $(M + \alpha) < 0$. *If* $\varphi, \psi \in cl\, D(A)$ *are such that* $(x_\varphi - x_\psi)$ *is bounded on* $\mathbb{R}^+$, *and if*

(i) $\lim_{s\to-\infty} v(s)(\varphi(s) - \psi(s)) = 0$, *or*

(ii) v *satisfies (v3*), then*

$$\lim_{t\to\infty} \left\| x_\varphi(t) - x_\psi(t) \right\| = 0 = \lim_{t\to\infty} \|S(t)\varphi - S(t)\psi\|_v. \tag{3.1}$$

Proof. We collect the following facts required for the proof. First, the results of [64, Lemma 2.2] imply that, given $\varphi \in cl\, D(A)$,

$$S(t)\varphi = \tilde{\varphi}(t) + g_\varphi(t), \tag{3.2}$$

with $\tilde{\varphi}$ and $g_\varphi : \mathbb{R}^+ \to C(\mathbb{R}^-, X)$ defined by

$$\tilde{\varphi}(t)(s) = \begin{cases} \varphi(t+s) - \varphi(0) & , s \le -t \\ 0 & , -t \le s \le 0, \end{cases}$$

and

$$g_\varphi(t)(s) = \begin{cases} \varphi(0) & , s \le -t \\ x_\varphi(t+s) & , -t \le s \le 0, \end{cases}$$

and that, according to either assumption (i) or (ii) of Theorem 3.1,

$$\lim_{t\to\infty} \left\| \tilde{\varphi}(t) - \tilde{\psi}(t) \right\|_v = 0. \tag{3.3}$$

Moreover, according to stability inequality (2.2) of Theorem 2.1 above, for $\varphi, \psi \in cl\, D(A)$,

$$\begin{aligned} & e^{-\alpha t}\|x_\varphi(t) - x_\psi(t)\| - e^{-\alpha r}\|x_\varphi(r) - x_\psi(r)\| \\ & \quad \le \int_r^t e^{-\alpha\tau}\|F(S(\tau)\varphi) - F(S(\tau)\psi)\| d\tau \end{aligned} \tag{3.4}$$

for all $0 \le r \le t$.

Proof of (3.1). Concerning the first assertion of (3.1), assuming the contrary, let $0 < t_n \uparrow \infty$ such that

$$\lim_{n\to\infty} \left\|x_\varphi(t_n) - x_\psi(t_n)\right\| = \beta := \limsup_{t\to\infty} \left\|x_\varphi(t) - x_\psi(t)\right\| > 0. \tag{3.5}$$

Combining (3.2) and (3.4) leads to

$$\begin{aligned} & \left\|x_\varphi(t) - x_\psi(t)\right\| - e^{\alpha(t-r)}\left\|x_\varphi(r) - x_\psi(r)\right\| \\ \le \;& Me^{\alpha t}\int_r^t e^{-\alpha\tau}\left\|\tilde{\varphi}(\tau) - \tilde{\psi}(\tau)\right\| d\tau + Me^{\alpha t}\int_r^t e^{-\alpha\tau}\left\|g_\varphi(\tau) - g_\psi(\tau)\right\|_v d\tau \end{aligned} \tag{3.6}$$

for all $0 \le r \le t$. Notice that, according to (3.3), the first integral term on the right tends to zero as $t \to \infty$. As for the second term, given $\epsilon > 0$, choose $r_0 > 0$ such that

$$v(s)\left\|(x_\varphi - x_\psi)_{|\mathbb{R}^+}\right\|_\infty < \epsilon \qquad \text{for all} \quad s \le (-r_0/2).$$

Given $\tau \geq r \geq r_0$, using the definition of g_φ, g_ψ, and separately considering the cases $s \leq -\tau$, $-\tau \leq s \leq (-\tau/2)$, and $(-\tau/2) \leq s \leq 0$, reveals that

$$v(s)\big\|g_\varphi(\tau)(s) - g_\psi(\tau)(s)\big\| \leq \begin{cases} \epsilon, & s \leq (-\tau/2) \\ \sup\limits_{\rho \geq (r/2)} \big\|x_\varphi(\rho) - x_\psi(\rho)\big\|, & (-\tau/2) \leq s \leq 0. \end{cases}$$

Thus,

$$\big\|g_\varphi(\tau) - g_\psi(\tau)\big\|_v \leq \max\{\epsilon, \sup_{\rho \geq (\tau/2)} \big\|x_\varphi(\rho) - x_\psi(\rho)\big\|\} \tag{3.7}$$

for all $\tau \geq r \geq r_0$. Invoking (3.7) into (3.6) with $t = t_n$ for n so large that $t_n > r > r_0$, and letting $n \to \infty$, we arrive at

$$\beta \leq (-M/\alpha)(\epsilon + \sup_{\rho \geq (r/2)} \big\|x_\varphi(\rho) - x_\psi(\rho)\big\|) \qquad \text{for all} \quad r \geq r_0.$$

This implies that $\beta \leq \epsilon + (-M/\alpha)\beta$ for all $\epsilon > 0$, and thus leads to the desired contradiction, for, by the assumptions, $(-M/\alpha) < 1$.

The final assertion of Theorem 3.1 now is a consequence of (3.2), (3.3) and (3.7), thus completing the proof.

Note. The proof of Theorem 3.1 crucially depends on inequality (2.2) of Theorem 2.1 holding for all $\varphi, \psi \in cl\,D(A)$, compare Remark 2.3.2 above. Earlier asymptotic stability results in the spirit of Theorem 3.1 – even for the finite-delay case, had been restricted to strong solutions, and thus to corresponding restrictions on the operator B, and the state space X being Hilbert or, at least, having a uniformly convex dual (cf. [71, Prop. 4] and, for the nonautonomous case, [17, Thm. 4]).

As a first consequence of Theorem 3.1, we deduce a stability result that lifts compactness and strong-solution assumptions from [64, Thm. 3.1].

Proposition 3.2. *Consider* (FDE) *in the infinite-delay case with initial-history space* E_v, (*with* v *satisfying* (v1) *and* (v2)), *and with the operator* $B \subset X \times X$ *closed. Assume that there exists a weight* $u : \mathbb{R}^- \to (0,1]$ *satisfying* (v1) *and* (v2), *and* $M_u > 0$, *such that*

(i) $u \leq v$,

(ii) $(u(s)/v(s)) \to 0$ *as* $s \to -\infty$, *and*

(iii) $\|F\varphi - F\psi\| \leq M_u\|\varphi - \psi\|_u$ *for all* $\varphi, \psi \in \hat{E}$, *with* $(M_u + \alpha) < 0$.

If $\varphi, \psi \in cl\, D(A)$ are such that $(x_\varphi - x_\psi)$ is bounded on $\mathbb{R}^+$, then

$$\lim_{t\to\infty} \left\| x_\varphi(t) - x_\psi(t) \right\| = 0 = \lim_{t\to\infty} \| S(t)\varphi - S(t)\psi \|_w, \tag{3.8}$$

with $w = \sqrt{uv}$.

Proof. If $w = \sqrt{uv}$, then (1) $u \le w \le v$, (2) w satisfies (v1)-(v3), and (3) $(w(s)/v(s)) \to 0$ as $s \longrightarrow -\infty$. According to (3), $\lim_{s\to-\infty} w(s)(\varphi(s) - \psi(s)) = 0$ for all $\varphi, \psi \in cl\, D(A)$. Finally, the evolution operator for (FDE) can be extended to the setting of $\hat{E}_w := cl_{E_w}(\hat{E})$. (For details, see *Step 1* of the proof of [64, Thm. 3.1]). Applying Theorem 3.1 completes the proof.

The following is a typical application of such change-of-the-weight stability results. (Compare [28, Example 3.4] for the scalar, and [64, Example 3.4, Remarks 3.5] for the finite-dimensional linear versions.)

Example 3.3. Consider the delay problem

$$\begin{cases} \dot{x}(t) + (\alpha I + B)x(t) \ni D(x(t-r)) + \int_{-\infty}^{t} C(t-s)x(s)ds, \quad t \ge 0 \\ x_{|\mathbb{R}^-} = \varphi, \end{cases} \tag{3.9}$$

with X a Banach space, $B \subset X \times X$ $m-$accretive, $0 \in B0$, $D \in B(X) =$ bounded linear operators on $X, C \in L^1(\mathbb{R}^+, B(X))$, such that $(\|D\| + \|C\|_1) < \alpha$, and $r > 0$. According to [1] and [28, Example 3.4], let v be a weight satisfying (v1) – (v3) and $v \equiv 1$ on $[-r, 0]$, such that

$$\|D\| + \int_{-\infty}^{0} \|C(-s)\| \frac{1}{v(s)}\, ds < \alpha.$$

Then, for all $\varphi \in E_v$ the unique mild solution x_φ to (3.9) satisfies $\lim_{t\to\infty} \left\| x_\varphi(t) \right\| = 0$.

Proof. First, the identical zero function is a solution, hence all solutions are bounded on $\mathbb{R}^+$, for the solution semigroup is nonexpansive ($\gamma = \max\{0, M + \alpha\} = 0$.) Then, once again from [28, Example 3.4], there exists a weight u^* satisfying (v1) - (v3) with $u^* \equiv 1$ on $[-r, 0]$ such that

$$\|D\| + \int_{-\infty}^{0} \|C(-s)\| \frac{1}{v(s)u^*(s)}\, ds < \alpha. \tag{3.10}$$

Letting $u := v\, u^*$, and observing (3.10), Proposition 3.2 applies to complete the proof.

In the context of Theorem 3.1, if the identical zero function is a solution to (FDE), then $\lim_{t\to\infty} \left\|x_\varphi(t)\right\| = 0$ for all initial histories $\varphi \in cl\, D(A)$ (subject to (i) or (ii) of Theorem 3.1). For a fixed-point-attractor result to hold, however, all that is required is the existence of a solution with relatively compact range over $\mathbb{R}^+$.

Theorem 3.4. *Consider* (FDE) *in the infinite-delay case with initial-history space E_v, with v satisfying (v3), and assume that $(M+\alpha) < 0$. If there exists $\varphi_1 \in cl\, D(A)$ such that the solution x_{φ_1} has relatively compact range over $\mathbb{R}^+$, then there exists a fixed point $\varphi_0 \in cl\, D(A)$ of $(S(t))_{t\geq 0}$ such that, with $x_0 = \varphi_0(0) \in cl\, D(B)$,*

$$\lim_{t\to\infty} \|S(t)\psi - \varphi_0\|_v = 0 = \lim_{t\to\infty} \|x_\psi(t) - x_0\| \tag{3.11}$$

(1) *for all $\psi \in cl\, D(A)$, with $v(\cdot)\psi(\cdot)$ vanishing at $-\infty$, provided the latter also holds for $v(\cdot)\varphi_1(\cdot)$, or* (2) *for all $\psi \in cl\, D(A)$, provided v satisfies* $(v3*)$.

Proof. As $\gamma = \max\{0, M+\alpha\} = 0$, the solution semigroup $(S(t))_{t\geq 0}$ is nonexpansive, hence all solutions are bounded. Moreover, by [64, Thm. 2.4, Cor. 2.8], the motion $S(\cdot)\varphi_1 : \mathbb{R}^+ \to E_v$ is asymptotically almost periodic (recalling that motions of semigroups of nonexpansive operators are such, provided they have relatively compact range), and decomposes uniquely in the form

$$S(\cdot)\varphi_1 = S(\cdot)\varphi_0 + \rho, \qquad \text{with} \quad \lim_{t\to\infty} \|\rho(t)\|_v = 0, \qquad \text{and} \tag{3.12}$$

$S(\cdot)\varphi_0$ almost periodic (= restriction to $\mathbb{R}^+$ of an almost periodic function from $\mathbb{R}$ to E_v). Given any $\tau > 0$, if $v(\cdot)\varphi_1(\cdot)$, vanishes at $-\infty$, then so does $v(\cdot)S(\tau)\varphi_1(\cdot)$. In any event, Theorem 3.1 applies to show that

$$\lim_{t\to\infty} \|S(t+\tau)\varphi_1 - S(t)\varphi_1\|_v = 0 \qquad \text{for all} \quad \tau \geq 0. \tag{3.13}$$

Letting $0 < t_n \nearrow \infty$ in (3.13) such that the corresponding sequence of translates $S(\cdot)_{t_n}\varphi_1$ converges to $S(\cdot)\varphi_0$ uniformly over $\mathbb{R}^+$, reveals that, in fact, $S(\tau)\varphi_0 = \varphi_0$ for all $\tau \geq 0$. A further appeal to Theorem 3.1 in conjunction with (3.12) completes the proof.

Remarks 3.5. 1. Theorem 3.3 essentially reduces the search for a fixed-point attractor to the existence of a solution with relatively compact range. In the finite-dimensional case, this trivially amounts to boundedness; but this also holds for infinite-dimensional state spaces X in a variety of cases: recall from [60, Thm. 3.1], that boundedness of x_φ on $\mathbb{R}^+$ implies relatively compact range over $\mathbb{R}^+, \varphi \in cl\, D(A)$, provided $((M+\alpha) \leq 0$ and) the operator B has compact resolvents J_λ^B, $\lambda > 0$ sufficiently small. This, in turn, holds for broad classes of diffusion/absorption operators set up in $L^p(\Omega)-$spaces $(1 \leq p < \infty)$ over bounded domains $\Omega \subset \mathbb{R}^N$, cf. [60, Section 4], and see section 5 below.

2. The following is further information on the fixed point φ_0 specified in Theorem 3.4. (a) If $B \subset X \times X$ is a closed operator, then $\varphi_0 \in D(A)$ (and is a constant function on $\mathbb{R}^-$), and $x_0 \in D(B)$(cf. [63, Prop. 2.9, 1.]). (b) If φ_1 is an element of $\hat{D}(A)$, then so is φ_0, and thus $x_0 \in \hat{D}(B)$ (compare Remark 2.3.1 (c) above).

4. Initial-history spaces of L^1-type. We place the results of sections 2 and 3 above in the context of an initial history space of type $E = L^1(I, \nu; X)$, and supplement the corresponding results of [44, 72, 74, 73], with a view on the nonautonomous results of [18, 22, 21, 42, 43]. In particular, (1) we extend the results of the cited works from the case of mostly global Lipschitz continuity (or local on norm-balls) assumptions (on F) and $m-$accretivity and single-valuedness assumptions (on B) and particular results on strong solutions for Hilbert or reflexive state spaces (except for the local approach of [21] as described in Remark 2.3, 4., in section 2) to the general local approach of assumptions (**A1**) and (**A2**) and global existence of mild solutions in the general context, and (2) improve on the respective stability results by concentrating on a particular (weighted Lebesgue-) measure ν on I.

Throughout this section, the initial-history space E is given by $E = L^1(\mathbb{R}^-, \nu; X)$ with $\nu = p(\cdot)d\lambda$, where the Lebesgue-measurable function $p : \mathbb{R}^- \to (0, 1]$ is chosen such that, for some $\mu \geq 0$,

$$p(s)e^{-\mu s} \text{ is nondecreasing on } \mathbb{R}^-, \text{ and } p(0) = 1. \tag{4.1}$$

(In particular, attention is restricted to infinite-delay. As specified in [65], even in our general local context, the finite-delay case can easily be reduced to this case.)

As usual, while dealing with equivalence classes of functions, for the semigroup approach via the initial-history space the initial value (in X) has to be added as an extra variable. Thus, we start from the space

$$E_X = L^1(\mathbb{R}^-, \nu; X) \times X, \quad \text{with norm } \|[\varphi, h]\| = \max\{\|\varphi\|_1, \|h\|\},$$

and denote by π_1 and π_2 the projections of E_X onto its first and second component, respectively.

Accordingly, assumptions (**A1**) and (**A2**) of section 2 are modified as follows:

(**A1***) The operator $B \subset X \times X$ is such that $(B + \alpha I)$ is accretive for some $\alpha \in \mathbb{R}$. $\hat{X} \subset X$ and $\hat{E} \subset E$ are closed subsets of X and E, respectively, such that

(1*) $F : \hat{E} \times \hat{X} \to X$ is continuous and bounded (i.e., transforms bounded sets of $\hat{E} \times \hat{X}$ to bounded sets in X), with the property that

(2*) there exists $M > 0$ such that, for all $\varphi, \psi \in \hat{E}$ continuous, with $\varphi(0), \psi(0) \in \hat{X}$ and

$$\|[\varphi, \varphi(0)] - [\psi, \psi(0)]\| = \|\varphi(0) - \psi(0)\|,$$

$$\|F(\varphi, \varphi(0)) - F(\psi, \psi(0))\| \leq M\|\varphi(0) - \psi(0)\|,$$

and, moreover,

(3*) for $x \in \hat{X}$, $\psi \in \hat{E}$, and $\lambda > 0$ with $\lambda\gamma < 1$, with $\gamma := \max\{1-\mu, M+\alpha\}$, if $\varphi_x \in E$ is the solution to

$$\varphi - \lambda\varphi' = \psi, \qquad \varphi(0) = x,$$

then $\varphi_x \in \hat{E}$.

(**A2***) If $\psi \in \hat{E}$, $k \in \hat{X}$ and $\lambda > 0$ with $\lambda\gamma < 1$, then

$$(k + \lambda F(\varphi_x, x)) \in (I + \lambda B)(D(B) \cap \hat{X}) \qquad \text{for each} \qquad x \in \hat{X}.$$

(FDE) will be studied in the following form.

$$(FDE)_1 \quad \begin{cases} \dot{x}(t) + Bx(t) \ni F(x_t, x(t)), & t \geq 0 \\ x_0 = \varphi \in \hat{E}, \quad x(0) = h \in X. \end{cases}$$

The following modified evolution operator approach will be considered:

Associate with $(FDE)_1$ the operator A in E_X defined by

$$\begin{cases} D(A) = \{[\varphi, h] \in \hat{E} \times \hat{X} : \varphi \text{ locally absolutely continuous on } \mathbb{R}^-, \\ \qquad \text{differentiable a.e. } s \in (-\infty, 0],\ \varphi' \in E, \varphi(0) = h \in D(B)\} \\ A[\varphi, h] := [-\varphi', -F(\varphi, h) + Bh], \quad [\varphi, h] \in D(A), \end{cases}$$

and consider the following statements:
(**S1***) $-A$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on $cl\,D(A) \subset \hat{E}\times\hat{X}$ of type $\gamma,$ with $\gamma = \max\{1-\mu, M+\alpha\}:$ $\|S(t)[\varphi,h] - S(t)[\psi,k]\| \leq e^{\gamma t}\|[\varphi,h]-[\psi,k]\|$, $t\geq 0$, $[\varphi,h],[\psi,k]\in cl\,D(A)$.
(**S2***) If $[\varphi,h]\in cl\,D(A)$, and $x(\varphi,h):\mathbb{R}\ \to X$ is defined by

$$x(\varphi,h)(t) = \begin{cases} \varphi(t) & t\in\mathbb{R}^{\,-} \\ \pi_2 S(t)[\varphi,h] & t\geq 0, \end{cases} \tag{4.2}$$

then $\pi_1 S(t)[\varphi,h] = x(\varphi,h)_t$ (i.e., $(S(t))_{t\geq 0}$ acts as a translation).
(**S3***) For $[\varphi,h]\in cl\,D(A)$, the function $x(\varphi,h)$ from (**S2***) solves $(FDE)_1$.

In the present context, the following are the complete analogs of Theorems 2.1 and 2.2.

Theorem 4.1. *Statements* (**S1***) *and* (**S2***) *hold, and, for all* $[\varphi,h]\in cl\,D(A)$, *the function* $x(\varphi,h)$ *of* (4.2) *from* (**S2***) *is a global mild solution to* $(FDE)_1$. *Moreover, all further statements of Theorem* 2.1 *hold with regard to these solutions.*

Proof. Let $\lambda>0$, $\lambda\gamma<1$.
Step 1.1. $R(I+\lambda A)\supset cl\,D(A)$: Given $[\psi,k]\in cl\,D(A)$, (**A2***) serves to define the mapping $T:\hat{X}\to\hat{X}$, $Tx = J_\lambda^B(k+\lambda F(\varphi_x,x))$ (with φ_x from (**A1***), (3*)). By (**A1***), (2*), this map is a strict contraction:

$$\|Tx-Ty\| \leq \frac{\lambda}{1-\lambda\alpha}\big\|F(\varphi_x,x)-F(\varphi_y,y)\big\| \leq \frac{\lambda M}{1-\lambda\alpha}\|x-y\|.$$

Thus, there exists a fixed point $h = J_\lambda^B(k+\lambda F(\varphi_h,h))\in\hat{X}\cap D(B)$, hence $k\in(I+\lambda B)h-\lambda F(\varphi_h,h)$, so that, in fact, $[\varphi_h,h]\in D(A)$, and $[\psi,k]\in(I+\lambda A)[\varphi_h,h]$.
Step 1.2. $(A+\gamma I)$ is accretive: Let $[\varphi_i,h_i] = J_\lambda^A[\psi_i,k_i]$, then

$$\varphi_i(0) = h_i = J_\lambda^B(k_i+\lambda F(\varphi_i,h_i)),$$
$$\varphi_i(s) = e^{s/\lambda}h_i + \frac{e^{s/\lambda}}{\lambda}\int_s^0 e^{-\xi/\lambda}\psi_i(\xi)d\xi,$$

$i\in\{1,2\}$. Hence, if $\|\varphi_1-\varphi_2\|_1\leq\|h_1-h_2\|$, then

$$\|h_1-h_2\| \leq \frac{1}{1-\lambda\alpha}\|k_1-k_2\| + \frac{\lambda M}{1-\lambda\alpha}\|h_1-h_2\|,$$

so that

$$(1-\lambda(\alpha+M))\|h_1-h_2\| \leq \|k_1-k_2\|. \tag{4.3}$$

If $\|h_1-h_2\| < \|\varphi_1-\varphi_2\|_1$, then

$$\begin{aligned}
&\|\varphi_1-\varphi_2\|_1 = \int_{-\infty}^{0} \Big\| e^{s/\lambda}(h_1-h_2) + \frac{e^{s/\lambda}}{\lambda}\int_s^0 e^{-\xi/\lambda}(\psi_1(\xi)-\psi_2(\xi))d\xi \Big\| p(s)ds \\
&\leq \|h_1-h_2\| \int_{-\infty}^{0} e^{s((1+\lambda\mu)/\lambda)} p(s)e^{-\mu s}ds \\
&+ \int_{-\infty}^{0} \Big(\int_{-\infty}^{\xi} \frac{e^{s((1+\lambda\mu)/\lambda)}}{\lambda} p(s)e^{-\mu s}ds \Big) e^{-\xi/\lambda}\|\psi_1(\xi)-\psi_2(\xi)\|d\xi \\
&\leq \frac{1}{1+\lambda\mu}\Big\{ \lambda\|h_1-h_2\| + \int_{-\infty}^{0} p(\xi)e^{-\mu\xi}e^{\xi((1+\lambda\mu)/\lambda)}e^{-\xi/\lambda}\|\psi_1(\xi)-\psi_2(\xi)\|d\xi \Big\} \\
&\leq \frac{\lambda}{1+\lambda\mu}\|\varphi_1-\varphi_2\|_1 + \frac{1}{1+\lambda\mu}\|\psi_1-\psi_2\|_1,
\end{aligned}$$

so that

$$(1-\lambda(1-\mu))\|\varphi_1-\varphi_2\|_1 \leq \|\psi_1-\psi_2\|_1. \tag{4.4}$$

Inequalities (4.3) and (4.4) serve to prove the claim. *Step 1.1* and *Step 1.2* imply (**S1***). The validity of (**S2***) is due to [56].
Step 2: The proof that the function $x(\varphi, h)$ of (4.2) is a mild solution to $(FDE)_1$ parallels *Step 2* of the proof of Theorem 2.1 with appropriate changes. Starting from

$$S(t)[\varphi,h] = \lim_{\lambda\to 0^+} (J_\lambda^A)^{[t/\lambda]}[\varphi,h], \quad \text{uniformly over bounded } t\text{-intervals},$$

we let $u_0 = h$, $u_k = \pi_2\{(J_\lambda^A)^k[\varphi,h]\}$, $k\in\mathbb{N}$, and observe that, in the present context,

$$\frac{1}{\lambda}(J_\lambda^A - I)(J_\lambda^A)^{k-1}[\varphi,h] = -A_\lambda (J_\lambda^A)^{k-1}[\varphi,h] \in -A(J_\lambda^A)^k[\varphi,h].$$

Hence,

$$\frac{1}{\lambda}(u_k-u_{k-1}) \in F(\pi_1\{(J_\lambda^A)^k[\varphi,h]\}, \pi_2\{(J_\lambda^A)^k[\varphi,h]\}) - Bu_k.$$

This implies that the systems $\{u_0, ..., u_{N_\lambda}\}$, $\{F(\pi_1\{J_\lambda^A[\varphi,h]\}, \pi_2\{J_\lambda^A[\varphi,h]\}),$ $..., F(\pi_1\{(J_\lambda^A)^{N_\lambda}[\varphi,h]\}, \pi_2\{(J_\lambda^A)^{N_\lambda}[\varphi,h]\})\}$ constitute an $\epsilon-$discretization of the evolution equation associated with $(FDE)_1$ with $u_0 = h$ and, by (**S2***), $f(t) = F(\pi_1\{S(t)[\varphi,h]\}, \pi_2\{S(t)[\varphi,h]\}) = F(x(\varphi,h)_t, x(\varphi,h)(t))$. The remaining arguments are now analogous to those of *Step 2* of the proof of Theorem 2.1. This completes the proof of Theorem 4.1.

Theorem 4.2. *If $(B + \alpha I)$ is $m-$accretive in X, and X has the Radon-Nikodym property then, for every $[\varphi,h] \in \hat{D}(A)$, the function $x(\varphi,h)$ from* (**S2***) *is a global strong solution to $(FDE)_1$, i.e., it is locally absolutely continuous on $\mathbb{R}^+$ and differentiable a.e. $\mathbb{R}^+$ such that $x(\varphi,h)(0) = h$, and $\dot{x}(\varphi,h)(t) + B(x(\varphi,h)(t)) \ni F(x(\varphi,h)_t, x(\varphi,h)(t))$ a.e. $t > 0$. Moreover, $x(\varphi,h)_t \in \hat{E}$ and $x(\varphi,h)(t) \in \hat{X} \cap \hat{D}(B)$ for all $t \geq 0$.*

Remarks 4.3. 1. If F is supposed to be Lipschitz continuous on $\hat{E} \times \hat{X}$, then the solutions to $(FDE)_1$ as asserted by Theorems 4.1 and 4.2 are unique among all mild solutions x to $(FDE)_1$ with $x_t \in \hat{E}$ and $x(t) \in \hat{X}$ for all $t \geq 0$.

2. Remarks 2.3.2 and 2.3.4 of section 2 apply analogously to Theorems 4.1 and 4.2. In particular, these results extend the corresponding results of [72, 74] (autonomous case) and [18, 22] (nonautonomous case) by dispensing with global Lipschitz assumptions on F and $m-$accretivity and single-valuedness of B, and by the extension to existence of mild solutions for general X. For a local approach of a different kind in the nonautonomous case, leading to local integral solutions to $(FDE)_1$, see [21]. Finally, compare [42] for X reflexive, $E = L^p(\mathbb{R}^-, \nu; X)$ with $1 < p < \infty$, $\nu = e^{\delta \cdot} d\lambda, \delta \geq 0$, and $(B + \alpha I)$ $m-$accretive (and densely defined), and $F : E \times X \to X$ locally Lipschitz.

3. The above L^1-setup provides a subtle further improvement over existing results: inclusion of the second (state space) variable in the history-responsive operator F allows to cover examples with F depending on the values of $x(\varphi,h)(\cdot)$ on $\mathbb{R}^+$ as well, and not just on the equivalence classes $x(\varphi,h)_t, t \in \mathbb{R}^+$ – such as (5.4), (5.5) and (5.7) - (5.9) in section 5. below. The history-responsive operators of the examples in [22, 74] are given in integral form, hence well-defined for equivalence classes.

4. A further achievement of the result of Theorem 4.1 concerns the question of stability: with $\gamma = max\{1-\mu, M+\alpha\}$, if $\mu \geq 1$ and $(M+\alpha) \leq 0$, the semigroup $(S(t))_{t\geq 0}$ associated to $(FDE)_1$ is stable ($\gamma = 0$). In the L^1-initial-history space setting, this result was not achieved so far, or partly achieved for weights $p(\cdot)$ depending on the parameter α of $(FDE)_1$, such as

in [74] with $\gamma = (\alpha+1)+\beta$ for the weight $p(s) = e^{(-\alpha+1)s}$. Clearly, while the stability result of Theorem 4.1 is independent of the parameters of $(FDE)_1$, it is at the expense of choosing a strong fading memory L^1-initial-history space as specified by a weight of the form (4.1). This particular choice will additionally allow for an analog of Theorem 3.1 above for the present L^1-context (Theorem 4.5 below).

In the following, given $[\varphi, h] \in cl\, D(A),\ x(\varphi, h) : \mathbb{R} \to X$ will denote the mild solution to $(FDE)_1$ as asserted by Theorem 4.1 above.

Turning to the problem of asymptotic stability for $(FDE)_1$, as a direct consequence of Step 1.2 of the proof of Theorem 4.1, we note the following exponential asymptotic stability result corresponding to those of [57, Cor. 4.1] and [64, Thm. 3.6] for initial-history spaces of continuous functions.

Corollary 4.4. *In the context of Theorem 4.1, assume that $\mu > 1$ and $(M + \alpha) < 0$, and let $\rho := min\{\mu - 1, -(M + \alpha)\} > 0$. Then, for all $[\varphi, h]$, $[\psi, k] \in cl\, D(A)$,*

$$\|x(\varphi, h)(t) - x(\psi, k)(t)\| \leq e^{-\rho t}\|[\varphi, h] - [\psi, k]\| \qquad \text{for all} \quad t \geq 0. \quad (4.5)$$

In order to prove an asymptotic stability analog of Theorem 3.1 for the present L^1-context, we shall need the following corresponding strengthening of (**A1***) (1*) and (2*) on the Lipschitz continuity of F in $\hat{E} \times \hat{X}$.

(**A1***) (1**) There exists $M > 0$ such that, for all $[\varphi, h], [\psi, k] \in \hat{E} \times \hat{X}$, $\|F(\varphi, h) - F(\psi, k)\| \leq M\|[\varphi, h] - [\psi, k]\|$.

Theorem 4.5. *In addition to the general assumptions of this section, let $\mu = 1$, i.e., $s \to p(s)e^{-s}$ nondecreasing on $\mathbb{R}^-$, and let (**A1***) (1**) above hold, and assume that $(M + \alpha) < 0$. If $[\varphi, h]$, $[\psi, k] \in cl\, D(A)$ are such that $(x(\varphi, h) - x(\psi, k))$ is bounded on $\mathbb{R}^+$, then*

$$\lim_{t\to\infty} \|x(\varphi, h)(t) - x(\psi, k)(t)\| = 0 = \lim_{t\to\infty} \|S(t)[\varphi, h] - S(t)[\psi, k]\|. \quad (4.6)$$

Proof. The analog of stability inequality (2.2) for the present context (as alluded to in Theorem 4.1) implies that

$$\begin{aligned} \|x(\varphi, h)(t) - x(\psi, k)(t)\| &\leq e^{(t-r)\alpha}\|x(\varphi, h)(r) - x(\psi, k)(r)\| \quad (4.7)\\ &+ Me^{t\alpha} \int_r^t e^{-\alpha\tau}\|[\pi_1(S(\tau)[\varphi, h]), \pi_2(S(\tau)[\varphi, h])]\\ &\qquad - [\pi_1(S(\tau)[\psi, k]), \pi_2(S(\tau)[\psi, k])]\|d\tau \end{aligned}$$

for all $0 \leq r \leq t$. Concerning the first assertion of (4.6), assuming the contrary, let $0 < t_n \uparrow \infty$ such that

$$\lim_{n\to\infty} \|x(\varphi,h)(t_n) - x(\psi,k)(t_n)\| = \beta \\ := \limsup_{t\to\infty} \|x(\varphi,h)(t) - x(\psi,k)(t)\| > 0. \tag{4.8}$$

Given $\epsilon > 0$, let $r_0 > 0$ such that

$$\sup_{\rho\geq r} \|x(\varphi,h)(\rho) - x(\psi,k)(\rho)\| < \beta + \epsilon \qquad \text{for all} \quad r \geq r_0. \tag{4.9}$$

With $r \geq 2r_0$ in (4.7), and using (**S2***) and (4.1), the L^1−norm in the integrand on the right-hand side can be estimated from above as follows:

$$\begin{aligned} &\int_{-\infty}^{-\tau} \|\pi_1(S(\tau)[\varphi,h])(s) - \pi_1(S(\tau)[\psi,k])(s)\| p(s)ds \\ &= \int_{-\infty}^{0} \|\varphi(s) - \psi(s)\| p(s-\tau)e^{s-\tau}e^{-(s-\tau)}ds \\ &\leq \int_{-\infty}^{0} \|\varphi(s) - \psi(s)\| p(s)e^{-\tau}ds = e^{-\tau}\|\varphi-\psi\|_1; \end{aligned} \tag{4.10}$$

$$\begin{aligned} &\int_{-\tau}^{-\tau/2} \|\pi_1(S(\tau)[\varphi,h])(s) - \pi_1(S(\tau)[\psi,k])(s)\| p(s)ds \\ &= \int_{-\tau}^{-\tau/2} \|\pi_2(S(\tau+s)[\varphi,h]) - \pi_2(S(\tau+s)[\psi,k])\| p(s)ds \\ &\leq \|[\varphi,h]-[\psi,k]\| \int_{-\tau}^{-\tau/2} e^s p(s)e^{-s}ds \leq e^{-\tau/2}\|[\varphi,h]-[\psi,k]\|; \end{aligned} \tag{4.11}$$

$$\begin{aligned} &\int_{-\tau/2}^{0} \|\pi_1(S(\tau)[\varphi,h])(s) - \pi_1(S(\tau)[\psi,k])(s)\| p(s)ds \\ &\leq \int_{-\tau/2}^{0} e^s \|x(\varphi,h)(\tau+s) - x(\psi,k)(\tau+s)\| ds \\ &\leq \sup_{\rho\geq r/2} \|x(\varphi,h)(\rho) - x(\psi,k)(\rho)\|. \end{aligned} \tag{4.12}$$

Thus, choosing $r \geq 2r_0$, and $t = t_n$ in (4.7), invoking (4.8) – (4.12), and letting $n \to \infty$, yields

$$\beta \leq (-M/\alpha)(\beta+\epsilon) \leq (-M/\alpha)\beta + \epsilon \qquad \text{for all} \quad \epsilon > 0.$$

As $(-M/\alpha) < 1$, this contradicts the assumption $\beta > 0$, and thus completes the proof of the first part of (4.6). It remains to show that

$$\lim_{t\to\infty} \|\pi_1(S(t)[\varphi, h]) - \pi_1(S(t)[\psi, k])\|_1 = 0.$$

Using (**S2***) and (4.1) once again, this follows from

$$\begin{aligned}
&\int_{-\infty}^{0} \|\pi_1(S(t)[\varphi, h])(s) - \pi_1(S(t)[\psi, k])(s)\| p(s) ds \\
&\quad = \int_{-\infty}^{t} \|x(\varphi, h)(s) - x(\psi, k)(s)\| p(s-t) e^{-(s-t)} e^{s-t} ds \\
&\quad \le \int_{-\infty}^{0} \|x(\varphi, h)(s) - x(\psi, k)(s)\| p(s) e^{-t} ds \\
&\quad\quad + \int_{0}^{t} \|x(\varphi, h)(s) - x(\psi, k)(s)\| e^{s-t} ds \\
&\quad = e^{-t} \|\varphi - \psi\|_1 + e^{-t} \int_0^t e^s \|x(\varphi, h)(s) - x(\psi, k)(s)\| ds,
\end{aligned}$$

and the fact that, from the first part of the proof, the last term above converges to zero as $t \to \infty$. This completes the proof of Theorem 4.5.

5. Examples. We present examples of diffusion/absorption processes with hereditary effects, arising in thermodynamics and biology/population dynamics.

Thermostat problem in materials with thermal memory. In the setting of a bounded open subset Ω of $\mathbb{R}^N$, $N \in \{2, 3\}$, the regulation of temperature through heat injection / extraction in the interior of Ω, controlled by a thermostat, can be modeled as follows: given prescribed temperatures $h_1(x) \le h_2(x)$, $x \in \Omega$, the temperature $u(t, x)$ at time $t \ge 0$ and $x \in \Omega$ is to deviate as little as possible from the interval $[h_1(x), h_2(x)]$ through regulation by heat sources with heat flux $\tilde{g}$, restricted by $-\tilde{g} \in [g_1, g_2]$ with $0 \in [g_1, g_2]$. In case heat is injected or extracted proportionally to the difference of the actual heat to the bounds $h_1(x)$ and $h_2(x)$, the function $-\tilde{g} = \beta(\cdot, u)$ is piecewise linear and monotonically non-decreasing (see [16, Ch. 2.3.1]). However, for the ideal case of keeping $u(t, x)$ within the interval $[h_1(x), h_2(x)]$ at any time and at every $x \in \Omega$, one has to choose (cf. [16, Ch. 2.3.1])

$$\beta(x; r) = \begin{cases} (-\infty, 0] &, \quad r = h_1(x) \\ 0 &, \quad h_1(x) < r < h_2(x) \\ [0, \infty) &, \quad r = h_2(x) \end{cases}$$

In [8], the process as described above has been considered in the $L^2(\Omega)-$ setting, and with Dirichlet boundary conditions, as being governed by the following multivalued operator B in $L^2(\Omega)$:

$$B = (-\Delta + \tilde{\beta}) \qquad \text{with} \qquad D(B) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \cap D(\tilde{\beta}),$$

where

$$\begin{cases} D(\tilde{\beta}) &= \{u \in L^2(\Omega) \mid \exists v \in L^2(\Omega) : v(\omega) \in \beta(\omega; u(\omega)) \text{ a.e. } \omega \in \Omega\}, \\ \tilde{\beta}(u) &= \{v \in L^2(\Omega) \mid v(\omega) \in \beta(\omega; u(\omega)) \text{ a.e. } \omega \in \Omega\}, \quad u \in D(\tilde{\beta}). \end{cases}$$

For materials with a suitable thermal memory, in the $L^2(\Omega)-$setting, the full initial-history problem thus is modeled by

$$\begin{cases} \dot{u}(t) - \Delta u(t) + \tilde{\beta}(u(t)) \ni F(u_t) \ , \ \ t \geq 0 \\ u_{|\mathbb{R}^-} = \varphi, \end{cases} \tag{5.1}$$

This example serves to underline the need for allowing the operator B in (FDE) to be multivalued in general.

However, rather than $L^2(\Omega)$, the space $L^1(\Omega)$ is the natural state space for this problem, and we wish to place it in the broader setting of general nonlinear diffusion/absorption processes associated with the differential expression

$$-div\, a(\cdot, grad\, u) + \tilde{\beta}(u) \ \ + \text{ Dirichlet boundary conditions.} \tag{5.2}$$

Assumptions ([76, 77, 78, 79]): Given an open subset Ω of $\mathbb{R}^N$,

1. the vector field $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following conditions:

   (H1) $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function, i.e., measurable in $x \in \Omega$ for all $\xi \in \mathbb{R}^N$, and continuous in $\xi \in \mathbb{R}^N$ for a.e. $x \in \Omega$; $a(\cdot, 0) = 0$;

   (H2) "monotonicity condition": $(a(x,\xi) - a(x,\hat{\xi})) \cdot (\xi - \hat{\xi}) \geq 0$ for all $\xi, \hat{\xi} \in \mathbb{R}^N$, a.e. $x \in \Omega$;

   (H3) "coerciveness condition": $a(x,\xi) \cdot \xi \geq \lambda_0 |\xi|^p - a_0(x)$ for all $\xi \in \mathbb{R}^N, |\xi| \geq R_0$, a.e. $x \in \Omega$, where $1 < p < \infty$, $a_0 \in L^1(\Omega)$, $\lambda_0 > 0$, $R_0 \geq 0$;

   (H4) $|a(x,\xi)| \leq b_0(x) + C_0 |\xi|^{p-1}$ for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$, where $b_0 \in L^{p'}(\Omega) + L^\infty(\Omega)$, $1/p + 1/p' = 1$, $C_0 \geq 0$.

2. $j : \Omega \times \mathbb{R} \to [0, \infty]$ is a function satisfying
   (a) $j(\cdot, r)$ is measurable for any $r \in \mathbb{R}$,
   (b) $j(x, \cdot)$ is convex, lower semicontinuous with $j(x, 0) = 0$ for a.e. $x \in \Omega$,

   and $\beta(x, r) := \partial j(x, \cdot)(r)$ for $r \in \mathbb{R}$, a.e. $x \in \Omega$, i.e., $\beta(x, \cdot)$ is the subdifferential of the convex function $j(x, \cdot)$, which is defined as usual by $t \in \partial j(x, r) \Leftrightarrow j(x, s) \geq j(x, r) + t(s - r)$ for all $s \in \mathbb{R}$.

Defining the operator A_0 in $L^{1\cap\infty}(\Omega)$ by

$$\begin{cases} D(A_0) = \{u \in L^{1\cap\infty}(\Omega) \mid \xi \cdot u \in W_0^{1,p}(\Omega) \text{ for all } \xi \in C_0^\infty(\Omega), \\ \qquad\qquad |a(\cdot, grad\, u)| \in L^{p'+\infty}(\Omega), -div\, a(\cdot, grad\, u) \in L^{1\cap\infty}(\Omega)\} \\ A_0 u := -div\, a(\cdot, grad\, u) \text{ in the sense of distributions, } u_0 \in D(A_0), \end{cases}$$

let $A := \bar{A_0}^{L^1(\Omega)}$ be the closure of A_0 in $L^1(\Omega)$. Moreover, with the absorption operator $C \subset L^1(\Omega) \times L^1(\Omega)$ defined by

$$w \in Cu \Leftrightarrow u, w \in L^1(\Omega),\, w(x) \in \partial\, j(x, u(x)) \qquad \mu - a.e.\ x \in \Omega,$$

the operator $B := A + C$ is a realization of the diffusion/absorption operator with Dirichlet boundary conditions of (5.2) above in $L^1(\Omega)$.

At this point, recall ([3]) that an m-accretive operator $B \subset L^1(\Omega) \times L^1(\Omega)$ is called *m-completely accretive (mca* for short) if its resolvents J_λ^B are (a) order-preserving, and (b) nonexpansive in both the L^1- and the $L^\infty-$norm. The operator $B = A + C$ just defined is m-completely accretive in case j does not depend on the space variable, or if $j(\cdot, r) \in L^{1+\infty}(\Omega)$ in the general case. (For more general conditions to this effect, cf. [5].) Since the general existence results of this paper do not impose special geometric conditions on the state space X, they can thus be applied to problems of the form (5.1) in the natural L^1- context, and for the Laplacian being replaced by more general nonlinear diffusion/absorption operators as specified above.

Corresponding statements hold for diffusion operators A as above with (generally nonlinear) Neumann boundary conditions such as

$$-div\ a(\cdot, grad\, u); \qquad -a(\cdot, grad\, u) \cdot n \in \beta(u) \quad \text{on} \quad \partial\Omega \tag{5.3}$$

with $\beta \subset \mathbb{R}^2$ a maximal monotone graph with $0 \in \beta(0)$, for they can also be realized by m-completely accretive operators in $L^1(\Omega)$ (cf. [80, 76, 77, 78, 79]).

Diffusive population models with delay in the birth process. Given an open subset $\Omega \subset \mathbb{R}^N$, the following is a particular model for a spatially distributed population with delay in the birth process:

$$\begin{cases} \dfrac{\partial u}{\partial t}u(x,t) - d\Delta u(x,t) = au(x,t)\Big[1 - bu(x,t) \\ \qquad\qquad\qquad\qquad\qquad\qquad - \int_{-1}^{0} u(x,t+r(s))d\eta(s)\Big] \\ u_{|[-R,0]} = \varphi \in C([-R,0],X) \\ + \text{ boundary conditions,} \end{cases} \tag{5.4}$$

where $a, b, d > 0, \eta$ is a positive bounded regular Borel measure on $[-1,0]$ with ($\|\eta\| > 0$ and) $b + \|\eta\| = 1$, and $r : [-1,0] \to [-R,0]$ is a continuous delay function, and X is an appropriate state space of real valued functions on Ω (to be specified later). In particular, this equation serves as a model for the density of red blood cells in an animal.

This model (and related ones with slightly different or more general history responsive functions) has been considered by a number of authors under both Dirichlet ([41, 54, 55]) and (linear) Neumann boundary conditions ([31, 45, 46, 47, 48, 49, 51, 66, 82]). Because of the quadratic terms arising in the history responsive function

$$F(\varphi) = a\varphi(0)\left[1 - b\varphi(0) - \int_{-1}^{0} \varphi(r(s))d\eta(s)\right],$$

and thus the problem of restricting existence results to function spaces on Ω invariant under products, in these references the state spaces have been restricted to Ω bounded and either spaces of functions continuous up to the closure of Ω ([41, 45, 46, 47, 51, 54, 55, 66]), or to 'small' Sobolev spaces and corresponding restricted initial histories such as $\varphi \in C([-R,0], W^{2,p}(\Omega))$ with $p > N$ ([31, 48, 49, 82]). However, with $u(x,t)$ representing a population and thus the L^1-norm of $u(\cdot,t)$ being a measure of the total population at time t, the natural state space obviously is $L^1(\Omega)$. Moreover, once again for natural reasons, restriction to nonnegative L^1-functions is appropriate.

We shall now indicate how (a) the local approach to (FDE) as developed in this paper, and (b) m-complete accretivity of the negative Laplacian and more general diffusion/absorption operators under Dirichlet or (generally, nonlinear) Neumann boundary conditions such as (5.2) and (5.3) above can be brought to bear on extending the solution theory for the above model (5.4) to the natural L^1- context as well as to more general models.

Proposition 5.1. *Let Ω be an open subset of $\mathbb{R}^N$, $X = L^1(\Omega)$, and let $B \subset L^1(\Omega) \times L^1(\Omega)$ be an m-completely accretive operator with $0 \in B(0)$. Then there exists $\beta_0 > 0$, such that, given any $\beta \geq \beta_0$, and any $\varphi \in C([-R,0],X)$ with $0 \leq \varphi(s)(x) \leq \beta$, $s \in [-R,0]$, a.e. $x \in \Omega$, and $\varphi(0) \in cl\, D(B)$, the delay equation*

$$\begin{cases} \dot{u}(t) + Bu(t) \ni au(t)\left[1 - bu(t) - \int_{-1}^{0} u(t + r(s))d\eta(s)\right], & t \geq 0 \\ u_{|[-R,0]} = \varphi \end{cases} \tag{5.5}$$

has a unique global (mild) solution $u \in C(\mathbb{R}^+, L^1(\Omega))$ such that $0 \leq u(t) \leq \beta$ a.e. Ω for all $t \geq 0$.

Proof. According to Theorem 2.1, it remains to check condition (**A2**) of section 2 for appropriate sets $\hat{X}$ and $\hat{E}$. Given $\beta > 0$, let $\hat{X}_\beta = \{x \in X \mid 0 \leq x \leq \beta \text{ a.e. } \Omega\}$, and $\hat{E}_\beta = \{\varphi \in C([-R,0],X) \mid \varphi(s) \in X_\beta, s \in [-R,0]\}$. Then, as in [65, section 4], we replace the operator B in (5.5) by $(B + \alpha I)$ (for $\alpha > 0$ to be specified later), and, accordingly, change the history-responsive operator F to $F_\alpha : \hat{E}_\beta \to X$, defined by $F_\alpha(\varphi) = a\varphi(0)\left[\frac{a+\alpha}{a} - b\varphi(0) - \int_{-1}^{0} \varphi(r(s))d\eta(s)\right]$. Then, if we choose

$$\alpha \geq \begin{cases} (\|\eta\|a)/b & , \text{ if } \quad 0 < b \leq \frac{1}{2} \\ a/(4b-1) & , \text{ if } \quad b > \frac{1}{2}, \end{cases}$$

and let $\beta = (a + \alpha)/a$, the following holds: for $\lambda > 0$, $x \in \hat{X}_\beta$, and $\psi \in \hat{E}_\beta$,

$$0 \leq \frac{1}{1 + \lambda\alpha}\left[\psi(0) + \lambda F_\alpha(\varphi_x)\right] \leq \beta \qquad \text{a.e.} \quad \Omega. \tag{5.6}$$

Using the fact that B has order-preserving resolvents (defined on all of X) that also contract in the L^∞−norm (together with $0 \in B(0)$), we conclude from (5.6) that

$$0 \leq \left(I + \frac{\lambda}{1 + \lambda\alpha}B\right)^{-1}\left\{\frac{1}{1 + \lambda\alpha}\left[\psi(0) + \lambda F_\alpha(\varphi_x)\right]\right\} \leq \beta \qquad \text{a.e.} \quad \Omega.$$

This shows that (**A2**) is fulfilled in our setting. The assertions of Proposition 5.1 can now be read from Theorem 2.1 and the above choices of the parameters α and β.

Note. The analogous statement to the one of Proposition 5.1 for initial histories φ out of $L^1([-R,0], L^1(\Omega))$ can similarly be deduced from Theorem

4.1 above, provided the measure $\eta \circ r^{-1}$ has a bounded Radon-Nikodym derivative with respect to Lebesgue measure on $[-R, 0]$.

Existence and flow-invariance results corresponding to those of Proposition 5.1 for the model (5.5) can also be achieved for the related model

$$\begin{cases} \dot{u}(t) + Bu(t) \ni u(t)\Big[1 + au(t) - b(u(t))^2 \\ \qquad\qquad -(1+a-b)\int_{-r}^{0} f(s)u(t+s)ds\Big], \quad t \geq 0 \\ u_{|[-r,0]} = \varphi \end{cases} \tag{5.7}$$

for Ω open in $\mathbb{R}^N$, $B \subset L^1(\Omega) \times L^1(\Omega)$ mca (such as (5.2) or (5.3) above), $a, b > 0$, $b < 1 + a$, and $f \in L^1((-r, 0))$ nonnegative with $\|f\|_1 = 1$. Once again, for Ω bounded, $B = -\Delta$ with $0-$Neumann boundary conditions, and state space $C(\bar{\Omega})$, see [26, 58].

Also, both models (5.5) and (5.7) can be extended to infinite delays and, more importantly, to temporal averages being replaced by spatio-temporal averages over the past history, such as

$$\begin{cases} \dot{u}(t) + Bu(t) \ni au(t)\Big[1 - bu(t) \\ \qquad\qquad - \int_{-\infty}^{t}\int_{\Omega} g(\cdot - y, t - s)u(s)(y)dyds\Big], \quad t \geq 0 \\ u_{|[-\infty,0]} = \varphi \end{cases} \tag{5.8}$$

and

$$\begin{cases} \dot{u}(t) + Bu(t) \ni u(t)\Big[1 + au(t) - b(u(t))^2 \\ \qquad -(1+a-b)\int_{-\infty}^{t}\int_{\Omega} g(\cdot - y, t - s)u(s)(y)dyds\Big], \quad t \geq 0 \\ u_{|[-\infty,0]} = \varphi \end{cases} \tag{5.9}$$

with $g \in L^1(\mathbb{R}^N \times (0, \infty))$ nonnegative suitably chosen, and Ω and B as in (5.5) and (5.7). For the discussion of model (5.9) for Ω bounded and $B = -\Delta$, together with further relevant references, the reader is referred to [9]. (The models (5.7), (5.8) and (5.9), as well as further nonlinear delay reaction-diffusion-absorption problems, will be considered in detail elsewhere.)

Remarks 5.2. 1. The models (5.5) and (5.7) – (5.9) above – together with Proposition 5.1 serve to demonstrate the novelty in the existence results of Sections 2 and 4:

(a) For existence results, state spaces need not be restricted by geometric/topological conditions. Also, 'bad' history-responsive functions can be

dealt with by using the general local setup of (**A1**) and (**A2**) to suitably restrict to 'thin' subsets of initial histories and/or to subsets naturally adapted to the problem, like nonnegative ones (that are, possibly, bounded above) as is natural for population models. Notice in particular that, for the natural state space $X = L^1(\Omega)$ for the population models (5.5) and (5.7) – (5.9), the history-responsive operator F is not globally defined, not even on norm-balls – much less locally Lipschitz.
(b) In the context of models (5.4), (5.5) and (5.7) – (5.9), the generality of Proposition 5.1 with respect to the state-responsive operator B just being m-completely accretive enlarges the range of applicability. For the environment Ω bounded, for instance, not only spatially diffusive models with zero population at the boundary ([41, 54, 55]) or with "reflecting walls" ([31, 48, 49, 66, 82]) or linear flux at the boundary ([45, 46, 47, 51]) can be considered, but more general models with diffusion and absorption processes combined, and more general "semipermeable walls" at the boundary as well, modeled by $-\partial u/\partial n \in \beta(u)$.
(c) Most importantly, the automatic flow-invariance inherent in the existence results of sections 2 and 4 is particularly useful for population models such as (5.4) and (5.5) (as well as (5.7) – (5.9)): nonnegativity and boundedness are automatically ensured globally, provided the same holds for the initial history. More precisely, Proposition 5.1 refers to the natural assumption of a given nonnegative and $L^\infty-$bounded initial history φ, and asserts that the solution u_φ to (5.5) exists globally with $0 \le u_\varphi(x,t) \le \max\{\beta_0, \|\varphi\|_\infty\}$ a.e. Ω for all $t \ge 0$.

2. As far as regularity of (mild) solutions is concerned, even though L^1-spaces are geometrically 'bad' (in particular, they fail the Radon-Nikodym property), complete accretivity of B comes to our rescue: in case B is m-completely accretive, the mild solution to the inhomogeneous Cauchy problem (CP) of section 2 above is actually a strong solution, provided $u_0 \in D(B)$ and $f \in W^{1,1}_{loc}(\mathbb{R}^+, X)$ ([75, §2.4]). This result can thus be applied to investigate regularity of the solutions to (5.5) asserted by Proposition 5.1. With regard to the original model (5.4), where mild solutions coincide with the corresponding variation-of-constants solutions, we refer to Remark 2.4 above for the usual linear regularity problem.

3. Even though the models (5.4) and (5.5), at first sight, may appear rather simple, beyond global existence, the asymptotic behavior of solutions is still wide open. The references [31, 41, 45, 46, 47, 48, 49, 51, 54, 55, 65, 66, 81, 82] contain various results for (5.4) on existence and stability of

equilibria for state spaces restricted as explained above, or for $L^2(\Omega)$. Note, for instance, that, for the above L^1-state space, the stability considerations of [41, 54, 55, 65], based on a corresponding principle of linearized stability for (FDE), cannot be applied, for the history-responsive operator F is not Fréchet differentiable in this space.

For bounded environments $\Omega \subset \mathbb{R}^N$, Proposition 5.1 asserts that L^∞-bounded initial histories lead to global solutions with weakly relatively compact range. In case the semigroup generated by B is compact, the ranges of solutions are even norm-relatively compact. (This problem will be considered elsewhere.) This applies, for instance, to the Dirichlet-Laplacian of (5.4). However, whether the ranges of solutions to (5.4) and (5.5) are (norm-) relatively compact in general, and whether, in that case, they would even be asymptotically almost periodic, seems open.

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REFERENCES

[1] F. V. Atkinson and J. R. Haddock, *On determining phase spaces for functional differential equations*, Funkcial. Ekvac. **31** (1988), 331–347.

[2] V. Barbu and S.I. Grossman, *Asymptotic behavior of linear integrodifferential systems*, Trans. Amer. Math. Soc. **173** (1972), 277–288.

[3] P. Bénilan and M.G. Crandall, *Completely accretive operators*, Semigroup Theory and Evolution Equations (E. Mitidieri P. Clément and B. de Pagter, eds.), Lecture Notes Pure Appl. Math., no. 135, Marcel-Dekker, 1991, pp. 41–75.

[4] P. Bénilan, M.G. Crandall, and A. Pazy, *Evolution Equations Governed by Accretive Operators*, Springer, in preparation.

[5] P. Bénilan and P. Wittbold, *Absorptions nonlinéaires*, J. Funct. Anal. **114** (1993), 59–96.

[6] D.W. Brewer, *A nonlinear semigroup for a functional differential equation*, Trans. Amer. Math. Soc. **236** (1978), 173–191.

[7] D.W. Brewer, *Locally Lipschitz continuous functional differential equations and nonlinear semigroups*, Illinois J. Math. **26** (1982), 374–381.

[8] H. Brézis, *Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations*, Contributions to Nonlinear Functional Analysis (E. Zarantonello, ed.), Acad.Press, 1971, pp. 101–156.

[9] N.F. Britton, *Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model*, SIAM J. Appl. Math. **50** (1990), 1663–1688.

[10] C. Corduneanu and V. Lakshmikantham, *Equations with unbounded delay: a survey*, Nonlinear Analysis TMA **4** (1979), 831–877.

[11] M.G. Crandall, *A generalized domain for semigroup generators*, Proc. Amer. Math. Soc. **37** (1973), 434–440.

[12] M.G. Crandall and T.M. Liggett, *Generation of semi-groups of nonlinear transformations on general Banach spaces*, Amer. J. Math. **93** (1971), 265–298.

[13] M.G. Crandall and A. Pazy, *Nonlinear evolution equations in Banach spaces*, Israel J. Math. **11** (1972), 57–94.

[14] M.C. Delfour, *The largest class of hereditary systems defining a C_0-semigroup on the product space*, Canad. J. Math. **32** (1980), 969–978.

[15] J. Diestel and J.J Uhl, *Vector Measures*, Surveys, no. 15, Amer. Math. Soc., Providence, R.I., 1977.

[16] G. Duvaut and J.L. Lions, *Inequalities in Mechanics and Physics*, Grundlehren math. Wiss., no. 219, Springer, Berlin, 1976.

[17] J. Dyson and R. Villella-Bressan, *Functional differential equations and non-linear evolution operators*, Proc. Royal Soc. Edinburgh **75A** (1975/76), 223–234.

[18] J. Dyson and R. Villella-Bressan, *Nonlinear functional differential equations in L^1-spaces*, Nonlinear Analysis TMA **1** (1977), 383–395.

[19] J. Dyson and R. Villella-Bressan, *Some remarks on the asymptotic behavior of a nonautonomous, nonlinear functional differential equation*, J. Diff. Eqns. **25** (1977), 275–287.

[20] J. Dyson and R. Villella-Bressan, *Semigroups of translation associated with functional and functional differential equations*, Proc. Royal Soc. Edinburgh **82A** (1979), 171–188.

[21] J. Dyson and R. Villella-Bressan, *Integral solutions of locally Lipschitz continuous functional differential equations*, Diff. Integral Eqns. **8** (1995), 1157–1166.

[22] J. Dyson and R. Villella-Bressan, *On integral solutions of nonautonomous delay equations and their propagation*, Nonlinear Analysis TMA **24** (1995), 693–710.

[23] J. Dyson and R. Villella-Bressan, *Nonautonomous locally Lipschitz continuous functional differential equations in spaces of continuous functions*, Nonlinear Diff. Eqns. Appl. **3** (1996), 127–147.

[24] L. Evans, *Nonlinear evolution equations in an arbitrary Banach space*, Israel J. Math. **26** (1977), 1–42.

[25] H. Flaschka and M.J. Leitman, *On semigroups of nonlinear operators and the solution of the functional differential equation $\dot{x}(t) = F(x_t)$*, J. Math. Anal. Appl. **49** (1975), 649–658.

[26] S.A. Gourley and N.F. Britton, *On a modified Volterra population equation with diffusion*, Nonlinear Analysis TMA **21** (1993), 389–395.

[27] J.R. Haddock and W.E. Hornor, *Precompactness and convergence in norm of positive orbits in a certain fading memory space*, Funkcial. Ekvac. **31** (1988), 349–361.

[28] J.R. Haddock and J. Terjéki, *On the location of positive limit sets for autonomous functional differential equations with infinite delay*, J. Differential Equations **86** (1990), 1–32.

[29] J.K. Hale, *Functional differential equations with infinite delays*, J. Math. Anal. Appl. **48** (1974), 276–283.

[30] J.K. Hale, *Theory of Functional Differential Equations*, Appl. Math. Sci., no. 3, Springer, New York, 1977.

[31] J.K. Hale, *Large diffusivity and asymptotic behavior in parabolic systems*, J. Math. Anal. Appl. **118** (1986), 455–466.

[32] J.K. Hale and J. Kato, *Phase space for retarded equations with infinite delay*, Funkcial. Ekvac. **21** (1978), 11–41.

[33] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Appl. Math. Sci., no. 99, Springer, New York, 1993.

[34] Y. Hino, S. Murakami, and T. Naito, *Functional Differential Equations with Infinite Delay*, Lecture Notes Math., no. 1473, Springer, Berlin, 1991.

[35] F. Kappel and W. Schappacher, *Autonomous nonlinear functional differential equations and averaging approximations*, Nonlinear Analysis TMA **2** (1978), 391–422.

[36] F. Kappel and W. Schappacher, *Some considerations to the fundamental theory of infinite delay equations*, J. Diff. Eqns. **37** (1980), 141–183.

[37] A.G. Kartsatos, *A direct method for the existence of evolution operators associated with functional evolutions in general Banach spaces*, Funkcial. Ekvac. **31** (1988), 89–102.

[38] A.G. Kartsatos and M.E. Parrott, *Global solutions of functional evolution equations involving locally defined Lipschitzian perturbations*, J. London Math. Soc. **(2) 27** (1983), 306–316.

[39] A.G. Kartsatos and M.E. Parrott, *A method of lines for a nonlinear abstract functional evolution equation*, Trans. Amer. Math. Soc. **286** (1984), 73–89.

[40] A.G. Kartsatos and M.E. Parrott, *The weak solution of a functional differential equation in a general Banach space*, J. Diff. Eqns. **75** (1988), 290–302.

[41] W. Kerscher and R. Nagel, *Positivity and stability for Cauchy problems with delay*, Partial Differential Equations, Lecture Notes Math., no. 1324, Springer, 1988, pp. 216–235.

[42] K. Kunisch and W. Schappacher, *Mild and strong solutions for partial differential equations with delay*, Annali Mat. Pura Appl. Ser. 4 **125** (1980), 193–219.

[43] K. Kunisch and W. Schappacher, *Variation of constants formulas for partial differential equations with delay*, Nonlinear Analysis TMA **5** (1981), 123–142.

[44] K. Kunisch and W. Schappacher, *Necessary conditions for partial differential equations with delay to generate C_0-semigroups*, J. Diff. Eqns. **50** (1983), 49–79.

[45] R.H. Martin and H.L. Smith, *Abstract functional differential equations and reaction-diffusion systems*, Trans. Amer. Math. Soc. **321** (1990), 1–44.

[46] R.H. Martin and H.L. Smith, *Convergence in Lotka-Volterra systems with diffusion and delay*, Differential Equations with Applications in Biology, Physics and Engineering (J.A. Goldstein, F. Kappel, and W. Schappacher, eds.), Lecture Notes Pure Appl. Math., no. 133, Marcel-Dekker, 1991, pp. 259–267.

[47] R.H. Martin and H.L. Smith, *Reaction-diffusion systems with time delays: monotonicity, invariance, comparison and convergence*, J. reine angew. Math. **413** (1991), 1–35.

[48] M.C. Memory, *Invariant manifolds for partial functional differential equations*, Proc. 2nd Int. Conf. Math. Population Dynamics, 1991, pp. 223–232.

[49] M.C. Memory, *Stable and unstable manifolds for partial functional differential equations*, Nonlinear Analysis TMA **16** (1991), 131–142.

[50] I. Miyadera, *Nonlinear Semigroups*, Transl. of Math. Monographs, no. 109, Amer. Math. Soc., Providence, RI, 1992.

[51] S. Murakami, *Stable equilibrium point of some diffusive functional differential equations*, Nonlinear Analysis TMA **25** (1995), 1037–1043.

[52] R. Nagel et al., *One-parameter Semigroups of Positive Operators*, Lecture Notes Math., no. 1184, Springer, Berlin, 1986.

[53] M.E. Parrott, *Representation and approximation of generalized solutions of a nonlinear functional differential equation*, Nonlinear Analysis TMA **6** (1982), 307–318.

[54] M.E. Parrott, *Positivity and a principle of linearized stability for delay-differential equations*, Diff. Integral Eqns. **2** (1989), 170–182.

[55] M.E. Parrott, *Linearized stability and irreducibility for a functional differential equation*, SIAM J. Math. Anal. **23** (1992), 649–661.

[56] A.T. Plant, *Nonlinear semigroups of translations in Banach space generated by functional differential equations*, J. Math. Anal. Appl. **60** (1977), 67–74.

[57] A.T. Plant, *Stability of nonlinear functional differential equations using weighted norms*, Houston J. Math. **3** (1977), 99–108.

[58] R. Redlinger, *On Volterra's population equation with diffusion*, SIAM J. Math. Anal. **16** (1985), 135–142.

[59] W.M. Ruess, *The evolution operator approach to functional differential equations with delay*, Proc. Amer. Math. Soc. **119** (1993), 783–791.

[60] W.M. Ruess, *Compactness and asymptotic stabilty for solutions of functional differential equations with infinite delay*, Evolution Equations (G. Ferreyra, G. Ruiz Goldstein, and F. Neubrander, eds.), Lecture Notes Pure Appl. Math., no. 168, Marcel-Dekker, 1994, pp. 361–374.

[61] W.M. Ruess, *Asymptotic behavior of solutions to delay functional differential equations*, Recent Developments in Evolution Equations (A.C. McBride and G.F. Roach, eds.), Pitman Research Notes Math., no. 324, Longman, 1995, pp. 220–230.

[62] W.M. Ruess, *Existence of solutions to partial functional differential equations with delay*, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type (A.G. Kartsatos, ed.), Lecture Notes Pure Appl. Math., no. 178, Marcel Dekker, 1996, pp. 259–288.

[63] W.M. Ruess and W.H. Summers, *Operator semigroups for functional differential equations with delay*, Trans. Amer. Math. Soc. **341** (1994), 695–719.

[64] W.M. Ruess and W.H. Summers, *Almost periodicity and stability for solutions to functional differential equations with delay*, Diff. Integral Eqns. **9** (1996), 1225–1252.

[65] W.M. Ruess and W.H. Summers, *Linearized stability for abstract differential equations with delay*, J. Math. Anal. Appl. **198** (1996), 310–336.

[66] A. Schiaffino, *On a diffusion Volterra equation*, Nonlinear Analysis TMA **3** (1979), 595–600.

[67] N. Tanaka, *On the existence of solutions for functional evolution equations*, Nonlinear Analysis TMA **12** (1988), 1087–1104.

[68] C.C. Travis and G.F. Webb, *Existence and stability for partial functional differential equations*, Trans. Amer. Math. Soc. **200** (1974), 395–418.

[69] C.C. Travis and G.F. Webb, *Partial differential equations with deviating arguments in the time variable*, J. Math. Anal. Appl. **56** (1976), 397–409.

[70] G.F. Webb, *Autonomous nonlinear functional differential equations and nonlinear semigroups*, J. Math. Anal. Appl. **46** (1974), 1–12.

[71] G.F. Webb, *Asymptotic stability for abstract nonlinear functional differential equations*, Proc. Amer. Math. Soc. **54** (1976), 225–230.

[72] G.F. Webb, *Functional differential equations and nonlinear semigroups in L^p- spaces*, J. Diff. Eqns. **20** (1976), 71–89.

[73] G.F. Webb, *Volterra integral equations and nonlinear semigroups*, Nonlinear Analysis TMA **1** (1977), 415–427.

[74] G.F. Webb, *Volterra integral equations as functional differential equations on infinite intervals*, Hiroshima J. Math. **7** (1977), 61–70.

[75] P. Wittbold, *Nonlinear evolution equations and completely accretive operators*, Diplomarbeit, Universität Essen, Essen, Germany, 1992.

[76] P. Wittbold, *Absorptions nonlinéaires*, Thèse doctorat, Université de Besançon, Besançon, France, 1994.

[77] P. Wittbold, *Diffusion nonlinéaire avec absorption*, C.R. Acad. Sci. Paris **319, Série I** (1994), 675–678.

[78] P. Wittbold, *Nonlinear diffusion with absorption*, Progress in partial differential eqns.: the Metz surveys 4 (M. Chipot and I. Shafrir, eds.), Pitman Res. Notes Math. Series, no. 345, Longman, 1996, pp. 142–157.

[79] P. Wittbold, *Nonlinear diffusion with absorption*, Potential Analysis **7** (1997), 437–465.

[80] P. Wittbold, *Surjectivity of perturbations of m-T-accretive operators*, in preparation.

[81] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Appl. Math. Sci., no. 119, Springer, New York, 1996.

[82] K. Yoshida, *The Hopf bifurcation and its stability for semilinear diffusion equations with time delay arising in ecology*, Hiroshima Math. J. **12** (1982), 321–348.