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ASYMPTOTIC BEHAVIOUR OF A NONLINEAR ELLIPTIC EQUATION WITH CRITICAL SOBOLEV EXPONENT: THE RADIAL CASE

Frédéric Robert
Université de Cergy-Pontoise, Département de Mathématiques, Site Saint-Martin
2, Avenue Adolphe Chauvin, F 95302 Cergy-Pontoise Cedex, France

(Submitted by: L.A. Peletier)

1. Introduction and statement of the results

Let B be the unit ball in $\mathbb{R}^N$, $N \geq 3$, and $a, f : \mathbb{R} \to \mathbb{R}$ two smooth functions. We regard $x \mapsto a(|x|)$ and $x \mapsto f(|x|)$ as functions of the variable $x \in \mathbb{R}^N$. As is easily seen, these functions are locally Lipschitz. In particular, they are locally in $C^{0,\alpha}$ for all $\alpha \in (0,1)$. In order to fix ideas, we suppose that $f > 0$ and that $f(0) = 1$. Then we consider the following problem:

$$(I)\ \begin{cases} \Delta u + a(|x|)u = N(N-2)f(|x|)u^p \ \text{ in } B \\ u > 0 \text{ in } B, \ \ u = 0 \text{ on } \partial B, \end{cases}$$

where $\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$ is the Laplacian with the minus sign convention, and $p = \frac{N+2}{N-2}$ is critical from the viewpoint of Sobolev embeddings. We let $H_0^1(B)$ be the standard Sobolev space, defined as the completion of $\mathcal{D}(B)$, the set of smooth functions with compact support in B, with respect to the norm $\|u\| = \sqrt{\int_B |\nabla u|^2\,dx}$. In the sequel, we suppose that the operator $u \mapsto \Delta u + a(|x|)$ is coercive on $H_0^1(B)$. This is the case when $a > -\lambda_1$, where λ_1 is the first eigenvalue of Δ for the Dirichlet problem.

Situations where (I) does not have a solution are in Pohozaev [14]. In particular, (I) does not possess a solution if $a \equiv 0$ and $f \equiv 1$. However, as it is subcritical from the viewpoint of Sobolev embeddings, the problem

$$(I_\epsilon)\left\{ \begin{array}{l} \Delta u_\epsilon + a(|x|)u_\epsilon = N(N-2)f(|x|)u_\epsilon^{p-\epsilon} \ \text{ in } B \\ u_\epsilon > 0 \text{ in } B, \ \ u_\epsilon = 0 \text{ on } \partial B \end{array}\right.$$

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has a solution $u_\epsilon \in C^2(\overline{B})$ for all $\epsilon \in (0, p-1)$. This solution can be assumed to be minimizing and radially symmetrical (MRS), where u_ϵ is said to be MRS if u_ϵ is radially symmetrical and

$$\frac{\int_B \left(|\nabla u_\epsilon|^2 + a(|x|)u_\epsilon^2\right) dx}{\left(\int_B f(|x|)u_\epsilon^{p-\epsilon+1} dx\right)^{\frac{2}{p-\epsilon+1}}} = \inf_{v\in\mathcal{D}(B)_R\setminus\{0\}} \frac{\int_B \left(|\nabla v|^2 + a(|x|)v^2\right) dx}{\left(\int_B f(|x|)|v|^{p-\epsilon+1} dx\right)^{\frac{2}{p-\epsilon+1}}}$$

where $\mathcal{D}(B)_R$ denotes the set of smooth, radially symmetrical functions with compact support in B. The arguments required for the proof of this result are by now classical.

On the one hand, we are concerned in this article with the existence of conditions on a and f for (I) to have a solution. On the other hand, we are concerned with the asymptotic behaviour of u_ϵ as $\epsilon \to 0$ when (I) does not have a solution. The existence of solutions for (I) has been studied by various authors. In particular, when $f \equiv 1$ and $a \equiv \lambda$, $\lambda \in \mathbb{R}$, Brézis and Nirenberg [2] got that (I) has a solution if and only if $\lambda \in (0, \lambda_1)$ when $N \geq 4$, and $\lambda \in (\frac{1}{4}\lambda_1, \lambda_1)$ when $N = 3$. Independently, asymptotic-type studies were first developed by Atkinson and Peletier [1]. With arguments from ODE theory, and assuming that $a \equiv 0$ and $f \equiv 1$, they got that

$$\lim_{\epsilon\to 0} \epsilon u_\epsilon^2(0) = \frac{4\Gamma(N)}{(N-2)\Gamma(\frac{N}{2})^2},$$

and that, for all $x \in B\setminus\{0\}$,

$$\lim_{\epsilon\to 0} \epsilon^{-1/2} u_\epsilon(x) = \frac{\sqrt{N-2}\,\Gamma(\frac{N}{2})}{2\sqrt{\Gamma(N)}}\Big(\frac{1}{|x|^{N-2}} - 1\Big).$$

Brézis and Peletier [3] returned to this problem, but with arguments from PDE theory, and they conjectured that a similar behaviour should occur in the nonradial case. This was proved to be true independently by Han [9] and Rey [15]. When $a \equiv 0$ and f is nonconstant, our problem was studied by Hebey [10], [13]. Existence results for (I) and the asymptotic behaviour of the u_ϵ's were given in these articles. An approach to the case where the Laplacian is the $p-$Laplace operator is in Garcìa Azorero and Peral Alonso [7]. We generalize in the present work what was done in [10]. In particular, we do not assume anymore that $a \equiv 0$. As one may easily check, the linear term au, and more precisely its negative part $a_- u$, leads to serious difficulties. We overcome these difficulties by assuming that a_- is small in a sense to be made precise below.

In what follows, we set

$$k_a \stackrel{\text{def}}{=} \inf\{l \geq 0 : a^{(l)}(0) \neq 0\}, \qquad k_f \stackrel{\text{def}}{=} \inf\{l \geq 1 : f^{(l)}(0) \neq 0\}$$

with the convention that $k_a = \infty$ (respectively $k_f = \infty$) if $a^{(l)}(0) = 0$ for all $l \geq 0$ (respectively $f^{(l)}(0) = 0$ for all $l \geq 1$). We denote by G the Green's function of the operator $\Delta + a$, so that G is such that

$$\Delta_y G(x,y) + a(|y|)G(x,y) = \delta_x$$

on $B \times B$ minus its diagonal, and $G(x,y) = 0$ for $y \in \partial B$ and $x \in B$. (As already mentioned, $\Delta + a$ is supposed to be coercive.) If $y \notin \partial B$, $G(x,y) > 0$, while $(x,y) \mapsto G(x,y)$ is symmetrical in (x,y). Moreover, $G(x,0)$ is radially symmetrical. We let $g(r) = G(x,0)$ where $r = |x|$. This function is defined on $(0,1]$. If $a \equiv 0$,

$$g(r) = \frac{1}{(N-2)\omega_{N-1}}\Big(\frac{1}{r^{N-2}} - 1\Big),$$

where ω_{N-1} denotes the volume of the standard sphere of $\mathbb{R}^N$. For $k \in \mathbb{N}$ and $q > 0$, we let

$$I_{k,q} = \int_0^\infty \frac{r^k}{(1+r^2)^{\frac{(N-2)q}{2}}}\, dr$$

when this integral makes sense, and we let ω_k be the volume of the standard sphere of $\mathbb{R}^{k+1}$. We also let

$$\alpha_k(N) = \frac{(k+1)(k+2)I_{k+N-1,2}}{(N-2)^2 I_{k+N+1,p+1}}, \qquad \alpha(N) = \frac{I_{2N-3,p+1}}{(N-3)!\omega_{N-1}^2}$$

and

$$\Phi(a) = \int_0^1 \Big(a(r) + \frac{1}{2} r a'(r)\Big) g(r)^2 r^{N-1} dr.$$

As is easily checked, Φ is defined as soon as $k_a > N-4$. Our first result is concerned with the existence of solutions to (I). This result generalizes previous results obtained by Demengel and Hebey [5] with another method.

Theorem 1. *There exists $\gamma = \gamma(N)$, $\gamma > 0$ depending only on N, such that if $||a_-||_{L^{\frac{N}{2}}(B)} < \gamma$, and if we are in one of the following cases,*

1. $k_a < N-4$,
 (a) $k_f < k_a + 2$, *and* $f^{(k_f)}(0) > 0$
 (b) $k_f = k_a + 2$, *and* $\alpha_{k_a}(N) a^{(k_a)}(0) < f^{(k_a+2)}(0)$
 (c) $k_f > k_a + 2$, *and* $a^{(k_a)}(0) < 0$

2. $k_a = N-4$,
 (a) $k_f < N-2$, *and* $f^{(k_f)}(0) > 0$
 (b) $k_f \geq N-2$, *and* $a^{(k_a)}(0) < 0$

3. $k_a > N-4$,
 (a) $k_f < N-2$, *and* $f^{(k_f)}(0) > 0$

(b) $k_f \geq N-2$, *and* $g'(1)^2 + 2\Phi(a) < \alpha(N) f^{(N-2)}(0)$,

then (I) *possesses an MRS solution, obtained as the limit of a subsequence of* u_ϵ *in* $C^2(\overline{B})$.

As already mentioned, there are situations in which the u_ϵ's do not converge, but develop a concentration. The concentration is characterized by one of the following properties: a subsequence of (u_ϵ) which converges almost everywhere converges to 0, or $u_\epsilon \to 0$ in $L^q(B)$ as soon as $q < p+1$. Such a situation occurs when $a \equiv 0$ and $f \equiv 1$. This follows from Hopf's maximum principle and the Pohozaev identity applied to (I),

$$\frac{(N-2)^2}{2}\int_B |x| f'(|x|) u^{p+1} dx - \int_B \Big(a(|x|) + \tfrac{1}{2}|x| a'(|x|)\Big) u^2 dx = \tfrac{1}{2}\int_{\partial B} |\nabla u|^2 d\sigma.$$

Still according to this identity, the u_ϵ's also develop a concentration when f is decreasing and $a + \frac{1}{2} r a' \geq 0$. As a first step, the concentration is ruled by the following classical result:

Theorem 2. *If the* u_ϵ*'s develop a concentration, then*

1. $\lim_{\epsilon\to 0} u_\epsilon = 0$ *in* $C^2_{loc}(\overline{B}\backslash\{0\})$ *and* $\lim_{\epsilon\to 0} ||u_\epsilon||_\infty = +\infty$

2. $\lim_{\epsilon\to 0} {u_\epsilon}^{p+1-\epsilon} = \frac{\omega_N}{2^N}\delta_0$ *in the sense of distributions*

3. $\lim_{\epsilon\to 0} \frac{u_\epsilon(0)}{||u_\epsilon||_\infty} = 1$

where $||u_\epsilon||_\infty$ *is the* L^∞*-norm of* u_ϵ.

Given $k \in \mathbb{N}$, we now set

$$\alpha^{(1)}_{k,N} = \frac{2^{N+1}(k+2)\omega_{N-1} I_{k+N-1,2}}{(N-2)^3 k! \omega_N}, \qquad \alpha^{(2)}_{k,N} = \frac{2^{N+1}\omega_{N-1} I_{k+N-1,p+1}}{(N-2)(k-1)!\omega_N}$$

and

$$\alpha^{(1)}_N = \frac{2^{N+2}\omega_{N-1}}{(N-4)!(N-2)^3\omega_N}, \qquad \alpha^{(2)}_N = \frac{2^{N+1}\omega^3_{N-1}}{(N-2)\omega_N}.$$

Generalizing the results of [1], [3], and [10], the asymptotic behaviour of the u_ϵ's is ruled by the following result:

Theorem 3. *There exists* $\gamma = \gamma(N)$, $\gamma > 0$ *depending only on* N, *such that if* $||a_-||_{L^{\frac{N}{2}}(B)} < \gamma$, *and if the* u_ϵ*'s develop a concentration, then*

$$\lim_{\epsilon\to 0} u_\epsilon(0) u_\epsilon(x) = (N-2)\omega_{N-1} G(x,0)$$

in $C^2_{loc}(\overline{B}\backslash\{0\})$, *and*

1. *If* $k_a < N-4$ *and*

(a) $k_f < k_a + 2$, *then* $\epsilon u_\epsilon(0)^{\frac{2k_f}{N-2}} \to -\alpha^{(2)}_{k_f,N} f^{(k_f)}(0)$

(b) $k_f = k_a+2$, *then* $\epsilon u_\epsilon(0)^{\frac{2k_f}{N-2}} \to \alpha^{(1)}_{k_a,N} a^{(k_a)}(0) - \alpha^{(2)}_{k_a+2,N} f^{(k_a+2)}(0)$

(c) $k_f > k_a + 2$, *then* $\epsilon u_\epsilon(0)^{\frac{2(k_a+2)}{N-2}} \to \alpha^{(1)}_{k_a,N} a^{(k_a)}(0)$

2. *If* $k_a = N-4$ *and*

(a) $k_f < N-2$, *then* $\epsilon u_\epsilon(0)^{\frac{2k_f}{N-2}} \to -\alpha^{(2)}_{k_f,N} f^{(k_f)}(0)$

(b) $k_f \geq N-2$, *then* $\epsilon \, \frac{u_\epsilon(0)^2}{\ln u_\epsilon(0)} \to \alpha^{(1)}_N a^{(k_a)}(0)$

3. *If* $k_a > N-4$ *and*

(a) $k_f < N-2$, *then* $\epsilon u_\epsilon(0)^{\frac{2k_f}{N-2}} \to -\alpha^{(2)}_{k_f,N} f^{(k_f)}(0)$

(b) $k_f = N-2$, *then* $\epsilon u_\epsilon(0)^2 \to -\alpha^{(2)}_{N-2,N} f^{(N-2)}(0) + \alpha^{(2)}_N g'(1)^2 + 2\alpha^{(2)}_N \Phi(a)$

(c) $k_f > N-2$, *then* $\epsilon u_\epsilon(0)^2 \to \alpha^{(2)}_N g'(1)^2 + 2\alpha^{(2)}_N \Phi(a)$

where $\alpha^{(1)}_{k,N}$, $\alpha^{(2)}_{k,N}$, $\alpha^{(1)}_N$, $\alpha^{(2)}_N$, *and* $\Phi(a)$ *are as above.*

The following sections are devoted to the proofs of these three theorems.

2. Elements from concentration theory

We let (u_ϵ) be a sequence of MRS solutions to (I_ϵ). In what follows, we suppose that

$$\lim_{\epsilon \to 0} \frac{\int_B (|\nabla u_\epsilon|^2 + a{u_\epsilon}^2)\, dx}{\left(\int_B f u_\epsilon^{p+1-\epsilon} dx\right)^{\frac{2}{p+1-\epsilon}}} = \frac{N(N-2)\omega_N^{\frac{2}{N}}}{4}\ . \tag{1}$$

Note that the right-hand side in this relation is the inverse of the square of the best constant $K(N,2)$ for the Sobolev inequality corresponding to the embedding of $H^1(\mathbb{R}^N)$ in $L^{p+1}(\mathbb{R}^N)$. We say that $x_0 \in \overline{B}$ is a concentration point of the u_ϵ's if for all $\delta > 0$,

$$\limsup_{\epsilon \to 0} \int_{B \cap B(x_0,\delta)} f(|x|) u_\epsilon^{p+1-\epsilon} dx > 0.$$

We suppose here that any subsequence of (u_ϵ) which converges almost everywhere converges to 0. Then, the u_ϵ's develop a concentration. Multiplying (I_ϵ) by u_ϵ and integrating by parts, we get that

$$\lim_{\epsilon \to 0} \int_B \left(|\nabla u_\epsilon|^2 + a{u_\epsilon}^2\right) dx = \frac{N(N-2)\omega_N}{2^N}$$

and

$$\lim_{\epsilon \to 0} \int_B f u_\epsilon^{p+1-\epsilon} dx = \frac{\omega_N}{2^N}.$$

Since the operator $\Delta + a$ is coercive, the sequence (u_ϵ) is bounded in $H^1(B)$.

Given $x_0 \in \overline{B}$ and $\delta > 0$, we let $\eta \in C^\infty(\mathbb{R}^N)$ be a cut-off function such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B(x_0, \delta/2)$, and $\eta = 0$ in $\mathbb{R}^N \backslash B(x_0, \delta)$. Multiplying (I_ϵ) by $\eta^2 {u_\epsilon}^k$, where $k \geq 1$, we easily obtain that

$$\frac{4k}{(k+1)^2}\int_B |\nabla(\eta {u_\epsilon}^{\frac{k+1}{2}})|^2 dx - \frac{2(k-1)}{(k+1)^2}\int_B \eta(\Delta\eta) u_\epsilon^{k+1} dx$$
$$-\frac{2}{k+1}\int_B |\nabla\eta|^2 u_\epsilon^{k+1} dx + \int_B a\eta^2 u_\epsilon^{k+1} dx = N(N-2)\int_B f(|x|)\eta^2 u_\epsilon^{k+p-\epsilon} dx.$$

The following result follows from this relation and our original assumption. It is by now classical, and we refer to [10] or [11] for its proof.

Lemma 2.1. *The following properties hold:*

1. $u_\epsilon \to 0$ *in* $L^q(B)$ *for all* $1 < q < p+1$, *in particular for* $q = 2$.
2. *If* $x_0 \in \overline{B}$ *is a concentration point, then for all* $\delta > 0$,
$$f(x_0)^{1-\frac{2}{N}}\left(\limsup_{\epsilon\to 0}\int_{B\cap B(x_0,\delta)} f(|x|) u_\epsilon^{p+1-\epsilon} dx\right)^{\frac{2}{N}} \geq \frac{{\omega_N}^{\frac{2}{N}}}{4}.$$
3. (u_ϵ) *possesses one and only one concentration point, the point* $x_0 = 0$.
4. $\lim_{\epsilon\to 0} ||u_\epsilon||_{L^\infty(B)} = +\infty$.
5. $\lim_{\epsilon\to 0} u_\epsilon = 0$ *in* $C^2_{loc}(\overline{B}\backslash\{0\})$.
6. $\lim_{\epsilon\to 0} {u_\epsilon}^{p+1-\epsilon} = \frac{\omega_N}{2^N}\delta_0$ *in the sense of distributions.*

In particular, if $x_\epsilon \in B$ *is such that* $u_\epsilon(x_\epsilon) = ||u_\epsilon||_{L^\infty(B)}$, *then* $\lim_{\epsilon\to 0} x_\epsilon = 0$.

Now we let $\mu_\epsilon^{-\frac{N-2}{2}} = ||u_\epsilon||_{L^\infty(B)}$, and, for $x \in B_\epsilon$, we set

$$V_\epsilon(x) = \mu_\epsilon^{\frac{N-2}{2}} u_\epsilon(x_\epsilon + k_\epsilon x)$$

where $k_\epsilon = \mu_\epsilon^{1-\frac{N-2}{4}\epsilon}$ and $B_\epsilon = B(\frac{-x_\epsilon}{k_\epsilon}, \frac{1}{k_\epsilon})$. Clearly, $0 \leq V_\epsilon \leq 1$, $V_\epsilon(0) = 1$ and $\cup B_\epsilon = \mathbb{R}^N$. Moreover, V_ϵ is such that

$$\Delta V_\epsilon(x) + k_\epsilon^2 a(x_\epsilon + k_\epsilon x)V_\epsilon(x) = N(N-2) f(x_\epsilon + k_\epsilon x) V_\epsilon(x)^{p-\epsilon}, \qquad (2)$$

where $x \in B_\epsilon$ and $a(x) = a(|x|)$, $f(x) = f(|x|)$. By standard elliptic theory (see [8], Theorem 3.9), ∇V_ϵ is uniformly bounded on any compact subset of $\mathbb{R}^N$. Together with Ascoli's theorem, it follows that the V_ϵ's converge in C^0 to a function v on any compact subset. From standard elliptic theory (see for instance [8]), the convergence is C^2 (on every compact subset), and

$$\begin{cases} \Delta v = N(N-2)v^p \quad \text{in } \mathbb{R}^N \\ 0 \leq v \leq 1\ ,\ v(0) = 1. \end{cases}$$

By Caffarelli, Gidas and Spruck [4], it follows that $v(x) = \left(\frac{1}{1+|x|^2}\right)^{\frac{N-2}{2}}$. Then we have the following result:

Lemma 2.2. *The two following properties hold:*

1. $\lim_{\epsilon\to 0} V_\epsilon = v$ *in* $L^{p+1}\left(\mathbb{R}^N\right)$.
2. $\lim_{\epsilon\to 0} \mu_\epsilon^\epsilon = 1$,

where μ_ϵ, V_ϵ, *and* v *are as above, and* V_ϵ *is extended by* 0 *outside* B_ϵ.

Proof. We first remark that

$$\int_{B_\epsilon} |\nabla V_\epsilon|^2\,dx = (\mu_\epsilon^\epsilon)^{\left(\frac{N-2}{2}\right)^2} \int_{B_\epsilon} |\nabla u_\epsilon|^2\,dx.$$

Let μ be the limit of a subsequence of the μ_ϵ^ϵ's. Then $0 \le \mu \le 1$, while

$$\int_{\mathbb{R}^N} |\nabla(V_\epsilon - v)|^2\,dx = \int_{B_\epsilon} |\nabla V_\epsilon|^2\,dx + \int_{B_\epsilon} |\nabla v|^2\,dx - 2\int_{B_\epsilon} \nabla V_\epsilon \nabla v\,dx$$

and

$$\int_{B_\epsilon} \nabla V_\epsilon \nabla v\,dx = \int_{B_\epsilon} V_\epsilon \Delta v\,dx = N(N-2)\int_{B_\epsilon} V_\epsilon v^p\,dx,$$

where

$$\nabla V_\epsilon \nabla v = \sum_{i=1}^N \frac{\partial V_\epsilon}{\partial x_i}\frac{\partial v}{\partial x_i}.$$

Since $0 \le V_\epsilon \le 1$, we have that $0 \le V_\epsilon v^p \le v^p$, and by Lebesgue's dominated convergence theorem

$$\int_{B_\epsilon} \nabla V_\epsilon \nabla v\,dx \to N(N-2)\int_{\mathbb{R}^N} v^{p+1}\,dx = \int_{\mathbb{R}^N} \Delta v\, v\,dx = \int_{\mathbb{R}^N} |\nabla v|^2\,dx.$$

As one easily checks,

$$\int_{\mathbb{R}^N} |\nabla v|^2\,dx = N(N-2)\int_{\mathbb{R}^N} v^{p+1}dx = \frac{N(N-2)\omega_N}{2^N}.$$

Independently,

$$\lim_{\epsilon\to 0}\int_B \left(|\nabla u_\epsilon|^2 + a{u_\epsilon}^2\right)dx = \frac{N(N-2)\omega_N}{2^N},$$

and since $2 < p+1$,

$$\int_B |\nabla u_\epsilon|^2 dx \to \frac{N(N-2)\omega_N}{2^N}.$$

Then,

$$\int_{\mathbb{R}^N} |\nabla(V_\epsilon - v)|^2 dx \to \left(\mu^{\left(\frac{N-2}{2}\right)^2} - 1\right)\frac{N(N-2)\omega_N}{2^N} \le 0$$

so that $\mu_\epsilon^\epsilon \to 1$ and

$$\int_{\mathbb{R}^N} |\nabla(V_\epsilon - v)|^2 dx \to 0.$$

The convergence of V_ϵ to v in $L^{p+1}(\mathbb{R}^N)$ follows from the standard Sobolev inequality

$$\Big(\int_{\mathbb{R}^N} |V_\epsilon - v|^{p+1}\, dx\Big)^{\frac{2}{p+1}} \le K(N,2)^2 \int_{\mathbb{R}^N} |\nabla(V_\epsilon - v)|^2 dx.$$

This ends the proof of the lemma. □

3. An asymptotic estimate

As in Section 2, we assume that the u_ϵ's develop a concentration. Our main goal here is to establish the following fundamental estimate:

Proposition 1. *There exists $\gamma = \gamma(N)$, $\gamma > 0$ depending only on N, such that if the negative part a_- of a is such that $||a_-||_{L^{\frac{N}{2}}(B)} < \gamma$, then for all x in B, and up to a subsequence,*

$$u_\epsilon(x) \le A\Big(\frac{\mu_\epsilon}{{\mu_\epsilon}^2 + |x - x_\epsilon|^2}\Big)^{\frac{N-2}{2}}, \qquad (3)$$

where $A > 0$ is a constant independent of x and ϵ.

Such an estimate was obtained by Han [9] and Hebey [10] when $a \equiv 0$. As already mentioned, the linear part au, and more precisely the negative part a_-u of au, makes us have to face a much more critical situation. Several steps that we detail in this section are involved in the proof of this result.

3.1. **A first estimate.** As a first step in the proof of the proposition, we prove the following:

Lemma 3.1. *Given (c_ϵ) a sequence of real numbers which has a limit as $\epsilon \to 0$,*

$$|x - x_\epsilon|^{\frac{N-2}{2} + c_\epsilon \epsilon} u_\epsilon(x) \le A \qquad (4)$$

for all $\epsilon > 0$, and all $x \in B$, where $A > 0$ is a constant which does not depend on ϵ and x.

Proof. We use arguments that were developed by Druet [6]. For $x \in B$, we set $w_\epsilon(x) = |x - x_\epsilon|^{\frac{N-2}{2} + c_\epsilon \epsilon} u_\epsilon(x)$ and let y_ϵ be a point such that $w_\epsilon(y_\epsilon) = ||w_\epsilon||_\infty$. We assume for the sake of contradiction that $w_\epsilon(y_\epsilon) \to \infty$. Then $y_\epsilon \to 0$. We write

$$w_\epsilon(y_\epsilon) = |y_\epsilon - x_\epsilon|^{\frac{N-2}{2} + c_\epsilon \epsilon} u_\epsilon(y_\epsilon) \le |y_\epsilon - x_\epsilon|^{\frac{N-2}{2} + c_\epsilon \epsilon} u_\epsilon(x_\epsilon)$$

$$\le |y_\epsilon - x_\epsilon|^{\frac{N-2}{2}+c_\epsilon\epsilon} \mu_\epsilon^{-\left(\frac{N-2}{2}+c_\epsilon\epsilon\right)} \mu_\epsilon^{c_\epsilon\epsilon}.$$

It follows that $\frac{|y_\epsilon - x_\epsilon|}{\mu_\epsilon} \to +\infty$. Let $k'_\epsilon = u_\epsilon(y_\epsilon)^{-\frac{2}{N-2}+\frac{\epsilon}{2}}$. Since $u_\epsilon(y_\epsilon) \to +\infty$, we get that $k'_\epsilon \to 0$. For $x \in B(-\frac{y_\epsilon}{k'_\epsilon}, \frac{1}{k'_\epsilon})$, we set $\overline{u}_\epsilon(x) = u_\epsilon(y_\epsilon)^{-1} u_\epsilon(y_\epsilon + k'_\epsilon x)$. As one easily checks,

$$\Delta \overline{u}_\epsilon(x) + {k'_\epsilon}^2 a(y_\epsilon + k'_\epsilon x)\overline{u}_\epsilon(x) = N(N-2) f(y_\epsilon + k'_\epsilon x)\bar{u}_\epsilon(x)^{p-\epsilon}$$

for all $x \in B(-\frac{y_\epsilon}{k'_\epsilon}, \frac{1}{k'_\epsilon})$. For ϵ small, $1 \le u_\epsilon(y_\epsilon) \le \mu_\epsilon^{-\frac{N-2}{2}}$, and then $u_\epsilon(y_\epsilon)^\epsilon \to 1$. Now, take $x \in B(0,2)$. For ϵ sufficiently small, $B(0,2) \subset B(-\frac{y_\epsilon}{k'_\epsilon}, \frac{1}{k'_\epsilon})$, and

$$|x_\epsilon - y_\epsilon - k'_\epsilon x| \ge |x_\epsilon - y_\epsilon| - |k'_\epsilon x| \ge |x_\epsilon - y_\epsilon|\big(1 - 2\frac{k'_\epsilon}{|x_\epsilon - y_\epsilon|}\big) \ge \frac{1}{2}|x_\epsilon - y_\epsilon|$$

since $\frac{k'_\epsilon}{|x_\epsilon - y_\epsilon|} \to 0$. Taking $x \in B(0,2)$,

$$\begin{aligned} u_\epsilon(y_\epsilon + k'_\epsilon x) &= \frac{w_\epsilon(y_\epsilon + k'_\epsilon x)}{|x_\epsilon - y_\epsilon - k'_\epsilon x|^{\frac{N-2}{2}+c_\epsilon\epsilon}} \\ &\le 2^{\frac{N-2}{2}+c_\epsilon\epsilon} \frac{w_\epsilon(y_\epsilon)}{|x_\epsilon - y_\epsilon|^{\frac{N-2}{2}+c_\epsilon\epsilon}} = 2^{\frac{N-2}{2}+c_\epsilon\epsilon} u_\epsilon(y_\epsilon). \end{aligned}$$

As a consequence, $\bar{u}_\epsilon(x) \le 2^{\frac{N}{2}}$ for ϵ small and all $x \in B(0,2)$. Independently,

$$\int_{B(0,2)} \bar{u}_\epsilon^{p+1} dx = u_\epsilon(y_\epsilon)^{-\epsilon\frac{N}{2}} \int_{B(y_\epsilon, 2k'_\epsilon)} u_\epsilon^{p+1} dx$$

while $B(y_\epsilon, 2k'_\epsilon) \cap B(x_\epsilon, R\mu_\epsilon) = \emptyset$ for all $R > 0$, as soon as ϵ is small enough. From Lemma 2.2, we easily get that

$$\int_{B(x_\epsilon, R\mu_\epsilon)^c} u_\epsilon^{p+1} dx \to \int_{B(0,R)^c} v^{p+1} dx.$$

It follows that for all $R > 0$,

$$\limsup \int_{B(0,2)} \bar{u}_\epsilon^{p+1} dx \le \int_{B(0,R)^c} v^{p+1} dx,$$

and then $\int_{B(0,2)} \bar{u}_\epsilon^{p+1} dx \to 0$. In other words, $\bar{u}_\epsilon \to 0$ in $L^{p+1}(B(0,2))$, and $(\bar{u}_\epsilon)$ is bounded. Coming back to the equation satisfied by $\bar{u}_\epsilon$, and by standard elliptic theory, it follows that $\bar{u}_\epsilon \to 0$ in $C^0(B(0,1))$, a contradiction with the relation $\bar{u}_\epsilon(0) = 1$. The lemma is proved. □

Note that one of the consequences of Lemma 3.1 is that $V_\epsilon(x) \le A|x|^{-\frac{N-2}{2}}$ for all $x \in B_\epsilon \backslash \{0\}$.

3.2. **An estimate for x_ϵ.** We prove in this subsection the following result:

Lemma 3.2. $|x_\epsilon| = o(k_\epsilon)$.

Proof. Since u_ϵ is radially symmetrical, $\int_B x^i {u_\epsilon}^k dx = 0$ for all $i = 1, \dots, N$ and all $k \in \mathbb{N}$. Noting that

$$\int_B x^i {u_\epsilon}^k dx = \frac{k_\epsilon^N}{{\mu_\epsilon}^{k\frac{N-2}{2}}} \int_{B_\epsilon} (x_\epsilon^i + k_\epsilon z^i) V_\epsilon^k dz$$

this leads to

$$\frac{x_\epsilon^i}{k_\epsilon} \int_{B_\epsilon} {V_\epsilon}^k dz + \int_{B_\epsilon} z^i {V_\epsilon}^k dz = 0.$$

By Lemma 3.1, $V_\epsilon(x) \le A|x|^{-\frac{N-2}{2}}$ for all $x \in B_\epsilon \backslash \{0\}$. Choosing k such that $k > \frac{2(N+1)}{N-2}$, and since v is radially symmetrical, we get with Lebesgue's dominated convergence theorem that

$$\int_{B_\epsilon} {V_\epsilon}^k dz \to \int_{\mathbb{R}^N} v^k dz > 0, \qquad \int_{B_\epsilon} z^i {V_\epsilon}^k dz \to \int_{\mathbb{R}^N} z^i v^k dz = 0.$$

It follows that $x_\epsilon^i = o(k_\epsilon)$ for all i, a relation from which the lemma easily follows. □

3.3. **A second estimate.** We let v_ϵ be defined by $v_\epsilon(x) = \mu_\epsilon^{\frac{N-2}{2}} u_\epsilon(k_\epsilon x)$. Clearly, v_ϵ is radially symmetrical. A priori, and contrary to V_ϵ, $v_\epsilon(0)$ does not equal 1. On the other hand, writing $v_\epsilon(x) = V_\epsilon(x - \frac{x_\epsilon}{k_\epsilon})$, and according to Lemma 3.2, we see that $v_\epsilon(0) \to 1$. In particular, this proves the third part of Theorem 2:

Lemma 3.3. $\lim_{\epsilon \to 0} \frac{u_\epsilon(0)}{||u_\epsilon||_\infty} = 1$.

More generally, $v_\epsilon \to v$ in $C^2(K)$ for all compact K in $\mathbb{R}^N$, where v_ϵ is extended by 0 outside $B(0, \frac{1}{k_\epsilon})$. Moreover, v_ϵ satisfies in $B(0, \frac{1}{k_\epsilon})$ the equation

$$\Delta v_\epsilon + k_\epsilon^2 a(k_\epsilon x) v_\epsilon = N(N-2) f(k_\epsilon x) v_\epsilon^{p-\epsilon}.$$

As is easily seen, V_ϵ has the same properties as v_ϵ. In particular, $v_\epsilon(x) \le A|x|^{-\frac{N-2}{2}}$ for all x in $B(0, \frac{1}{k_\epsilon}) \backslash \{0\}$. We prove here the following result:

Lemma 3.4. *Let $\nu > 0$ be such that $\nu < N-2$. There exists a positive constant $\gamma = \gamma(N, \nu)$ depending only on N and ν, and there exists a positive constant A which does not depend on ϵ, such that if $||a_-||_{L^{\frac{N}{2}}(B)} < \gamma$, then*

$$v_\epsilon(x) \le \frac{A}{|x|^{N-2-\nu}}$$

for all $x \in B(0, \frac{1}{k_\epsilon})\backslash\{0\}$, *and all* $\epsilon > 0$.

Proof. We let ϕ be the map

$$\begin{array}{rcl} \phi : \mathbb{R}^N\backslash\{0\} & \to & \mathbb{R}^N\backslash\{0\} \\ x & \mapsto & \frac{x}{|x|^2} \end{array}$$

and we let w_ϵ be the Kelvin transform of v_ϵ, given by

$$w_\epsilon(x) = \begin{cases} \frac{1}{|x|^{N-2}} v_\epsilon(\frac{x}{|x|^2}) & \text{if } \phi(x) \in B(0, \frac{1}{k_\epsilon}) \\ 0 & \text{otherwise .} \end{cases}$$

We set $C_\epsilon = \phi(B(0, \frac{1}{k_\epsilon})) = \mathbb{R}^N\backslash\overline{B}(0, k_\epsilon)$. As one easily checks, w_ϵ satisfies in C_ϵ the equation

$$\Delta w_\epsilon(x) + A_\epsilon(x) w_\epsilon(x) = f_\epsilon(x) w_\epsilon(x)^{p-\epsilon} \tag{5}$$

where $A_\epsilon(x) = \frac{k_\epsilon^2 a(\frac{k_\epsilon x}{|x|^2})}{|x|^4}$ and $f_\epsilon(x) = \frac{N(N-2)}{|x|^{(N-2)\epsilon}} f(\frac{k_\epsilon x}{|x|^2})$. In particular, according to Lemma 2.2, f_ϵ is uniformly bounded. We define $\Omega = B(0, \delta)$, where $\delta > 0$ will be chosen later, and we extend w_ϵ by 0 in $B(0, k_\epsilon)$. For $t \geq 2$,

$$\int_\Omega \Delta w_\epsilon {w_\epsilon}^{t-1} dx + \int_\Omega A_\epsilon {w_\epsilon}^t dx = \int_\Omega f_\epsilon w_\epsilon^{p+t-1-\epsilon} dx.$$

Since w_ϵ equals 0 on the boundary of C_ϵ, an integration by parts gives

$$\int_\Omega \Delta w_\epsilon {w_\epsilon}^{t-1} dx = \int_\Omega \nabla w_\epsilon \nabla w_\epsilon^{t-1} dx - \int_{\partial B(0,\delta)} \frac{\partial w_\epsilon}{\partial n} w_\epsilon^{t-1} d\sigma.$$

The second term in the right-hand side of this relation is bounded for $\delta > 0$ fixed. It follows that

$$\frac{4(t-1)}{t^2} \int_\Omega \left|\nabla w_\epsilon^{\frac{t}{2}}\right|^2 dx + \int_\Omega A_\epsilon w_\epsilon^t \, dx = \int_\Omega f_\epsilon w_\epsilon^{p+t-1-\epsilon} \, dx + O(1).$$

By the standard Sobolev inequality (see for instance [12]),

$$\Big(\int_\Omega {w_\epsilon}^{\frac{p+1}{2}t} \, dx\Big)^{\frac{2}{p+1}} \leq A_1 \int_\Omega \left|\nabla w_\epsilon^{\frac{t}{2}}\right|^2 dx + A_2 \int_\Omega w_\epsilon^t \, dx,$$

where A_1 depends only on N and $A_2 = A_2(\delta)$ depends only on N and δ. Here, we just need to take $A_1 > 2^{2/N} K(N, 2)$ in order to get the existence of A_2. Independently, by Hölder's inequality,

$$-\int_\Omega A_\epsilon w_\epsilon^t \, dx \leq ||A_\epsilon^-||_{L^{\frac{N}{2}}(\Omega\backslash B(0,k_\epsilon))} ||w_\epsilon||^t_{L^{t\frac{p+1}{2}}(\Omega)},$$

where A_ϵ^- denotes the negative part of A_ϵ. In the same way,

$$\int_\Omega f_\epsilon w_\epsilon^{p+t-1-\epsilon} dx \leq ||f_\epsilon||_\infty ||w_\epsilon||^{p-1-\epsilon}_{L^{p+1}(\Omega)} Vol(\Omega)^{\frac{\epsilon}{p+1}} ||w_\epsilon||^t_{L^{t\frac{p+1}{2}}(\Omega)}$$

while

$$\frac{1}{A_1}\Big(\int_\Omega w_\epsilon^{\frac{p+1}{2}t}dx\Big)^{\frac{2}{p+1}} - \frac{A_2}{A_1}\int_\Omega w_\epsilon^t dx \le \int_\Omega \big|\nabla w_\epsilon^{\frac{t}{2}}\big|^2 dx.$$

Defining $\varphi(t) = \frac{t^2}{4(t-1)}$, it follows that

$$\Big[\frac{1}{A_1} - \varphi(t)||A_\epsilon^-||_{L^{\frac{N}{2}}(\Omega - B(0,k_\epsilon))}\Big]||w_\epsilon||^t_{L^{t\frac{p+1}{2}}(\Omega)}$$
$$\le \frac{A_2}{A_1}\int_\Omega {w_\epsilon}^t dx + \varphi(t)||f_\epsilon||_\infty ||w_\epsilon||^{p-1-\epsilon}_{L^{p+1}(\Omega)} Vol(\Omega)^{\frac{\epsilon}{p+1}}||w_\epsilon||^t_{L^{t\frac{p+1}{2}}(\Omega)} + O\left(\varphi(t)\right).$$

As is easily seen,

$$\int_\Omega w_\epsilon^{p+1}(x)\, dx \le \int_{|x|\ge\frac{1}{\delta}} v_\epsilon^{p+1}(x)\, dx \le \int_{|x|\ge\frac{1}{2\delta}} V_\epsilon^{p+1}(x)\, dx.$$

Then, with Lemma 2.2 we obtain that for all $\eta > 0$, there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$, and all $\epsilon > 0$, $||w_\epsilon||_{L^{p+1}(\Omega)} < \eta$. Now, let $q > 2$ be given. In what follows, we assume that

$$||A_\epsilon^-||_{L^{\frac{N}{2}}(\Omega\setminus B(0,k_\epsilon))} \le \frac{1}{2A_1\varphi(q)} \tag{6}$$

and we choose $\delta > 0$ sufficiently small such that

$$\varphi(q)||f_\epsilon||_\infty||w_\epsilon||^{p-1-\epsilon}_{L^{p+1}(\Omega)} Vol(\Omega)^{\frac{\epsilon}{p+1}} \le \frac{1}{4A_1}.$$

Since the map $t \mapsto \varphi(t)$ is increasing on $[2, +\infty)$, there exists a constant $K > 0$ such that for all $2 \le t \le q$,

$$\frac{1}{4A_1}||w_\epsilon||^t_{L^{t\frac{p+1}{2}}(\Omega)} \le \frac{A_2}{A_1}||w_\epsilon||^t_{L^t(\Omega)} + K\varphi(t).$$

Since $||w_\epsilon||_{L^{p+1}(\Omega)}$ is bounded, it follows by induction that $||w_\epsilon||_{L^q(\Omega)} = O(1)$, and $||w_\epsilon||_{L^q(\Omega)}$ is bounded. Actually, w_ϵ is even bounded in $L^{s_k}(\Omega)$ where $s_k = (p+1)^{k+1}/2^k$ and k is the smallest k for which $s_k \ge q$. We now borrow ideas from Zheng-Chao Han (personal communication). We let $D \subset B(0, \delta)$ be an open subset of $\mathbb{R}^N$. Then

$$\int_D w_\epsilon^q(x)\, dx = \int_{\phi(D)} |x|^{(N-2)q-2N} v_\epsilon^q(x)\, dx.$$

We set $D = \phi\left(B(x, 1)\right)$ where x is such that $|x| > 1+\frac{1}{\delta}$. Clearly, $D \subset B(0, \delta)$, and

$$\int_\Omega w_\epsilon^q(x)\, dx \ge \int_D w_\epsilon^q(y)\, dy = \int_{B(x,1)} |y|^{(N-2)q-2N} v_\epsilon^q(y)\, dy$$

$$\geq (|x|-1)^{(N-2)q-2N} \int_{B(x,1)} v_\epsilon^q(y)\,dy.$$

It follows that for x such that $|x| > 1 + \frac{1}{\delta}$,

$$||v_\epsilon||_{L^q(B(x,1))} \leq \frac{A}{|x|^{N-2-\frac{2N}{q}}}\ ,$$

where $A > 0$ does not depend on ϵ. Let L be the operator

$$Lu = \Delta u + k_\epsilon^2 a(k_\epsilon x)u - N(N-2)f(k_\epsilon x)v_\epsilon^{p-1-\epsilon}u.$$

Since $Lv_\epsilon = 0$, we can apply the Harnack inequality to v_ϵ, as is stated for example in [8] (Theorem 8.20 and Corollary 8.21). Since the coefficients of L are bounded, it follows that

$$v_\epsilon(x) \leq \frac{A}{|x|^{N-2-\frac{2N}{q}}}$$

for all x such that $|x| > 1 + \frac{1}{\delta}$. Taking $\nu = \frac{2N}{q}$, $q \gg 1$, and since v_ϵ is bounded, we get the desired inequality, that of Lemma 3.4. The proof then reduces to the proof of (6). To obtain (6), we note that

$$\int_{\Omega\setminus B(0,k_\epsilon)} |A_\epsilon^-(x)|^{\frac{N}{2}}\,dx \leq \int_{\phi\left(B(0,\frac{1}{k_\epsilon})\right)} |A_\epsilon^-(x)|^{\frac{N}{2}}\,dx$$

$$= k_\epsilon^N \int_{B(0,\frac{1}{k_\epsilon})} |a_-(k_\epsilon x)|^{\frac{N}{2}}\,dx = \int_B |a_-(x)|^{\frac{N}{2}}\,dx.$$

Then,

$$||A_\epsilon^-||_{L^{\frac{N}{2}}(\Omega\setminus B(0,k_\epsilon))} \leq ||a_-||_{L^{\frac{N}{2}}(B)},$$

and if $||a_-||_{L^{\frac{N}{2}}(B)} < \frac{(2N-\nu)\nu}{2N^2A_1}$, where $\nu = \frac{2N}{q}$, we get (6). This ends the proof of the lemma. □

Concerning Lemma 3.4, note that if $\nu < \frac{2}{p}$, then $(p-\epsilon)(N-2-\nu) > N$ for $\epsilon \ll 1$. It follows that there exists $\gamma = \gamma(N)$, $\gamma > 0$ depending only on N, such that if $||a_-||_{L^{\frac{N}{2}}(B)} < \gamma$, then $||v_\epsilon||_{L^{p-\epsilon}(\mathbb{R}^N)} \leq A$ where A does not depend on ϵ.

3.4. **Proof of Proposition** 1. As one may easily check, the estimate (3) is equivalent to the existence of a constant A such that for all $\epsilon > 0$ and all $x \in B$,

$$|x|^{N-2}u_\epsilon(x_\epsilon)u_\epsilon(x) \leq A. \tag{7}$$

(Here, we use the fact that $x_\epsilon = o(k_\epsilon)$.) Let $y_\epsilon \in B$ be a point where $x \mapsto |x|^{N-2}u_\epsilon(x)$ achieves its maximum. In order to prove (7), we assume for the sake of contradiction that $|x|^{N-2}u_\epsilon(x_\epsilon)u_\epsilon(x)$ is unbounded. Up to a subsequence, we get that

$$|y_\epsilon|^{N-2}u_\epsilon(x_\epsilon)u_\epsilon(y_\epsilon) \to +\infty. \tag{8}$$

Without loss of generality, up to another subsequence, we can assume that $y_\epsilon \to y_0$ in $\overline{B}$. As a first remark, we claim that $|y_0| < 1$. For this purpose, let $z_\epsilon(x) = \frac{u_\epsilon(x)}{u_\epsilon(y_\epsilon)}$. The equation satisfied in B by z_ϵ is

$$\Delta z_\epsilon + a(x)z_\epsilon = N(N-2)f(x)u_\epsilon(y_\epsilon)^{p-1-\epsilon}z_\epsilon^{p-\epsilon},$$

and z_ϵ is radially symmetrical. Since $|x|^{N-2}u_\epsilon(x)$ achieves its maximum at $x = y_\epsilon$, we get that

$$z_\epsilon(x) \le \frac{|y_\epsilon|^{N-2}}{|x|^{N-2}}$$

and z_ϵ is bounded on any compact subset of $\overline{B}\backslash\{0\}$. By standard elliptic theory (see for instance [8]), it follows that (z_ϵ) is actually $C^{1,\alpha}$-bounded in any compact subset of $\overline{B}\backslash\{0\}$. In particular, if $y_0 \in \partial B$, and since $z_\epsilon = 0$ on ∂B,

$$|z_\epsilon(y_\epsilon)| = |z_\epsilon(y_\epsilon) - z_\epsilon(y_0)| \le A|y_\epsilon - y_0|$$

where $A > 0$ does not depend on ϵ. But $z_\epsilon(y_\epsilon) = 1$, and hence $|y_0| < 1$. This proves the above claim.

Now we set $y_\epsilon = k_\epsilon \hat{x}_\epsilon$. As another remark, we claim that $|\hat{x}_\epsilon| \to +\infty$. If not, then, up to another subsequence,

$$\begin{aligned} |y_\epsilon|^{N-2}u_\epsilon(x_\epsilon)u_\epsilon(y_\epsilon) &= k_\epsilon^{N-2}|\hat{x}_\epsilon|^{N-2}\mu_\epsilon^{-\frac{N-2}{2}}u_\epsilon(k_\epsilon\hat{x}_\epsilon) \\ &\approx |\hat{x}_\epsilon|^{N-2}\mu_\epsilon^{\frac{N-2}{2}}u_\epsilon(k_\epsilon\hat{x}_\epsilon) = |\hat{x}_\epsilon|^{N-2}v_\epsilon(\hat{x}_\epsilon), \end{aligned}$$

which is bounded since v_ϵ uniformly converges on any compact subset of $\mathbb{R}^N$. This proves the claim.

Now, let G be the Green's function for the operator $\Delta + a$, as defined in the Introduction. In addition to being radially symmetrical, one of its classical properties is that for all compact subsets $K \subset B$, there exists a constant $A > 0$ such that for all $x \in K$ and all $y \in B$,

$$|y - x|^{N-2}G(x, y) \le A.$$

Then, we write

$$u_\epsilon(y_\epsilon) = \int_B G(y_\epsilon, \tilde{x})\left(\Delta u_\epsilon(\tilde{x}) + a(\tilde{x})u_\epsilon(\tilde{x})\right) d\tilde{x}.$$

From the equation satisfied by u_ϵ, the equivalence of k_ϵ and μ_ϵ, and the change of variable $\tilde{x} = k_\epsilon x$, it follows that

$$u_\epsilon(y_\epsilon) \approx N(N-2)\mu_\epsilon^{\frac{N-2}{2}} \int_{B(0,\frac{1}{k_\epsilon})} f(k_\epsilon x) v_\epsilon^{p-\epsilon}(x) G(y_\epsilon, k_\epsilon x)\, dx$$

and then that

$$u_\epsilon(x_\epsilon)u_\epsilon(y_\epsilon) \le A \int_{B(0,\frac{1}{k_\epsilon})} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x)\, dx,$$

where A does not depend on ϵ. Let us now define

$$\Omega_\epsilon^1 = \Big\{x \in B\big(0, \frac{1}{k_\epsilon}\big) : |y_\epsilon - k_\epsilon x| \ge \frac{1}{2}|y_\epsilon|\Big\}$$
$$\Omega_\epsilon^2 = \Big\{x \in B\big(0, \frac{1}{k_\epsilon}\big) : |y_\epsilon - k_\epsilon x| < \frac{1}{2}|y_\epsilon|\Big\}.$$

We write

$$\int_{B(0,\frac{1}{k_\epsilon})} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x) dx = \int_{\Omega_\epsilon^1} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x) dx + \int_{\Omega_\epsilon^2} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x) dx.$$

According to the above-mentioned property of the Green's function, and since $|y_0| < 1$ so that the y_ϵ's are in a compact subset of B,

$$\int_{\Omega_\epsilon^1} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x) dx \le A \int_{\Omega_\epsilon^1} \frac{v_\epsilon^{p-\epsilon}(x)}{|y_\epsilon - k_\epsilon x|^{N-2}} dx \le \frac{2^{N-2}}{|y_\epsilon|^{N-2}} A \int_{B(0,\frac{1}{k_\epsilon})} v_\epsilon^{p-\epsilon}(x) dx.$$

Together with the remark we made at the end of Subsection 3.3, and under the assumption that $||a_-||_{L^{\frac{N}{2}}(B)} < \gamma$, where $\gamma > 0$ depends only on N and is as in this remark, we get that

$$\int_{\Omega_\epsilon^1} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x)\, dx \le \frac{A}{|y_\epsilon|^{N-2}}.$$

Similarly,

$$\int_{\Omega_\epsilon^2} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x)\, dx \le A \int_{\Omega_\epsilon^2} \frac{v_\epsilon^{p-\epsilon}(x)}{|y_\epsilon - k_\epsilon x|^{N-2}}\, dx,$$

and if $\Omega_\epsilon = \big\{x : |x| < \frac{1}{2}|y_\epsilon|\big\}$, then, with the change of variable $y = k_\epsilon x - y_\epsilon$,

$$\int_{\Omega_\epsilon^2} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x)\, dx \le \frac{A}{k_\epsilon^N} \int_{\Omega_\epsilon} \frac{1}{|y|^{N-2}} v_\epsilon^{p-\epsilon}\big(\frac{y + y_\epsilon}{k_\epsilon}\big)\, dy.$$

Since $|\frac{y+y_\epsilon}{k_\epsilon}| \ge \frac{1}{2}|\hat{x}_\epsilon|$, and by Lemma 3.4,

$$\frac{1}{k_\epsilon^N} \int_{\Omega_\epsilon} \frac{1}{|y|^{N-2}} v_\epsilon^{p-\epsilon}\big(\frac{y + y_\epsilon}{k_\epsilon}\big)\, dy \le \frac{A}{|\hat{x}_\epsilon|^{(N-2-\nu)(p-\epsilon)} k_\epsilon^N} \int_{\Omega_\epsilon} \frac{1}{|y|^{N-2}}\, dy$$

$$\leq \frac{A}{|\hat{x}_\epsilon|^{(N-2-\nu)(p-\epsilon)} k_\epsilon^N} \int_0^{\frac{1}{2}|y_\epsilon|} t\, dt \leq \frac{A|y_\epsilon|^2}{|\hat{x}_\epsilon|^{(N-2-\nu)(p-\epsilon)} k_\epsilon^N}.$$

Since $k_\epsilon \leq |y_\epsilon| \leq 1$, we get with Lemma 2.2 that $|y_\epsilon|^\epsilon \to 1$. It follows that $|\hat{x}_\epsilon|^\epsilon \to 1$, and we can write that

$$\frac{|y_\epsilon|^2}{|\hat{x}_\epsilon|^{(N-2-\nu)(p-\epsilon)} k_\epsilon^N} \leq \frac{A}{|\hat{x}_\epsilon|^{2-p\nu}|y_\epsilon|^{N-2}} \ .$$

Choosing ν such that $\nu < \frac{2}{p}$ (this was done at the end of Section 3.3), we obtain that

$$\int_{\Omega_\epsilon^2} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x)\, dx \leq \frac{o(1)}{|y_\epsilon|^{N-2}}.$$

It follows that

$$\begin{aligned} |y_\epsilon|^{N-2} u_\epsilon(x_\epsilon) u_\epsilon(y_\epsilon) &\leq A|y_\epsilon|^{N-2} \int_{B(0,\frac{1}{k_\epsilon})} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x)\, dx \\ &\leq A|y_\epsilon|^{N-2} \int_{\Omega_\epsilon^1} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x)\, dx \\ &+ A|y_\epsilon|^{N-2} \int_{\Omega_\epsilon^2} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x)\, dx \leq A + o(1), \end{aligned}$$

which contradicts (8). It follows that (7) is true, and then (3) is also true. The proposition is proved. □

Now that Proposition 1 is proved, we go on with the study of the asymptotic behaviour of the u_ϵ's. This is the aim of the following section, where the first assertion in Theorem 3 is proved.

4. Convergence to the Green's function

Here again, we assume that the u_ϵ's develop a concentration. First, we recall a result obtained by Brézis and Peletier [3]:

Lemma 4.1. *Let u be a C^2 solution of*

$$\begin{cases} \Delta u = f & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

and let ω be a neighbourhood of ∂B. Then

$$||u||_{W^{1,q}(B)} + ||\nabla u||_{C^{0,\beta}(\omega')} \leq A(||f||_{L^1(B)} + ||f||_{L^\infty(\omega)})$$

for all $q < \frac{N}{N-1}$, all $0 < \beta < 1$, and all $\omega' \subset\subset \omega$.

Note that it follows from this result that

$$\int_{\partial B} |\nabla u_\epsilon|^2 d\sigma = O({\mu_\epsilon}^{N-2}).$$

By Lemma 4.1 we indeed just need to get estimates for the L^1 norm in B and the L^∞ norm in a neighbourhood of ∂B, of the function g_ϵ given by

$$g_\epsilon(x) = N(N-2)f(x)u_\epsilon(x)^{p-\epsilon} - a(x)u_\epsilon(x).$$

As is easily seen, these estimates follow from Proposition 1.

Now we prove the first assertion in Theorem 3. This is the aim of the following lemma where, as in the Introduction, G denotes the Green's function of the operator $\Delta + a$.

Lemma 4.2. $\lim_{\epsilon\to 0} u_\epsilon(x_\epsilon)u_\epsilon(x) = (N-2)\omega_{N-1}G(x,0)$ *in* $C^2_{loc}(\overline{B}\backslash\{0\})$.

Proof. Let K be a compact subset of $B\backslash\{0\}$, and $x \in K$. It follows from the equation satisfied by the u_ϵ's that

$$\begin{aligned} u_\epsilon(x) &= N(N-2)\int_B f(y)u_\epsilon^{p-\epsilon}(y)G(x,y)\,dy \\ &= N(N-2)\frac{{k_\epsilon}^N}{{\mu_\epsilon}^{(p-\epsilon)\frac{N-2}{2}}}\int_{\mathbb{R}^N} g_\epsilon(z)\,dz, \end{aligned}$$

where $g_\epsilon(z) = f(k_\epsilon z)v_\epsilon^{p-\epsilon}(z)G(x,k_\epsilon z)$. By classical properties of the Green's function, there exists a constant $A>0$ such that for all $x\in K$, and all $y \in B$, $G(x,y) \le A|x-y|^{-N+2}$. Dealing distinctly with the cases $|x-k_\epsilon z| \le \delta$ and $|x-k_\epsilon z| > \delta$, where $\delta > 0$ is such that for all $x \in K$, $|x| \ge 2\delta$, and, according to Proposition 1, we see that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^N} g_\epsilon(z)\,dz = f(0)G(x,0)\int_{\mathbb{R}^N} v^p(z)\,dz,$$

where the limit is uniform with respect to $x \in K$. As is easily checked,

$$\int_{\mathbb{R}^N} v^p(z)\,dz = \frac{\omega_{N-1}}{N}$$

and

$$\lim_{\epsilon\to 0} u_\epsilon(x_\epsilon)u_\epsilon(x) = (N-2)\omega_{N-1}G(x,0)$$

in $C^0_{loc}(B\backslash\{0\})$. The convergence in $C^0_{loc}(\overline{B}\backslash\{0\})$ then follows from Lemma 4.1 and the equation satisfied by $w_\epsilon = u_\epsilon(x_\epsilon)u_\epsilon$, that is,

$$\Delta w_\epsilon + a(x)w_\epsilon = N(N-2)f(x){\mu_\epsilon}^{\frac{N-2}{2}(\frac{4}{N-2}-\epsilon)}w_\epsilon^{p-\epsilon}.$$

The convergence in $C^2_{loc}(\overline{B}\backslash\{0\})$ is easily obtained by classical results of elliptic theory; see for instance [8]. The lemma is proved. □

5. Convergence to a solution

In this section, we consider a sequence of functions $(\tilde{u_\epsilon})$ such that

$$\begin{cases} \Delta\tilde{u_\epsilon} + a\tilde{u_\epsilon} = N(N-2)\lambda_\epsilon f(x)\tilde{u_\epsilon}^{p-\epsilon} \quad \text{in } B \\ \tilde{u_\epsilon} > 0 \ \text{ in } B \ \text{ and } \ \tilde{u_\epsilon} = 0 \ \text{ on } \partial B \\ N(N-2)\int_B f(x)\tilde{u_\epsilon}^{p+1-\epsilon}dx = 1, \end{cases}$$

where

$$\lambda_\epsilon = \inf_{v\in\mathcal{D}(B)_R\backslash\{0\}} \frac{\int_B \left(|\nabla v|^2 + av^2\right) dx}{\left(N(N-2)\int_B f|v|^{p+1-\epsilon}dx\right)^{\frac{2}{p+1-\epsilon}}}.$$

We set

$$\lambda = \inf_{v\in\mathcal{D}(B)_R\backslash\{0\}} \frac{\int_B \left(|\nabla v|^2 + av^2\right) dx}{\left(N(N-2)\int_B f|v|^{p+1}dx\right)^{\frac{2}{p+1}}}.$$

The following results are by now classical. We therefore restrict ourselves to brief comments on their proofs. For details, see for instance [10].

Lemma 5.1. $\lim_{\epsilon\to 0}\lambda_\epsilon = \lambda$.

Proof. Let $u \in \mathcal{D}(B)_R\backslash\{0\}$. By Hölder's inequality,

$$\begin{aligned} &\left(N(N-2)\int_B f|u|^{p+1-\epsilon}dx\right)^{\frac{2}{p+1-\epsilon}} \\ &\leq Vol(B)^{\frac{2\epsilon}{(p+1)(p+1-\epsilon)}}\left(N(N-2)\int_B f|u|^{p+1}dx\right)^{\frac{2}{p+1}}. \end{aligned}$$

It follows that $\lambda \leq \liminf_{\epsilon\to 0}\lambda_\epsilon$. Conversely, let $\alpha > 0$ be any positive real number, and let $u \in \mathcal{D}(B)_R\backslash\{0\}$ be such that

$$\frac{\int_B \left(|\nabla u|^2 + au^2\right) dx}{\left(N(N-2)\int_B f|u|^{p+1}dx\right)^{\frac{2}{p+1}}} < \lambda + \alpha.$$

Clearly, when $\epsilon \to 0$,

$$\frac{\int_B \left(|\nabla u|^2 + au^2\right) dx}{\left(N(N-2)\int_B f|u|^{p+1-\epsilon}dx\right)^{\frac{2}{p+1-\epsilon}}} \longrightarrow \frac{\int_B \left(|\nabla u|^2 + au^2\right) dx}{\left(N(N-2)\int_B f|u|^{p+1}dx\right)^{\frac{2}{p+1}}}.$$

We then obtain that $\limsup_{\epsilon\to 0}\lambda_\epsilon \leq \lambda + \alpha$. Since $\alpha > 0$ is arbitrary, the result follows. □

We now state the following result.

Lemma 5.2. *Assume that a subsequence of $(\tilde{u_\epsilon})$ converges almost everywhere to a function $\tilde{u} \neq 0$. Then*

1. *$\tilde{u}$ is an MRS solution of the problem*

$$(\star)\ \begin{cases} \Delta u + a(x)u = N(N-2)\lambda f(x)u^p \ \text{in } B \\ u > 0 \ \text{in } B\ ,\ \text{and } u = 0 \ \text{on } \partial B \end{cases}$$

2. $\lim_{\epsilon\to 0} \tilde{u_\epsilon} = \tilde{u}$ *in* $C^2(\overline{B})$.

Proof. Point 1 easily follows from classical arguments of variational theory, like the ones developed, for example, in the study of the Yamabe problem. We first prove that $\tilde{u}$ is a solution of $(\star)$, and then that $\tilde{u}$ is minimizing. Point 2 follows from classical arguments of elliptic theory. □

At last, we state the following result.

Lemma 5.3. *We always have* $\lambda \le \frac{1}{4}\left(N(N-2)\omega_N\right)^{\frac{2}{N}}$*, and if this inequality is strict, then up to a subsequence,* $\tilde{u_\epsilon}$ *converges almost everywhere to a function* $\tilde{u} \neq 0$*. Together with Lemma* 5.2*, the convergence is then* C^2*, and* $\tilde{u}$ *is an MRS solution of problem* $(\star)$*.*

Proof. Here again, the result follows from classical variational arguments. We obtain the first assertion with the function z_ϵ given by

$$z_\epsilon(x) = \frac{\phi(|x|)}{\left(\epsilon^2 + |x|^2\right)^{\frac{N-2}{2}}}\ ,$$

where ϕ is a cut-off function that equals 1 around 0. As $\epsilon \to 0$, we get indeed that

$$\frac{\int_B \left(|\nabla z_\epsilon|^2 + a z_\epsilon^2\right) dx}{\left(N(N-2)\int_B f|z_\epsilon|^{p+1}dx\right)^{\frac{2}{p+1}}} \longrightarrow \frac{\left(N(N-2)\omega_N\right)^{\frac{2}{N}}}{4}.$$

For the second assertion, the energy associated to the problem goes under the critical energy. The fact that the $\tilde{u_\epsilon}$'s do not develop a concentration under such an assumption is by now classical. □

6. Proof of the theorems

Theorem 2 immediately follows from what we said in Section 2, and from Lemma 3.3. The first assertion of Theorem 3 was proved in Section 4. Only Theorem 1 and points 1, 2 and 3 of Theorem 3 remain to be proved. Everything here comes from the estimate obtained in Proposition 1, and from the Pohozaev identity [14]. When applied to the functions u_ϵ, this identity gives

$$\underbrace{\frac{N(N-2)^2\epsilon}{2(p+1-\epsilon)}\int_B f(|x|)u_\epsilon^{p+1-\epsilon}dx}_{I_\epsilon}+\underbrace{\frac{N(N-2)}{p+1-\epsilon}\int_B |x|f'(|x|)u_\epsilon^{p+1-\epsilon}dx}_{II_\epsilon}$$
$$-\underbrace{\int_B\left(a(|x|)+\frac{1}{2}|x|a'(|x|)\right)u_\epsilon{}^2dx}_{III_\epsilon}=\underbrace{\frac{1}{2}\int_{\partial B}|\nabla u_\epsilon|^2d\sigma}_{IV_\epsilon}.$$

In what follows, we assume that the u_ϵ's develop a concentration. With the notation of Section 5, this gives that $\lambda=\frac{1}{4}\left(N(N-2)\omega_N\right)^{\frac{2}{N}}$. In particular, we recover the results of Sections 2, 3, and 4. We estimate in what follows the terms I_ϵ, II_ϵ, III_ϵ, and IV_ϵ of the Pohozaev identity.

The terms I_ϵ and IV_ϵ are the easiest to estimate. We straightforwardly obtain that

$$I_\epsilon=\frac{(N-2)^3\omega_N}{2^{N+2}}\left(1+o(1)\right)\epsilon,$$

and it follows from Lemma 4.2 that

$$IV_\epsilon=\frac{1}{2}(N-2)^2\omega_{N-1}^3g'(1)^2\mu_\epsilon^{N-2}+o\left(\mu_\epsilon^{N-2}\right),$$

where g is as in the Introduction.

Concerning the term II_ϵ, we write that $f'(r)=\frac{f^{(k_f)}(0)}{(k_f-1)!}r^{k_f-1}+O(r^{k_f})$. Then

$$\begin{aligned}&\int_B|x|f'(|x|)u_\epsilon^{p+1-\epsilon}dx\\&=\frac{f^{(k_f)}(0)}{(k_f-1)!}\int_B|x|^{k_f}u_\epsilon^{p+1-\epsilon}dx+O\Big(\int_B|x|^{k_f+1}u_\epsilon^{p+1-\epsilon}dx\Big)\\&=\frac{f^{(k_f)}(0)}{(k_f-1)!}(1+o(1))\mu_\epsilon^{k_f}\underbrace{\int_{B(0,\frac{1}{k_\epsilon})}|x|^{k_f}v_\epsilon^{p+1-\epsilon}dx}_{II_\epsilon^1}\\&+O\Big(\mu_\epsilon^{k_f+1}\underbrace{\int_{B(0,\frac{1}{k_\epsilon})}|x|^{k_f+1}v_\epsilon^{p+1-\epsilon}dx}_{II_\epsilon^2}\Big).\end{aligned}$$

If $k_f<N$, and together with Proposition 1, II_ϵ^1 converges by the dominated convergence theorem. This holds also for II_ϵ^2 if $k_f+1<N$. When $k_f=$

$N-1$, II_ϵ^2 diverges, but is bounded by $|\ln k_\epsilon|$. This leads to

$$\int_B |x| f'(|x|) u_\epsilon^{p+1-\epsilon} dx \approx \frac{f^{(k_f)}(0)}{(k_f-1)!}\ \mu_\epsilon^{k_f} \int_{\mathbb{R}^N} |x|^{k_f} v^{p+1} dx$$

as soon as $k_f < N$. In the same way,

$$\int_B |x| f'(|x|) u_\epsilon^{p+1-\epsilon} dx = O\left(\mu_\epsilon^N |\ln \mu_\epsilon|\right)$$

if $k_f = N$, and

$$\int_B |x| f'(|x|) u_\epsilon^{p+1-\epsilon} dx = O\left(\mu_\epsilon^N\right)$$

if $k_f > N$. Then,

$$II_\epsilon = \frac{(N-2)^2}{2}\ \frac{f^{(k_f)}(0)}{(k_f-1)!}\ \mu_\epsilon^{k_f} \int_{\mathbb{R}^N} |x|^{k_f} v^{p+1} dx + o\left(\mu_\epsilon^{k_f}\right)$$

if $k_f \le N-2$, while $II_\epsilon = o\left(\mu_\epsilon^{N-2}\right)$ if $k_f > N-2$.

We are finally concerned with the term III_ϵ. The study there is more intricate, and we separate the cases $k_a < N-4$, $k_a > N-4$, and $k_a = N-4$. We first write that

$$a(r) = \frac{a^{(k_a)}(0)}{k_a!} r^{k_a} + O(r^{k_a+1}), \quad a'(r) = \frac{a^{(k_a)}(0)}{(k_a-1)!} r^{k_a-1} + O(r^{k_a}).$$

If $k_a < N-4$, we obtain with the same kind of arguments as the ones used above that

$$III_\epsilon = \frac{a^{(k_a)}(0)}{k_a!}(1+\frac{k_a}{2})\mu_\epsilon^{k_a+2} \int_{\mathbb{R}^N} |x|^{k_a} v^2 dx + o(\mu_\epsilon^{k_a+2}).$$

Since $\mu_\epsilon^{-1} = (1+o(1))\, u_\epsilon(0)^{\frac{2}{N-2}}$ we get point 1 of Theorem 3 with what has been said before. If, for example, $k_a < N-4$ and $k_f < k_a+2$, multiplying the Pohozaev identity by $\mu_\epsilon^{-k_f}$, we obtain that

$$\frac{(N-2)\omega_N}{2^{N+1}}\left(\epsilon \mu_\epsilon^{-k_f}\right) + \frac{f^{(k_f)}(0)}{(k_f-1)!}\,(1+o(1)) \int_{\mathbb{R}^N} |x|^{k_f} v^{p+1} dx = 0,$$

which straightforwardly leads to point $1(a)$ of Theorem 3. The same arguments are valid for the points $1(b)$ and $1(c)$ of Theorem 3.

We now assume that $k_a > N-4$, and we let h be the function

$$h(x) = a(|x|) + \frac{1}{2}|x| a'(|x|).$$

There exists a constant $C > 0$ such that $|h(x)| \leq C|x|^{k_a}$. Let $\delta > 0$. We write that

$$\Big| \int_{B(0,\delta)} h(x) u_\epsilon^2(x_\epsilon) u_\epsilon^2(x)\, dx \Big| \leq A \int_{B(0,\delta)} \frac{|x|^{k_a}}{(\mu_\epsilon^2 + |x|^2)^{N-2}}\, dx$$
$$\leq A \int_0^\delta \frac{r^{k_a+N-1}}{(\mu_\epsilon^2 + r^2)^{N-2}}\, dr \leq A \mu_\epsilon^{k_a-(N-4)} \int_0^{\frac{\delta}{\mu_\epsilon}} \frac{s^{k_a+N-1}}{(1+s^2)^{N-2}}\, ds$$
$$\leq A \mu_\epsilon^{k_a-(N-4)} \Big(O(1) + \int_1^{\frac{\delta}{\mu_\epsilon}} s^{k_a-(N-4)-1}\, ds \Big) \leq A \Big(\delta^{k_a-(N-4)} + \mu_\epsilon^{k_a-(N-4)} \Big),$$

where A does not depend on ϵ and δ. Independently, $|x|^{N-2}|G(x,0)| \leq A$. It follows that for $k_a > N-4$, $|x|^{k_a} G(x,0)^2$ is integrable. We let

$$H_\delta(\epsilon) = \Big| \int_{B \setminus B(0,\delta)} h(x) u_\epsilon^2(x_\epsilon) u_\epsilon^2(x)\, dx - \int_{B\setminus B(0,\delta)} h(x) \left((N-2)\omega_{N-1} G(x,0)\right)^2 dx \Big|.$$

By Lemma 4.2, $H_\delta = o(1)$. We then write that

$$\Big| \int_B h(x) u_\epsilon^2(x_\epsilon) u_\epsilon^2(x) dx - \int_B h(x) \left((N-2)\omega_{N-1} G(x,0)\right)^2 dx \Big|$$
$$\leq \Big| \int_{B(0,\delta)} h(x) u_\epsilon^2(x_\epsilon) u_\epsilon^2(x) dx \Big| + \Big| \int_{B(0,\delta)} h(x) ((N-2)\omega_{N-1} G(x,0))^2 dx \Big| + H_\delta(\epsilon)$$
$$\leq A \Big| \int_{B(0,\delta)} |x|^{k_a} u_\epsilon^2(x_\epsilon) u_\epsilon^2(x) dx \Big| + A \Big| \int_{B(0,\delta)} |x|^{k_a} ((N-2)\omega_{N-1} G(x,0))^2 dx \Big| + H_\delta(\epsilon)$$
$$\leq A \delta^{k_a-(N-4)} + o(1).$$

Since $\delta > 0$ is arbitrary, it follows that

$$\frac{1}{\mu_\epsilon^{N-2}} \int_B \Big(a(|x|) + \frac{1}{2}|x| a'(|x|) \Big) u_\epsilon^2(x)\, dx$$
$$= (N-2)^2 \omega_{N-1}^2 \int_B \Big(a(|x|) + \frac{1}{2}|x| a'(|x|) \Big) G(x,0)^2\, dx + o(1)$$

and then that $III_\epsilon = (N-2)^2 \omega_{N-1}^3 \Phi(a) \mu_\epsilon^{N-2} + o\left(\mu_\epsilon^{N-2}\right)$. Multiplying the Pohozaev identity by $\mu_\epsilon^{-k_f}$, we then obtain the points $3(a)$ and $3(b)$ of Theorem 3. Point $3(c)$ is obtained similarly, multiplying now the Pohozaev identity by $\mu_\epsilon^{-(N-2)}$.

At last, we assume that $k_a = N-4$. By Proposition 1, we easily obtain that

$$\int_B \Big(a(|x|) + \frac{1}{2}|x| a'(|x|) \Big) u_\epsilon^2\, dx$$

$$= \frac{a^{(k_a)}(0)}{k_a!}(1+\frac{k_a}{2})\mu_\epsilon^{k_a+2}\int_{\mathbb{R}^N}|x|^{k_a}v_\epsilon^2\,dx + O(\mu_\epsilon^{k_a+2})$$
$$= \frac{(N-2)a^{(N-4)}(0)}{2(N-4)!}\mu_\epsilon^{N-2}\int_{\mathbb{R}^N}|x|^{N-4}v_\epsilon^2\,dx + O(\mu_\epsilon^{N-2}),$$

and we are now left with getting an estimate for the term

$$III_\epsilon^1 = \int_{\mathbb{R}^N}|x|^{N-4}v_\epsilon^2\,dx.$$

Let us consider $\delta \in (0,1)$ to be chosen later. By Proposition 1,

$$III_\epsilon^1 = \int_{B(0,\frac{\delta}{k_\epsilon})}|x|^{N-4}v_\epsilon^2\,dx + O\left(1\right).$$

Let $(\hat{x}_\epsilon)$ be a sequence of points such that $|\hat{x}_\epsilon| \le \frac{\delta}{k_\epsilon}$. We set $R_\epsilon = \frac{v_\epsilon(\hat{x}_\epsilon)}{v(\hat{x}_\epsilon)}$. If $|\hat{x}_\epsilon|$ is bounded, then $R_\epsilon \to 1$ since $v_\epsilon \to v$ uniformly on every compact subset of $\mathbb{R}^N$. Otherwise, $|\hat{x}_\epsilon| \to +\infty$, and, up to a subsequence, two cases occur: Either there exists $\delta_0 > 0$ such that $k_\epsilon|\hat{x}_\epsilon| \to \delta_0$, or $k_\epsilon|\hat{x}_\epsilon| \to 0$. In the first case, we set $y_\epsilon = k_\epsilon\hat{x}_\epsilon$. Then $|y_\epsilon| \le \delta$ and $R_\epsilon \approx |y_\epsilon|^{N-2}u_\epsilon(x_\epsilon)u_\epsilon(y_\epsilon)$. It follows from Lemma 4.2 that $R_\epsilon \to (N-2)\omega_{N-1}\delta_0^{N-2}g(\delta_0)$. In the second case, where $|\hat{x}_\epsilon| \to +\infty$ and $k_\epsilon|\hat{x}_\epsilon| \to 0$, we use the Green's formula. Setting $y_\epsilon = k_\epsilon\hat{x}_\epsilon$,

$$R_\epsilon \approx N(N-2)|y_\epsilon|^{N-2}\mu_\epsilon^{-\frac{N-2}{2}}\int_{B(0,1)}f(x)u_\epsilon^{p-\epsilon}(x)G(y_\epsilon,x)\,dx$$
$$\approx N(N-2)|y_\epsilon|^{N-2}\int_{B\left(0,\frac{1}{k_\epsilon}\right)}f(k_\epsilon x)v_\epsilon^{p-\epsilon}(x)G(y_\epsilon,k_\epsilon x)\,dx.$$

We let $\delta_\epsilon = C|y_\epsilon|$ where $C \in (0,1)$, and we write that

$$|y_\epsilon|^{N-2}\int_{B\left(0,\frac{1}{k_\epsilon}\right)}f(k_\epsilon x)v_\epsilon(x)^{p-\epsilon}G(y_\epsilon,k_\epsilon x)\,dx$$
$$= \underbrace{|y_\epsilon|^{N-2}\int_{\Omega_\epsilon^1}f(k_\epsilon x)v_\epsilon^{p-\epsilon}(x)G(y_\epsilon,k_\epsilon x)\,dx}_{III_\epsilon^2}$$
$$+ \underbrace{|y_\epsilon|^{N-2}\int_{\Omega_\epsilon^2}f(k_\epsilon x)v_\epsilon^{p-\epsilon}(x)G(y_\epsilon,k_\epsilon x)\,dx}_{III_\epsilon^3},$$

where $\Omega_\epsilon^1 = \{x \in B(0, \frac{1}{k_\epsilon}) : |y_\epsilon - k_\epsilon x| > \delta_\epsilon\}$ and $\Omega_\epsilon^2 = \{x \in B(0, \frac{1}{k_\epsilon}) : |y_\epsilon - k_\epsilon x| \leq \delta_\epsilon\}$. We then study III_ϵ^2 and III_ϵ^3 separately. Concerning III_ϵ^2,

$$|G(y_\epsilon, k_\epsilon x)| \leq \frac{A}{|y_\epsilon - k_\epsilon x|^{N-2}} \leq \frac{A}{\delta_\epsilon^{N-2}}.$$

As a consequence, if $x \in \Omega_\epsilon^1$,

$$\left||y_\epsilon|^{N-2} f(k_\epsilon x) v_\epsilon^{p-\epsilon}(x) G(y_\epsilon, k_\epsilon x)\right| \leq A\Big(\frac{|y_\epsilon|}{\delta_\epsilon}\Big)^{N-2} v^{p-\epsilon}(x) \leq \frac{A v^{p-\epsilon_0}(x)}{C^{N-2}}$$

for $\epsilon \leq \epsilon_0$, $\epsilon_0 > 0$ small. In particular,

$$h_\epsilon(x) = |y_\epsilon|^{N-2} 1_{\Omega_\epsilon^1}(x) f(k_\epsilon x) v_\epsilon^{p-\epsilon}(x) G(y_\epsilon, k_\epsilon x)$$

is bounded from above by an integrable function, where $1_{\Omega_\epsilon^1}$ denotes the characteristic function of Ω_ϵ^1. Clearly,

$$\frac{|y_\epsilon - k_\epsilon x|}{\delta_\epsilon} = \frac{|y_\epsilon - \frac{|y_\epsilon|}{|\hat{x}_\epsilon|} x|}{C|y_\epsilon|} = \frac{1}{C}\left|\frac{y_\epsilon}{|y_\epsilon|} - \frac{x}{|\hat{x}_\epsilon|}\right| \to \frac{1}{C},$$

which is greater than 1. Moreover,

$$G(y_\epsilon, k_\epsilon x) \approx \frac{1}{(N-2)\omega_{N-1}|y_\epsilon - k_\epsilon x|^{N-2}}$$

so that

$$|y_\epsilon|^{N-2} G(y_\epsilon, k_\epsilon x) \to \frac{1}{(N-2)\omega_{N-1}}.$$

Then, and since $f(0) = 1$, h_ϵ converges almost everywhere to the function $\frac{v^p}{(N-2)\omega_{N-1}}$. By the dominated convergence theorem,

$$III_\epsilon^2 \to \frac{1}{(N-2)\omega_{N-1}} \int_{\mathbb{R}^N} v^p dx = \frac{1}{N(N-2)}.$$

Concerning the term III_ϵ^3, a rough estimate is that

$$\left|III_\epsilon^3\right| \leq A|y_\epsilon|^{N-2} \int_{\Omega_\epsilon^2} v^p(x) G(y_\epsilon, k_\epsilon x)\, dx \leq A|y_\epsilon|^{N-2} \int_{\Omega_\epsilon^2} \frac{v^p(x)}{|y_\epsilon - k_\epsilon x|^{N-2}}\, dx.$$

Together with the change of variable $k_\epsilon x = y + y_\epsilon$, we obtain

$$\left|III_\epsilon^3\right| \leq A\frac{|y_\epsilon|^{N-2}}{k_\epsilon^N} \int_{|y| \leq \delta_\epsilon} \frac{1}{|y|^{N-2}} v^p\left(\frac{y + y_\epsilon}{k_\epsilon}\right) dy.$$

Clearly, if $|y| \leq \delta_\epsilon$,

$$\left|\frac{y + y_\epsilon}{k_\epsilon}\right| \geq \frac{|y_\epsilon| - |y|}{k_\epsilon} \geq \frac{|y_\epsilon| - \delta_\epsilon}{k_\epsilon} = (1 - C)\frac{|y_\epsilon|}{k_\epsilon} = (1 - C)|\hat{x}_\epsilon|$$

while $v(x) \le A|x|^{-N+2}$. As a consequence,

$$\left|III_\epsilon^3\right| \le \frac{A|y_\epsilon|^{N-2}\omega_{N-1}}{(1-C)^{N+2}|\hat{x}_\epsilon|^{N+2}k_\epsilon^N}\int_0^{\delta_\epsilon} t\,dt = \frac{AC^2\omega_{N-1}}{2(1-C)^{N+2}|\hat{x}_\epsilon|^2}$$

and $III_\epsilon^3 \to 0$. In particular, $R_\epsilon \approx N(N-2)III_\epsilon^2$, and $R_\epsilon \to 1$. Summarizing: either $k_\epsilon|\hat{x}_\epsilon| \to 0$, and then $R_\epsilon \to 1$, or $k_\epsilon|\hat{x}_\epsilon| \to \delta_0$, where $\delta_0 > 0$, and then $R_\epsilon \to (N-2)\omega_{N-1}\delta_0^{N-2}g(\delta_0)$. Let $\alpha \in (0,1)$ be given. We note that

$$\lim_{\delta_0\to 0^+}(N-2)\omega_{N-1}\delta_0^{N-2}g(\delta_0) = 1,$$

and we choose $\delta > 0$ such that for all $\delta_0 \in (0,\delta)$,

$$1-\alpha \le (N-2)\omega_{N-1}\delta_0^{N-2}g(\delta_0) \le 1+\alpha.$$

Then $1-\alpha \le R_\epsilon \le 1+\alpha$. We now set

$$m_\epsilon = \min_{0\le|x|\le\frac{\delta}{k_\epsilon}}\frac{v_\epsilon(x)}{v(x)} \text{ and } M_\epsilon = \max_{0\le|x|\le\frac{\delta}{k_\epsilon}}\frac{v_\epsilon(x)}{v(x)}.$$

According to what we just said, $1-\alpha \le m_\epsilon \le M_\epsilon \le 1+\alpha$, and then

$$(1-\alpha)\int_{B(0,\frac{\delta}{k_\epsilon})}|x|^{N-4}v^2\,dx \le \int_{B(0,\frac{\delta}{k_\epsilon})}|x|^{N-4}v_\epsilon^2\,dx \le (1+\alpha)\int_{B(0,\frac{\delta}{k_\epsilon})}|x|^{N-4}v^2\,dx.$$

Therefore, as is easily checked,

$$\frac{1}{|\ln k_\epsilon|}\int_{B\left(0,\frac{\delta}{k_\epsilon}\right)}|x|^{N-4}v^2dx \to \omega_{N-1}.$$

Since $\alpha \in (0,1)$ is arbitrary,

$$\frac{1}{|\ln k_\epsilon|}III_\epsilon^1 \to \omega_{N-1},$$

and we thus proved that

$$III_\epsilon = \frac{(N-2)\omega_{N-1}a^{(N-4)}(0)}{2(N-4)!}\mu_\epsilon^{N-2}|\ln\mu_\epsilon| + o\left(\mu_\epsilon^{N-2}|\ln k_\epsilon|\right).$$

Multiplying the Pohozaev identity by $\mu_\epsilon^{-k_f}$, and according to the preceding estimates, we obtain point $2(a)$ of Theorem 3. Similarly, multiplying the Pohozaev identity by $\mu_\epsilon^{-N+2}|\ln\mu_\epsilon|^{-1}$, we obtain point $2(b)$ of Theorem 3. In particular, Theorem 3 is proved. □

We are now left with the proof of Theorem 1. According to the results of Section 5, it suffices to show that, under the assumptions of this theorem, at least one subsequence of (u_ϵ) converges almost everywhere to a nonzero function. If not, the u_ϵ's develop a concentration and we are back to one of the situations described in Theorem 3. Noting that the assumptions of Theorem 1 are those that make the limits of the different points of Theorem 3 negative, Theorem 1 is proved. □

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