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MULTIPLE SOLUTIONS OF H-SYSTEMS ON SOME MULTIPLY-CONNECTED DOMAINS

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Abstract. In this note, we consider the following problem:

$$\begin{cases} \Delta u = 2u_x \wedge u_y \quad \text{in } \Omega, \quad u \in H_0^1(\Omega; \mathbf{R}^3), \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $\Omega \subset \mathbf{R}^2$ is a smooth bounded domain. We show that if the domain Ω is conformal equivalent to a $(K+1)$-ply connected domain satisfying some conditions, then the problem has at least K distinct non-trivial solutions.

1. Introduction

Let Ω be a bounded smooth domain in $\mathbf{R}^2$ with generic point $z = (x, y)$. In this paper we consider the following problem (HD):

$$\Delta u = 2u_x \wedge u_y \quad \text{in } \Omega, \tag{1.1}$$

$$u|_{\partial\Omega} = 0, \tag{1.2}$$

where $u \in H_0^1(\Omega; \mathbf{R}^3)$, "$\wedge$" denotes the usual vector product in $\mathbf{R}^3$, and, for example, $u_x = \frac{\partial}{\partial x} u$.

(1.1) is the equation of surfaces of prescribed constant mean curvature $H \equiv 1$ in conformal representation and called the *H-system* [2] [3].

Note that (1.1) is invariant under a conformal change of variables in the parameter domain Ω. Since $u \equiv 0$ is always the solution of (HD), we call it the trivial solution.

On the existence and the non-existence of non-trivial solutions of (HD), H. Wente proved the following results in 1975 [17].

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(a) If Ω is a simply-connected domain (without loss of generality, we may assume Ω is the unit disc in $\mathbf{R}^2$ by conformal invariance), then the problem (HD) has the only solution $u \equiv 0$.

(b) If Ω is a doubly-connected domain (conformal equivalent to an annulus), then there exists at least one non-trivial solution of (HD).

The proof of part (b) was done by analyzing some ODE system equivalent to the equation (1.1).

Thus the topology of the domain has the influence on the existence of non-trivial solutions of (HD). Similar phenomenon is observed for the Dirichlet problem of the nonlinear elliptic equation involving critical Sobolev exponent (SD):

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} & \text{in } \Omega \subset \mathbf{R}^N (N \geq 3), \\ u > 0 \quad \text{in } \Omega, & u|_{\partial\Omega} = 0, \end{cases}$$

where $\Omega \subset \mathbf{R}^N (N \geq 3)$ is a bounded smooth domain.

Now, the effect of the topology and the geometry of the domain on the existence of non-trivial solutions of (SD) has been extensively studied ; See [1] [4] [13] and references therein. These results say that, in principle, the richer is the topology of the domain, the more we can find non-trivial solutions.

As for the problem (HD), we may expect the same principle holds; that is, if Ω is a multiply-connected domain, then we would have multiple solutions of (HD). It is proposed to investigate the effect of topology on the existence of non-trivial solutions of (HD) in [16] (Chapter III. Remarks 5.9).

In this note, we shall prove that there exists a $(K+1)$-ply connected domain $(K \geq 1)$, on which the problem (HD) admits K distinct non-trivial solutions. More precisely, we need the following definition.

Definition. (condition $(A^{z_1,z_2,\dots,z_K}_{R_1,R_2,\dots,R_K})$) For $z_1, z_2, \dots, z_K \in \mathbf{R}^2$ and positive numbers $R_1, R_2, \dots, R_K$ $(R_i > 1, \forall i)$, we say that a domain $\Omega \subset \mathbf{R}^2$ satisfies the condition $(A^{z_1,z_2,\cdots,z_K}_{R_1,R_2,\dots,R_K})$ if the following holds :

(1) $B_{R_i}(z_i) := \{z \in \mathbf{R}^2 : |z - z_i| < R_i\}$ are disjoint discs,

(2) $A_{R_i^{-1},R_i}(z_i) := \{z \in \mathbf{R}^2 : R_i^{-1} < |z - z_i| < R_i\} \subset \Omega$ $(\forall i = 1, 2, \dots, K)$,

(3) $B_{(2R_i)^{-1}}(z_i) := \{z \in \mathbf{R}^2 : |z - z_i| < (2R_i)^{-1}\} \subset \Omega^c$ $(\forall i = 1, 2, \dots, K)$.

Theorem. *For every $K \in \mathbf{N}$, there exists a bounded smooth domain $\Omega \subset \mathbf{R}^2$ satisfying the condition $(A^{z_1,z_2,\dots,z_K}_{R_1,R_2,\dots,R_K})$ for some points $z_1, z_2, \dots, z_K \in \mathbf{R}^2$ and constants $4 < R_1 < R_2 < \cdots < R_K$, on which the problem (HD) admits at least K distinct non-trivial solutions.*

We remark that the same multiplicity result holds on any domain which is conformal equivalent to the domain Ω in the above theorem by conformal invariance of the equation (1.1).

Our proof of the theorem will be done by the variational method based on Morse theory, as in [4]. Coron treated the problem (SD) in [4]. The same method to look for distinct non-trivial solutions in different contexts has been investigated by several authors, for example, [5, 12, 14, 15] and references therein. Coron's strategy has been used for H-systems in another approach, see [9].

2. Variational formulation of the problem

Solutions of (HD) correspond to critical points of the (free) functional

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{2}{3}Q(u), \qquad u \in H_0^1(\Omega; \mathbf{R}^3),$$

where

$$Q(u) := \int_\Omega u \cdot u_x \wedge u_y$$

is called the oriented volume functional. We know Q is a smooth functional on $H_0^1(\Omega;\mathbf{R}^3)$, for example, by the facts that the wedge product $u_x \wedge u_y$ for $u \in H_0^1(\Omega;\mathbf{R}^3)$ belongs to the Hardy space $\mathcal{H}^1$, $H_0^1(\Omega;\mathbf{R}^3) \subset BMO(\mathbf{R}^2;\mathbf{R}^3)$, the space of functions of bounded mean oscillations on $\mathbf{R}^2$, and the famous duality theorem that $(\mathcal{H}^1)^* = BMO$. We prefer to formulate the problem in constrained variational form. Denote $M := \{u \in H_0^1(\Omega;\mathbf{R}^3) : Q(u) = 1\}$, which is a smooth Hilbert submanifold of $H_0^1(\Omega;\mathbf{R}^3)$, and

$$S(u) := \frac{\int_\Omega |\nabla u|^2}{|Q(u)|^{2/3}}, \quad \text{for } Q(u) \neq 0.$$

It is easy to verify that

- **(1)** $u \in H_0^1(\Omega;\mathbf{R}^3)$ is a critical point of E and $Q(u) \neq 0$, then $\overline{u} := \frac{1}{(Q(u))^{1/3}}u$ is a critical point of S and $\overline{u} \in M$.
- **(2)** $\overline{u} \in M$ is a critical point of S, then $u := \left(\frac{-1}{2}\int_\Omega |\nabla \overline{u}|^2\right)\overline{u}$ is a critical point of E.

From now on, we will find multiple critical points of S on the manifold M. Our argument relies on the fundamental theorem of the Morse theory: that is, if the smooth functional S satisfies the Palais-Smale (PS) compactness condition on the set $S^{-1}(\alpha,\beta) := \{u \in M : \alpha < S(u) < \beta\}$ and there is no critical point of S in $S^{-1}(\alpha,\beta)$, then, the sub-level set $M^\alpha := \{u \in M :$

$S(u) < \alpha\}$ is a strong deformation retract of M^β, so there are no topological differences between M^α and M^β.

We will argue by using the contraposition of this theorem successively, and will find multiple critical points of S by detecting topological (homotopical) differences between appropriate two sub-level sets. Hereafter, we denote

$$\bar{S} := \inf_{u \in M} S(u) = (32\pi)^{1/3}$$

is the best constant of *the isoperimetric inequality for H_0^1-mappings*:

$$\bar{S}|Q(u)|^{\frac{2}{3}} \le \int_\Omega |\nabla u|^2 \qquad \text{for all} \quad u \in H_0^1(\Omega; \mathbf{R}^3). \tag{2.1}$$

We know that $\bar{S}$ is never attained for $\Omega \neq \mathbf{R}^2$, and when $\Omega = \mathbf{R}^2$, the family of functions of the form

$$\varphi^\varepsilon(x, y) = \frac{2\varepsilon}{\varepsilon^2 + x^2 + y^2} \begin{pmatrix} x \\ y \\ \varepsilon \end{pmatrix}, \qquad \varepsilon > 0 \tag{2.2}$$

are extremals for $\bar{S}$ in $D^{1,2}(\mathbf{R}^2; \mathbf{R}^3)$.

3. Preliminary results

In this section, let $\Omega \subset \mathbf{R}^2$ be any bounded smooth domain and we use the same symbol E, Q, S etc. for functions from $\mathbf{R}^2$ to $\mathbf{R}^3$ (with trivial change of definition).

First of all, we recall the theorem of Brezis and Coron on the behavior of the Palais-Smale sequences for the functional E in $H_0^1(\Omega; \mathbf{R}^3)$.

Global Compactness Theorem. ([3]) *Let $\{u^n\} \subset H_0^1(\Omega; \mathbf{R}^3)$ be a Palais-Smale sequence for E, that is,* $\sup_{n \in \mathbf{N}} E(u^n) < \infty$ *and* $E'(u^n) \to 0$, H^{-1} *strongly. Then there exist*

- $u^0 \in H_0^1(\Omega; \mathbf{R}^3)$*: a solution of (HD),*
- $\omega^1, \cdots, \omega^p$ $(0 \le p < \infty)$*: a finite number of nonconstant solutions of*

$$\begin{cases} \Delta\omega = 2\omega_x \wedge \omega_y \quad \textit{on } \mathbf{R}^2, \\ \int_{\mathbf{R}^2} |\nabla \omega|^2 < \infty, \end{cases}$$

- *sequences* $(a_n^1), \cdots, (a_n^p) \subset \Omega$,
- *sequences* $(\varepsilon_n^1), \cdots, (\varepsilon_n^p) \subset \mathbf{R}_+$, $\lim_{n\to\infty} \varepsilon_n^i = 0$ $(\forall i)$,

such that, up to a subsequence, we have

(1) $u^n \rightharpoonup u^0$ *weakly in* $H_0^1(\Omega; \mathbf{R}^3)$,

(2) $\left\| u^n - u^0 - \sum_{i=1}^{p} \omega^i\left(\frac{\cdot - a_n^i}{\varepsilon_n^i}\right) \right\|_{H^1} \to 0 \quad (n \to \infty)$,
(3) $\int_\Omega |\nabla u^n|^2 = \int_\Omega |\nabla u^0|^2 + \sum_{i=1}^{p} \int_{\mathbf{R}^2} |\nabla \omega^i|^2 + o(1)$,
(4) $Q(u^n) = Q(u^0) + \sum_{i=1}^{p} Q(\omega^i) + o(1)$,
(5) $E(u^n) = E(u^0) + \sum_{i=1}^{p} E(\omega^i) + o(1)$.

In addition, we have

$$\sum_{i=1}^{p} E(\omega^i) = \frac{4\pi}{3} k \quad \textit{and} \quad \sum_{i=1}^{p} \int_{\mathbf{R}^2} |\nabla \omega^i|^2 = 8\pi k$$

for some nonnegative integer k.

This theorem precisely describes the obstruction for a Palais-Smale sequence of E to convergent strongly in H_0^1.

From the above theorem, we obtain the local compactness results for the functional E and S.

Lemma 1. *Let $\Omega \subset \mathbf{R}^2$ be a bounded smooth domain. Then we have*

(a) *every sequence $(u^n) \subset H_0^1(\Omega; \mathbf{R}^3)$ such that*

$$\begin{cases} \lim_{n\to\infty} E(u^n) = \beta \in (\frac{4\pi}{3}, \frac{8\pi}{3}), \\ E'(u^n) \to 0 \quad H^{-1} \textit{ strongly}, \end{cases}$$

is relatively compact in $H_0^1(\Omega; \mathbf{R}^3)$; that is, E satisfies the $(PS)_\beta$ condition for $\beta \in (\frac{4\pi}{3}, \frac{8\pi}{3})$.

(b) *every sequence $(\overline{u}^n) \subset M = \{u \in H_0^1(\Omega; \mathbf{R}^3); Q(u) = 1\}$ such that*

$$\begin{cases} \lim_{n\to\infty} S(\overline{u}^n) = \overline{\beta} \in (\bar{S}, 2^{1/3}\bar{S}), \\ S'(\overline{u}^n) \to 0 \quad H^{-1} \textit{ strongly}, \end{cases}$$

is relatively compact in $H_0^1(\Omega; \mathbf{R}^3)$; that is, S satisfies the $(PS)_{\overline{\beta}}$ condition on M for $\overline{\beta} \in (\bar{S}, 2^{1/3}\bar{S})$.

Proof. (a) By our assumptions, we can apply the Global Compactness Theorem to (u^n), so we obtain $u^0 \in H_0^1(\Omega; \mathbf{R}^3)$ which is a solution of (HD), and a nonnegative integer k such that

$$u^n \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega; \mathbf{R}^3), \tag{3.1}$$

$$E(u^n) = E(u^0) + \frac{4\pi}{3} k + o(1), \tag{3.2}$$

$$\int_\Omega |\nabla u^n|^2 = \int_\Omega |\nabla u^0|^2 + 8\pi k + o(1). \tag{3.3}$$

It suffices to prove that $k = 0$, by (3.1) and (3.3). Now, by our assumption $\lim_{n\to\infty} E(u^n) \in (\frac{4\pi}{3}, \frac{8\pi}{3})$ and (3.2), it is impossible that $E(u^0) = 0$. So we conclude $u^0 \not\equiv 0$. Since u^0 is a solution of (HD), by integration by parts we have

$$2Q(u^0) = -\int_\Omega |\nabla u^0|^2 \neq 0 \quad \text{and} \quad E(u^0) = \frac{1}{24}(S(u^0))^3,$$

which implies that $E(u^0) > \frac{4\pi}{3}$, because $S(u^0) > \bar{S} = (32\pi)^{1/3}$ on any bounded domain Ω. Therefore, if $k \neq 0$, it cannot be hold true that

$$E(u^0) + \frac{4\pi}{3}k = \lim_{n\to\infty} E(u^n) \in (\frac{4\pi}{3}, \frac{8\pi}{3}),$$

and so we obtain $k = 0$.

(b) From the assumption $\|S'(\overline{u}^n)\|_{H^{-1}} \to 0$, we have $\exists(\lambda_n)$: a sequence of Lagrange multipliers, and $\exists(f^n) \in H^{-1}$ such that

$$-\Delta\overline{u}^n = \lambda_n \overline{u}^n_x \wedge \overline{u}^n_y + f^n, \quad f^n \to 0 \quad \text{in } H^{-1} \text{ strongly.}$$

Integration by parts implies that

$$\lambda_n = \int_\Omega |\nabla\overline{u}^n|^2 + o(1) = \overline{\beta} + o(1).$$

Let $u^n := (-\frac{\lambda_n}{2})\overline{u}^n \in H^1_0(\Omega; \mathbf{R}^3)$, then we easily obtain

$$\Delta u^n = 2u^n_x \wedge u^n_y + \frac{\lambda_n}{2}f^n, \quad \frac{\lambda_n}{2}f^n \to 0 \quad \text{in } H^{-1} \text{ strongly,}$$
$$E(u^n) = \frac{1}{24}(\lambda_n)^3 + o(1) \to \frac{1}{24}(\overline{\beta})^3.$$

Now our assumption $\overline{\beta} \in (\bar{S}, 2^{1/3}\bar{S})$ implies that $(u^n) = (-\frac{\lambda_n}{2}\overline{u}^n)$ satisfies the assumptions of (a), so is relatively compact in $H^1_0(\Omega; \mathbf{R}^3)$. Since (λ_n) is away from 0 uniformly in n, then $(\overline{u}^n) = (-\frac{2}{\lambda_n}u^n)$ is also relatively compact. □

Next lemma describes the precise behavior of a minimizing sequence for $\bar{S}$ on any bounded domain Ω, and will be derived from the general Concentration Compactness Theorem for the isoperimetric inequality by P.L. Lions [10, 11]. In Appendix, we recall the Concentration-Compactness Theorem for the isoperimetric inequality for reader's convenience.

Lemma. *Let $\{u^n\} \subset M$ be a sequence such that $S(u^n) = \bar{S} + o(1)$ as $n \to \infty$. Then there exists a subsequence (still denoted by u^n) and $z = (x, y) \in \overline{\Omega}$ such that*

$$|\nabla u^n|^2 \overset{*}{\rightharpoonup} \bar{S}\delta_z \quad \textit{in } \mathcal{M}(\overline{\Omega})$$

in the sense of Radon measures of $\overline{\Omega}$.

We define *the barycentric map* $F : H_0^1(\Omega; \mathbf{R}^3) \setminus \{0\} \to \mathbf{R}^2$ as

$$F(u) = \frac{\int_\Omega z|\nabla u|^2 dz}{\int_\Omega |\nabla u|^2 dz}, \quad z = (x, y).$$

As a direct consequence of the above lemma, we have the following:

Lemma 2. *Let V be a neighborhood of $\overline{\Omega}$. Then there exists a constant $\varepsilon > 0$ such that $F(M^{\bar{S}+\varepsilon}) \subset V$ holds, where $M^{\bar{S}+\varepsilon} = \{u \in M : S(u) < \bar{S} + \varepsilon\}$.*

Proof. We argue by contradiction and suppose for every $m \in \mathbf{N}$, there exists $u^m \in M$ such that $S(u^m) < \bar{S} + \frac{1}{m}$ and $F(u^m) \notin V$. Since (u^m) is a minimizing sequence for $\bar{S}$ in M, the above lemma implies that it certainly blows up at exactly one point; that is, $\exists z_0 \in \overline{\Omega}$ such that

$$|\nabla u^m|^2 dxdy \overset{*}{\rightharpoonup} \bar{S}\delta_{z_0} \quad \text{in } \mathcal{M}(\overline{\Omega}).$$

Therefore, we have

$$F(u^m) = \frac{\int_\Omega z|\nabla u^m|^2 dz}{\int_\Omega |\nabla u^m|^2 dz} \to z_0 \in \overline{\Omega} \subset V$$

as $m \to \infty$. This is a contradiction. $\square$

Now, we define a certain family of functions in M which will be used to detect the topological difference between two appropriate sub-level sets of the functional S.

For $\tilde{z} = (\tilde{x}, \tilde{y}) \in \mathbf{R}^2, t \in [0, 1)$, and $\sigma = (x_0, y_0) \in \Sigma := \{z \in \mathbf{R}^2 : |z| = 1\}$, define

$$u_t^{\sigma,\tilde{z}}(z) = u_t^\sigma(z; \tilde{z}) := \frac{2(1-t)}{(1-t)^2 + |z - \tilde{z} - t\sigma|^2} \begin{pmatrix} x - \tilde{x} - tx_0 \\ y - \tilde{y} - ty_0 \\ t - 1 \end{pmatrix} \tag{3.4}$$

where $z = (x, y)$. When $t = 0$,

$$u_0^{\sigma,\tilde{z}}(z) = u_0(z; \tilde{z}) := \frac{2}{1 + |z - \tilde{z}|^2} \begin{pmatrix} x - \tilde{x} \\ y - \tilde{y} \\ -1 \end{pmatrix}$$

is a function independent of $\sigma \in \Sigma$, and as $t \to 1, u_t^{\sigma,\tilde{z}}$ "concentrates" at $\tilde{z} + \sigma$ in some sense. Note that on $\mathbf{R}^2$, $\bar{S}$ is attained by any such function $u_t^{\sigma,\tilde{z}}$ (see (2.2)).

Let η_R be the following log type cut-off function for $R > 0$.

$$\eta_R(s) := \begin{cases} 0, & s \le \frac{1}{R}, s \ge R, \\ \frac{2\log Rs}{\log R}, & \frac{1}{R} \le s \le \frac{1}{\sqrt{R}}, \\ 1, & \frac{1}{\sqrt{R}} \le s \le \sqrt{R}, \\ \frac{2\log(\frac{s}{R})}{\log(\frac{1}{R})}, & \sqrt{R} \le s \le R. \end{cases}$$

Denote $w^{\sigma,\tilde{z}}_{t,R}(z) := \eta_R(|z-\tilde{z}|)u^{\sigma,\tilde{z}}_t(z) \in H^1_0(\mathbf{R}^2;\mathbf{R}^3)$ and also consider the normalized function

$$v^{\sigma,\tilde{z}}_{t,R}(z) := \frac{w^{\sigma,\tilde{z}}_{t,R}(z)}{(Q(w^{\sigma,\tilde{z}}_{t,R}))^{1/3}}.$$

Now, we then have the following estimates for $v^{\sigma,\tilde{z}}_{t,R}$.

Lemma 3.

(1) *For $\forall\delta > 0$, there exists $R_0 \ge 4$ sufficiently large such that if $R \ge R_0$, then*

$$S(v^{\sigma,\tilde{z}}_{t,R}) \le \bar{S} + \delta$$

holds uniformly for $\sigma \in \Sigma, t \in [0,1)$ and $\tilde{z} \in \mathbf{R}^2$.

(2) *For $\forall\varepsilon > 0$ and $\forall R \ge 4$, there exists $\eta \in [0,1)$ such that if $t \in [\eta, 1)$, then*

$$S(v^{\sigma,\tilde{z}}_{t,R}) \le \bar{S} + \varepsilon, \quad |F(v^{\sigma,\tilde{z}}_{t,R}) - (\tilde{z}+\sigma)| \le \varepsilon$$

holds uniformly for $\sigma \in \Sigma$ and $\tilde{z} \in \mathbf{R}^2$.

Proof. We may assume without loss of generality that $\tilde{z} = 0$, and we abbreviate $u^\sigma_t = u^{\sigma,0}_t, w^\sigma_{t,R} = w^{\sigma,0}_{t,R}$ and so on.

(1) Direct calculation using the definition (3.4) shows that

$$|\nabla u^\sigma_t|^2(z) = \frac{8(1-t)^2}{[(1-t)^2 + |z-t\sigma|^2]^2}, \tag{3.5}$$

$$\int_{\mathbf{R}^2} |\nabla u^\sigma_t|^2 = 8\pi, \tag{3.6}$$

$$u^\sigma_t \cdot (u^\sigma_t)_x \wedge (u^\sigma_t)_y = \frac{-8(1-t)^4}{[(1-t)^2 + |z-t\sigma|^2]^3}, \tag{3.7}$$

$$Q(u^\sigma_t) = \int_{\mathbf{R}^2} u^\sigma_t \cdot (u^\sigma_t)_x \wedge (u^\sigma_t)_y = -4\pi. \tag{3.8}$$

Note also

$$|\nabla u^\sigma_t|^2 \overset{*}{\rightharpoonup} 8\pi\delta_\sigma \quad \text{and} \quad u^\sigma_t \cdot (u^\sigma_t)_x \wedge (u^\sigma_t)_y \overset{*}{\rightharpoonup} (-4\pi)\delta_\sigma$$

as $t \to 1$ in $\mathcal{M}(\mathbf{R^2})$. We claim that

$$\int_{\mathbf{R}^2} |\nabla w_{t,R}^\sigma - \nabla u_t^\sigma|^2 \to 0 \quad (R \to \infty) \tag{3.9}$$

and

$$Q(w_{t,R}^\sigma) = \int_{\mathbf{R}^2} w_{t,R}^\sigma \cdot (w_{t,R}^\sigma)_x \wedge (w_{t,R}^\sigma)_y \to -4\pi \quad (R \to \infty) \tag{3.10}$$

uniformly in $\sigma \in \Sigma,\ t \in [0,1)$. Indeed, we have

$$\int_{\mathbf{R}^2} |\nabla w_{t,R}^\sigma - \nabla u_t^\sigma|^2 = \int_{\mathbf{R}^2} |\nabla((\eta_R - 1)u_t^\sigma)|^2$$
$$\leq C \int_{|z|\leq \frac{1}{\sqrt{R}}} |\nabla u_t^\sigma|^2 + C \int_{|z|\geq \sqrt{R}} |\nabla u_t^\sigma|^2 + C \int_{\mathbf{R}^2} |\nabla \eta_R|^2 =: I_1 + I_2 + I_3$$

where C is an absolute constant.

Now, when we estimate I_1, we divide the proof in two cases:

First case: $t \in [0, 3/4)$. We estimate $|\nabla u_t^\sigma|^2$ in (3.5) as

$$|\nabla u_t^\sigma|^2 \leq \frac{8}{(1-t)^2} < 8 \times 16.$$

So, we have

$$\int_{|z|\leq \frac{1}{\sqrt{R}}} |\nabla u_t^\sigma|^2 \leq 8 \cdot 16 \cdot \pi (\frac{1}{\sqrt{R}})^2 \to 0 \quad (R \to \infty)$$

uniformly in $\sigma \in \Sigma, t \in [0, 3/4)$.

Second case: $t \in [3/4, 1)$. In this case, we estimate as

$$|\nabla u_t^\sigma|^2 \leq \frac{8}{|z - t\sigma|^4}.$$

Note that if $|z| \leq \frac{1}{\sqrt{R}}$ for $R > 4$, then $|z - t\sigma| > \frac{1}{4}$ holds for $\forall \sigma \in \Sigma, \forall t \in [3/4, 1)$. Therefore,

$$\int_{|z|\leq \frac{1}{\sqrt{R}}} |\nabla u_t^\sigma|^2 \leq 8 \cdot 4^4 \cdot \pi (\frac{1}{\sqrt{R}})^2 \to 0 \quad (R \to \infty)$$

uniformly in $\sigma \in \Sigma, t \in [3/4, 1)$. So we conclude: $I_1 \to 0 \quad (R \to \infty)$ uniformly in $\sigma \in \Sigma, t \in [0, 1)$.

For the estimate of I_2, we note that if $|z| \geq \sqrt{R}$ for $R > 4$, then we have $|z - t\sigma| > \frac{1}{2}|z|$ for $\forall \sigma \in \Sigma, \forall t \in [0, 1)$. Therefore,

$$\int_{|z|\geq \sqrt{R}} |\nabla u_t^\sigma|^2 \leq 8 \cdot 2^4 \int_{|z|\geq \sqrt{R}} (\frac{1}{|z|^4}) = 2^7 \cdot \pi (\frac{1}{R}) \to 0$$

as $R \to \infty$ uniformly in $\sigma \in \Sigma$ and $t \in [0,1)$.

As for I_3, by the definition of η_R, we calculate

$$\int_{\mathbf{R}^2} |\nabla \eta_R|^2 = \Big(\int_{\frac{1}{R} \le |z| \le \frac{1}{\sqrt{R}}} + \int_{\sqrt{R} \le |z| \le R} \Big) (\frac{2}{\log R})^2 \frac{1}{|z|^2}$$
$$= 2\pi \Big(\int_{r=1/R}^{r=1/\sqrt{R}} + \int_{r=\sqrt{R}}^{r=R} \Big) (\frac{2}{\log R})^2 \frac{1}{r^2} r dr$$
$$= 2 \cdot \frac{4\pi}{\log R} \to 0 \quad (R \to \infty) \quad \text{for } \forall \sigma \in \Sigma, \ \forall t \in [0,1).$$

Therefore we obtain (3.9).

To conclude (3.10), we write

$$Q(w^\sigma_{t,R}) = \int_{\mathbf{R}^2} \eta_R^3 u^\sigma_t \cdot (u^\sigma_t)_x \wedge (u^\sigma_t)_y$$
$$= \int_{\mathbf{R}^2} u^\sigma_t \cdot (u^\sigma_t)_x \wedge (u^\sigma_t)_y + \int_{\mathbf{R}^2} (\eta_R^3 - 1) u^\sigma_t \cdot (u^\sigma_t)_x \wedge (u^\sigma_t)_y := II_1 + II_2.$$

We know $II_1 = -4\pi$ by (3.8). By using (3.7), we have

$$0 \le II_2 \le \Big(\int_{|z| \le \frac{1}{\sqrt{R}}} + \int_{|z| \ge \sqrt{R}} \Big) \frac{8(1-t)^4}{((1-t)^2 + |z - t\sigma|^2)^3}.$$

Now, as in the estimation of I_1, I_2, we can conclude

$$II_2 \to 0 \quad (R \to \infty) \quad \text{uniformly in } \sigma \in \Sigma, \ t \in [0,1).$$

So, we have the claim (3.10).

From (3.9) and (3.10), we obtain:

$$S(w^\sigma_{t,R}) \to S(u^\sigma_t) = \frac{8\pi}{|-4\pi|^{2/3}} = \bar{S} \quad (R \to \infty)$$

uniformly in $\sigma \in \Sigma, t \in [0,1)$. This implies the assertion (1).

(2) For fixed $R > 4$, we will show that:

$$\int_{\mathbf{R}^2} |\nabla w^\sigma_{t,R}|^2 \to 8\pi \quad (t \to 1), \tag{3.11}$$

and

$$Q(w^\sigma_{t,R}) = \int_{\mathbf{R}^2} w^\sigma_{t,R} \cdot (w^\sigma_{t,R})_x \wedge (w^\sigma_{t,R})_y \to -4\pi \quad (t \to 1) \tag{3.12}$$

uniformly in $\sigma \in \Sigma$.

First, we observe that there exists $\exists t_0 \in [0,1), \exists\delta > 0$ such that for $\forall R > 4$, $\forall t \in [t_0, 1)$ and $\forall \sigma \in \Sigma$, it holds

$$\eta_R(z) \equiv 1 \quad \text{on } B_\delta(t\sigma). \tag{3.13}$$

Here, $B_\delta(z)$ is a ball of center z and radius δ. ($t_0 = \frac{3}{4}$ and $\delta = \frac{1}{8}$ will suffice.)

We compute:

$$\begin{aligned}\int_{\mathbf{R}^2} |\nabla w^\sigma_{t,R}|^2 &= \int_{\mathbf{R}^2} \eta_R^2 |\nabla u^\sigma_t|^2 + \int_{\mathbf{R}^2} 2\eta_R u^\sigma_t \cdot ((\eta_R)_x (u^\sigma_t)_x + (\eta_R)_y (u^\sigma_t)_y) \\ &\quad + \int_{\mathbf{R}^2} |\nabla \eta_R|^2 |u^\sigma_t|^2 =: I + II + III.\end{aligned}$$

Then, we estimate as

$$I = \int_{\mathbf{R}^2} \eta_R^2 |\nabla u^\sigma_t|^2 \le \int_{B_\delta(t\sigma)} \eta_R^2 |\nabla u^\sigma_t|^2 + \int_{B_R(0)\setminus B_\delta(t\sigma)} \eta_R^2 |\nabla u^\sigma_t|^2 =: I_1 + I_2.$$

For $t \in [t_0, 1)$, we know by (3.13) and (3.5),

$$\begin{aligned} I_1 &= \int_{B_\delta(t\sigma)} |\nabla u^\sigma_t|^2 = \int_{B_\delta(t\sigma)} \frac{8(1-t)^2}{[(1-t)^2 + |z - t\sigma|^2]^2} dxdy \\ &= \int_0^{2\pi} \int_{r=0}^{r=\delta} \frac{8(1-t)^2}{((1-t)^2 + r^2)^2} r dr d\theta = 8\pi \Big(1 - \frac{(1-t)^2}{(1-t)^2 + \delta^2}\Big) \to 8\pi \end{aligned}$$

as $t \to 1$ uniformly in $\sigma \in \Sigma$. Similarly, we have

$$\begin{aligned} I_2 &\le \int_{\mathbf{R}^2 \setminus B_\delta(t\sigma)} |\nabla u^\sigma_t|^2 \\ &= \int_0^{2\pi} \int_{r=\delta}^{r=\infty} \frac{8(1-t)^2}{((1-t)^2 + r^2)^2} r dr d\theta = 8\pi \frac{(1-t)^2}{(1-t)^2 + \delta^2} \to 0 \end{aligned}$$

as $t \to 1$ uniformly in $\sigma \in \Sigma$.

As for III, we see $III \to 0$ uniformly in $\sigma \in \Sigma$ as $t \to 1$, because

$$\begin{aligned} III &= \int_{B_\delta(t\sigma)} |\nabla \eta_R|^2 |u^\sigma_t|^2 + \int_{B_R(0)\setminus B_\delta(t\sigma)} |\nabla \eta_R|^2 |u^\sigma_t|^2 \\ &\le 0 + \|\nabla \eta_R\|^2_{L^\infty} \cdot \frac{4(1-t)^2}{(1-t)^2 + \delta^2} \cdot \pi R^2. \end{aligned}$$

Finally, the estimate of II is almost the same as in the calculations of I,III: for example, we have

$$\Big| \int_{\mathbf{R}^2} 2\eta_R (\eta_R)_x u^\sigma_t \cdot (u^\sigma_t)_x \Big| = \Big| \Big(\int_{B_\delta(t\sigma)} + \int_{B_R(0)\setminus B_\delta(t\sigma)} \Big) 2\eta_R (\eta_R)_x u^\sigma_t \cdot (u^\sigma_t)_x \Big|$$

$$\le 0 + 2\|\nabla \eta_R\|_{L^\infty} \cdot \int_{B_R(0)\setminus B_\delta(t\sigma)} \frac{4(1-t)^2|x - tx_0|}{(1-t)^2 + |z - t\sigma|^2} dxdy$$
$$\le 2\|\nabla \eta_R\|_{L^\infty} \cdot \frac{4(1-t)^2}{(1-t)^2 + \delta^2}(R+1) \cdot \pi R^2 \to 0 \quad (t \to 1).$$

Therefore, we obtain the claim (3.11).

To show the claim (3.12), we rewrite

$$Q(w^\sigma_{t,R}) = \int_{\mathbf{R}^2} \eta_R^3 u_t^\sigma \cdot (u_t^\sigma)_x \wedge (u_t^\sigma)_y$$
$$= \int_{B_\delta(t\sigma)} \eta_R^3 u_t^\sigma \cdot (u_t^\sigma)_x \wedge (u_t^\sigma)_y + \int_{B_R(0)\setminus B_\delta(t\sigma)} \eta_R^3 u_t^\sigma \cdot (u_t^\sigma)_x \wedge (u_t^\sigma)_y =: I + II.$$

as in the proof of (3.11). Calculation shows:

$$I = \int_{B_\delta(t\sigma)} u_t^\sigma \cdot (u_t^\sigma)_x \wedge (u_t^\sigma)_y = \int_{B_\delta(t\sigma)} \frac{-8(1-t)^4}{((1-t)^2 + |z - t\sigma|^2)^3} dxdy$$
$$= 2\pi(-8)(1-t)^4 \int_{r=0}^{r=\delta} \frac{r}{((1-t)^2 + r^2)^3} dr$$
$$= 4\pi \frac{(1-t)^4}{((1-t)^2 + \delta)^2} - 4\pi \to -4\pi \quad (t \to 1),$$

and

$$|II| \le \int_{\mathbf{R}^2\setminus B_\delta(t\sigma)} \frac{8(1-t)^4}{((1-t)^2 + |z - t\sigma|^2)^3} dxdy$$
$$= 2\pi \int_{r=\delta}^{r=\infty} \frac{8(1-t)^4}{((1-t)^2 + r^2)^3} rdr = 4\pi \frac{(1-t)^4}{((1-t)^2 + \delta)^2} \to 0 \quad (t \to 1).$$

From these, we obtain the claim (3.12).

Finally, to complete the lemma, we must show that

$$\sup_{\sigma \in \Sigma} |F(w^\sigma_{t,R}) - \sigma| \to 0 \quad (t \to 1).$$

We calculate again:

$$F(w^\sigma_{t,R}) - \sigma = \frac{1}{\|\nabla w^\sigma_{t,R}\|^2_{L^2}} \int_{B_R(0)} |\nabla w^\sigma_{t,R}|^2 (z - \sigma) dz$$
$$= \frac{1}{\|\nabla w^\sigma_{t,R}\|^2_{L^2}} \Big(\int_{B_R(0)} |\nabla u_t^\sigma|^2 |\eta_R|^2 (z - \sigma) dz$$
$$+ \int_{B_R(0)} 2\eta_R u_t^\sigma \cdot ((\eta_R)_x (u_t^\sigma)_x + (\eta_R)_y (u_t^\sigma)_y)(z - \sigma) dz$$

$$+ \int_{B_R(0)} |\nabla \eta_R|^2 |u_t^\sigma|^2 (z-\sigma) dz\Big) =: \frac{1}{\|\nabla w_{t,R}^\sigma\|_{L^2}^2}(I + II + III).$$

For estimating I, we divide the integral as before:

$$\begin{aligned} I &= \int_{B_\delta(t\sigma)} |\nabla u_t^\sigma|^2 |\eta_R|^2 (z-\sigma) dz + \int_{B_R(0)\setminus B_\delta(t\sigma)} |\nabla u_t^\sigma|^2 |\eta_R|^2 (z-\sigma) dz \\ &=: I_1 + I_2 \end{aligned}$$

Now by (3.5) and (3.13),

$$\begin{aligned} I_1 &= \int_{B_\delta(t\sigma)} \frac{8(1-t)^2}{((1-t)^2 + |z-t\sigma|^2)^2} (z - t\sigma + (t-1)\sigma) dz \\ &= \int_0^{2\pi} \int_{r=0}^{r=\delta} \frac{8(1-t)^2}{((1-t)^2 + r^2)^2} \left\{ r \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + (t-1)\sigma \right\} r dr d\theta \\ &= 2\pi \int_{r=0}^{r=\delta} \frac{8(1-t)^2 (t-1)\sigma}{((1-t)^2 + r^2)^2} r dr \\ &= 8\pi \frac{(1-t)^3 \sigma}{(1-t)^2 + \delta^2} - 8\pi (1-t)\sigma \to 0 \quad (t \to 1). \end{aligned}$$

For I_2, we have

$$\begin{aligned} |I_2| &\le \int_{B_R(0)\setminus B_\delta(t\sigma)} \frac{8(1-t)^2}{((1-t)^2 + \delta^2)^2} (|z| + |\sigma|) dz \\ &\le \frac{8(1-t)^2}{((1-t)^2 + \delta^2)^2} \cdot (R+1) \cdot \pi R^2 \to 0 \quad (t \to 1). \end{aligned}$$

So, we conclude $|I| \to 0$ uniformly in σ as $t \to 1$.

The remaining terms II and III are estimated as before: Then we complete the proof of Lemma. □

4. Proof of the theorem

In this section, we will construct a bounded smooth domain Ω satisfying the condition $(A_{R_1,\cdots,R_K}^{z_1,\cdots,z_K})$ for some points $z_1, z_2, \cdots, z_K \in \mathbf{R}^2$ and appropriate constants $R_1 < R_2 < \cdots < R_K$, on which we can prove the existence of multiple solutions for the problem (HD).

First, let $\delta_1 := \frac{1}{2}(2^{1/3}\bar{S} - \bar{S})$ and we use Lemma 3 (1) for $\delta = \delta_1$ and $\tilde{z} = z_1$ where z_1 is any point in $\mathbf{R}^2$. Then we have $R_1 (\ge 4)$ such that

$$S(v_{t,R_1}^{\sigma,z_1}) \le \bar{S} + \delta_1, \quad \forall \sigma \in \Sigma,\ \forall t \in [0,1). \tag{4.1}$$

From now on, we fix this R_1 and assume that, for the moment, the domain Ω satisfies

$$A_{R_1^{-1},R_1}(z_1) \subset \Omega, \quad B_{(2R_1)^{-1}}(z_1) \subset \Omega^c. \tag{4.2}$$

Next, take a neighborhood V_1 of $\overline{\Omega}$ such that

$$B_{R_1^{-1}}(z_1) \not\subset V_1. \tag{4.3}$$

For this V_1, we apply Lemma 2 and deduce that there exists an $\varepsilon_1 > 0$ such that

$$F(M^{\bar{S}+\varepsilon_1}) \subset V_1. \tag{4.4}$$

Then we apply Lemma 3 (2) for this choice of $\varepsilon = \varepsilon_1$ and $R = R_1$; There exists $t_1 \in (0,1)$ such that

$$v_{t_1,R_1}^{\sigma,z_1} \in M^{\bar{S}+\varepsilon_1} \quad (\forall \sigma \in \Sigma), \tag{4.5}$$

$$|F(v_{t_1,R_1}^{\sigma,z_1}) - (z_1 + \sigma)| \leq \varepsilon_1 \quad (\forall \sigma \in \Sigma). \tag{4.6}$$

Now, define the map $f : \Sigma \to M \subset H_0^1(\Omega; \mathbf{R}^3)$ as $f(\sigma) := v_{t_1,R_1}^{\sigma,z_1}(\cdot)$. We see f is continuous with respect to H_0^1-topology so $\{f(\sigma) : \sigma \in \Sigma\}$ is a continuous curve in $M^{\bar{S}+\varepsilon_1}$ by (4.5).

By (4.3), (4.4) and (4.6), we conclude:

$$\{f(\sigma) : \sigma \in \Sigma\} \text{ is not homotopic to } \{\text{a point}\} \text{ in } M^{\bar{S}+\varepsilon_1}. \tag{4.7}$$

On the other hand, if we set $\eta : [0,1) \times \Sigma \to M$ such that $\eta(t,\sigma) := v_{t,R_1}^{\sigma,z_1}(\cdot)$, then we have

$$\eta(t,\sigma) \in M^{\bar{S}+\delta_1}, \quad \forall t \in [0,1), \forall \sigma \in \Sigma \quad \text{by (4.1)},$$
$$\eta(0,\sigma) = v_{0,R_1}^{z_1} \text{ is independent of } \sigma \in \Sigma,$$

and

$$\eta(t_1,\sigma) = v_{t_1,R_1}^{\sigma,z_1} = f(\sigma).$$

Therefore, we obtain

$$\{f(\sigma) : \sigma \in \Sigma\} \text{ is homotopic to a point } \{v_{0,R_1}^{z_1}(\cdot)\} \text{ in } M^{\bar{S}+\delta_1}. \tag{4.8}$$

Recall Lemma 1 (2) that the functional S satisfies the $(PS)_{\overline{\beta}}$ condition on M for $\overline{\beta} \in (\bar{S}, 2^{1/3}\bar{S})$. So by the fundamental theorem of Morse Theory, if there are no critical points of S in M with critical values in $(\bar{S}+\varepsilon_1, \bar{S}+\delta_1)$, then we would have the conclusion that $M^{\bar{S}+\varepsilon_1}$ is a strong deformation retract of $M^{\bar{S}+\delta_1}$. But this is a contradiction to (4.7) and (4.8).

Therefore, we must have at least one critical point $v_1 \in M$ for S such that

$$\bar{S} + \varepsilon_1 < S(v_1) < \bar{S} + \delta_1.$$

Let $u_1 = (-\frac{1}{2}\int_\Omega |\nabla v_1|^2)v_1$, then u_1 is a solution of (HD), $u_1 \not\equiv 0$.

If we want to have a second solution, let $\delta_2 := \frac{\varepsilon_1}{2}$ and we use Lemma 3 (1) for $\delta = \delta_2$ and $\tilde{z} = z_2$ where z_2 is any point in $\mathbf{R}^2$. Then we have $R_2 (\geq R_1)$ such that

$$S(v^{\sigma,z_2}_{t,R_2}) \leq \bar{S} + \delta_2, \quad \forall \sigma \in \Sigma,\ \forall t \in [0,1).$$

We may assume that, in addition to (4.2), Ω satisfies:

$A_{R_2^{-1},R_2}(z_2) \subset \Omega$, $\quad B_{(2R_2)^{-1}}(z_2) \subset \Omega^c$, $\quad B_{R_1}(z_1), B_{R_2}(z_2)$ are disjoint discs,

by using the condition $(A^{z_1,\cdots,z_K}_{R_1,\cdots,R_K})$.

We can repeat the above argument and we will find a second solution $u_2 \not\equiv 0$. Continuing this way, we find at least K distinct solutions of (HD) successively. This proves the theorem. □

5. Appendix

Concentration-Compactness Theorem for the isoperimetric inequality.

Let $\Omega \subset \mathbf{R}^2$ be a bounded smooth domain. Consider a sequence $\{v^n\} \subset H^1_0(\Omega;\mathbf{R}^3) \subset H^1_0(\mathbf{R}^2;\mathbf{R}^3)$ (extended by 0 outside Ω) with

$$\sup_{n\in\mathbf{N}} \int_{\mathbf{R}^2} |\nabla v^n|^2 < \infty,$$
$$v^n \rightharpoonup v^0 \text{ weakly in } H^1_0(\Omega;\mathbf{R}^3),$$
$$|\nabla v^n|^2 \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\overline{\Omega}), \quad T^n \to T \quad \text{in } \mathcal{D}'(\mathbf{R}^2)$$

as $n \to \infty$, where $T^n \in \mathcal{D}'(\mathbf{R}^2)$ is a compactly supported distribution, defined by

$$T^n(\varphi) := \int_{\mathbf{R}^2} (\varphi v^n)\cdot v^n_x \wedge v^n_y\, dxdy \quad \text{for } \forall \varphi \in \mathcal{D}(\mathbf{R}^2).$$

Then μ is a finite nonnegative Radon measure with $\mu(\overline{\Omega}) < \infty$ and T is a compactly supported distribution with supp$(T) \subset \overline{\Omega}$ and the followings hold:
(part 1) (*forms of the limit measure and the limit distribution*)

There exist $\exists J \in \mathbf{N}\cup\{\infty\}$, nonnegative numbers $\{\mu_j\}_{j=1}^J$, real numbers $\{\nu_j\}_{j=1}^J$ and points $\{z_j\}_{j=1}^J \subset \overline{\Omega}$ such that

(1)

$$\mu = |\nabla v^0|^2 dxdy + \sum_{j=1}^J \mu_j \delta_{z_j} + \tilde{\mu},$$

where $\tilde{\mu} \in \mathcal{M}(\overline{\Omega})$ is a nonnegative, nonatomic measure.

(2)

$$T = T_0 + \sum_{j=1}^{J} \nu_j \delta_{z_j} \quad in\ \mathcal{D}'(\mathbf{R}^2),$$

where T_0 *is a distribution defined by*

$$T_0(\varphi) := \int_{\mathbf{R}^2} (\varphi v^0) \cdot v_x^0 \wedge v_y^0 dxdy \quad for\ \forall \varphi \in \mathcal{D}(\mathbf{R}^2).$$

(3) (*the isoperimetric inequality for atoms*)

$$|\nu_j| \le (\frac{1}{\bar{S}})^{3/2} \mu_j^{3/2}, \quad \forall j \in \{1, \cdots, J\}.$$

(4) (*the isoperimetric inequality for total mass*)

$$|T(1)| \le (\frac{1}{\bar{S}})^{3/2} \mu(\overline{\Omega})^{3/2}.$$

(part 2) (*concentration-compactness alternative*)

If $\mu(\overline{\Omega}) = \bar{S} = (32\pi)^{1/3}$ *and* $|T(1)| = 1$, *then one and only one of the following statements holds true.*

(a) (*concentration*) *there exists* $z_0 \in \overline{\Omega}$ *such that* $\mu = \bar{S}\delta_{z_0}$ *and* $T = \delta_{z_0}$.
(b) (*compactness*) $v^n \to v^0$ *strongly in* $H_0^1(\Omega; \mathbf{R}^3)$. *In this case,* $\mu = |\nabla v^0|^2 dxdy$ *and* $T = T_0$.

The Proof of this general theorem will be done, as indicated in ([11], p.102): See also [6] and [8] for more sophisticated treatment of the subject.

Here, we give some details for reader's convenience.

Assume first that $v^0 \equiv 0$. Since T^n has compact support, we note that we may test T^n by any $\varphi \in C^\infty(\mathbf{R}^2)$. Then, the isoperimetric inequality for $\varphi v^n \in H_0^1(\mathbf{R}^2; \mathbf{R}^3)$ implies

$$|T^n(\varphi^3)|^{2/3} = \Big| \int_{\mathbf{R}^2} (\varphi v^n) \cdot (\varphi v^n)_x \wedge (\varphi v^n)_y \Big|^{2/3} \le \frac{1}{\bar{S}} \int_{\mathbf{R}^2} |\nabla(\varphi v^n)|^2. \quad (5.1)$$

Now we estimate the right hand side of (5.1) as follows:

$$\Big| \int_{\mathbf{R}^2} |\nabla(\varphi v^n)|^2 dxdy - \int_{\mathbf{R}^2} \varphi^2 |\nabla v^n|^2 dxdy \Big|$$
$$\le \Big| \int_\Omega |\nabla \varphi|^2 |v^n|^2 \Big| + \Big| \int_\Omega 2\varphi v^n \cdot (\varphi_x v_x^n + \varphi_y v_y^n) \Big|.$$

Since $v^n \to v^0$ strongly in $L^2(\Omega)$ by Rellich theorem, we see that this bound goes to 0 as $n \to \infty$.

Then by passing to the limit in (5.1), we obtain

$$|T(\varphi^3)|^{2/3} \leq \frac{1}{\bar{S}} \int_{\mathbf{R}^2} \varphi^2 d\mu, \tag{5.2}$$

which we call *the reverse Hölder inequality between* T *and* μ.

Now we claim that T is a compactly supported signed Radon measure on $\mathbf{R}^2$. Indeed, let $K \subset \mathbf{R}^2$ be any compact set and let $\{\psi^j\} \subset \mathcal{D}(\mathbf{R}^2)$ be any sequence such that

$$\mathrm{supp}(\psi^j) \subset K \ (\forall j), \quad \sup_{z \in \mathbf{R}^2} |\psi^j(z)| \to 0 \ (j \to \infty).$$

We will show that $T(\psi^j) \to 0$ as $j \to \infty$ and conclude that T is a signed measure.

In fact, for ψ^j as above, we can choose a positive number C_j such that $C_j \to 0$ as $j \to \infty$ and $\psi^j + C_j > 0$ on $\mathbf{R}^2$. ($C_j = \max\{0, -\min_{z \in \mathbf{R}^2}(\psi^j(z))\} + 1/j$ will suffice.)

Applying the reverse Hölder inequality for $(\psi^j + C_j)^{1/3} \in C^\infty(\mathbf{R}^2)$, we obtain

$$|T(\psi^j + C_j)|^{2/3} \leq \frac{1}{\bar{S}} \int_{\mathbf{R}^2} (\psi^j + C_j)^{2/3} d\mu.$$

Note that $\mu(\mathbf{R}^2) = \mu(\overline{\Omega}) < \infty$ and $\sup_{z \in \mathbf{R}^2} |\psi^j(z) + C_j| \to 0$ as $j \to \infty$, we have $\limsup_{j \to \infty} |T(\psi^j)| = 0$ and the claim.

By approximation in (5.2), we have

$$|T(E)|^{2/3} \leq \frac{1}{\bar{S}} \mu(E) \quad (\forall E \subset \mathbf{R}^2, \text{ Borel}). \tag{5.3}$$

Now since μ is a finite measure supported by $\overline{\Omega}$, the set of atoms of μ :

$$D := \{z \in \mathbf{R}^2 : \mu(\{z\}) > 0\} \subset \overline{\Omega}$$

is at most countable. We can therefore write $D = \{z_j\}_{j=1,\cdots,J}$, and if we denote $\mu_j := \mu(\{z_j\})$, then

$$\mu = \sum_{j=1}^{J} \mu_j \delta_{z_j} + \tilde{\mu}, \tag{5.4}$$

where $\tilde{\mu} \in \mathcal{M}(\overline{\Omega})$ is a nonnegative, nonatomic measure.

Because (5.3) implies that T is absolutely continuous with respect to μ; So we can write

$$T(E) = \int_E D_\mu T d\mu, \quad (\forall E \subset \mathbf{R}^2, \text{ Borel}),$$

where

$$D_\mu T(z) := \lim_{r\to 0} \frac{T(B_r(z))}{\mu(B_r(z))}$$

is a symmetric derivative of Radon measure T with respect to μ at z. Note that this limit exists for $\mu-$a.e. $z \in \mathbf{R}^2$ (see [7] §1.6).

From (5.3), we have

$$0 \le \frac{|T(B_r(z))|}{\mu(B_r(z))} \le \left(\frac{1}{\bar{S}}\right)^{3/2} \mu(B_r(z))^{1/2}, \tag{5.5}$$

provided $\mu(B_r(z)) \neq 0$. Therefore we deduce that

$$D_\mu T(z) = 0 \quad \mu - \text{a.e. } z \in \mathbf{R}^2 \setminus D. \tag{5.6}$$

Now define $\nu_j := D_\mu T(z_j)\mu_j$ $(j = 1, \cdots, J)$. Then we have

$$T(E) = \int_E D_\mu T d\mu = \int_{E\cap D} D_\mu T d\mu = \sum_{z_j \in E\cap D} D_\mu T(z_j)\mu(\{z_j\}) = \sum_{z_j\in E\cap D} \nu_j$$

for $\forall E \subset \mathbf{R}^2$, Borel set. So we conclude that

$$T = \sum_{j=1}^{J} \nu_j \delta_{z_j} \quad \text{in } \mathcal{M}(\mathbf{R}^2). \tag{5.7}$$

(5.4) and (5.7) implies the assertions (1) and (2) for $v^0 \equiv 0$.

Now (5.5) for $z = z_j \in D$ implies

$$|\nu_j| \le (\frac{1}{\bar{S}})^{3/2} \mu_j^{3/2},$$

which is the assertion (3).

The assertion (4) is a direct consequence of the reverse Hölder inequality (5.2) for $\varphi \equiv 1$. We have proved (part 1) of the Theorem for $v^0 \equiv 0$.

Next assume $v^0 \not\equiv 0$ and write $w^n := v^n - v^0$. Then $w^n \rightharpoonup 0$ weakly in H_0^1 and

$$|\nabla w^n|^2 = |\nabla v^n|^2 - 2\nabla v^n \cdot \nabla v^0 + |\nabla v^0|^2 \overset{*}{\rightharpoonup} \mu - |\nabla v^0|^2 dxdy \quad \text{in } \mathcal{M}(\overline{\Omega}).$$

Moreover, define $\tilde{T}^n \in \mathcal{D}'(\mathbf{R}^2)$ as

$$\tilde{T}^n(\varphi) := \int_{\mathbf{R}^2} (\varphi w^n) \cdot w_x^n \wedge w_y^n dxdy \quad \text{for } \forall \varphi \in \mathcal{D}(\mathbf{R}^2),$$

then we also have

$$T^n(\varphi) = \tilde{T}^n(\varphi) + T_0(\varphi) + o(1)$$

as $n \to \infty$ for any $\varphi \in \mathcal{D}'(\mathbf{R}^2)$, according to a lemma of Brezis-Coron [2], Lemma A.12. Now we may apply the above proof to w^n and $\tilde{T}^n$, and we obtain (part 1) of the Theorem for $v^0 \not\equiv 0$.

As noticed in [8], the key point in the proof of (part 2) is the following convexity argument. Let

$$\mu_0 := \int_{\mathbf{R}^2} |\nabla v^0|^2 dxdy.$$

We have

$$\Big(\sum_{j=0}^{J} \mu_j\Big)^{\frac{3}{2}} \le (\mu(\overline{\Omega}))^{\frac{3}{2}} = \bar{S}^{\frac{3}{2}}|T(1)| \le \bar{S}^{\frac{3}{2}}\Big(|T_0(1)| + \sum_{j=1}^{J}|\nu_j|\Big) \le \Big(\sum_{j=0}^{J} \mu_j^{\frac{3}{2}}\Big)$$

by the isoperimetric inequality for the weak limit $v^0 \in H_0^1(\Omega; \mathbf{R}^3)$ and for the atoms. Strict convexity of the function $t \mapsto t^{\frac{3}{2}}$ for $t > 0$ implies that one and only one of the μ_j's can be nonzero.

If $\mu_0 = 0$, then we have $v^0 \equiv 0$ and there exists $\exists 1 j_0 \in \{1, \cdots, J\}$ such that $\mu = \mu_{j_0}\delta_{z_{j_0}} + \tilde{\mu}$ and $T = \nu_{j_0}\delta_{z_{j_0}}$.

If $\tilde{\mu}(\overline{\Omega}) > 0$, then $\mu_{j_0} < \bar{S}$ and the isoperimetric inequality for atoms implies that $|T(1)| = |\nu_{j_0}| < 1$, a contradiction. So in this case, concentration (a) must occur.

If $\mu_0 \neq 0$, then $\mu_j = \nu_j = 0$ $(1 \le \forall j \le J)$, so we have $\mu = |\nabla v^0|^2 dxdy + \tilde{\mu}$ and $T = T_0$. As in the former case, we deduce that $\tilde{\mu} \equiv 0$. We then have $\mu_0 = \mu(\overline{\Omega}) = \bar{S}$, so we obtain

$$\int_\Omega |\nabla v^n|^2 \to \int_\Omega |\nabla v^0|^2 \quad \text{as } n \to \infty.$$

Since we know also $v^n \rightharpoonup v^0$ weakly in $H_0^1(\Omega; \mathbf{R}^3)$, therefore we conclude that $v^n \to v^0$ strongly in $H_0^1(\Omega; \mathbf{R}^3)$ and compactness (b) occurs. □

We finally remark that the Concentration-Compactness alternative for the isoperimetric inequality applies to minimizing sequences for the best isoperimetric constant $\bar{S} = (32\pi)^{1/3}$.

Let $\{v^n\} \subset M$ be a minimizing sequence for $\bar{S}$. Then $|T(1)| = 1$ and $\mu(\overline{\Omega}) = \bar{S}$. If alternative (b) would hold, then the best isoperimetric constant $\bar{S}$ would be attained on a bounded domain Ω, which contradicts the fact $\bar{S}$ is never attained on a domain $\Omega \subset \mathbf{R}^2$, $\Omega \neq \mathbf{R}^2$.

So alternative (a) must hold for minimizing sequences for $\bar{S}$, that is, a subsequence of $\{v^n\}$ concentrates at a single point $z_0 \in \overline{\Omega}$.

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