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SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE*

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Abstract. We study a linear singular first-order integro-differential Cauchy problems in Banach spaces. Singular here means that the integro differential equation is *not* in normal form neither can it be reduced to such a form. We generalize to this context some existence and uniqueness theorems known for differential equations. Particular attention is given to single out the optimal regularity properties of solutions as well as to point out several explicit applications related to singular partial integro-differential of parabolic type.

1. Introduction

In this paper we will be concerned with the following integro-differential initial value problem in the complex Banach space X, with norm $\|\cdot\|$:

$$\text{(P)}\quad \begin{cases} D_t(Mu(t)) + Lu(t) = \displaystyle\int_0^t k(t-s)L_1u(s)\mathrm{d}s + f(t), \qquad 0 \le t \le \tau, \\ Mu(0) = Mu_0. \end{cases} \tag{1.1}$$

We assume that L, L_1, M are *closed* linear operators from X into itself, M being *not* necessarily *invertible*, whose domains are related by the relationship $\mathcal{D}(L) \subseteq \mathcal{D}(L_1) \cap \mathcal{D}(M)$. Moreover, we assume that L *admits a continuous*

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inverse operator. As far as the triplet (k, f, u_0) is concerned, we assume for the time being that

$$k \in L^1((0,\tau);\mathbf{R}) = L^1((0,\tau)), \qquad f \in C([0,\tau];X), \qquad u_0 \in \mathcal{D}(L). \quad (1.2)$$

Notice that $T = ML^{-1} \in \mathcal{L}(X)$, the space of all bounded linear operators from X into itself, endowed with the uniform norm.

In general, only strict solutions to (P) shall be investigated, where by a strict solution u to (P) we mean that $u \in C([0,\tau];\mathcal{D}(L))$, $Mu \in C^1([0,\tau];X)$ and (P) holds.

In this paper we shall confine ourselves to the parabolic case, where the multivalued linear operator $-A = -LM^{-1} := -T^{-1}$, T^{-1} denoting the inverse graph to T, generates an infinitely differentiable semigroup $\{e^{-tA}\}_{t>0}$ (see [2, 1]) given by

$$e^{-tA} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I + A)^{-1}\, d\lambda, \quad \text{for } t > 0, \quad (1.3)$$

with

$$\Gamma = \{\lambda = a - c(1+|y|)^\alpha + iy : -\infty < y < \infty\} \quad (a, c > 0,\ a > c,\ 0 < \alpha \le 1) \quad (1.4)$$

and

$$\|(\lambda I + A)^{-1}\|_{\mathcal{L}(X)} \le C(1+|\lambda|)^{-\beta}, \qquad \text{for all } \lambda \in \Sigma_\alpha \quad (1.5)$$

where

$$\Sigma_\alpha = \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \ge -c(1+|\operatorname{Im}\lambda|)^\alpha\}, \quad (0 < \beta \le \alpha \le 1,\ \alpha+\beta > 1). \quad (1.6)$$

Notice that (1.5) reads equivalently either

$$\|M(\lambda M + L)^{-1}\|_{\mathcal{L}(X)} \le C(1+|\lambda|)^{-\beta}, \qquad \text{for all } \lambda \in \Sigma_\alpha \quad (1.7)$$

or

$$\|L(\lambda M + L)^{-1}\|_{\mathcal{L}(X)} = \|(\lambda T + I)^{-1}\|_{\mathcal{L}(X)} \le C(1+|\lambda|)^{1-\beta}, \quad \text{for all } \lambda \in \Sigma_\alpha. \quad (1.8)$$

We begin our investigation by observing that the change of unknown $Lu(t) = v(t)$ transforms problem (P) into the equivalent one

$$\text{(Q)} \quad \begin{cases} D_t(Tv(t)) + v(t) = \displaystyle\int_0^t k(t-s)Sv(s)\, ds + f(t), & 0 \le t \le \tau, \\ Tv(0) = Tv_0, \end{cases} \quad (1.9)$$

where

$$v_0 = Lu_0 \quad \text{and} \quad S = L_1 L^{-1} \in \mathcal{L}(X). \quad (1.10)$$

Let $w(\cdot)$ be the strict solution to

$$(\mathrm{Q}') \quad \begin{cases} D_t(Tw(t)) + w(t) = \displaystyle\int_0^t k(t-s)Sw(s)\,\mathrm{d}s + g(t), & 0 \le t \le \tau, \\ Tw(0) = 0, \end{cases} \tag{1.11}$$

where

$$g(t) = f(t) - f(0) + \int_0^t k(s)Sf(0)\,\mathrm{d}s. \tag{1.12}$$

Moreover, let $z(\cdot)$ be the solution to

$$(\mathrm{Q}'') \quad \begin{cases} D_t(Tz(t)) + z(t) = \displaystyle\int_0^t k(t-s)Sz(s)\,\mathrm{d}s, & 0 \le t \le \tau, \\ Tz(0) = T[v_0 - f(0)]. \end{cases} \tag{1.13}$$

Then it is readily seen that $v(t) = w(t)+z(t)+f(0)$ is in fact a strict solution to (Q).

Section 2 is devoted to the homogeneous problem (P) corresponding to $f(t) \equiv 0$, while in section 3 we first treat the nonhomogeneous problem and then the general case. We in fact show that, if (1.8) holds and the data f and v_0 are suitably regular, then problem (P) has a (unique) strict solution. In section 4 the regularity in time of the solutions is studied. Section 5 provides two applications to the case where M, L and L_1 are partial differential operators. At last, in sections 6 and 7, after introducing real interpolation spaces, it shall be shown that the hypotheses on the kernel k and the regularity assumptions ensuring existence and maximal regularity can be weakened a bit. This shall be useful in further investigations related to the identification of kernel k in equation (1.1).

2. The homogeneous problem

Let us consider the equation

$$D_t(Tz(t)) + z(t) = \int_0^t k(t-s)Sz(s)\,\mathrm{d}s, \qquad 0 \le t \le \tau, \tag{2.1}$$

with the initial condition

$$Tz(0) = Tz_0, \tag{2.2}$$

where $z_0 \in X$.

In Sections 2–7 we shall suppose that the growth of the kernel $k \in L^1_{loc}([0,+\infty))$ obeys the inequality

$$|k(t)| \le ct^{\gamma_1 - 1}(1+t^{\gamma_2})\mathrm{e}^{-\nu t} \quad \text{for some } \gamma_1 > 0,\ \gamma_2,\ \nu \ge 0, \tag{2.3}$$

and that its Laplace transform can be extended by an holomorphic function $\widehat{k}$ satisfying the estimate

$$|\widehat{k}(\lambda)| \le c_0|\lambda|^{-\sigma},\ \operatorname{Re}\lambda \ge a - c(1+|\operatorname{Im}\lambda|),\ (c_0>0,\ \sigma>1-\beta), \tag{2.4}$$

a and c being the *same* positive constants as in (1.4). Let

$$P(\lambda) = \lambda T + I - \widehat{k}(\lambda)S, \qquad \lambda \in \Sigma_\alpha. \tag{2.5}$$

Observe that

$$P(\lambda) = (\lambda T + I)[I - \widehat{k}(\lambda)(\lambda T + I)^{-1}S]. \tag{2.6}$$

Then hypotheses (1.8) and (2.4) assure that $P(\lambda)$ has a bounded inverse satisfying

$$\|P(\lambda)^{-1}\|_{\mathcal{L}(X)} \le C(1+|\lambda|)^{1-\beta}, \qquad \lambda \in \Sigma_\alpha,\ |\lambda| \ge \lambda_0 \tag{2.7}$$

for some large enough λ_0 (more exactly $\lambda_0^\sigma(1+\lambda_0)^{-1+\beta} \ge 2cC\|S\|_{\mathcal{L}(X)}$).

It is no restriction to assume a suitably large so that any $\lambda \in \Gamma$ implies $|\lambda| \ge \lambda_0$.

We seek now for a solution $z(\cdot)$ to problem (2.1), (2.2) in the form

$$\begin{aligned} z(t) &= ⨍_\Gamma e^{\lambda t} P(\lambda)^{-1} T z_0 \, d\lambda \\ &= z_0 + ⨍_\Gamma \frac{e^{\lambda t}}{\lambda} P(\lambda)^{-1} \widehat{k}(\lambda) S z_0 \, d\lambda - ⨍_\Gamma \frac{e^{\lambda t}}{\lambda} P(\lambda)^{-1} z_0 \, d\lambda. \end{aligned} \tag{2.8}$$

As we shall recognize in the sequel, the key role is played by the family of linear operators

$$R(t) = ⨍_\Gamma \frac{e^{\lambda t}}{\lambda} P(\lambda)^{-1} \, d\lambda, \qquad t > 0. \tag{2.9}$$

where $⨍_\Gamma$ denotes $(2\pi i)^{-1} \int_\Gamma$. Observe that without any additional assumption on z_0, $R(0)z_0$ may have no sense. To estimate $R(t)$ we perform the change of variable $c(1+y)^\alpha t = r$, i.e., $y = -1 + r^{1/\alpha}(ct)^{-1/\alpha}$. From definition (1.4) and estimate (1.8) we deduce

$$\begin{aligned} \|R(t)\|_{\mathcal{L}(X)} &\le 2Ce^{at} \int_0^{+\infty} \frac{e^{-ct(1+y)^\alpha}}{(1+y)^\beta} \, dy \\ &= \frac{2}{\alpha} Ce^{at} t^{(\beta-1)/\alpha} \int_{ct}^{+\infty} e^{-r} r^{(1-\alpha-\beta)/\alpha} \, dr \\ &\le C' e^{at} t^{(\beta-1)/\alpha} \int_0^{+\infty} e^{-r} r^{(1-\alpha-\beta)/\alpha} \, dr = C'' e^{at} t^{(\beta-1)/\alpha},\ \text{if } \beta < 1. \end{aligned} \tag{2.10}$$

If $\beta(=\alpha)=1$, then $\|R(t)\|_{\mathcal{L}(X)} \le C\mathrm{e}^{at}$ holds. Hence

$$\|R(t)\| \le C\mathrm{e}^{at}t^{(\beta-1)/\alpha}, \qquad t>0 \tag{2.11}$$

for any pair (α,β) satisfying (1.6).

Introduce now the functions

$$z_1(t) = \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda}P(\lambda)^{-1}\widehat{k}(\lambda)Sz_0\,\mathrm{d}\lambda, \quad z_2(t) = -\frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda}P(\lambda)^{-1}z_0\,\mathrm{d}\lambda. \tag{2.12}$$

According to (2.4), which implies $\beta+\sigma>1$, we deduce that $z_1(\cdot)$ is bounded by

$$\|z_1(t)\| \le C\|Sz_0\|\mathrm{e}^{at}\int_0^{+\infty} \mathrm{e}^{-ct(1+y)^\alpha}(1+y)^{-\sigma-\beta}\,\mathrm{d}y \tag{2.13}$$

and is continuous at $t=0$, too. More attention must be paid to $z_2(t)$ to show its continuity at $t=0$. For this purpose we observe that if $z_0=Tw_0$, $w_0\in X$, then

$$z_2(t) = -tw_0 + \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda^2}P(\lambda)^{-1}w_0\,\mathrm{d}\lambda - \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda^2}P(\lambda)^{-1}\widehat{k}(\lambda)Sw_0\,\mathrm{d}\lambda. \tag{2.14}$$

Consequently, $z_2(\cdot)$ is continuous at $t=0$ and $z_2(0)=0$. Moreover, it is a trivial task to check that (cf. (1.2))

$$\begin{aligned}
D_tTz_1(t) &= \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda}\widehat{k}(\lambda)Sz_0\,\mathrm{d}\lambda - \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda}\widehat{k}(\lambda)P(\lambda)^{-1}Sz_0\,\mathrm{d}\lambda \\
&\quad + \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda}\widehat{k}(\lambda)^2SP(\lambda)^{-1}Sz_0\,\mathrm{d}\lambda \\
&= -z_1(t) + \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda}\widehat{k}(\lambda)Sz_0\,\mathrm{d}\lambda + \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda}\widehat{k}(\lambda)^2SP(\lambda)^{-1}Sz_0\,\mathrm{d}\lambda \\
&= -z_1(t) + \int_0^t k(t-s)Sz_0\,\mathrm{d}s + \frac{1}{2\pi i}\int_\Gamma \mathrm{e}^{\lambda t}\widehat{k}(\lambda)S\widehat{k}(\lambda)\frac{P(\lambda)^{-1}}{\lambda}Sz_0\,\mathrm{d}\lambda \\
&= -z_1(t) + \int_0^t k(t-s)Sz_0\,\mathrm{d}s + \frac{1}{2\pi i}\int_\Gamma \mathrm{e}^{\lambda t}\widehat{k}(\lambda)\widehat{Sz_1}(\lambda)\,\mathrm{d}\lambda \\
&= -z_1(t) + \int_0^t k(t-s)Sz_0\,\mathrm{d}s + \int_0^t k(t-s)Sz_1(s)\,\mathrm{d}s \\
&= -z_1(t) + \int_0^t k(t-s)S[z_0+z_1(s)]\,\mathrm{d}s, \qquad 0\le t\le\tau.
\end{aligned} \tag{2.15}$$

Notice that $z_1(t)=\int_0^t k(t-s)R(s)Sz_0\,\mathrm{d}s$ since $\widehat{R}(\lambda)=\lambda^{-1}P(\lambda)^{-1}$.

From (2.14) we easily deduce the formula

$$\begin{aligned} Tz_2(t) = -tTw_0 + \frac{t^2}{2}w_0 - \int_\gamma \frac{e^{\lambda t}}{\lambda^3}P(\lambda)^{-1}w_0\,d\lambda \\ + \int_\Gamma \frac{e^{\lambda t}}{\lambda^3}\widehat{k}(\lambda)SP(\lambda)^{-1}w_0\,d\lambda - \int_\Gamma \frac{e^{\lambda t}}{\lambda^3}\widehat{k}(\lambda)Sw_0\,d\lambda \\ - \int_\Gamma \frac{e^{\lambda t}}{\lambda^3}\widehat{k}(\lambda)SP(\lambda)^{-1}\widehat{k}(\lambda)Sw_0\,d\lambda + \int_\Gamma \frac{e^{\lambda t}}{\lambda^3}P(\lambda)^{-1}\widehat{k}(\lambda)Sw_0\,d\lambda. \end{aligned} \tag{2.16}$$

In turn, (2.16) implies

$$\begin{aligned} D_tTz_2(t) &= -Tw_0 + tw_0 - \int_\Gamma \frac{e^{\lambda t}}{\lambda^2}P(\lambda)^{-1}w_0\,d\lambda + \int_\Gamma \frac{e^{\lambda t}}{\lambda^2}\widehat{k}(\lambda)SP(\lambda)^{-1}w_0\,d\lambda \\ &\quad - \int_\Gamma \frac{e^{\lambda t}}{\lambda^2}\widehat{k}(\lambda)Sw_0\,d\lambda - \int_\Gamma \frac{e^{\lambda t}}{\lambda^2}\widehat{k}(\lambda)SP(\lambda)^{-1}\widehat{k}(\lambda)Sw_0\,d\lambda \\ &\quad + \int_\Gamma \frac{e^{\lambda t}}{\lambda^2}P(\lambda)^{-1}\widehat{k}(\lambda)Sw_0\,d\lambda \\ &= -z_2(t) - Tw_0 + \int_\Gamma \frac{e^{\lambda t}}{\lambda^2}\widehat{k}(\lambda)SP(\lambda)^{-1}w_0\,d\lambda - \int_\Gamma \frac{e^{\lambda t}}{\lambda^2}\widehat{k}(\lambda)Sw_0\,d\lambda \\ &\quad - \int_\Gamma \frac{e^{\lambda t}}{\lambda^2}\widehat{k}(\lambda)SP(\lambda)^{-1}\widehat{k}(\lambda)Sw_0\,d\lambda. \end{aligned} \tag{2.17}$$

On the other hand, since

$$z_2(t) = -tw_0 + \int_0^t R(s)w_0\,ds - \int_0^t (k*R)(s)Sw_0\,ds, \tag{2.18}$$

where $(k*R)(t) = \int_0^t k(t-s)R(s)\,ds$, the Laplace transform of $z_2(\cdot)$ is given by

$$\widehat{z}_2(\lambda) = -\lambda^{-2}w_0 + \lambda^{-2}P(\lambda)^{-1}w_0 - \lambda^{-2}\widehat{k}(\lambda)P(\lambda)^{-1}Sw_0, \quad \lambda \in \Gamma. \tag{2.19}$$

Hence the identity

$$\begin{aligned} \widehat{k}(\lambda)S\widehat{z}_2(\lambda) = &-\lambda^{-2}\widehat{k}(\lambda)Sw_0 + \lambda^{-2}\widehat{k}(\lambda)SP(\lambda)^{-1}w_0 \\ &- \lambda^{-2}\widehat{k}(\lambda)S\widehat{k}(\lambda)P(\lambda)^{-1}Sw_0 \end{aligned} \tag{2.20}$$

and the assumption $z_0 = Tw_0$ show that

$$D_tTz_2(t) = -z_2(t) - z_0 + \int_0^t k(t-s)Sz_2(s)\,ds, \qquad 0 \le t \le \tau. \tag{2.21}$$

Therefore,

$$\begin{aligned} D_t(Tz(t)) &= D_t\left[T(z_0 + z_1(t) + z_2(t)\right] \\ &= -z_1(t) + \int_0^t k(t-s)S[z_0 + z_1(s)]\,\mathrm{d}s - z_2(t) - z_0 \\ &+ \int_0^t k(t-s)Sz_2(s)\,\mathrm{d}s \\ &= -(z_0 + z_1(t) + z_2(t)) + \int_0^t k(t-s)S[z_0 + z_1(s) + z_2(s)]\,\mathrm{d}s \\ &= -z(t) + \int_0^t k(t-s)Sz(s)\,\mathrm{d}s, \qquad 0 \le t \le \tau. \end{aligned} \tag{2.22}$$

Moreover, from $z_1(0) = z_2(0) = 0$, it follows that (2.2) holds, too.

We have thus proved

Proposition 2.1. *Under assumptions* (1.8), (2.3), (2.4) *with* $\alpha + \beta > 1$, *problem* (2.1), (2.2) *has one strict solution* z *for any* $z_0 = Tw_0 \in \mathcal{R}(T)$. *Moreover* z *admits the representation*

$$\begin{aligned} z(t) = z_0 - tw_0 &+ \int_0^t R(s)w_0\,\mathrm{d}s \\ &+ \int_0^t k(t-s)R(s)Sz_0\,\mathrm{d}s - \int_0^t (k*R)(s)Sw_0\,\mathrm{d}s. \end{aligned} \tag{2.23}$$

3. The nonhomogeneous problem

We first observe that for any Laplace transformable function f the natural candidate for a solution to the Cauchy problem

$$D_t(Tw(t)) + w(t) = \int_0^t k(t-s)Sw(s)\,\mathrm{d}s + f(t), \qquad 0 \le t \le \tau, \tag{3.1}$$

$$Tw(0) = 0, \tag{3.2}$$

is given by

$$w(t) = \int\!\!\!\!\!-_\Gamma \mathrm{e}^{\lambda t} P(\lambda)^{-1} \widehat{f}(\lambda)\,\mathrm{d}\lambda, \qquad t > 0. \tag{3.3}$$

Now, without any further assumption, w may be *not* defined at $t = 0$. On the other hand, if $f(0) = 0$, formally we get

$$w(t) = \int\!\!\!\!\!-_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda} P(\lambda)^{-1} \lambda \widehat{f}(\lambda)\,\mathrm{d}\lambda$$

$$= \quad \fint_\Gamma \mathrm{e}^{\lambda t}\widehat{R}(\lambda)\widehat{f'}(\lambda)\,\mathrm{d}\lambda = \int_0^t R(t-s)f'(s)\,\mathrm{d}s \tag{3.4}$$

where $R(\cdot)$ is the operator–valued function introduced in (2.9).

We can now get rid of the assumption that f should be Laplace transformable and we can simply assume that a candidate solution w to the Cauchy problem (3.1), (3.2) is defined by the last side in (3.4). To show that w is actually a solution we need the following lemma.

Lemma 3.1. *Under assumptions* (1.8), (2.3), (2.4) *and* $\alpha+\beta>1$*,*

$$D_tTR(t) = I - R(t) + (k * SR)(t), \quad \text{in } \mathcal{L}(X), \quad t>0, \tag{3.5}$$

$$\|D_tTR(t)\|_{\mathcal{L}(X)} \le C\mathrm{e}^{at}t^{(\beta-1)/\alpha} \qquad t>0. \tag{3.6}$$

Proof. Equation (3.5) easily follows from the identities

$$\begin{aligned} TR(t) &= \fint_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda^2}[P(\lambda)^{-1} - I + \widehat{k}(\lambda)S]P(\lambda)^{-1}\,\mathrm{d}\lambda \\ &= tI - \fint_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda^2}P(\lambda)^{-1}\,\mathrm{d}\lambda + \fint_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda^2}\widehat{k}(\lambda)SP(\lambda)^{-1}\,\mathrm{d}\lambda,\ t>0. \end{aligned} \tag{3.7}$$

Estimate (3.6) is an immediate consequence of representation (3.5).

Let $f \in C^1([0,\tau],X)$, $f(0)=0$. In view of the previous lemma, since $\alpha+\beta>1$, we deduce that the function $w(\cdot)$ defined by the last side in (3.4) is an X–valued strongly continuous function on $[0,\tau]$ satisfying $w(0)=0$. Moreover,

$$\begin{aligned} Tw(t) &= \int_0^t TR(t-s)f'(s)\,\mathrm{d}s \\ &= \int_0^t (t-s)f'(s)\,\mathrm{d}s - \int_0^t \mathrm{d}s \fint_\Gamma \frac{\mathrm{e}^{\lambda(t-s)}}{\lambda^2}P(\lambda)^{-1}f'(s)\,\mathrm{d}\lambda \\ &\quad + \int_0^t \mathrm{d}s \fint_\Gamma \frac{\mathrm{e}^{\lambda(t-s)}}{\lambda^2}\widehat{k}(\lambda)SP(\lambda)^{-1}f'(s)\,\mathrm{d}\lambda. \end{aligned} \tag{3.8}$$

Since $t \to \|R(t)\|_{\mathcal{L}(X)}$ is integrable on $(0,\tau)$, it is easy to check (cf. (3.7)) that

$$\begin{aligned} D_tTw(t) &= D_t\int_0^t TR(t-s)f'(s)\,\mathrm{d}s = TR(0)f'(t) + \int_0^t D_tTR(t-s)f'(s)\,\mathrm{d}s \\ &= \int_0^t [I - R(t-s) + (k*SR)(t-s)]f'(s)\,\mathrm{d}s \\ &= f(t) - w(t) + [(k*SR)*f'](t) = f(t) - w(t) + [k*(SR*f')](t) \\ &= f(t) - w(t) + (k*Sw)(t), \qquad 0 \le t \le \tau. \end{aligned} \tag{3.9}$$

Thus we have

Proposition 3.1. *Under assumptions* (1.8), (2.3), (2.4), *with* $\alpha+\beta>1$, *problem* (3.1), (3.2) *has at least one solution for any* $f \in C^1([0,\tau];X)$ *satisfying* $f(0)=0$.

Taking into account the reduction of problem (P) to problems (Q') and (Q") we obtain

Theorem 3.1. *Let* $k \in C([0,\tau])$ *satisfy properties* (2.3), (2.4) *and let* (1.8), $\alpha+\beta>1$, $0<\beta\le\alpha\le 1$ *be fulfilled. Then for any pair* $(f,u_0) \in C^1([0,\tau];X)\times\mathcal{D}(L)$ *such that* $f(0)-Lu_0 \in \mathcal{R}(T)=\mathcal{R}(ML^{-1})$, *problem* (P) *has at least one strict solution* u *represented by*

$$\begin{aligned} u(t) &= u_0 - tL^{-1}w_0 + \int_0^t L^{-1}R(s)w_0\,\mathrm{d}s \\ &+ \int_0^t k(t-s)L^{-1}R(s)L_1[u_0-L^{-1}f(0)]\,\mathrm{d}s - \int_0^t k*(L^{-1}R)(s)L_1L^{-1}w_0\,\mathrm{d}s \\ &+ \int_0^t L^{-1}R(t-s)[f'(s)+k(s)L_1L^{-1}f(0)]\,\mathrm{d}s \end{aligned} \tag{3.10}$$

where

$$Lu_0 - f(0) = ML^{-1}w_0. \tag{3.11}$$

The uniqueness of the solution follows from the growth condition (1.8) and well–known properties of Laplace transforms.

Remark 3.1. Let Γ' be the path

$$\Gamma' = \{\lambda = a - c(1+|y|) + iy : -\infty < y < \infty\} \tag{3.12}$$

oriented from $\infty\,\mathrm{e}^{-i\eta}$ to $\infty\,\mathrm{e}^{i\eta}$ ($\tan\eta = -1/c$). Under assumption (2.4), with $\sigma>1$, from the identity

$$\begin{aligned} k(t)-k(r) &= \frac{1}{2\pi i}\int\!\!\!\!-_{\Gamma'} \frac{\mathrm{e}^{\lambda t}-\mathrm{e}^{\lambda r}}{\lambda}\lambda\widehat{k}(\lambda)\,\mathrm{d}\lambda = \int\!\!\!\!-_{\Gamma'} \lambda\widehat{k}(\lambda)\,\mathrm{d}\lambda \int_r^t \mathrm{e}^{\lambda s}\,\mathrm{d}s \\ &= \int_r^t \mathrm{d}s\Big(\int\!\!\!\!-_{\Gamma'} \mathrm{e}^{\lambda s}\lambda\widehat{k}(\lambda)\,\mathrm{d}\lambda\Big) \qquad 0\le r\le t \end{aligned} \tag{3.13}$$

we derive the inequality

$$|k(t)-k(r)| \le C\int_r^t \mathrm{d}s\int_{\Gamma'} \mathrm{e}^{s\mathrm{Re}\,\lambda}|\lambda|^{1-\sigma}|\,\mathrm{d}\lambda|. \tag{3.14}$$

Observe now that

$$\int_{\Gamma'} \mathrm{e}^{s\mathrm{Re}\,\lambda}|\lambda|^{1-\sigma}|\,\mathrm{d}\lambda| \le C\mathrm{e}^{as}\int_0^{+\infty} \mathrm{e}^{-cs(1+y)}(1+y)^{1-\sigma}\,\mathrm{d}y. \tag{3.15}$$

If $\sigma > 2$, then

$$|k(t)-k(r)| \le C' \int_r^t \mathrm{e}^{as}\,\mathrm{d}s \le C'\mathrm{e}^{a\tau}|t-r| \le C''|t-r|,\ 0 \le r < t \le \tau. \quad (3.16)$$

where $C'' = C'\mathrm{e}^{a\tau}$.

If $\sigma = 2$ we are in a situation like that of analytic semigroups. Changing Γ' to $s^{-1}\Gamma'$ we see again that $|k(t) - k(r)| \le C''|t - r|$.

If $1 < \sigma < 2$, the change of variable $c(1 + y)s = \nu$ yields

$$\int_0^{+\infty} \mathrm{e}^{-cs(1+y)}(1 + y)^{1-\sigma}\,\mathrm{d}y = (cs)^{\sigma-2} \int_{cs}^{+\infty} \nu^{1-\sigma}\mathrm{e}^{-\nu}\,\mathrm{d}\nu \le (cs)^{\sigma-2} \int_0^{+\infty} \nu^{1-\sigma}\mathrm{e}^{-\nu}\,\mathrm{d}\nu = Cs^{\sigma-2}. \quad (3.17)$$

Hence, by the means of the change of variable $s = \tau + \rho(t - \tau)$ we obtain, for any $0 \le r \le t$,

$$|k(t) - k(r)| \le C \int_r^t s^{\sigma-2}\mathrm{e}^{as}\,\mathrm{d}s \le C\mathrm{e}^{at} \int_r^t s^{\sigma-2}\,\mathrm{d}s \le C\mathrm{e}^{a\tau}(t - r)^{\sigma-1} \int_0^1 \rho^{\sigma-2}\,\mathrm{d}\rho \le C'\mathrm{e}^{a\tau}(t - r)^{\sigma-1} = C''(t - r)^{\sigma-1}. \quad (3.18)$$

Therefore, for any $\sigma > 1$ we have that, under (2.3),

$$|k(t) - k(r)| \le C(\tau)|t - r|^{K(\sigma)} \qquad \forall t, r \in [0, \tau] \quad (3.19)$$

where $K(\sigma) = \min(\sigma - 1, 1)$. In particular, k is continuous on $[0, +\infty)$.

4. Regularity in time of the solutions

Henceforth we shall suppose thet the kernel k satisfies (2.3) and (2.4) with $\sigma > 1$, so that k is continuous on $[0, \infty)$. If $z_0 = Lu_0 - f(0)$, from

$$z_1(t) = \fint_\Gamma e^{\lambda t}\widehat{R}(\lambda)\widehat{k}(\lambda)Sz_0\,\mathrm{d}\lambda = \int_0^t k(s)R(t-s)Sz_0\,\mathrm{d}s = \int_0^t k(t-s)R(s)Sz_0\,\mathrm{d}s, \quad (4.1)$$

it follows that

$$z_1(t)-z_1(\bar{t}) = \int_0^{\bar{t}} [k(t-s)-k(\bar{t}-s)]R(s)Sz_0\,\mathrm{d}s + \int_{\bar{t}}^t k(t-s)R(s)Sz_0\,\mathrm{d}s. \quad (4.2)$$

According to (3.19) this implies for any $\tau \ge t > \bar{t} \ge 0$,

$$\|z_1(t) - z_1(\bar{t})\| \le C\|Sz_0\| \int_0^{\bar{t}} |t - \bar{t}|^{K(\sigma)}\|R(s)\|_{\mathcal{L}(X)}\,\mathrm{d}s + C(\tau)\|Sz_0\| \int_{\bar{t}}^t s^{\frac{\beta-1}{\alpha}}\,\mathrm{d}s$$

$$\leq C\|Sz_0\|\Big[|t-\bar{t}|^{K(\sigma)}\int_0^{\bar{t}} s^{\frac{\beta-1}{\alpha}}\,\mathrm{d}s + C(\tau)\int_{\bar{t}}^{t} s^{\frac{\beta-1}{\alpha}}\,\mathrm{d}s\Big]. \tag{4.3}$$

Since $\alpha+\beta>1$, the change of variable $s=(1-\nu)\bar{t}+\nu t$ in the last integral yields

$$\int_{\bar{t}}^{t} s^{(\beta-1)/\alpha}\,\mathrm{d}s = (t-\bar{t})\int_0^1 [\bar{t}+\nu(t-\bar{t})]^{(\beta-1)/\alpha}\,\mathrm{d}\nu$$
$$\leq (t-\bar{t})^{(\alpha+\beta-1)/\alpha}\int_0^1 \nu^{(\beta-1)/\alpha}\,\mathrm{d}\nu = C(t-\bar{t})^{(\alpha+\beta-1)/\alpha}. \tag{4.4}$$

Therefore, there exists $C'=C'(\tau)>0$ such that

$$\|z_1(t)-z_1(\bar{t})\|\leq C'|t-\bar{t}|^{\rho}, \tag{4.5}$$

where

$$\rho = \min\,[K(\sigma),(\alpha+\beta-1)/\alpha] = \min\,[\sigma-1,(\alpha+\beta-1)/\alpha]. \tag{4.6}$$

We will now estimate $z_2(t)-z_2(\bar{t})$, where $z_0 = Tw_0$, $w_0\in X$. For this purpose we exploit the representation

$$z_2(t)-z_2(\bar{t}) = (\bar{t}-t)w_0 + \int\!\!\!\!\!\!-_{\Gamma} \frac{\mathrm{e}^{\lambda t}-\mathrm{e}^{\lambda\bar{t}}}{\lambda^2}P(\lambda)^{-1}w_0\,\mathrm{d}\lambda$$
$$- \int\!\!\!\!\!\!-_{\Gamma} \frac{\mathrm{e}^{\lambda t}-\mathrm{e}^{\lambda\bar{t}}}{\lambda^2}P(\lambda)^{-1}\widehat{k}(\lambda)Sw_0\,\mathrm{d}\lambda$$
$$= \zeta_1(t,\bar{t})+\zeta_2(t,\bar{t})+\zeta_3(t,\bar{t}) \qquad t,\bar{t}\in[0,\tau]. \tag{4.7}$$

Observe that

$$\zeta_2(t,\bar{t}) = \int\!\!\!\!\!\!-_{\Gamma}\Big(\int_{\bar{t}}^{t}\mathrm{e}^{\lambda s}\,\mathrm{d}s\Big)\frac{P(\lambda)^{-1}}{\lambda}w_0\,\mathrm{d}\lambda = \int_{\bar{t}}^{t} R(s)w_0\,\mathrm{d}s. \tag{4.8}$$

Hence,

$$\|\zeta_2(t,\bar{t})\|\leq C\|w_0\|\int_{\bar{t}}^{t} s^{(\beta-1)/\alpha}\,\mathrm{d}s \leq C\|w_0\|(t-\bar{t})^{(\alpha+\beta-1)/\alpha}. \tag{4.9}$$

Since

$$\zeta_3(t,\bar{t}) = -\int_{\bar{t}}^{t}\mathrm{d}s\int\!\!\!\!\!\!-_{\Gamma}\mathrm{e}^{\lambda s}\widehat{R}(\lambda)\widehat{k}(\lambda)Sw_0\,\mathrm{d}\lambda \tag{4.10}$$
$$= -\int_{\bar{t}}^{t}\Big[\int_0^s k(s-\nu)R(\nu)\,d\nu\Big]Sw_0\,\mathrm{d}s,$$

we already know that

$$\|\zeta_3(t,\bar{t})\|\leq C(\tau)\|w_0\|(t-\bar{t}). \tag{4.11}$$

Therefore, we get again

$$\|z_2(t) - z_2(\bar{t})\| \le C(\tau)\|w_0\|(t-\bar{t})^\rho. \tag{4.12}$$

Let now

$$\begin{aligned} w(t) &= \int_0^t R(t-s)[f'(s) + k(s)Sf(0)]\,\mathrm{d}s \\ &= \int_0^t R(s)[f'(t-s) + k(t-s)Sf(0)]\,\mathrm{d}s. \end{aligned} \tag{4.13}$$

Suppose $f \in C^{1+\theta}([0,\tau];X)$ for some $\theta \in (0,1)$. Then for $0 \le t' < t \le \tau$, we get (cf. (4.6))

$$\begin{aligned} &\|w(t) - w(t')\| \\ &= \Big\| \int_0^{t'} R(s)[f'(t-s) - f'(t'-s) + (k(t-s) - k(t'-s))Sf(0)]\,\mathrm{d}s \\ &+ \int_{t'}^t R(s)[f'(t-s) + k(t-s)Sf(0)]\,\mathrm{d}s\Big\| \\ &\le \big\{C\|f'\|_{C^\theta([0,\tau];X)}|t-t'|^\theta + C\|Sf(0)\|\|k\|_{C^{K(\sigma)}([0,\tau])}|t-t'|^{K(\sigma)}\big\} \\ &\times \int_0^{t'} s^{\frac{\beta-1}{\alpha}}\,\mathrm{d}s + \Big\{\|f'\|_{C([0,\tau];X)} + \|k\|_{C([0,\tau])}\|Sf(0)\|\Big\}\int_{t'}^t \|R(s)\|_{\mathcal{L}(X)}\,\mathrm{d}s \\ &\le C\|f'\|_{C^\theta([0,\tau];X)}|t-t'|^\theta + C|t-t'|^{K(\sigma)}\|f(0)\|\|k\|_{C^{K(\sigma)}([0,\tau])} \\ &+ C''\Big[\|k\|_{C([0,\tau])}\|f(0)\| + \|f'\|_{C([0,\tau];X)}\Big]|t-t'|^{(\alpha+\beta-1)/\alpha} \\ &= C\big(|t-t'|^\theta + |t-t'|^\rho\big)\big(\|f\|_{C^{1+\theta}([0,\tau];X)} + \|k\|_{C^{K(\sigma)}([0,\tau])}\|f(0)\|\big). \end{aligned} \tag{4.14}$$

Hence, if $0 < \theta \le \rho$, where $\rho = \min[\sigma - 1, (\alpha+\beta-1)/\alpha]$, $\sigma > 1$, then $Lu \in C^\theta([0,\tau];X)$ for any pair $(f, u_0) \in C^{1+\theta}([0,\tau];X) \times \mathcal{D}(L)$ with $f(0) - Lu_0 \in R(T)$. On the other hand, from $w = Lu \in C^\theta([0,\tau];X)$ it follows that

$$\begin{aligned} &\Big\| \int_0^t k(s)Sw(t-s)\,\mathrm{d}s - \int_0^{\bar{t}} k(s)Sw(\bar{t}-s)\,\mathrm{d}s\Big\| \\ &= \Big\| \int_0^{\bar{t}} k(s)S[w(t-s) - w(\bar{t}-s)]\,\mathrm{d}s + \int_{\bar{t}}^t k(s)Sw(t-s)\,\mathrm{d}s\Big\| \\ &\le C\|S\|\|w\|_{C^\theta([0,\tau];X)}\Big(|t-\bar{t}|^\theta \int_0^{\bar{t}} |k(s)|\,\mathrm{d}s + \|k\|_{C([0,\tau])}|t-\bar{t}|\Big) \\ &\le C'|t-\bar{t}|^\theta\Big(\|w\|_{C^\theta([0,\tau];X)} + \|k\|_{C([0,\tau])}\Big). \end{aligned} \tag{4.15}$$

Likewise we can show that $D_t(Mu) \in C^\theta([0,\tau];X)$. Therefore:

Theorem 4.1. *Let k enjoy properties* (2.3), (2.4) *with $\sigma > 1$ and let operators L, M satisfy* (1.8) *with $\alpha + \beta > 1$. If $0 < \theta \le \min[\sigma - 1, (\alpha+\beta-1)/\alpha]$, $\theta < 1$, then for any pair $(f, u_0) \in C^{1+\theta}([0,\tau];X) \times \mathcal{D}(L)$ such that $f(0) - Lu_0 \in \mathcal{R}(T) = \mathcal{R}(ML^{-1})$, problem (P) admits a unique solution u with regularity Lu, $D_t(Mu) \in C^\theta([0,\tau];X)$. Moreover, u is represented by* (3.10).

5. Applications

Application 1. Consider the degenerate integrodifferential parabolic equation

$$\begin{aligned} D_t u(t,x) &= a(x)\Delta u + \int_0^t k(t-s)a(x)^m \Delta u(x,s)\,\mathrm{d}s + f(x,t), \\ &\qquad (x,t) \in \Omega \times (0,\tau), \\ u(x,t) &= 0, \quad (x,t) \in \partial\Omega \times (0,\tau), \\ u(x,0) &= u_0(x), \quad x \in \Omega. \end{aligned} \tag{5.1}$$

We assume here that $m > 1$ and $a \in L^\infty(\Omega)$ satisfies $a(x) \ge 0$ on $\overline{\Omega}$, $a(x) > 0$ almost everywhere in Ω, Ω being a bounded domain in $\mathbf{R}^n$, $n \ge 1$, with a boundary $\partial\Omega$ of class C^2. Moreover, let $M_1 : L^2(\Omega) \to H^{-1}(\Omega)$ be the multiplication operator defined by $m(x) = a(x)^{-1}$ and let $L_1 u = \Delta u$ in $H^{-1}(\Omega)$ with $\mathcal{D}(L_1) = H_0^1(\Omega)$. Then it is shown in [1], Example 3.9, that under the assumption

$$1/a \in L^r(\Omega) \begin{cases} \text{for some } r \ge 2 \text{ when } n = 1, \\ \text{for some } r > 2 \text{ when } n = 2, \\ \text{for some } r \ge n \text{ when } n \ge 3, \end{cases} \tag{5.2}$$

the linear operator L defined by $L = M_1^{-1}L_1 : H_0^1(\Omega) \to L^2(\Omega)$, $\mathcal{D}(L) = \{u \in H_0^1(\Omega) : a\Delta u \in L^2(\Omega)\}$, satisfies

$$\|(\lambda I + L)^{-1}\|_{\mathcal{L}(X)} \le C|\lambda|^{-(2r-n)/(2r)}, \text{ for any } \operatorname{Re}\lambda \ge -c(1+|\operatorname{Im}\lambda|),\ (c>0), \tag{5.3}$$

i.e., (1.8) is satisfied with $\alpha = 1, \beta = (2r-n)/(2r)$.

Let $M = I$, the identity operator in $L^2(\Omega)$, and let $L_1 = a^m\Delta = a^{m-1}L$. Then Theorems 1 and 2 work for any kernel k continuous on $[0, +\infty)$ satisfying (2.4), with $\sigma > n/(2r)$ and $\sigma > 1$, respectively.

Application 2. Let K, H be two closed operators in the complex Banach space X such that $\mathcal{D}(K) \subseteq \mathcal{D}(H)$ and -1 is an eigenvalue of K of multiplicity one, so that

$$\|[(\lambda+1)I+K]^{-1}\|_{\mathcal{L}(X)} \le \frac{c}{|\lambda|} \qquad \text{for} \quad 0<|\lambda|\le\varepsilon,\ (\varepsilon>0). \tag{5.4}$$

Observe now that by means of the change of the unknown function $u = e^{-(\nu+1)t}v$ the equation

$$D_t((1+K)v)(t) = Kv(t) + \int_0^t l(t-s)Hv(s)\,ds + f(t), \quad 0\le t\le\tau, \tag{5.5}$$

becomes

$$D_t((I+K)u)(t) = -[\nu(K+I)+I]u(t) + \int_0^t l(t-s)e^{-(\nu+1)(t-s)}Hu(s)\,ds + f_\nu(t), \tag{5.6}$$

where $f_\nu(t) = e^{-(\nu+1)t}f(t)$. Choosing $\nu = 2/\varepsilon$ we easily deduce that $M = I+K$, $L=\nu(K+I)+I$ and $L_1 = H$, satisfy assumption (1.8) with $\alpha=\beta=1$. Of course in this case the kernel k is given by $k(t) = e^{-(\nu+1)t}l(t)$.

Various partial equations of interest in applied sciences can be described in the above abstract form. For example, in the Banach space $X = C([0,\pi])$ endowed with the uniform norm we can choose $H=K$ and

$$\mathcal{D}(K) = \{u\in C^2([0,\pi]) : u(0)=u(\pi)=u''(0)=u''(\pi)=0\}, \quad Ku=u''. \tag{5.7}$$

6. Maximal regularity: A further approach

In this section we show that the results in the sections 3 and 4 can be improved in some sense both with respect to the initial conditions and the nonhomogeneous term.

First we consider the nonhomogeneous problem (3.1), (3.2), where f is supposed to be only an element of

$$C_0^\theta([0,\tau];X) = \{f\in C^\theta([0,\tau];X) : f(0)=0\}, \tag{6.1}$$

where $\theta\in(0,1)$ will be more carefully determined later on.

To show that the solution w to problem (3.1), (3.2) enjoys the maximal regularity $w, D_tw \in C^\omega([0,\tau],X)$ for some optimal $\omega\in(0,1)$ we will proceed by modifying (and improving) the operator technique developed in [FY1] for the equation $BTv+v=f$. For this purpose we suppose that B is a closed linear operator from the complex Banach space E into itself such that

$$\|(B-zI)^{-1}\|_{\mathcal{L}(E)} \le C(1+|\mathrm{Re}\ z|)^{-1}, \quad \mathrm{Re}\ z\le a_0,\ a_0>0 \tag{6.2}$$

Observe now that T is a bounded linear operator from E into itself satisfying

$$\|(zT+I)^{-1}\|_{\mathcal{L}(E)} \le C(1+|z|)^{1-\beta} \qquad \text{for all } z\in\Sigma_\alpha \tag{6.3}$$

where (cf. section 1)

$$\Sigma_\alpha = \{z \in \mathbf{C} : \mathrm{Re}\, z \geq -c(1+|\mathrm{Im}\, z|)^\alpha\} \quad (0<\beta\leq\alpha\leq 1). \tag{6.4}$$

Finally, assume that S is another bounded linear operator from E to E, while $k(\cdot)$ is a kernel with properties (2.3) and (2.4).

If $P(\lambda) = \lambda T - I - \widehat{k}(\lambda)S$ and Γ is the contour $\lambda = a - c(1+|y|)^\alpha + iy$, $-\infty < y < \infty$ $(a > c > 0)$, let us choose $a_0 > a - c$ suitably large so that $P(\lambda)$ has a bounded inverse for any large $\lambda \in \Sigma_\alpha$:

$$\|P(\lambda)^{-1}\|_{\mathcal{L}(X)} \leq C(1+|\lambda|)^{1-\beta}, \quad \lambda \in \Sigma_\alpha,\ |\lambda| \geq \lambda_0 \text{ for some } \lambda_0 > 0. \tag{6.5}$$

Of course, having in mind our application, we are going to assume that B^{-1} commutes with T and S:

$$B^{-1}T = TB^{-1}, \qquad B^{-1}S = SB^{-1}. \tag{6.6}$$

In our concrete case we have $B = D_t$, $\mathcal{D}(B) = \{u \in C^1([0,\tau];X) : u(0) = 0\}$, $E = C([0,\tau];X)$. Observe that in this case condition (6.2) is satisfied since the function $\mu \to (1+|\mu|)\int_0^T \mathrm{e}^{s\mu}\,\mathrm{d}s$ is bounded on $(-\infty, 0]$. Moreover, we still denote by T, S the operators induced by T and S in $C([0,\tau];X)$, respectively. Notice that in our specific case operator B commutes with the integral operator $k * S$. Let us now associate with any f in the real interpolation space $(E, \mathcal{D}(B))_{\theta,\infty}$, $0<\theta<1$ the element

$$w = \int_\Gamma \lambda^{-1} P(\lambda)^{-1} B(B-\lambda I)^{-1} f\,\mathrm{d}\lambda. \tag{6.7}$$

To estimate w first we consider the following identity holding for any $\lambda \in \Sigma_\alpha$:

$$\begin{aligned}(\lambda I - B)^{-1} = &-(|\mathrm{Im}\,\lambda| I + B)^{-1}\\ &+ (\lambda + |\mathrm{Im}\,\lambda|)(\lambda I - B)^{-1}(|\mathrm{Im}\,\lambda| I + B)^{-1},\end{aligned} \tag{6.8}$$

which implies

$$B(\lambda I - B)^{-1} = \left\{-I + (\lambda + |\mathrm{Im}\,\lambda|)(\lambda I - B)^{-1}\right\} B(|\mathrm{Im}\,\lambda| I + B)^{-1} \quad \forall \lambda \in \Sigma_\alpha. \tag{6.9}$$

Whence we easily derive the inequality

$$\begin{aligned}\|w\|_E &\leq C\|f\|_{\theta,\infty} \int_\Gamma |\lambda|^{-1}(1+|\lambda|)^{1-\beta}\Big(1 + \frac{|\lambda + |\mathrm{Im}\,\lambda|\,|}{|\mathrm{Re}\,\lambda| + 1}\Big)(|\mathrm{Im}\,\lambda| + 1)^{-\theta}|\,\mathrm{d}\lambda|\\ &\leq C'\|f\|_{\theta,\infty} \int_{-\infty}^{+\infty} (1+|y|)^{-\beta+1-\alpha-\theta}\,\mathrm{d}y.\end{aligned} \tag{6.10}$$

The last integral in (6.10) converges provided that $1 > \theta > 2 - \alpha - \beta$ and $\alpha + \beta > 1$. Furthermore,

$$\begin{aligned} Tw &= \fint_\Gamma \lambda^{-2}[(\lambda T + I - \widehat{k}(\lambda)S) - I + \widehat{k}(\lambda)S]P(\lambda)^{-1}B(B-\lambda I)^{-1}f\,\mathrm{d}\lambda \\ &= B^{-1}f - \fint_\Gamma \lambda^{-2}P(\lambda)^{-1}B(B-\lambda I)^{-1}f\,\mathrm{d}\lambda \\ &+ \fint_\Gamma \lambda^{-2}\widehat{k}(\lambda)SP(\lambda)^{-1}B(B-\lambda I)^{-1}f\,\mathrm{d}\lambda \\ &= B^{-1}f - \fint_\Gamma \lambda^{-1}P(\lambda)^{-1}(B-\lambda I)^{-1}f\,\mathrm{d}\lambda \\ &+ \fint_\Gamma \lambda^{-1}\widehat{k}(\lambda)SP(\lambda)^{-1}(B-\lambda I)^{-1}f\,\mathrm{d}\lambda. \end{aligned} \tag{6.11}$$

Hence, $Tw \in \mathcal{D}(B)$ and

$$BTw = f - w + \fint_\Gamma \lambda^{-1}\widehat{k}(\lambda)SP(\lambda)^{-1}B(B-\lambda I)^{-1}f\,\mathrm{d}\lambda. \tag{6.12}$$

Notice that for all $\lambda \in \Gamma$, we have

$$\|B(B-\lambda I)^{-1}f\|_E \le C(1+|\mathrm{Im}\,\lambda|)^{1-\alpha-\theta}\|f\|_{\theta,\infty} \tag{6.13}$$

so that

$$\begin{aligned} &\int_\Gamma |\lambda|^{-1}|\widehat{k}(\lambda)|\|P(\lambda)^{-1}\|_{\mathcal{L}(E)}\|B(B-\lambda I)^{-1}f\|_E|\,\mathrm{d}\lambda| \\ &\le C\|f\|_{\theta,\infty}\fint_\Gamma |\lambda|^{-1-\sigma}(1+|\lambda|)^{1-\beta}(1+|\mathrm{Im}\,\lambda|)^{1-\alpha-\theta}|\,\mathrm{d}\lambda| \\ &\le C'\|f\|_{\theta,\infty}\int_0^{+\infty}(1+y)^{1-\alpha-\beta-\theta-\sigma}\,\mathrm{d}y. \end{aligned} \tag{6.14}$$

Since $\theta > 2 - \alpha - \beta$ and $\sigma > 0$, we conclude that the last integral converges.

Let us now focus our attention on the regularity of w. From

$$\|P(z)^{-1}\|_{\mathcal{L}(E)} \le C(1+|z|)^{1-\beta} \qquad \forall z \in \Gamma, \tag{6.15}$$

for any $s > 0$ we get that the function

$$s^\omega B(B+sI)^{-1}w = s^\omega B(B+sI)^{-1}\fint_\Gamma z^{-1}P(z)^{-1}B(B-zI)^{-1}f\,\mathrm{d}z \tag{6.16}$$

can be estimated in E precisely as in [2]. Since B commutes with T and S, we also have

$$s^\omega B(B+sI)^{-1}w$$

$$
\begin{aligned}
&= s^\omega \fint_\Gamma z^{-1}P(z)^{-1}B(B+sI)^{-1}(B-zI+zI)(B-zI)^{-1}f\,\mathrm{d}z \\
&= s^\omega \fint_\Gamma z^{-1}P(z)^{-1}B(B+sI)^{-1}f\,\mathrm{d}z \\
&+ s^\omega \fint_\Gamma P(z)^{-1}B(B+sI)^{-1}(B-zI)^{-1}f\,\mathrm{d}z \\
&= s^\omega \fint_\Gamma z^{-1}P(z)^{-1}B(B+sI)^{-1}f\,\mathrm{d}z \\
&- s^\omega \fint_\Gamma (z+s)^{-1}P(z)^{-1}B[(B+sI)^{-1}f-(B-zI)^{-1}f]\,\mathrm{d}z \\
&= s^\omega \fint_\Gamma (z+s)^{-1}P(z)^{-1}B(B-zI)^{-1}f\,\mathrm{d}z. \qquad (6.17)
\end{aligned}
$$

To estimate $s^\omega B(B+sI)^{-1}w$ we need the following inequality

$$|z+s|^2 \geq C(|\mathrm{Im}\,z|^2+s^2) \qquad \forall (z,s)\in\Gamma\times[0,+\infty). \tag{6.18}$$

For this purpose we introduce the function $\psi : \mathbf{R}\times[0,+\infty)\to\mathbf{R}_+$ defined by

$$\psi(y,s) = \frac{[s+a-c(1+|y|)^\alpha]^2+y^2}{y^2+s^2}. \tag{6.19}$$

Observe that for all $(y,s)\in\mathbf{R}\times[0,+\infty)$

$$D_s\psi(y,s) = 2(y^2+s^2)^{-2}[a-c(1+|y|)^\alpha]\{s^2+[s+a-c(1+|y|)^\alpha]s-y^2\}. \tag{6.20}$$

Consequently, $D_s\psi$ vanishes if

$$s_0(y) = \frac{1}{2}\left\{c(1+|y|)^\alpha - a + \sqrt{[c(1+|y|)^\alpha-a]^2+4y^2}\right\} \quad y\in\mathbf{R}, \tag{6.21}$$

In conclusion we deduce the lower bound

$$\psi(y,s) \geq \min[\psi(y,0),\psi(y,s_0(y)),\psi(y,+\infty)] \text{ for all } (y,s)\in\mathbf{R}\times[0,+\infty). \tag{6.22}$$

The assertion immediately follows from the assumption $a>c$ and the explicit expressions of the three y–functions in the right–hand side in (6.22).

From (6.17) and (6.18), taking advantage of the change of variable $y = -1+s\nu$, we easily derive the following inequalities

$$
\begin{aligned}
&s^\omega\|B(B+s)^{-1}w\|_E \leq C\|f\|_{\theta,\infty}s^\omega \fint_\Gamma |z+s|^{-1}(1+|\mathrm{Im}\,z|)^{2-\alpha-\beta-\theta}\,|\mathrm{d}z| \\
&\leq C'\|f\|_{\theta,\infty}s^\omega \int_0^{+\infty} \frac{(1+y)^{2-\alpha-\beta-\theta}}{1+y+s}\,\mathrm{d}y
\end{aligned}
$$

$$= C''\|f\|_{\theta,\infty} s^{\omega+2-\alpha-\beta-\theta} \int_{1/s}^{+\infty} \frac{\nu^{2-\alpha-\beta-\theta}}{1+\nu}\,\mathrm{d}\nu$$
$$\leq C''\|f\|_{\theta,\infty} s^{\omega-(\theta+\alpha+\beta-2)} \int_{0}^{+\infty} \frac{\nu^{2-\alpha-\beta-\theta}}{\nu+1}\,\mathrm{d}\nu. \tag{6.23}$$

Choose $\omega = \theta+\alpha+\beta-2$ and observe that $1 > \theta > 2-\alpha-\beta$ implies the convergence of the last integral. Hence $w \in (E, \mathcal{D}(B))_{\omega,\infty}$. Define then

$$\Psi = -\!\!\!\!\!\!\int_{\Gamma} \lambda^{-1}\widehat{k}(\lambda) S P(\lambda)^{-1} B(B-\lambda I)^{-1} f\,\mathrm{d}\lambda. \tag{6.24}$$

Since B commutes with T and S, one readily sees that

$$s^{\omega} B(B-sI)^{-1}\Psi = -\!\!\!\!\!\!\int_{\Gamma} s^{\omega}(z+s)^{-1}\widehat{k}(z) S P(z)^{-1} B(B-zI)^{-1} f\,\mathrm{d}z. \tag{6.25}$$

As σ is strictly positive, we get

$$s^{\omega}\|B(B-sI)^{-1}\Psi\|_E \leq C\|f\|_{\theta,\infty} s^{\omega} \int_{\Gamma} |z+s|^{-1}|z|^{-\sigma}(1+|\mathrm{Im}\,z|)^{2-\alpha-\beta-\theta}|\mathrm{d}z|. \tag{6.26}$$

Therefore, Ψ has the same regularity as w. Hence, under our last assumption the unique solution w to the operator equation

$$BTw + w = f + -\!\!\!\!\!\!\int_{\Gamma} \lambda^{-1}\widehat{k}(\lambda) S P(\lambda)^{-1} B(B-\lambda I)^{-1} f\,\mathrm{d}\lambda \tag{6.27}$$

with $f \in (E, \mathcal{D}(B))_{\theta,\infty}$, $\alpha+\beta > 1$, $2-\alpha-\beta < \theta < 1$, has the additional properties $w, BTw \in (E, \mathcal{D}(B))_{\omega,\infty}$, where $\omega = \theta+\alpha+\beta-2$.

If we choose

$$E = C([0,\tau];X),\ \mathcal{D}(B) = \{u \in C^1([0,\tau];X) : u(0)=0\},\ Bu = D_t u, \tag{6.28}$$

we get

$$(E, \mathcal{D}(B))_{\omega,\infty} = C_0^{\omega}([0,\tau];X). \tag{6.29}$$

Extend now any function $f \in C_0^{\theta}([0,\tau];X)$ to $[0,+\infty)$ according to the rule

$$f_0(t) = \begin{cases} f(t) & 0 \leq t \leq \tau \\ f(\tau)\mathrm{e}^{-(t-\tau)} & t > \tau. \end{cases} \tag{6.30}$$

Consider then the formulae

$$w(t) = -\!\!\!\!\!\!\int_{\Gamma} \lambda^{-1} P(\lambda)^{-1} D_t \int_0^t \mathrm{e}^{\lambda(t-s)} f(s)\,\mathrm{d}s\,\mathrm{d}\lambda$$
$$= D_t \int_0^t \mathrm{d}s -\!\!\!\!\!\!\int_{\Gamma} \mathrm{e}^{\lambda(t-s)}\lambda^{-1}P(\lambda)^{-1} f(s)\,\mathrm{d}\lambda = D_t \int_0^t R(t-s) f(s)\,\mathrm{d}s \tag{6.31}$$

and

$$\widehat{k}(\lambda)S\lambda^{-1}P(\lambda)^{-1}B(B-\lambda I)^{-1}f = \widehat{k}(\lambda)\widehat{SR}(\lambda)B(B-\lambda I)^{-1}f. \tag{6.32}$$

Consequently, the function

$$\varphi = \fint_\Gamma \widehat{k}(\lambda)S\lambda R(\lambda)^{-1}B(B-\lambda I)^{-1}f\,\mathrm{d}\lambda \tag{6.33}$$

satisfies

$$\begin{aligned}(B^{-1}\varphi)(t) &= \left(\fint_\Gamma \widehat{k}(\lambda)S\lambda^{-1}P(\lambda)^{-1}(B-\lambda I)^{-1}f\,\mathrm{d}\lambda\right)(t)\\ &= \fint_\Gamma \widehat{k}(\lambda)S\widehat{R}(\lambda)\,\mathrm{d}\lambda \int_0^t \mathrm{e}^{\lambda(t-s)}f(s)\,\mathrm{d}s\\ &= \int_0^t \Big(\fint_\Gamma \mathrm{e}^{\lambda(t-s)}\widehat{k}(\lambda)S\widehat{R}(\lambda)\,\mathrm{d}\lambda\Big)f(s)\,\mathrm{d}s\\ &= [(k*SR)*f](t) = [k*S(R*f)](t).\end{aligned} \tag{6.34}$$

Therefore,

$$\varphi(t) = D_t[k*S(R*f)](t) = [k*SD_t(R*f)](t) = (k*Sw)(t). \tag{6.35}$$

We have thus proved (cf. (6.12))that under the present assumptions the nonhomogeneous problem (3.1), (3.2) has a *unique* strict solution w for every $f \in C_0^\theta([0,\tau];X)$ if $2-\alpha-\beta<\theta<1$, $\alpha+\beta>1$, with the regularities

$$w, D_t(Tw) \in C_0^\omega([0,\tau];X), \qquad \omega = \theta+\alpha+\beta-2. \tag{6.36}$$

Remark 6.1. Under the assumption $2-\alpha-\beta<\theta<1$ and proceeding as in the usual semigroup theory, we could prove that w admits the explicit representation

$$w(t) = \int_0^t R'(t-s)[f(s)-f(t)]\,\mathrm{d}s + R(t)f(t), \qquad t\in[0,\tau]. \tag{6.37}$$

Unfortunately, we are unable to show directly that each term in (6.37) belongs to $C_0^\omega([0,\tau];X)$. Moreover, the estimate

$$\|R'(t-s)\|_{\mathcal{L}(X)} \le C\mathrm{e}^{a(t-s)}(t-s)^{(\beta-2)/\alpha} \qquad 0\le s\le t\le\tau \tag{6.38}$$

implies that

$$\|R'(t-s)[f(s)-f(t)]\| \le C_1\|f\|_{C^\theta([0,\tau];X)}\mathrm{e}^{a(t-s)}(t-s)^{(\alpha\theta+\beta-2)/\alpha} \tag{6.39}$$

is summable over $(0,t)$ if and only if $\theta > (2-\alpha-\beta)/\alpha$ (instead of $\theta > 2-\alpha-\beta$).

7. More on maximal regularity

In this section, using the optimal Hölder exponents determined in Section 6, we show that the results in the Sections 3 and 4 can be improved also with respect to the initial conditions.

First we consider the function

$$z_1(t) = \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda} P(\lambda)^{-1}\widehat{k}(\lambda) S z_0 \,\mathrm{d}\lambda. \tag{7.1}$$

Then for any pair (t,s) such that $t > s \geq 0$ we have

$$\begin{aligned} z_1(t) - z_1(s) &= ⨍_\Gamma \frac{\mathrm{e}^{\lambda t} - \mathrm{e}^{\lambda s}}{\lambda}\widehat{k}(\lambda)P(\lambda)^{-1} S z_0 \,\mathrm{d}\lambda \\ &= \int_s^t \mathrm{d}r ⨍_\Gamma \mathrm{e}^{\lambda r}\widehat{k}(\lambda)P(\lambda)^{-1} S z_0 \,\mathrm{d}\lambda, \end{aligned} \tag{7.2}$$

so that

$$\|z_1(t) - z_1(s)\| \leq C\|Sz_0\| \int_s^t \mathrm{d}r ⨍_\Gamma \frac{\mathrm{e}^{r\,\mathrm{Re}\,\lambda}}{|\lambda|^\sigma}(1+|\lambda|)^{1-\beta}|\,\mathrm{d}\lambda|. \tag{7.3}$$

Taking into account that $\lambda = a - c(1+|\lambda|)^\alpha + iy$, $-\infty < y < \infty$ and assuming $\beta + \sigma > 2$, we deduce

$$\begin{aligned} \|z_1(t) - z_1(s)\| &\leq C\|Sz_0\| \int_s^t \mathrm{e}^{ar}\,\mathrm{d}r \int_0^{+\infty} \mathrm{e}^{-cr(1+y)^\alpha}(1+y)^{1-\beta-\sigma}\,\mathrm{d}y \\ &\leq C\|Sz_0\|\mathrm{e}^{at}\int_s^t \mathrm{d}r \int_0^{+\infty}(1+y)^{1-\beta-\sigma}\,\mathrm{d}y. \end{aligned} \tag{7.4}$$

Therefore, if $\beta+\sigma > 2,\ 0 \leq s < t \leq \tau$, we get that the best possible estimate

$$\|z_1(t) - z_1(s)\| \leq C'\|z_0\|\,|t-s|, \qquad (C' = C\mathrm{e}^{a\tau}). \tag{7.5}$$

On the other hand, if $2 - \alpha < \beta + \sigma < 2$, from

$$\int_0^{+\infty} \mathrm{e}^{-cr(1+y)^\alpha}(1+y)^{1-\beta-\sigma}\,\mathrm{d}y \leq Cr^{(\sigma+\beta-2)/\alpha} \tag{7.6}$$

we deduce

$$\|z_1(t) - z_1(s)\| \leq C\int_s^t \mathrm{e}^{ar} r^{(\sigma+\beta-2)/\alpha}\,\mathrm{d}r \tag{7.7}$$

$$\leq C'\|z_0\|(t-s)^{(\alpha+\beta+\sigma-2)/\alpha}\int_0^1 \rho^{(-2+\sigma+\beta)/\alpha}\,\mathrm{d}\rho = C''\|z_0\|(t-s)^{(\alpha+\beta+\sigma-2)/\alpha}.$$

If $\beta+\sigma=2$ and $\alpha=1$, by a change of variable and arguing as in the theory of analytic semigroups, we obtain

$$\|z_1(t)-z_1(s)\|\le C|t-s| \qquad s,t\in[0,\tau]. \tag{7.8}$$

If $\beta+\sigma=2$ and $\alpha\in(0,1)$, then

$$\|z_1(t)-z_1(s)\|\le C\|Sz_0\|\int_s^t \mathrm{d}r\int_0^{+\infty}\frac{\mathrm{e}^{-cr(1+y)^\alpha}}{1+y}\,\mathrm{d}y. \tag{7.9}$$

Since

$$\int_0^{+\infty}\frac{\mathrm{e}^{-cr(1+y)^\alpha}}{1+y}\,\mathrm{d}y\le\frac{c}{\alpha}\int_{cr}^{+\infty}\frac{\mathrm{e}^{-\xi}}{\xi}\,\mathrm{d}\xi\le\frac{c}{\alpha}\Big[C_1+\log\Big(\frac{1}{r}\Big)\Big] \tag{7.10}$$

we deduce

$$\|z_1(t)-z_1(s)\|\le C'\|Sz_0\|\Big[t\log\Big(\frac{1}{t}\Big)-s\log\Big(\frac{1}{s}\Big)+t-s\Big]. \tag{7.11}$$

As a consequence, if $\beta+\sigma=2$ and $\alpha\in(0,1)$, we have

$$\|z_1(t)-z_1(s)\|\le C|t-s|^\theta \qquad s,t\in[0,\tau], \tag{7.12}$$

θ being any real number in $(0,1)$.

By a simple change of variable from formula (2.15) we obtain

$$D_tTz_1(t)=-z_1(t)+\int_0^t k(s)S[z_0+z_1(t-s)]\,\mathrm{d}s, \qquad 0\le t\le\tau. \tag{7.13}$$

Observe now that the equality

$$\begin{aligned}&\int_0^t k(s)S[z_0+z_1(t-s)]\,\mathrm{d}s-\int_0^{t'} k(s)S[z_0+z_1(t'-s)]\,\mathrm{d}s\\&=\int_0^{t'} k(s)S[z_1(t-s)-z_1(t'-s)]\,\mathrm{d}s+\int_{t'}^t k(s)S[z_0+z_1(t-s)]\,\mathrm{d}s\end{aligned} \tag{7.14}$$

and estimates (7.7), (7.8), (7.11) imply

$$\begin{aligned}&\Big\|\int_0^t k(s)S[z_0+z_1(t-s)]\,\mathrm{d}s-\int_0^{t'} k(s)S[z_0+z_1(t'-s)]\,\mathrm{d}s\Big\|\\&=C|t-t'|^\psi\int_0^{t'}|k(s)|\,\mathrm{d}s+\|Sz_0\|\int_{t'}^t|k(s)|\,\mathrm{d}s+\|S\|_{\mathcal{L}(X)}\int_{t'}^t|k(s)|\,\|z_1(t-s)\|\,\mathrm{d}s\end{aligned} \tag{7.15}$$

where ψ is either 1 or $(\alpha+\beta+\sigma-2)/\alpha$ or any number in $(0,1)$. We need now to show that the function $t\to\int_0^t k(s)\,\mathrm{d}s$ belongs to $C^{\min(\sigma,1)}([0,\tau])$, where $C^1[0,\tau])$ stands here for the vector space of Lipschitz continuous scalar functions. For this purpose first we observe that

$$|k(\rho)| = \Big| \int_{\Gamma'} e^{\rho\lambda} \widehat{k}(\lambda)\, d\lambda \Big| \le C \int_{\Gamma'} \frac{e^{\rho \mathrm{Re}\,\lambda}}{|\lambda|^\sigma}\, d\lambda \le C\rho^{\min(\sigma-1,0)} \tag{7.16}$$

where the contour Γ' is defined by (3.12). Whence we easily deduce the bounds

$$\Big| \int_s^t k(r)\, dr \Big| \le C \int_s^t r^{\min(\sigma-1,0)}\, dr \le C'(t-s)^{\min(\sigma,1)}. \tag{7.17}$$

Consequently, from (7.15) and (7.17) we immediately infer the bounds

$$\begin{aligned} &\Big\| \int_0^t k(s)S[z_0 + z_1(t-s)]\, ds - \int_0^{t'} k(s)S[z_0 + z_1(t'-s)]\, ds \Big\| \\ &\le C\big[|t-t'|^\psi + |t-t'|^{\min(\sigma,1)}\big] + C\int_{t'}^t s^{\min(\sigma-1,0)}|t-s|^\psi\, ds \\ &\le C'|t-t'|^{\min(\psi,\sigma,1)}. \end{aligned} \tag{7.18}$$

Indeed, we have

$$\int_{t'}^t s^{\min(\sigma-1,0)}(t-s)^\psi\, ds \le (t-t')^\psi \int_{t'}^t s^{\min(\sigma-1,0)}\, ds \le C''(t-t')^{\psi+\min(\sigma,1)}. \tag{7.19}$$

We now want to estimate the Hölder constant of the function

$$z_2(t) = \int_\Gamma \frac{e^{\lambda t}}{\lambda} P(\lambda)^{-1} z_0\, d\lambda \tag{7.20}$$

without assuming $z_0 = Tw_0$. For this we introduce the intermediate spaces (cf. [2])

$$W_\theta = \{u \in X : \sup_{t>0} (1+t)^\theta \|L(tM+L)^{-1}u\| < +\infty\},\ \theta \in (1-\beta, 1). \tag{7.21}$$

Notice that W_θ endowed with the norm

$$\|u\|_\theta = \sup_{t>0}\ t^\theta \|L(tM+L)^{-1}u\| = \sup_{t>0}\ t^\theta \|(tT+I)^{-1}u\| \tag{7.22}$$

is a Banach space since L and $tM + L$ $(t > 0)$ are continuously invertible. Moreover, it is shown in ([2], Theorem 1.12) that the following continuous embeddings hold:

$$W_\theta \hookrightarrow (X, \mathcal{D}(A))_{\theta,\infty} = (X, \mathcal{D}(LM^{-1}))_{\theta,\infty} \hookrightarrow W_{\theta+\beta-1},\ 1-\beta < \theta < 1. \tag{7.23}$$

We can now prove the following lemma.

Lemma 7.1. *For any $z_0 \in W_\theta$ with $\theta \in (1-\beta, 1)$ the following estimate holds*

$$|\lambda|^{\theta+\beta-1}\|(\lambda T+I)^{-1}z_0\| \le C\|z_0\|_{W_\theta} \qquad \forall \lambda \in \Gamma. \tag{7.24}$$

Proof. First we consider the following chain of identities that hold for any $\lambda \in \Sigma_\alpha$:

$$\begin{aligned}&(\lambda T+I)^{-1} - (|\operatorname{Im}\lambda|T+I)^{-1}\\ &= \frac{|\operatorname{Im}\lambda|-\lambda}{\lambda}(\lambda T+I)^{-1}\lambda T(|\operatorname{Im}\lambda|T+I)^{-1}\\ &= \frac{|\operatorname{Im}\lambda|-\lambda}{\lambda}[I-(\lambda T+I)^{-1}](|\operatorname{Im}\lambda|T+I)^{-1},\end{aligned} \tag{7.25}$$

i.e.,

$$(\lambda T+I)^{-1} = \big\{I + \frac{|\operatorname{Im}\lambda|-\lambda}{\lambda}\big[I-(\lambda T+I)^{-1}\big]\big\}(|\operatorname{Im}\lambda|T+I)^{-1}, \text{ for all } \lambda \in \Sigma_\alpha. \tag{7.26}$$

This implies

$$\begin{aligned}&|\lambda|^{\theta+\beta-1}\|(\lambda T+I)^{-1}z_0\|\\ &\le |\lambda|^{\theta+\beta-1}[1+C(1+|\lambda|)^{1-\beta}]\|(|\operatorname{Im}\lambda|T+I)^{-1}z_0\|\\ &\le C_1(1+|y|)^{\theta+\beta-1}(1+|y|)^{1-\beta}\|(|y|T+I)^{-1}z_0\| \le C_2\|z_0\|_{W_\theta},\end{aligned} \tag{7.27}$$

as desired. □

Let us now fix $\phi \in (1-\beta, 1)$ and assume $z_0 \in W_\phi$. We now want to estimate

$$\begin{aligned}z_2(t)-z_2(s) &= \int_\Gamma \frac{e^{\lambda s}-e^{\lambda t}}{\lambda}Q(\lambda)^{-1}(\lambda T+I)^{-1}z_0\,d\lambda\\ &= \int_s^t dr\int_\Gamma \frac{e^{\lambda\tau}}{|\lambda|^{\phi+\beta-1}}Q(\lambda)^{-1}|\lambda|^{\phi+\beta-1}(\lambda T+I)^{-1}z_0\,d\lambda\end{aligned} \tag{7.28}$$

where $Q(\lambda) = I - \widehat{k}(\lambda)(\lambda T+I)^{-1}S$ (cf. (2.6)).

From Lemma 7.1 we get

$$\begin{aligned}&\Big\|\int_\Gamma \frac{e^{r\lambda}}{|\lambda|^{\phi+\beta-1}}|\lambda|^{\phi+\beta-1}(\lambda T+I)^{-1}z_0\,d\lambda\Big\| \le C\|z_0\|_\phi\int_\Gamma \frac{e^{r\operatorname{Re}\lambda}}{|\lambda|^{\phi+\beta-1}}|\,d\lambda|\\ &\le e^{ar}\int_0^{+\infty}\frac{e^{-cr(1+y)^\alpha}}{(1+y)^{\phi+\beta-1}}\,dy = C'e^{ar}r^{(\phi+\beta-2)/\alpha}\int_{cr}^{+\infty}e^{-\rho}\rho^{(2-\alpha-\beta-\phi)/\alpha}\,d\rho\\ &\le Ce^{ar}r^{(\phi+\beta-2)/\alpha} \quad \text{for all } r \in [s,t].\end{aligned} \tag{7.29}$$

Consequently, from (7.28) and (7.29), we deduce

$$\|z_2(t)-z_2(s)\| \le C\|z_0\|_\phi \mathrm{e}^{a\tau}\Big|\int_s^t \nu^{(\phi+\beta-2)/\alpha}\,\mathrm{d}\nu\Big| \le C'\|z_0\|_\phi |t-s|^{(\phi+\alpha+\beta-2)/\alpha}. \tag{7.30}$$

Choosing $\phi=\alpha\theta+(1-\alpha)(2-\alpha-\beta)$ we obtain

$$\|z_2(t)-z_2(s)\| \le C|t-s|^{\theta+\alpha+\beta-2}\|z_0\|_\theta \tag{7.31}$$

for any $z_0\in W_\theta$, $\alpha+\beta>1$ and $2-\alpha-\beta<\theta<1$. Indeed, it suffices to observe that our ϕ belongs to $(1-\beta,1)$, since

$$\begin{aligned}\alpha\theta+(1-\alpha)(2-\alpha-\beta) &> \alpha(2-\alpha-\beta)+(1-\alpha)(2-\alpha-\beta)\\ &= 1-\alpha+1-\beta \ge 1-\beta\end{aligned} \tag{7.32}$$

and

$$\alpha\theta+(1-\alpha)(2-\alpha-\beta) < \alpha+1-\alpha = 1. \tag{7.33}$$

Then we notice that, if $\alpha\in(0,1)$, $\theta\in(2-\alpha-\beta,1)$ implies $\alpha<\theta$, because it reads equivalently $(1-\alpha)(2-\alpha-\beta)<(1-\alpha)\theta$, while $\alpha=1$ implies $\phi=\theta$. Hence we easily derive our assertion:

$$\begin{aligned}\|z_2(t)-z_2(s)\| &\le C\|z_0\|_{\alpha\theta+(1-\alpha)(2-\alpha-\beta)}|t-s|^{\theta+\alpha+\beta-2}\\ &\le C'\|z_0\|_\theta |t-s|^{\theta+\alpha+\beta-2}, \qquad z_0\in W_\theta.\end{aligned} \tag{7.34}$$

Now we observe that

$$\begin{aligned}Tz_2(t) &= -\int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda^2}[\lambda T+I-\widehat{k}(\lambda)S-I+\widehat{k}(\lambda)S]P(\lambda)^{-1}z_0\,\mathrm{d}\lambda\\ &= -tz_0+\int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda^2}P(\lambda)^{-1}z_0\,\mathrm{d}\lambda-\int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda^2}\widehat{k}(\lambda)SP(\lambda)^{-1}z_0\,\mathrm{d}\lambda.\end{aligned} \tag{7.35}$$

Recalling that $z_0\in W_\theta$ and $\beta+\sigma>1$, we derive that Tz_2 is differentiable for $t\ge 0$ and

$$\begin{aligned}D_tTz_2(t) &= -z_0+\int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda}P(\lambda)^{-1}z_0\,\mathrm{d}\lambda-\int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda}\widehat{k}(\lambda)SP(\lambda)^{-1}z_0\,\mathrm{d}\lambda\\ &= -z_0+z_2(t)-x(t),\end{aligned} \tag{7.36}$$

where

$$x(t)=\int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda}\widehat{k}(\lambda)SP(\lambda)^{-1}z_0\,\mathrm{d}\lambda=\int_\Gamma \frac{\mathrm{e}^{\lambda r}}{\lambda}\widehat{k}(\lambda)SQ(\lambda)^{-1}(\lambda T+I)^{-1}z_0\,\mathrm{d}\lambda. \tag{7.37}$$

From the formula, where $0 \le s \le t \le \tau$,

$$x(t) - x(s) = \int_s^t \mathrm{d}r \fint_\Gamma \mathrm{e}^{\lambda r}\widehat{k}(\lambda) S Q(\lambda)^{-1}(\lambda T + I)^{-1} z_0 \,\mathrm{d}\lambda \tag{7.38}$$

we get the following inequalities with $\phi = \alpha\theta + (1-\alpha)(2-\alpha-\beta)$

$$\begin{aligned}\|x(t) - x(s)\| &\le \|z_0\|_\phi \int_s^t \mathrm{d}r \fint_\Gamma \mathrm{e}^{r\mathrm{Re}\,\lambda}|\lambda|^{-(\phi+\beta-1)}|\,\mathrm{d}\lambda| \\ &\le C\|z_0\|_\theta (t-s)^{\theta+\alpha+\beta-2}.\end{aligned} \tag{7.39}$$

Summing up, for any $z_0 \in W_\theta$ and $\theta \in (2-\alpha-\beta, 1)$ with $\alpha+\beta > 1$ we have shown that the function z_2 enjoys the properties

$$z_2, D_t T z_2 \in C^{\theta+\alpha+\beta-2}([0,\tau]; X). \tag{7.40}$$

We come back to the solution $z = z_0 + z_1 - z_2$ to the initial value problem

$$(\mathrm{Q}''') \quad \begin{cases} D_t(Tz) + z = k * Sz, & \text{on } [0,\tau] \\ Tz(0) = Tz_0. \end{cases} \tag{7.41}$$

From now on we will assume

$$0 < \beta \le \alpha \le 1, \quad \alpha+\beta > 1, \quad 2-\alpha-\beta < \theta < 1, \quad \sigma > 0. \tag{7.42}$$

Let us now consider the following cases 1–4.

1. If $\beta + \sigma > 2$ and $z_0 \in W_\theta$, then z has the optimal regularity

$$z, D_t T z \in C^{\theta+\alpha+\beta-2}([0,\tau]; X).$$

2. If $\beta + \sigma = 2$, $\alpha = 1$ and $z_0 \in W_\theta$, then z has the regularity

$$z, D_t T z \in C^{\theta+\beta-1}([0,\tau]; X).$$

3. If $\beta+\sigma = 2$, $\alpha < 1$ and $z_0 \in W_\theta$, since $z_1 \in C^\nu([0,\tau]; X)$ for any $\nu \in (0,1)$ and $D_t T z_1 \in C^{\min\,(\nu,\sigma,1)}([0,\tau]; X)$, we again derive the regularity

$$z, D_t T z \in C^{\theta+\alpha+\beta-2}([0,\tau]; X).$$

4. If $2-\alpha < \beta+\sigma < 2$, $\alpha < 1$ and $z_0 \in W_\theta$, since $z_1 \in C^{\theta+\alpha+\beta-2}([0,\tau]; X)$ and $D_t T z_1 \in C^{\min\,[(\alpha+\beta+\sigma-2)/\alpha,\sigma,1]}([0,\tau]; X)$, we deduce that z has the regularity

$$z, D_t T z \in C^{\min\,[(\alpha+\beta+\sigma-2)/\alpha,\sigma,\theta+\alpha+\beta-2]}([0,\tau]; X).$$

We now turn to the original problem (P). In view of (Q'), (Q''), we have to estimate function g defined by (1.12), i.e.,

$$g(t) = f(t) - f(0) + \Big(\int_0^t k(s)\,\mathrm{d}s\Big) S f(0) \qquad 0 \le t \le \tau. \tag{7.43}$$

Since

$$g(t') - g(t'') = f(t') - f(t'') + \Big(\int_{t''}^{t'} k(s)\,\mathrm{d}s\Big) S f(0), \quad 0 \le t'' < t' \le \tau \tag{7.44}$$

and (cf. (7.16))

$$|k(t)| \le C t^{\min(\sigma-1,0)} \tag{7.45}$$

we deduce

$$\begin{aligned} \|g(t') - g(t'')\| &\le C\Big[|t'-t''|^{\theta} + \int_{t''}^{t'} |k(s)|\,\mathrm{d}s\Big] \\ &\le C|t'-t''|^{\theta} + C\int_{t''}^{t'} s^{\min(\sigma-1,0)}\,\mathrm{d}s \le C'|t'-t''|^{\min(\theta,\sigma,1)} \end{aligned} \tag{7.46}$$

provided $f \in C^{\theta}([0,\tau];X)$. Therefore, if $\sigma \ge 1$, then $g \in C_0^{\theta}([0,\tau];X)$.

On the contrary, if $1-\beta < \sigma < 1$, we must assume $2-\alpha-\beta < \theta \le \sigma$ to derive the maximal regularity

$$w, D_t(Tw) \in C_0^{\omega}([0,\tau];X), \qquad \omega = \theta + \alpha + \beta - 2.$$

Consequently, if $\beta + \sigma \ge 2$ (which implies $\sigma \ge 1$) and $f \in C^{\theta}([0,\tau];X)$, $f(0) - Lu_0 \in W_{\theta}$, problem (P) admits a unique strict solution u with the regularity

$$Lu, D_t(Mu) \in C^{\omega}([0,\tau];X) \quad \text{with} \quad \omega = \theta + \alpha + \beta - 2.$$

If $0 < \sigma < 1$ and, in addition, $2-\alpha < \beta+\sigma < 2$, then the solution z to the homogeneous problem (Q$'''$) (see (7.41)) with $z_0 = Lu_0 - f(0) \in W_{\theta}$ has the regularity

$$z, D_t(Tz) \in C^{\nu}([0,\tau];X) \quad \text{with} \quad \nu = \min\Big(\frac{\alpha+\beta+\sigma-2}{\alpha},\ \sigma,\ \theta+\alpha+\beta-2\Big).$$

But, if $2-\alpha-\beta < \theta \le \sigma$ (which implies $1-\beta < \sigma$), we get

$$w, D_t(Tw) \in C_0^{\omega}([0,\tau];X) \quad \omega = \theta + \alpha + \beta - 2.$$

On the other hand, in this case we have firstly $\sigma \ge \omega$. Further from the relationships

$$\omega \le \frac{\alpha+\beta-2+\sigma}{\alpha} \quad \Longleftrightarrow \quad \theta \le \frac{\sigma - (1-\alpha)(2-\alpha-\beta)}{\alpha}$$

and

$$\frac{\sigma - (1-\alpha)(2-\alpha-\beta)}{\alpha} - \sigma = \frac{(1-\alpha)[\sigma - (2-\alpha-\beta)]}{\alpha} \ge 0$$

provided $2-\alpha-\beta \le \sigma$, this inequality implying in turn $\theta \le [\sigma - (1-\alpha)(2-\alpha-\beta)]/\alpha$. Therefore, we conclude that $\nu = \omega$.

Consequently, we have proved the following maximal regularity result

Theorem 7.1. *Let* $\sigma \in (2-\alpha-\beta, +\infty)$, $\theta \in (2-\alpha-\beta, \min\{\sigma, 1\})$, $f \in C^\theta([0,\tau]; X)$, $u_0 \in \mathcal{D}(L)$ *and* $f(0) - Lu_0 \in W_\theta$. *Then the solution* u *to problem* (P) *satisfies*

$$Lu, D_t(Mu) \in C^\omega([0,\tau]; X), \quad \omega = \theta + \alpha + \beta - 2.$$

Of course, the applications described in Section 5 can be reformulated according to this latter approach. Further we can give here another application, which, owing to its length, we postpone until the end of this section.

We also get the following sharp version of the maximal regularity property of the solution to the problem

$$(\widetilde{\mathrm{P}}) \quad \begin{cases} D_t(Mu(t)) + Lu(t) = f(t), & 0 \le t \le \tau, \\ Mu(0) = Mu_0. \end{cases}$$

Theorem 7.2. *Let assumption* (7.42) *be satisfied. Then for any* $f \in C^\theta([0,\tau]; X)$ *with* $f(0) - Lu_0 \in W_\theta$ *problem* $(\widetilde{P})$ *admits a unique strict solution* u *with the regularity* $u \in C^{\theta+\alpha+\beta-2}([0,\tau]; X)$ *and*

$$D_t Mu \in C^{\theta+\alpha+\beta-2}([0,\tau]; X).$$

Application 3. Let

$$k(t) = \sum_{n=0}^{+\infty} a_n t^{\sigma-1+\mu_n} \mathrm{e}^{-t\nu_n} \qquad t > 0, \tag{7.47}$$

where $\sigma > 0$, and let the sequences $\{a_n\}, \{\mu_n\}, \{\nu_n\} \subseteq \mathbf{R}$ enjoy the properties

$$\sum_{n=0}^{+\infty} |a_n| < +\infty, \quad 0 \le \mu_n \le \mu, \quad 0 \le \nu \le \nu_n \quad \text{for all } n \in \mathbf{N}. \tag{7.48}$$

Then k is continuous in $\mathbf{R}_+$ and satisfies

$$|k(t)| \le t^{\sigma-1} \max(1, t^\mu) \mathrm{e}^{-t\nu} \sum_{n=0}^{+\infty} |a_n| \qquad \text{for all } t \in \mathbf{R}_+. \tag{7.49}$$

Moreover, the Laplace transform $\widehat{k}(\cdot)$ of k is given by

$$\widehat{k}(\lambda) = \sum_{n=0}^{+\infty} a_n \int_0^{+\infty} t^{\sigma-1+\mu_n} \mathrm{e}^{-t(\nu_n+\lambda)} \, \mathrm{d}t = \sum_{n=0}^{+\infty} a_n \Gamma(\sigma+\mu_n)(\nu_n+\lambda)^{-\sigma-\mu_n}, \tag{7.50}$$

$\operatorname{Re}\lambda > 0$, where Γ denotes the gamma function.

To show that $\widehat{k}(\cdot)$ admits an analytic extension to the complex plane cut along the negative real axis we begin by considering the inequality

$$|\nu_n + \lambda|^{-(\sigma+\mu_n)} \leq |\nu_n + \lambda|^{-\sigma} \max(1, |\nu_n + \lambda|^{-\mu}). \tag{7.51}$$

Moreover, for any $\lambda \in \widetilde{\Sigma}_\theta = \{z \in \mathbf{C} : \operatorname{Re} z \geq a - c(1 + |\operatorname{Im} z|)\}$ ($\theta \in (\pi/2, \pi)$, $\tan\theta = -1/c$), we have

$$\begin{aligned} |\nu_n + \lambda|^2 &= \nu_n^2 + 2\nu_n|\lambda|\cos(\arg\lambda) + |\lambda|^2 \\ &\geq \nu_n^2 + 2\nu_n|\lambda|\cos(\theta) + |\lambda|^2 \geq [1 + \cos(\theta)](\nu_n^2 + |\lambda|^2) \\ &\geq 2[\cos(\theta/2)]^2(\nu^2 + |\lambda|^2) \geq [\cos(\theta/2)]^2|\lambda|^2. \end{aligned} \tag{7.52}$$

Consequently,

$$\begin{aligned} |\nu_n + \lambda|^{-\sigma-\mu_n} &\leq [\cos(\theta/2)]^{-\sigma}|\lambda|^{-\sigma} \max\{1, [\cos(\theta/2)]^{-\mu}|\lambda|^{-\mu}\} \\ &\leq C|\lambda|^{-\sigma}, \qquad \text{for all } \lambda \in \widetilde{\Sigma}_\theta. \end{aligned} \tag{7.53}$$

Finally, from formula (7.50) and bound (7.53) we deduce our assertion. Consequently properties (2.3) and (2.4) hold.

Assume now that the linear operators L, L_1, M in the complex Banach space X satisfy (1.8) and let (σ, θ) be a pair such that $\sigma \in [2 - \alpha - \beta, +\infty)$, $\theta \in (2 - \alpha - \beta, 1)$. Then, according to Theorem 7.1, for any pair $(f, u_0) \in C^\theta([0, \tau]; X) \times \mathcal{D}(L)$ with $f(0) - Lu_0 \in W_\theta$ the integrodifferential equation

$$D_t(M(u(t)) + Lu(t) = \int_0^t k(t-s)L_1u(s)\,\mathrm{d}s + f(t), \quad 0 \leq t \leq \tau, \tag{7.54}$$

admits a unique strict solution u such that $Mu(0) = Mu_0$ enjoying the properties $D_t(Mu), Lu \in C^{\theta+\alpha+\beta-2}([0, \tau]; X)$.

For example, let $X = L^2(\Omega)$, Ω being a bounded domain in $\mathbf{R}^n$ with a boundary $\partial\Omega$ of class C^2. Further let L, M, L_1 be the operators in X defined by

$$\begin{aligned} &\mathcal{D}(L) = H_0^1(\Omega) \cap H^2(\Omega), \qquad Lu = -\Delta u, \qquad u \in \mathcal{D}(L) \\ &L_1 = -L \\ &\mathcal{D}(M) = H_0^1(\Omega), \qquad Mu = mu, \qquad u \in H_0^1(\Omega), \end{aligned} \tag{7.55}$$

where m is a nonnegative measurable bounded function in Ω. Then it is shown in [2] that (1.8) holds with $\alpha = 1$, $\beta = 1/2$. Therefore, Theorem 7.1 applies to the kernel k with $\sigma \in (1/2, +\infty)$, $\theta \in (1/2, \min\{\sigma, 1\})$ and any pair $(f, u_0) \in C^\theta([0, \tau]; L^2(\Omega)) \times [H_0^1(\Omega) \cap H^2(\Omega)]$ such that $f(\cdot, 0) + \Delta u_0 \in W_\theta$. Under such assumptions we have Δu, $D_t(mu) \in C^{\theta-1/2}([0, \tau]; L^2(\Omega))$.

Of course, the most favourable situation for regularity occurs when the differential operators are in variational forms, i.e.,

$$X = H^{-1}(\Omega), \qquad \mathcal{D}(L) = \mathcal{D}(L_1) = H_0^1(\Omega) = \mathcal{D}(M) \tag{7.56}$$

because then $\alpha = \beta = 1$ and thus it suffices to assume $\sigma > 0$ to be able to apply Theorem 7.1. The data f, u_0 are supposed to satisfy

$$f \in C^\theta([0,\tau]; H^{-1}(\Omega)), \quad u_0 \in H_0^1(\Omega), \quad f(\cdot,0) + \Delta u_0 \in W_\theta. \tag{7.57}$$

Further, it is possible to show (see [1] Example 3.6) that similar results hold in $L^p(\Omega)$, $1 < p < \infty$, when $\alpha = 1, \beta = 1/p$, in the case where $\mathcal{D}(L) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $Lu = -\Delta u + cu$ and $c > 0$.

Finally, we note that from (7.23) and $\beta = 1$ we immediately obtain the equation $W_\theta = (H^{-1}(\Omega), \mathcal{D}(LM^{-1}))_{\theta,\infty}$. Since W_θ has a difficult characterization in this concrete case, we single out an explicit vector subspace in W_θ. Observe first that any $g \in \mathcal{D}(LM^{-1})$ admits the representation

$$g = mv = Mv \quad \text{with} \quad v \in H_0^1(\Omega) \tag{7.58}$$

and

$$\begin{aligned}\|g\|_{\mathcal{D}(LM^{-1})} &= \inf_{f \in LM^{-1}g} \|f\|_{H^{-1}(\Omega)} \\ &= \inf_{f = Lv,\, Mv = g} \|f\|_{H^{-1}(\Omega)} = \inf_{v \in H_0^1(\Omega),\, Mv = g} \|Lv\|_{H^{-1}(\Omega)}.\end{aligned} \tag{7.59}$$

We also assume something more on m, specifically $m \in C^1(\overline{\Omega})$. Then operator M defines, by duality, a bounded operator in $H^{-1}(\Omega)$. On the other hand, $M \in \mathcal{L}(H_0^1(\Omega); \mathcal{D}(LM^{-1}))$. Hence, by interpolation, $M \in \mathcal{L}((H^{-1}(\Omega); H_0^1(\Omega))_{\theta,\infty}; (H^{-1}(\Omega); \mathcal{D}(A))_{\theta,\infty})$, $0 < \theta < 1$. As a consequence, we get $M \in \mathcal{L}((H^{-1}(\Omega); H_0^1(\Omega))_{\theta,2}; W_\theta)$. Recall now the following equalities (for the last one cf. [4, pp. 60 and 79])

$$\begin{aligned}&(H^{-1}(\Omega); H_0^1(\Omega))_{\theta,2} = [H^{-1}(\Omega); H_0^1(\Omega)]_\theta \\ &= [H_0^1(\Omega); H^{-1}(\Omega)]_{1-\theta} = \begin{cases} H^{2\theta-1}(\Omega) & 1/2 \le \theta \le 3/4 \\ H_0^{2\theta-1}(\Omega) & 3/4 < \theta < 1 \end{cases}.\end{aligned} \tag{7.60}$$

Consequently, for any $\theta \in [1/2, 1)$ the assumption $f(\cdot,0) + \Delta u_0 \in W_\theta$ in (7.57) can be replaced by the explicit one

$$f(\cdot,0) + \Delta u_0 = mw_0, \qquad w_0 \in H_0^{2\theta-1}(\Omega). \tag{7.61}$$

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