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SHARP ESTIMATES AND FINITE SPEED OF PROPAGATION FOR A NEUMANN PROBLEM IN DOMAINS NARROWING AT INFINITY

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Abstract. We investigate the connection between the geometry of an unbounded domain Ω and the existence and qualitative behaviour of solutions to a degenerate doubly non linear parabolic equation posed in Ω. The domain Ω is assumed to be "narrowing" at infinity in a suitable sense, so that it has infinite volume. On the boundary of Ω we prescribe a homogeneous Neumann condition. Among other results, we prove sharp estimates for the finite speed of propagation of the support of positive solutions originating from initial data with bounded support. This is done by means of a new approach, which is flexible enough to be applied to the geometry at hand, and to cover the case of initial data measures. We also show that even if the initial datum has finite mass, the solution need not be globally bounded over Ω for a fixed positive time. We provide sharp estimates for such solutions. Our main tool is a new embedding inequality connected with the geometry of Ω.

1. Introduction. We study the behaviour of nonnegative solutions of a class of degenerate parabolic Neumann problems posed in $Q_T = \Omega \times (0, T)$, where $\Omega \subset \boldsymbol{R}^N$, $N \geq 2$, is an unbounded domain "narrowing" at infinity, and $T > 0$. A precise definition of the class of spatial domains we deal with

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is given below. A typical element of this class is the domain

$$\Omega^{\varepsilon} = \{x = (x', x_N) \in \boldsymbol{R}^N : |x'| < x_N^{-\varepsilon}\,,\ x_N > d\}\,, \quad d > 0\,,$$

for $0 < \varepsilon < 1/(N-1)$, so that volume $(\Omega) = \infty$. The model problem we look at is

$$u_t - \operatorname{div}(u^{\alpha}|Du|^{m-1}Du) = 0\,, \qquad \text{in } Q_T\,, \tag{1.1}$$

$$u^{\alpha}|Du|^{m-1}Du \cdot \mathbf{n} = 0\,, \qquad \text{on } \partial\Omega \times (0,T)\,, \tag{1.2}$$

$$u(x,0) = u_0(x)\,, \qquad \text{in } \Omega\,, \tag{1.3}$$

where $\alpha \geq 0$, $m > 0$, and $u_0 \geq 0$ is, in general, a locally finite measure. We assume throughout $m + \alpha > 1$, so that (1.1) is degenerate parabolic. The vector $\mathbf{n}$ denotes the outer normal to the boundary $\partial\Omega$. An alternative formulation of equation (1.1) is

$$\left(v^{\frac{m}{m+\alpha}}\right)_t - \frac{m^m}{(m+\alpha)^m}\operatorname{div}(|Dv|^{m-1}Dv) = 0\,,$$

where $v = u^{(m+\alpha)/m}$.

We think the interest and the novelty of our results are in the precise connection they establish between the shape of Ω and the properties of solutions to the problem above, like optimal supremum estimates and finite speed of propagation. This also shows under which respects our problem is similar to the Cauchy problem in $\boldsymbol{R}^N$, and in what, on the contrary, they diverge. Due to the fact that in domains narrowing at infinity all sup estimates must be local, even if the datum is globally integrable, we have to introduce a new approach to the problem of finite speed of propagation.

This technique is quite general, and it has been applied by us to geometries different from the one at hand, in [3], and (with suitable modifications) to the Cauchy problem for higher order parabolic equations, in [2]. We also note that the estimates developed here are employed in [5] in the investigation of a blow up problem in narrowing domains.

Neumann problems in a class of domains close to the one we consider here, but just for linear uniformly parabolic equations, were considered by Gushchin [13] and Lezhnev [18]. They investigated qualitative properties of solutions, mainly their rate of decay in time. Existence and uniqueness of solutions are dealt with in [14]. Their approaches are different from ours, relying mostly on the linearity of the equation, and on Green's function methods, while we employ integral techniques, based on suitable embedding results we introduce below. As usual, the embedding inequalities are connected with relative isoperimetrical inequalities in Ω. More specifically, let us define

$$\ell(v,\rho) = \inf\{|\partial G \cap \Omega_\rho|_{N-1} : G \subset \Omega_\rho\,, |G| = v\,, \partial G \text{ Lipschitz}\}\,, \tag{1.4}$$

for all $\rho > 0$, and for all $0 < v \le |\Omega_\rho|/2$, where (at least provisionally, see (1.9)–(1.10) below) we denote $\Omega_\rho = \Omega \cap \{|x| < \rho\}$, and Ω_ρ is assumed to be non-empty. The symbol $|\cdot|$ denotes the N dimensional Lebesgue measure, while the $N-1$ dimensional Hausdorff measure in $\boldsymbol{R}^N$ is denoted by $|\cdot|_{N-1}$. As an expository remark for readers who are not familiar with this kind of ideas, let us consider, for definiteness, sets G sharing much of ∂G with $\partial\Omega_\rho$. The shared part does not belong to $\partial G \cap \Omega_\rho$, and hence does not contribute to the surface area $|\partial G \cap \Omega_\rho|_{N-1}$.

Note that, as an essential difference with the case of "expanding domains" (e.g., domains like $\{|x'| < x_N^h\}$, $h \in [0,1]$, $x' = (x_1, \dots, x_{N-1})$), treated in the forthcoming papers [3], [4], we need here localize the definition of ℓ, i.e., we need introduce the restriction $G \subset \Omega_\rho$, $\rho < \infty$. In fact it is easy to see that for infinite cusps with infinite volume (like Ω^ε above), one has $\ell(v,\infty) \equiv 0$, for all $v > 0$. As a consequence of this fact, the embedding results are going to be local, and all our arguments must be devised accordingly. We may note here that infinite cylinders belong both to the class of narrowing domains, and to the class of expanding domains, and can be treated by both approaches. In the case of cylinders, of course, $\ell(v,\infty)$ does not vanish identically.

Let us define precisely the class of domains we are dealing with. Explicit examples of domains in this class can be found in [12], [18] for $N = 2$; we present some more examples for general $N \ge 2$ in Remark 1.1 and in Section 7 below. Assume we can find a continuous non decreasing function $f : [1,\infty) \to (0,\infty)$, such that

$$c_0 \frac{\rho}{f(\rho)} \le V(\rho) := |\Omega_\rho| \le c_1 \frac{\rho}{f(\rho)}\,, \qquad \text{for all } \rho \ge 1. \tag{1.5}$$

Here c_0, $c_1 > 0$ are suitable constants. Moreover, f is also required to fulfill

$$g(v,\rho) := c_2 \min\big(v^{\frac{N-1}{N}}, \frac{1}{f(\rho)}\big) \le \ell(v,\rho)\,, \tag{1.6}$$

for all $\rho \geq 1$, $0 < v \leq V(\rho)/2$, and a suitable constant $c_2 > 0$. The term $v^{\frac{N-1}{N}}$ clearly must appear in $g(v,\rho)$, at least if Ω_ρ has a Lipschitz boundary, while, heuristically, $1/f(\rho)$ takes the geometrical meaning of the area of $\Omega \cap \partial\Omega_\rho$ (see Remark 1.1).

Finally, we also require that for all $\delta > 0$

$$\nu_0(\delta)V(\rho) \leq V(\delta\rho) \leq \nu_1(\delta)V(\rho)\,, \qquad \text{for all } \rho \geq \max(1, 1/\delta), \tag{1.7}$$

where ν_0, ν_1 are two given nondecreasing positive functions, such that $\nu_1(\delta) < 1$ for $\delta < 1$.

Definition 1.1. We say that an open unbounded connected set $\Omega \subset \boldsymbol{R}^N$, $N \geq 2$, belongs to the class $\mathcal{N}(f)$ if its boundary $\partial\Omega$ is locally Lipschitz continuous, and if (1.5)–(1.7) are satisfied.

It follows from (1.7) that

$$|\Omega| = \infty\,. \tag{1.8}$$

Otherwise, we would get a contradiction by fixing $\delta < 1$ in (1.7), and then letting $\rho \to \infty$ there.

Let us also stress that all our results and proofs do not use the special definition $\Omega_\rho = \Omega \cap \{|x| < \rho\}$, stated above for the sake of clarity. In fact, ρ may stand for any 'privileged coordinate' in Ω, therefore allowing us to consider domains in a variety of shapes, see Remark 1.1 below. Rigorously, let us define a family $\{\Omega_\rho\}_{\rho\geq 1}$ of open subsets of Ω with Lipschitz boundary, satisfying the properties

$\Omega_\rho \subset \Omega_r$ if $\rho < r$, and every compact subset of $\overline{\Omega}$ is contained in $\overline{\Omega_\rho}$ for large enough ρ. (1.9)

For any $\rho_2 > \rho_1 \geq 1$, there exists a cutoff function $\zeta \in C^1(\overline{\Omega})$ such that $\operatorname{supp}\zeta \cap \Omega \subset \Omega_{\rho_2}$, with $\zeta \equiv 1$ in Ω_{ρ_1}, and $|D\zeta| \leq c_3/(\rho_2 - \rho_1)$, where c_3 is a fixed constant depending on Ω. (1.10)

Then, from now on, we may understand that in Definition 1.1, i.e., in (1.4)–(1.7), Ω_ρ takes the more general meaning just introduced.

We also note that condition $v \leq V(\rho)/2$ in (1.6) could be replaced with $v \leq \theta V(\rho)$ for any fixed $\theta < 1$.

Remark 1.1. Examples. The domain

$$\Omega^\varepsilon := \{x = (x', x_N) \in \boldsymbol{R}^N : |x'| < x_N^{-\varepsilon}\,,\, x_N > d\} \subset \boldsymbol{R}^N\,, \quad d > 0\,,$$

for $0 \le \varepsilon < 1$ such that $\beta = \varepsilon(N-1) < 1$, belongs to $\mathcal{N}(f)$, with $f(\rho) = \rho^\beta$, $\rho = x_N > 2d$. The proof of (1.6) (the other requirements being trivially satisfied) is rather lengthy, though not difficult, and we only say here that, essentially, we reduce to the case of the domain Ω defined in Section 7, by mapping Ω^ε onto (a suitable specialization of) Ω, by a subareal mapping (in the sense of [19], Chapter 3). Intuitively, one expects that $\ell(v, \rho) \simeq 1/f(\rho)$ if v is large enough to make convenient, in order to minimize $a = |\partial G \cap \Omega_\rho^\varepsilon|_{N-1}$, to choose G adjoining $\partial\Omega_\rho^\varepsilon \cap \Omega^\varepsilon$, so that, roughly, $a \simeq |\Omega^\varepsilon \cap \partial\Omega_\rho^\varepsilon|_{N-1}$.

By the same technique we can consider the "spiral-like" domain

$$\Sigma := \{(x, y) \in \boldsymbol{R}^2 : -\theta^{-\varepsilon} < R - \theta < \theta^{-\varepsilon}\,,\, \theta > 4\} \subset \boldsymbol{R}^2\,, \quad 0 < \varepsilon < 2\,,$$

where $R = \sqrt{x^2 + y^2}$, and $\theta \in (-\infty, +\infty)$ are the standard polar coordinates. We can show that Σ belongs to $\mathcal{N}(f)$, with the coordinate ρ given as the length measured along the curve $R = \theta$. Namely, $\rho = \rho(\theta) = \int_4^\theta \sqrt{1 + s^2}\, \mathrm{d}s$, $f(\rho) = \rho^{\frac{\varepsilon}{2}}$, $\Omega_\rho := \{(x, y) \in \Sigma \mid \theta(x, y) < \theta^{-1}(\rho)\}$. In fact (1.6) follows by mapping Σ onto $\Omega^{\frac{\varepsilon}{2}} \subset \boldsymbol{R}^2$ (defined as above) by the subareal mapping $\Phi(x, y) = (\theta^2, R - \theta)$, $(x, y) \in \Sigma$. Note that $\rho \simeq \theta^2 \simeq R^2$ in Σ, for large values of ρ. Thus, for example, if u_0 is compactly supported in Σ, from Theorem 1.2 below, we infer that the support of $u(\cdot, t)$, for large t, is sharply approximated, in the sense of (1.15), by sets $\{\theta \le Ct^{1/h}\} \subset \Sigma$, for suitable $C > 0$, and for $h = 4m + 2\alpha - \varepsilon(m + \alpha - 1)$.

Definition 1.2. A solution u to (1.1)–(1.3) is a non negative $u \in L^\infty_{\mathrm{loc}}(\overline{\Omega} \times (0, T))$, such that $u \in C((0, T); L^2_{\mathrm{loc}}(\overline{\Omega}))$, $u^\alpha |Du|^{m+1} \in L^1_{\mathrm{loc}}(\overline{\Omega} \times (0, T))$, and that

$$\int_0^T \int_\Omega [-u\zeta_t + u^\alpha |Du|^{m-1} Du \cdot D\zeta]\, \mathrm{d}x\, \mathrm{d}t = -\int_{\overline{\Omega}} \zeta(x, 0) u_0(x)\, \mathrm{d}x\,,$$

for all $\zeta \in C^1(\boldsymbol{R}^N \times [0, T])$, such that $\zeta \equiv 0$ out of $\{|x| \le K < \infty\}$, for a suitable $K > 0$, and $\zeta(x, T) = 0$. Moreover, we require that $u(\cdot, t)$ approaches u_0 as $t \to 0$, in the weak star sense of measures.

Notation. For any measurable set G, and positive numbers $q > 0$ and $r \geq 1$, we use the notation

$$\|u\|_{q,G} = \Big(\int_G |u|^q \,\mathrm{d}x \Big)^{1/q} , \qquad \fint_G u \,\mathrm{d}x = \frac{1}{|G|} \int_G u \,\mathrm{d}x \,,$$

$$|||u|||_r = \sup_{\rho \geq r} \rho^{-\frac{m+1}{m+\alpha-1}} \fint_{\Omega_\rho} \mathrm{d}u(x) \,, \quad u \text{ positive Radon measure in } \overline{\Omega}.$$

Throughout this paper we denote by γ, γ_0, $\gamma_1 \dots$, generic positive constants, depending on the quantities that will be indicated in each instance, and, implicitly, on the constants c_i appearing in the definition of $\mathcal{N}(f)$, as well as on $f(1)$, $\nu_1(1/2)$, and on N, m, α.

Let us state our general existence result. Please note that in the following, we understand all data measures to be *positive* measures.

Theorem 1.1. *Let u_0 be a positive Radon measure in $\overline{\Omega}$, such that $|||u_0|||_{\bar\rho} < \infty$, $\bar\rho \geq 1$. Then problem* (1.1)–(1.3) *has a solution defined in $\Omega \times (0, T_0)$, where $T_0 = \gamma_0 |||u_0|||_{\bar\rho}{}^{1-m-\alpha}$, and*

$$|||u(\cdot,t)|||_{\bar\rho} \leq \gamma |||u_0|||_{\bar\rho} \,, \tag{1.11}$$

$$\|u(\cdot,t)\|_{\infty,\Omega_\rho} \leq \gamma \rho^{\frac{m+1}{m+\alpha-1}} t^{-\frac{N}{\mathcal{K}}} |||u_0|||_{\bar\rho}{}^{\frac{m+1}{\mathcal{K}}} \,, \tag{1.12}$$

for $0 < t < T_0$, $\rho \geq \bar\rho$, where $\mathcal{K} = N(m+\alpha-1) + m + 1$.

Results of this type have been proven for the Cauchy problem (i.e., $\Omega = \boldsymbol{R}^N$), by DiBenedetto and Herrero in [11] when $\alpha = 0$; we also recall the recent work [15] for the case $\alpha \neq 0$, as well as [1] (whose general approach we follow in proving Theorem 1.1), [7], about porous-media equations.

We stress the fact that the admissible growth of the initial datum as $\rho \to \infty$ is the same in narrowing domains as in the case of the Cauchy problem. It is also clear from estimate (1.12) that the decay rate of solutions for small times, is the same for the Neumann problem in Ω as for the N-dimensional Cauchy problem. This was to be expected, since this phenomenon is essentially local. Also note that in (1.12) the shape of Ω does not appear explicitly (though it is of course involved in the definition of $|||\cdot|||_{\bar\rho}$). Anyway, the results below show that, under many respects, solutions to the Neumann problem in narrowing domains behave very differently from solutions to the Cauchy problem.

Next, we deal with the problem of finite speed of propagation. In the cases when the solution $u(\cdot,t)$ is compactly supported for all $t>0$, we introduce the function

$$Z(t)=\inf\{\rho>1 : u(x,t)=0 \text{ in } \Omega\setminus\Omega_\rho\}\,, \qquad t>0\,. \tag{1.13}$$

Obviously $Z(t)$ gives a measure of the speed of propagation of the support of u. In the power case of the domain Ω^ε discussed in Remark 1.1, our next result yields that for large times $Z(t)$ behaves like the power $P(t)=t^b$, where

$$b=[2m+\alpha-\varepsilon(N-1)(m+\alpha-1)]^{-1}\,. \tag{1.14}$$

The general result reads as follows:

Theorem 1.2. *Let u_0, u be as in Theorem 1.1, and assume moreover that* $\operatorname{supp} u_0\subset\overline{\Omega_{r_0}}$, $1<r_0<\infty$. *Then, for $t\geq\bar t>0$, we have*

$$\gamma_1 P(t)\leq Z(t)\leq\gamma_2 P(t)\,, \tag{1.15}$$

where $\rho=P(t)$ denotes the largest solution of

$$\rho f(\rho)^{-\frac{m+\alpha-1}{2m+\alpha}}=\|u_0\|_{1,\Omega}^{\frac{m+\alpha-1}{2m+\alpha}}\,t^{\frac{1}{2m+\alpha}}\,. \tag{1.16}$$

We also have for $t>\bar t$, the two-sided estimate

$$\gamma_1\frac{\|u_0\|_{1,\Omega}}{V(P(t))}\leq\|u(\cdot,t)\|_{\infty,\Omega}\leq\gamma_2\frac{\|u_0\|_{1,\Omega}}{V(P(t))}\,. \tag{1.17}$$

Here $\bar t$ depends on N, m, α, $f(1)$, $\|u_0\|_{1,\Omega}$, r_0.

The left hand side of (1.16) goes to ∞ as $\rho\to\infty$, because of (1.5), (1.8), so that $P(t)$ is well defined. In fact, as a consequence of Lemma 2.3 below, $P(t)$ in (1.15) may be any solution of (1.16). Note also that, recalling the definition of $P(t)$ and (1.5), we may recast (1.17) as

$$\gamma_1 t^{-\frac{1}{2m+\alpha}}\|u_0\|_{1,\Omega}^{\frac{m+1}{2m+\alpha}}f(P(t))^{\frac{m+1}{2m+\alpha}}\leq\|u(\cdot,t)\|_{\infty,\Omega}$$
$$\leq\gamma_2 t^{-\frac{1}{2m+\alpha}}\|u_0\|_{1,\Omega}^{\frac{m+1}{2m+\alpha}}f(P(t))^{\frac{m+1}{2m+\alpha}}\,.$$

When $f(\rho)\equiv$constant, i.e., Ω is a cylinder, the speed of propagation, for large times, is the same as in the one dimensional Cauchy problem, as we can see setting $\varepsilon = 0$ in (1.14) above. The intuitive interpretation of this fact is obvious: for large t diffusion is essentially one dimensional in cylinders. Therefore, Neumann problem in cylinders is similar to the N-dimensional Cauchy problem for small times (see above), and to the one dimensional Cauchy problem for large t.

In the general case, propagation is faster in thinner domains, in a sense made precise by (1.16) (and (1.14)).

We remark that in our approach we may let u_0 be a measure, where previous methods required some higher integrability of it (see [6], [9], [8]). Let us also point out that, relying on the local estimates given in Section 3 below, our approach circumvents the difficulty posed by the lack of an embedding theorem valid in the whole domain Ω.

A further typical feature of diffusion in narrowing domains is pointed out by:

Theorem 1.3. *Assume u_0 is a finite measure. Then problem* (1.1)–(1.3) *has a solution defined in $\Omega \times (0,\infty)$, such that for all $t > 0$, $\sigma \in (0, 1/2)$,*

$$\|u(\cdot,t)\|_{1,\Omega} = \|u_0\|_{1,\Omega}\,, \tag{1.18}$$

$$\|u(\cdot,t)\|_{\infty,\Omega_{(1+\sigma)\rho}\setminus\Omega_{(1-\sigma)\rho}} \le \gamma \max\left(t^{-\frac{N}{\mathcal{K}}}\Gamma(t)^{\frac{m+1}{\mathcal{K}}}, t^{-\frac{1}{\mathcal{H}}}\Gamma(t)^{\frac{m+1}{\mathcal{H}}} f(\rho)^{\frac{m+1}{\mathcal{H}}}\right), \tag{1.19}$$

provided $\rho \ge \gamma P(t)$, with $\mathcal{H} = 2m + \alpha$ and

$$\Gamma(t) = \sup_{0<\tau<t} \|u(\cdot,\tau)\|_{1,\Omega_{(1+2\sigma)\rho}\setminus\Omega_{(1-2\sigma)\rho}}\,.$$

Here $P(t)$ is defined by (1.16)*, or by $P(t) = 1$ if t is too small for* (1.16) *to admit solutions.*

Let us stress the following point: the sup estimate in (1.19) is not uniform on ρ, i.e., it is not uniform over Ω. This is a substantial difference with the case of the Cauchy problem, and also with the more general case of Ω being an "expanding" domain (see [4]). Indeed, in those cases, if the initial datum is a finite measure, the solution is uniformly bounded over the whole spatial domain for any fixed positive time. Such a regularizing effect does not take place in a domain Ω from the class $\mathcal{N}(f)$ (excepting of course the limiting case of $f \equiv$ constant), as suggested by (1.19). In fact, estimate (1.19) is optimal, as it is shown by the simple exact solution we exhibit in Section 7 below. Here we just add that a completely explicit estimate can be obtained from (1.19) simply by majorizing $\Gamma(t) \le \|u_0\|_{1,\Omega}$, owing to (1.18). It was shown in [13], [18], for linear equations, that uniform boundedness of u over Ω is connected with finiteness of suitable integral moments of the initial datum.

Remark 1.2. Our methods apply to equations of the more general type

$$u_t - \operatorname{div} \mathbf{a}(x, t, u, |u|^\alpha |Du|^{m-1} Du) = 0 \,,$$

under standard structure conditions. Also, theorems 1.1 and 1.3, as well as the estimates above for u and Z in Theorem 1.2, still hold for solutions of general sign (when u_0 is a signed measure); indeed our estimation procedures apply separately to the positive and negative parts of u.

Let us finally recall that the paper [23] is concerned with asymptotic L^∞ estimates of solutions to Neumann problems similar to (1.1)–(1.3), but only in a special class of narrowing domains, and under extra assumptions on the exponent m.

Section 2 contains preliminary material, and the already quoted embedding inequality in Ω_ρ. Section 3 is devoted to the proof of the main local a priori estimates. The proofs of theorems 1.1, 1.2, 1.3 are given in sections 4, 5, 6, in this order. Finally, Section 7 displays an explicit solution to a model problem of the class considered here.

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2. An embedding result and other preliminary material. In the following, we need introduce the function

$$\omega(z, \rho) = \frac{z^{\frac{N-1}{N}}}{g(z, \rho)} = \gamma \max\left(1, z^{\frac{N-1}{N}} f(\rho)\right), \qquad z \ge 0 \,, \rho \ge 1 \,. \tag{2.1}$$

Theorem 2.1. *Let $\Omega \in \mathcal{N}(f)$, $\rho \ge 1$, $v : \Omega_\rho \to \boldsymbol{R}$ be given, and let*

$$E_q = \int_{\Omega_\rho} |v|^q \,\mathrm{d}x < \infty, \quad E_\beta = \int_{\Omega_\rho} |v|^\beta \,\mathrm{d}x < \infty, \quad J_p = \int_{\Omega_\rho} |Dv|^p \,\mathrm{d}x < \infty \,,$$

for some $p > 1$, $\beta > 0$, $q \ge 1$, $q > \beta$. Assume moreover that v satisfies

$$E := E_\beta^{\frac{q}{q-\beta}} E_q^{-\frac{\beta}{q-\beta}} \le \theta_0 V(\rho) \,, \tag{2.2}$$

where $\theta_0 = \theta_0(q, p, \beta) \in (0, 1)$ *is a suitable constant. Then, for all* q *such that* $q(N-p) \le Np$, *we have*

$$\|v\|_{q,\Omega_\rho} \le \gamma\omega(E,\rho)E^{\frac{1}{q}-\frac{1}{p}+\frac{1}{N}}\|Dv\|_{p,\Omega_\rho}\,, \tag{2.3}$$

where γ *depends on* q, p, β, *and* ω *has been defined in* (2.1).

Proof. 1) *Case* $q > p$. Without loss of generality, we may assume $v \in C^\infty(\overline{\Omega_\rho})$. We denote by v^* the standard rearrangement of v in Ω_ρ, i.e., $v^*(t) = \inf\{\lambda : \mu(\lambda) < t\}$, where $\mu(\lambda) = |A(\lambda)|$, $A(\lambda) = \{x \in \Omega_\rho \mid |v(x)| > \lambda\}$. Using a well known inequality from the theory of Lorentz spaces (see [21], Thm. 3.11 chapter V), we may write for $k > 0$

$$\begin{aligned} E_q &= \int_{A(k)} |v|^q \,\mathrm{d}x + \int_{\Omega_\rho\setminus A(k)} |v|^q \,\mathrm{d}x \le \int_0^{\mu(k)} v^*(t)^q \,\mathrm{d}t + k^{q-\beta}E_\beta \\ &\le \Big(\frac{p}{q}\int_0^{\mu(k)} v^*(t)^p t^{\frac{p}{q}-1}\,\mathrm{d}t\Big)^{\frac{q}{p}} + k^{q-\beta}E_\beta\,. \end{aligned}$$

Thus, also employing Chebychev's inequality

$$\int_A u(x)\,\mathrm{d}x \ge t|A\cap\{u>t\}|\,, \qquad \text{if } u \ge 0,\ t > 0,$$

we obtain

$$\begin{aligned} E_q^{\frac{p}{q}} &\le \gamma\int_0^{\mu(k)} (v^*(t)-k)^p t^{\frac{p}{q}-1}\,\mathrm{d}t + \gamma k^p \mu(k)^{\frac{p}{q}} + \gamma k^{(q-\beta)\frac{p}{q}} E_\beta^{\frac{p}{q}} \\ &\le \gamma\int_0^{\mu(k)} (v^*(t)-k)^p t^{\frac{p}{q}-1}\,\mathrm{d}t + \gamma k^{(q-\beta)\frac{p}{q}} E_\beta^{\frac{p}{q}}\,. \end{aligned} \tag{2.4}$$

Next we select k so as to have in (2.4)

$$\gamma k^{(q-\beta)\frac{p}{q}} E_\beta^{\frac{p}{q}} = \frac{1}{2}E_q^{\frac{p}{q}}\,, \tag{2.5}$$

so that this term can be absorbed into the left hand side. Note that

$$[(v^*(t)-k)^p t^{\frac{p}{q}}]_t = p v^*_t (v^*-k)^{p-1} t^{\frac{p}{q}} + \frac{p}{q} t^{\frac{p}{q}-1}(v^*(t)-k)^p\,,$$

so that,

$$\frac{p}{q}\int_0^{\mu(k)} t^{\frac{p}{q}-1}(v^*(t)-k)^p\,\mathrm{d}t = -p\int_0^{\mu(k)} v_t^*(v^*-k)^{p-1}t^{\frac{p}{q}}\,\mathrm{d}t$$
$$\leq p\Big(\int_0^{\mu(k)}(-v_t^*)^p t^{\frac{p}{q}+p-1}\,\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^{\mu(k)} t^{\frac{p}{q}-1}(v^*(t)-k)^p\,\mathrm{d}t\Big)^{1-\frac{1}{p}}.$$

Thus, collecting the estimates above,

$$E_q^{\frac{p}{q}} \leq \gamma\int_0^{\mu(k)}(v^*(t)-k)^p t^{\frac{p}{q}-1}\,\mathrm{d}t \leq \gamma\int_0^{\mu(k)}[-(v^*)_t]^p t^{\frac{p}{q}+p-1}\,\mathrm{d}t$$
$$\leq \gamma\int_0^{\mu(k)}[-(v^*)_t]^p\omega(t,\rho)^p t^{\frac{p}{q}+\frac{p}{N}-1}g(t,\rho)^p\,\mathrm{d}t$$
$$\leq \gamma\omega(\mu(k),\rho)^p\mu(k)^{\frac{p}{q}+\frac{p}{N}-1}\int_0^{\mu(k)}[-(v^*)_t]^p g(t,\rho)^p\,\mathrm{d}t\,,$$

where we have used $p/q+p/N-1\geq 0$, and the increasing character of $\omega(\cdot,\rho)$. Note that, by Chebychev's inequality, and from (2.5),

$$\mu(k)\leq k^{-\beta}E_\beta\leq\gamma E\leq\gamma\theta_0 V(\rho)\leq\frac{1}{2}V(\rho)\,,$$

provided θ_0 in (2.2) is chosen suitably small. Then, reasoning as in [22], and using the isoperimetrical type inequality (see (1.6))

$$|\partial A(k)\cap\Omega_\rho|_{N-1}\geq g(\mu(k),\rho)\,,$$

we have $E_q^{\frac{p}{q}}\leq\gamma\omega(E,\rho)^p E^{\frac{p}{q}+\frac{p}{N}-1}J_p$, whence we obtain (2.3).

2) *Case* $q\leq p$. For a $k>0$ to be chosen, we have

$$E_q=\int_0^{V(\rho)} v^*(t)^q\,\mathrm{d}t\leq\gamma\int_0^{\mu(k)}(v^*(t)-k)^q\,\mathrm{d}t+\gamma\mu(k)^{1-\frac{q}{\beta}}E_\beta^{\frac{q}{\beta}}\,. \tag{2.6}$$

By Hölder's inequality we have

$$\int_0^{\mu(k)}(v^*(t)-k)^q\,\mathrm{d}t\leq\mu(k)^{1-\frac{q}{p}}\Big(\int_0^{\mu(k)}(v^*(t)-k)^p\,\mathrm{d}t\Big)^{\frac{q}{p}}. \tag{2.7}$$

Choose now k so that (for γ as in (2.6))

$$\gamma\mu(k)^{1-\frac{q}{\beta}}E_\beta^{\frac{q}{\beta}}=\frac{1}{2}E_q \qquad \text{i.e.,}\quad \mu(k)=(2\gamma)^{\frac{\beta}{q-\beta}}E\,; \tag{2.8}$$

this is possible because of (2.2), perhaps redefining θ_0 there, if necessary. In fact, obviously one may assume $v \ge 0$, and, by approximation, $\max \mu = |\operatorname{supp} v| = V(\rho)$. Then we have, reasoning as in [22],

$$E_q \le \gamma \mu(k)^{1-\frac{q}{p}} \Big(\int_0^{\mu(k)} (v^*(t) - k)^p \,\mathrm{d}t \Big)^{\frac{q}{p}}$$
$$\le \gamma \mu(k)^{1-\frac{q}{p}} \Big(\int_0^{\mu(k)} [-(v^*)_t]^p \frac{t^p}{g(t,\rho)^p} g(t,\rho)^p \,\mathrm{d}t \Big)^{\frac{q}{p}} \le \gamma \frac{\mu(k)^{1-\frac{q}{p}+q}}{g(\mu(k),\rho)^q} J_p^{\frac{q}{p}} ,$$

whence the claimed estimate (2.3), on applying again (2.8).

Remark 2.1. A simple application of Hölder's inequality gives

$$E \le |\operatorname{supp} v| \,, \tag{2.9}$$

so that (2.2) can be interpreted as an assumption on the measure of the support of v.

Corollary 2.1. *Let $v \in L^\infty((0,T); L^r(\Omega_\rho))$, $Dv \in (L^p(\Omega_\rho \times (0,T)))^N$, with $p > 1$, $r \ge 1$, and assume that*

$$\sup_{(0,T)} |\operatorname{supp} v(\cdot,t)| \le \theta_0 V(\rho) \,, \tag{2.10}$$

for the same θ_0 as in Theorem 2.1. Then

$$\int_0^T \int_{\Omega_\rho} |v|^{p+\frac{pr}{N}} \,\mathrm{d}x \,\mathrm{d}t \tag{2.11}$$
$$\le \gamma \sup_{0<t<T} \Big[\omega(|\operatorname{supp} v(\cdot,t)|,\rho)^p \Big(\int_{\Omega_\rho} |v(x,t)|^r \,\mathrm{d}x \Big)^{\frac{p}{N}} \Big] \int_0^T \int_{\Omega_\rho} |Dv|^p \,\mathrm{d}x \,\mathrm{d}t \,,$$

where the function ω has been defined in (2.1)*, and $\gamma = \gamma(p,r,N)$.*

Proof. We start from the embedding proven in Theorem 2.1, keeping in mind assumption (2.10) and Remark 2.1.

To prove (2.11) we discriminate between the cases $p < N$ and $p \ge N$. If $p < N$, we choose $q = Np/(N-p)$ in (2.3), to find

$$\int_0^T \int_{\Omega_\rho} |v(t)|^{p+\frac{pr}{N}} \,\mathrm{d}x \,\mathrm{d}t \le \int_0^T \Big(\int_{\Omega_\rho} |v(t)|^q \,\mathrm{d}x \Big)^{\frac{N-p}{N}} \Big(\int_{\Omega_\rho} |v(t)|^r \,\mathrm{d}x \Big)^{\frac{p}{N}} \,\mathrm{d}t$$
$$\le \gamma \int_0^T \omega(|\operatorname{supp} v(t)|,\rho)^p \Big(\int_{\Omega_\rho} |v(t)|^r \,\mathrm{d}x \Big)^{\frac{p}{N}} \Big(\int_{\Omega_\rho} |Dv(t)|^p \,\mathrm{d}x \Big) \,\mathrm{d}t \,,$$

and then (2.11). If $p \geq N$, selecting in (2.3) $q = p + pr/N > r$, $\beta = r$, we have

$$\int_{\Omega_\rho} |v|^q \,\mathrm{d}x \leq \gamma\omega(|\operatorname{supp} v|, \rho)^p \Big(\int_{\Omega_\rho} |v|^r \,\mathrm{d}x \Big)^{\frac{p}{N}} \int_{\Omega_\rho} |Dv|^p \,\mathrm{d}x \,.$$

Then we integrate over $(0, T)$ to get (2.11) again.

Remark 2.2. It is perhaps worth mentioning that if $v = 0$ on $\partial\Omega_\rho$, which we can not assume, embedding results of the type above are well known (see, e.g, [10]). See also [12] for a related inequality (with $p = 2$), involving moments of v over Ω.

We conclude this section with some technical lemmata.

Lemma 2.1. *Let $\Omega \in \mathcal{N}(f)$. For any $0 < \varepsilon < 1$, there exists $C_\varepsilon > 1$, such that*

$$V(\rho) \leq \varepsilon V(C_\varepsilon \rho)\,, \qquad \textit{for all } \rho \geq 1. \tag{2.12}$$

Here C_ε depends on ε and on $\nu_1(1/2)$.

Proof. Assume $r \geq 2^n$, $n \geq 0$. A repeated application of (1.7) yields

$$V(2^{-n} r) \leq \nu_1(1/2)^n V(r) \leq \varepsilon V(r)\,,$$

where last inequality clearly holds for $n \geq n_\varepsilon$, for a suitable $n_\varepsilon \geq 1$, owing to $\nu_1(1/2) < 1$, which we assumed in (1.7). Estimate (2.12) follows on defining $C_\varepsilon = 2^{n_\varepsilon}$, $r = C_\varepsilon \rho$.

In fact, reasoning as in Lemma 2.1, one proves $\nu_1(\delta) \to 0$ as $\delta \to 0$, provided $\nu_1(\delta') < 1$ for some $\delta' < 1$.

Lemma 2.2. *Let $\Omega \in \mathcal{N}(f)$. Then there exist two positive functions $\tilde\nu_0$, $\tilde\nu_1$ such that for all $\delta > 0$*

$$\tilde\nu_0(\delta) f(\rho) \leq f(\delta\rho) \leq \tilde\nu_1(\delta) f(\rho)\,, \qquad \textit{for all } \rho \geq \max(1, 1/\delta). \tag{2.13}$$

Proof. Claim (2.13) follows immediately from combining (1.7) and (1.5); one gets

$$\tilde\nu_0(\delta) = \frac{c_0}{c_1} \frac{\delta}{\nu_1(\delta)}\,, \qquad \tilde\nu_1(\delta) = \frac{c_1}{c_0} \frac{\delta}{\nu_0(\delta)}\,, \qquad \delta > 0\,.$$

Lemma 2.3. *Let $\Omega \in \mathcal{N}(f)$, and define $\xi(\rho) = \rho f(\rho)^{-\beta}$, $\rho \geq 1$, $0 < \beta \leq 1$. Then for any $b \geq 1$ there exists $c \geq 1$ such that for all ρ_1, $\rho_2 \geq 1$, $\xi(\rho_1) = b\xi(\rho_2)$ implies $\rho_1 \leq c\rho_2$.*

Proof. Clearly we may assume $\rho_1 > \rho_2$. Then, from (1.5),

$$c_1^{-1} V(\rho_1) f(\rho_1)^{1-\beta} \leq \xi(\rho_1) = b\xi(\rho_2) \leq b c_0^{-1} V(\rho_2) f(\rho_2)^{1-\beta}.$$

Whence, owing to the monotonic character of f, and to Lemma 2.1,

$$V(\rho_1) \leq b c_1 c_0^{-1} V(\rho_2) \leq b c_1 c_0^{-1} (b c_1 c_0^{-1})^{-1} V(\gamma\rho_2) = V(\gamma\rho_2),$$

for a suitable $\gamma > 0$. Thus $\rho_1 \leq \gamma\rho_2$.

3. Local estimates. In this Section, we prove a priori sup and gradient estimates for u, which will be needed in the following.

Proposition 3.1. *Let u be a continuous solution to (1.1)–(1.3) in $\overline{\Omega}\times(0,T)$, and let $t > 0$ be such that*

$$\frac{\tau}{\rho^{m+1}} u(x,\tau)^{m+\alpha-1} \leq 1, \qquad 0 < \tau < t, x \in \Omega_{(1+\sigma)\rho}, \tag{3.1}$$

where $\rho \geq 1$, $0 < \sigma < 1$, $0 < t < T$ are given. Then for all $q > 0$ we have, setting $Q_\infty = \Omega_\rho \times (t/2, t)$, $Q_0 = \Omega_{(1+\sigma)\rho} \times (t(1-\sigma)/2, t)$,

$$\begin{aligned} \|u\|_{\infty,Q_\infty} &\leq \gamma \max\left(t^{-\frac{N}{\mathcal{K}_q}} G_q(t,\rho+\sigma\rho)^{\frac{m+1}{\mathcal{K}_q}}, t^{-\frac{1}{\mathcal{H}_q}} G_q(t,\rho+\sigma\rho)^{\frac{m+1}{\mathcal{H}_q}} f(\rho)^{\frac{m+1}{\mathcal{H}_q}}\right) \\ &=: M_q(t,\rho+\sigma\rho), \end{aligned} \tag{3.2}$$

where

$$G_q(t,\rho) = \sup_{0<\tau<t} \int_{\Omega_\rho} u(x,\tau)^q \, \mathrm{d}x, \qquad t > 0, \rho \geq 1, \tag{3.3}$$

$\mathcal{K}_q = N(m+\alpha-1) + q(m+1)$, $\mathcal{H}_q = m+\alpha-1+q(m+1)$, *and* $\gamma = \gamma(\sigma, q)$.

Proof. Let $q > 0$ be fixed, and define for $n \geq 0$ $\rho_n = \rho + \sigma 2^{-n}\rho$, $t_n = t(1-\sigma 2^{-n})/2$, $k_n = k(1-2^{-n-1})$, $Q_n = \Omega_{\rho_n} \times (t_n, t)$, for a $k > 0$ to be chosen. Let us also introduce a sequence of smooth cutoff functions $\{\zeta_n\}$, $n \geq 1$, such that $\zeta_n \equiv 1$ in Q_n, $\zeta_n \equiv 0$ outside of Q_{n-1}, $0 \leq \zeta_n \leq 1$, and

$$0 \leq \zeta_{nt} \leq \gamma 2^n (\sigma t)^{-1}, \qquad |D\zeta_n| \leq \gamma 2^n (\sigma\rho)^{-1}.$$

By means of standard calculations, using in the weak formulation of the problem the testing function $(u-k_{n+1})_+^q\zeta_{n+1}^{m+1}$, we get for $n\geq 0$

$$\sup_{0<\tau<t}\int_{\Omega(\tau)}(u-k_{n+1})_+^{q+1}\zeta_{n+1}^{m+1}\,\mathrm{d}x+k^\alpha\int_0^t\!\!\int_\Omega|D(u-k_{n+1})_+^{\frac{q+m}{m+1}}\zeta_{n+1}|^{m+1}\,\mathrm{d}x\,\mathrm{d}\tau \leq \gamma\frac{b^n}{t\sigma^{m+1}}\left\{1+\frac{t}{\rho^{m+1}}\|u\|_{\infty,Q_0}^{m+\alpha-1}\right\}\iint_{Q_n}(u-k_n)_+^{q+1}\,\mathrm{d}x\,\mathrm{d}\tau \quad (3.4)$$

(here and below γ and $b>1$ denote suitable constants depending on q, m, N, α). We are going to apply the embedding result of Corollary 2.1 to $v=(u-k_{n+1})_+^{\frac{q+m}{m+1}}\zeta_{n+1}^{\frac{q+m}{q+1}}$, on setting $p=m+1$, $r=(q+1)(m+1)/(q+m)$. To this end, we remark that, for all $n\geq 0$, $\tau>t_n$, and θ_0 as in (2.2),

$$|\operatorname{supp}(u-k_{n+1})_+\zeta_{n+1}(\tau)|\leq V(2\rho)\leq\theta_0 V(C\rho)\,, \quad (3.5)$$

provided $C=C(\theta_0)>0$ is chosen large enough. We are using here Lemma 2.1. In other words, we may satisfy (2.10), in this context, by substituting the spatial domain $\Omega_{(1+\sigma)\rho}$ with a larger domain $\Omega_{C\rho}$, in which we apply the embedding inequality (note that v vanishes outside of $\Omega_{(1+\sigma)\rho}$). Then we infer from Corollary 2.1, that

$$\begin{aligned}
&Y_{n+1}\equiv\iint_{Q_{n+1}}(u-k_{n+1})_+^{q+1}\,\mathrm{d}x\,\mathrm{d}\tau\leq\iint_{Q_n}(u-k_{n+1})_+^{q+1}\zeta_{n+1}^{m+1}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leq|A_{n+1}|^{1-\frac{q+1}{\lambda}}\Big(\iint_{Q_n}(u-k_{n+1})_+^{\lambda}\zeta_{n+1}^{\frac{\lambda(m+1)}{q+1}}\,\mathrm{d}x\,\mathrm{d}\tau\Big)^{\frac{q+1}{\lambda}} \qquad (3.6)\\
&\leq\gamma|A_{n+1}|^{1-\frac{q+1}{\lambda}}\Big(\Big[\sup_{t_n<\tau<t}\omega(|A_{n+1}(\tau)|,C\rho)^{m+1}\\
&\times\Big(\int_{\Omega(\tau)}(u-k_{n+1})_+^{q+1}\zeta_{n+1}^{m+1}\,\mathrm{d}x\Big)^{\frac{m+1}{N}}\Big]\Big[\int_0^t\!\!\int_\Omega|D(u-k_{n+1})_+^{\frac{q+m}{m+1}}\zeta_{n+1}|^{m+1}\,\mathrm{d}x\,\mathrm{d}\tau\\
&+\frac{b^n}{\rho^{m+1}\sigma^{m+1}}\|u\|_{\infty,Q_0}^{\alpha+m-1}k^{-\alpha}Y_n\Big]\Big)^{\frac{q+1}{\lambda}},
\end{aligned}$$

for $\lambda=q+m+\frac{(m+1)(q+1)}{N}$, $A_{n+1}=\{(x,\tau)\in Q_n:u(x,\tau)>k_{n+1}\}\subset\boldsymbol{R}^{N+1}$, $A_{n+1}(\tau)=\{x\in\Omega_{\rho_n}:u(x,\tau)>k_{n+1}\}\subset\boldsymbol{R}^N$. Next we bound above the measures $|A_{n+1}|$, $|A_{n+1}(\tau)|$ by means of Chebychev's inequality, and, again, of (3.4); for $t_n < \tau < t$ we have for all $n \geq 1$

$$|A_{n+1}| \leq \gamma 2^{n(q+1)} k^{-q-1} Y_n \,, \tag{3.7}$$

$$|A_{n+1}(\tau)| \leq \gamma 2^{n(q+1)} k^{-q-1} \int_{\Omega_{\rho_n}(\tau)} (u-k_n)_+^{q+1} \,\mathrm{d}x \leq \gamma \frac{b^n k^{-q-1}}{t\sigma^{m+1}} Y_0 \,, \; n \geq 1 \,. \tag{3.8}$$

Thus we have, from (3.4)–(3.8),

$$Y_{n+1} \leq \gamma b^n k^{-\frac{q+1}{\lambda}(\alpha+\lambda-q-1)} \times \frac{Y_n^{1+\frac{(q+1)(m+1)}{\lambda N}}}{(t\sigma^{m+1})^{\frac{q+1}{\lambda}\left(1+\frac{m+1}{N}\right)}} \omega\Big(\gamma_1 b^n \frac{k^{-q-1}}{t\sigma^{m+1}} Y_0, C\rho\Big)^{(m+1)\frac{q+1}{\lambda}} \,, \tag{3.9}$$

for $n \geq 1$. Note that $\omega(b^n z, C\rho) \leq \gamma(C) b^n \omega(z,\rho)$, for $z \geq 0$, owing to the explicit representation (2.1), and to (2.13). Then we reason as follows: from Lemma 3.3 Chapter II of [17], we have that $Y_n \to 0$, i.e., $\|u\|_{\infty,Q_\infty} \leq k$, if

$$\gamma k^{-\frac{q+1}{\lambda}(\alpha+\lambda-q-1)} \frac{\|u\|_{q+1,Q_0}^{(q+1)\frac{(q+1)(m+1)}{\lambda N}}}{(t\sigma^{m+1})^{\frac{q+1}{\lambda}\left(1+\frac{m+1}{N}\right)}} \omega\Big(\frac{k^{-q-1}}{t\sigma^{m+1}} \|u\|_{q+1,Q_0}^{q+1}, \rho\Big)^{(m+1)\frac{q+1}{\lambda}} \leq 1 \,. \tag{3.10}$$

Let us select k so that (3.10) holds with an equality sign (this can be done a priori). Then we have, using the fact that the left hand side of (3.10) is decreasing in k,

$$\gamma \|u\|_{\infty,Q_\infty}^{-(\alpha+\lambda-q-1)} \frac{\|u\|_{q+1,Q_0}^{(q+1)\frac{m+1}{N}}}{(t\sigma^{m+1})^{1+\frac{m+1}{N}}} \omega\Big(\frac{\|u\|_{\infty,Q_\infty}^{-q-1}}{t\sigma^{m+1}} \|u\|_{q+1,Q_0}^{q+1}, \rho\Big)^{m+1} \geq 1 \,. \tag{3.11}$$

We now apply a second (finite) iteration procedure. Define a sequence of interpolating cylinders $Q^i = \Omega_{r_i} \times (t_i, t)$, $r_{i+1} = r_i + 2^{-i-1}\sigma\rho$, $r_0 = \rho$, $t_{i+1} = t_i - 2^{-i-2}\sigma t$, $t_0 = t/2$, $i \geq 0$, so that $Q^i \subset Q^{i+1} \subset Q_0$ for all $i \geq 0$, and $Q^0 = Q_\infty$. Let us also set, for a $1 > \delta > 0$ to be chosen, $U_i = \|u\|_{\infty,Q^i}$,

$$\begin{cases} j = 0 \,, & \text{if } U_0 \geq \delta U_1, \\ j = \max\{k \geq 1 \mid U_{i-1} < \delta U_i \,, \text{ for } 1 \leq i \leq k\} \,, & \text{if } U_0 < \delta U_1. \end{cases}$$

Clearly, $j < \infty$ unless we have $U_0 < \delta^i \|u\|_{\infty,Q_0}$ for all $i > 0$, i.e., $u \equiv 0$ in Q_∞. Moreover, as $U_{j+1} \le \delta^{-1} U_j$, we infer

$$\|u\|_{q+1,Q^{j+1}}^{q+1} \le U_{j+1} \|u\|_{q,Q^{j+1}}^{q} \le \delta^{-1} U_j \|u\|_{q,Q_0}^{q} .$$

Then we apply estimate (3.11) to the pair of cylinders Q^j, Q^{j+1}, together with the just stated bounds, and with $U_0 < \delta^j U_j$, getting

$$\gamma (2^a \delta)^{cj} U_0^{-(\alpha+m-1+\frac{m+1}{N}q)} \frac{\|u\|_{q,Q_0}^{q\frac{m+1}{N}}}{(t\sigma^{m+1})^{1+\frac{m+1}{N}}} \omega\Big((2^a\delta)^{cj} \frac{U_0^{-q}}{t\sigma^{m+1}} \|u\|_{q,Q_0}^{q}, \rho\Big)^{m+1} \ge 1, \tag{3.12}$$

where γ, a, $c > 0$ are suitable constants depending on m, N, q, α; γ depends on δ too. Finally we choose δ so that $2^a \delta = 1$. Then we can see that the bound for U_0 in (3.12) does not actually depend on j. Estimate (3.2) is now found by making explicit the majorization (3.12).

Proposition 3.2. *Under the assumptions of Proposition* 3.1 (*where we set formally* $\sigma = 3$ *in* (3.1)), *we have*

$$\int_0^t\!\!\int_{\Omega_\rho} |Du|^\mu u^\theta \,\mathrm{d}x\,\mathrm{d}\tau \le \gamma G_1(t, 4\rho) t^{1-\frac{\mu}{m+1}} M_q(t, 4\rho)^s , \tag{3.13}$$

provided $q > 0$, $0 < \mu < m+1$, *and*

$$s = \mu(2-\alpha)(m+1)^{-1} + \theta - 1 > 0, \quad \theta < \alpha - 1 + (m+1-\mu)(1+qN^{-1}). \tag{3.14}$$

Here γ *depends on the same quantities as in* (3.2), *and also on* μ, θ; M_q, G_1 *have the same meaning as in Proposition* 3.1.

Proof. We begin with the following application of Hölder's inequality

$$\begin{aligned}\int_0^t\!\!\int_{\Omega_\rho} |Du|^\mu u^\theta \,\mathrm{d}x\,\mathrm{d}\tau \le \Big(\int_0^t\!\!\int_{\Omega_\rho} \tau^{\lambda\frac{m+1}{\mu}} |Du|^{m+1} u^{(\theta-\varepsilon)\frac{m+1}{\mu}} \,\mathrm{d}x\,\mathrm{d}\tau \Big)^{\frac{\mu}{m+1}} \\ \times \Big(\int_0^t\!\!\int_{\Omega_\rho} \tau^{-\frac{\lambda(m+1)}{m+1-\mu}} u^{\frac{\varepsilon(m+1)}{m+1-\mu}} \,\mathrm{d}x\,\mathrm{d}\tau \Big)^{1-\frac{\mu}{m+1}} ,\end{aligned} \tag{3.15}$$

where we assume $0 < \mu < m+1$, and

$$\varepsilon(m+1) = m+1-\mu , \qquad 0 < \lambda(m+1) < m+1-\mu . \tag{3.16}$$

In order to estimate the first term on the right hand side of (3.15), we choose $\tau^{\lambda\frac{m+1}{\mu}} u^r \zeta^{m+1}$ as a testing function in problem (1.1)–(1.3), with

$$r = 1 - \alpha + (\theta - \varepsilon)\frac{m+1}{\mu} > 0 \,,$$

as a consequence of (3.14) and of (3.16). Here $\zeta(x)$ is a smooth cutoff function in $\Omega_{2\rho}$, with $\zeta \equiv 1$ in Ω_ρ, and $|D\zeta| \le \gamma\rho^{-1}$. After standard calculations we find

$$\begin{aligned} J_0 &:= \int_0^t\!\!\int_{\Omega_\rho} \tau^{\lambda\frac{m+1}{\mu}} |Du|^{m+1} u^{(\theta-\varepsilon)\frac{m+1}{\mu}} \,\mathrm{d}x\,\mathrm{d}\tau \\ &\le \gamma \int_0^t\!\!\int_{\Omega_{2\rho}} \tau^{\lambda\frac{m+1}{\mu}-1} u^{r+1} \,\mathrm{d}x\,\mathrm{d}\tau + \frac{\gamma}{\rho^{m+1}} \int_0^t\!\!\int_{\Omega_{2\rho}} \tau^{\lambda\frac{m+1}{\mu}} u^{m+\alpha+r} \,\mathrm{d}x\,\mathrm{d}\tau \,. \end{aligned}$$

By taking into account (3.1) and (3.2), we get

$$J_0 \le \gamma \int_0^t\!\!\int_{\Omega_{2\rho}} \tau^{\lambda\frac{m+1}{\mu}-1} u^{r+1} \,\mathrm{d}x\,\mathrm{d}\tau \le \gamma G_1(t, 2\rho) \int_0^t \tau^{\lambda\frac{m+1}{\mu}-1} M_q(\tau, 4\rho)^r \,\mathrm{d}\tau \,. \tag{3.17}$$

The last integral in (3.17) can be calculated explicitly, due to the definition of M_q in (3.2), provided we choose $\lambda > 0$ so that $\lambda\frac{m+1}{\mu} - \frac{Nr}{\mathcal{K}_q} > 0$, and the integral takes a finite value. In turn, this is possible, while keeping (3.16) in force, owing to (3.14). Finally, we substitute this bound for J_0 in (3.15), and we also majorize

$$\int_0^t\!\!\int_{\Omega_\rho} \tau^{-\frac{\lambda(m+1)}{m+1-\mu}} u^{\frac{\varepsilon(m+1)}{m+1-\mu}} \,\mathrm{d}x\,\mathrm{d}\tau \le \gamma G_1(t,\rho) t^{1-\frac{\lambda(m+1)}{m+1-\mu}}$$

(see (3.16) again). The sought after gradient estimate follows at once.

4. Proof of Theorem 1.1. Fix $\bar\rho \ge 1$. We construct a sequence of approximating solutions

$$\begin{aligned} u_{nt} - \operatorname{div}(u_n^\alpha |Du_n|^{m-1} Du_n) &= 0\,, && \text{in } \Omega_n \times (0,\infty), \\ u_n(x,t) &= 0\,, && \text{on } (\partial\Omega_n \cap \Omega) \times (0,\infty), \\ u_n^\alpha |Du_n|^{m-1} Du_n \cdot \mathbf{n} &= 0\,, && \text{on } (\partial\Omega_n \cap \partial\Omega) \times (0,\infty), \\ u_n(x,0) &= u_{0n}(x) && \text{in } \Omega_n, \end{aligned}$$

where $u_{0n} \in C^\infty(\overline{\Omega_n})$, and $u_{0n} \to u_0$ in the sense of measures, with $|||u_{0n}|||_{\bar\rho} \le \gamma |||u_0|||_{\bar\rho}$. Global existence of a nonnegative solution to the problem above follows, e.g., from the methods of [24]. In the following we drop for the sake of simplicity the index n, also defining $u_n \equiv 0$ in $\Omega \setminus \Omega_n$. Define

$$t_1 = \sup\{t > 0 : \sup_{0<\tau<t} \frac{\tau}{\rho^{m+1}} \|u(\cdot,\tau)\|_{\infty,\Omega_{4\rho}}^{m+\alpha-1} \le 1\,,\ \text{for all } \rho \ge \bar\rho\}\,.$$

Note that $t_1 > 0$ because of the regularity of u. Multiply the equation satisfied by u against $\zeta \in C^1(\overline{\Omega})$, $\zeta \equiv 1$ in Ω_ρ, $\zeta \equiv 0$ in $\Omega \setminus \Omega_{2\rho}$, $|D\zeta| \le \gamma/\rho$, $0 \le \zeta \le 1$; here we take $\rho \ge \bar\rho$. By integrating by parts we get

$$\begin{aligned}\int_{\Omega_\rho} u(x,t)\,\mathrm{d}x &\le \int_{\Omega_{2\rho}} u_0(x)\,\mathrm{d}x + \frac{\gamma}{\rho}\int_0^t\int_{\Omega_{2\rho}} |Du|^m u^\alpha \,\mathrm{d}x\,\mathrm{d}\tau \\ &\le \int_{\Omega_{2\rho}} u_0(x)\,\mathrm{d}x + \frac{\gamma}{\rho} G_1(t,4\rho) t^{\frac{1}{m+1}} M_1(t,4\rho)^{\frac{m+\alpha-1}{m+1}}\,, \end{aligned} \qquad (4.1)$$

provided $t < t_1$, so that we can make use of Proposition 3.2 with $\mu = m$, $\theta = \alpha$, $q = 1$ (it is clear that in (3.13) Ω_ρ can be replaced with $\Omega_{2\rho}$). Dividing both sides of (4.1) by $V(\rho)\rho^{\frac{m+1}{m+\alpha-1}}$, we find

$$\begin{aligned}\rho^{-\frac{m+1}{m+\alpha-1}} &⨍_{\Omega_\rho} u(x,t)\,\mathrm{d}x \le \gamma \rho^{-\frac{m+1}{m+\alpha-1}} ⨍_{\Omega_{2\rho}} u_0(x)\,\mathrm{d}x \\ &+ \gamma\Big[\sup_{0<\tau<t} (4\rho)^{-\frac{m+1}{m+\alpha-1}} ⨍_{\Omega_{4\rho}} u(x,\tau)\,\mathrm{d}x\Big] \frac{t^{\frac{1}{m+1}}}{\rho} M_1(t,4\rho)^{\frac{m+\alpha-1}{m+1}}\,. \end{aligned} \qquad (4.2)$$

Next we define $t_2 = \sup\{t > 0 : \sup_{\rho\ge\bar\rho} \frac{t^{\frac{1}{m+1}}}{\rho} M_1(t,4\rho)^{\frac{m+\alpha-1}{m+1}} \le \delta\} > 0$, for a $\delta > 0$ to be chosen; in this connection, we have to note that $t \mapsto tM_1(t,4\rho)^{m+\alpha-1}$ is increasing. Let us also set $t_0 = \min(t_1,t_2)$. For $t < t_0$ we get from (4.2)

$$\sup_{\rho\ge\bar\rho} \rho^{-\frac{m+1}{m+\alpha-1}} ⨍_{\Omega_\rho} u(x,t)\,\mathrm{d}x \le \gamma |||u_0|||_{\bar\rho} + \gamma\delta \sup_{0<\tau<t} \sup_{\rho\ge 4\bar\rho} \rho^{-\frac{m+1}{m+\alpha-1}} ⨍_{\Omega_\rho} u(x,t)\,\mathrm{d}x,$$

i.e., for small enough δ,

$$|||u(\cdot,t)|||_{\bar\rho} \le \gamma |||u_0|||_{\bar\rho}\,, \qquad 0 < t < t_0\,. \qquad (4.3)$$

As a further step in the proof, we show that, in fact, $t_0 = t_2$, if δ is suitably chosen. Indeed for $\rho \geq \bar{\rho}$, $t < t_0$ we have, applying Proposition 3.1 (note that (3.1) follows from $t < t_1$),

$$\frac{t}{\rho^{m+1}} \|u(\cdot,\tau)\|_{\infty,\Omega_{4\rho}}^{m+\alpha-1} \leq \gamma \frac{t}{\rho^{m+1}} M_1(t, 8\rho)^{m+\alpha-1} \leq \gamma \delta^{m+1} < \frac{1}{2} \,,$$

redefining δ if required. Finally, we bound away $t_0 = t_2$ from $t = 0$. This is done simply by exploiting the integral estimate (4.3) in the definition of t_2. For all $t < t_2$ we have by elementary calculations, recalling that $\mathcal{H} = 2m+\alpha$,

$$\begin{aligned} &\sup_{\rho\geq\bar{\rho}} \frac{t^{\frac{1}{m+1}}}{\rho} M_1(t,4\rho)^{\frac{m+\alpha-1}{m+1}} \leq \\ &\gamma \sup_{\rho\geq\bar{\rho}} \max \Big\{ t^{\frac{1}{\mathcal{K}}} \sup_{0<\tau<t} \Big(\frac{1}{\rho^{\frac{m+1}{m+\alpha-1}+N}} \int_{\Omega_{4\rho}} u(x,\tau)\, \mathrm{d}x \Big)^{\frac{m+\alpha-1}{\mathcal{K}}} \,, \\ &\qquad t^{\frac{1}{\mathcal{H}}} \sup_{0<\tau<t} \Big(\frac{f(\rho)}{\rho} \frac{1}{\rho^{\frac{m+1}{m+\alpha-1}}} \int_{\Omega_{4\rho}} u(x,\tau)\, \mathrm{d}x \Big)^{\frac{m+\alpha-1}{\mathcal{H}}} \Big\} . \end{aligned}$$

Moreover, $|\Omega_{4\rho}| \leq \gamma \rho^N$ and $f(\rho)/\rho \leq \gamma / V(4\rho)$ follow from the definition of the class $\mathcal{N}(f)$, so that, if $t_0 < \infty$, and for δ fixed as above,

$$\delta = \sup_{\rho\geq\bar{\rho}} \frac{t_0^{\frac{1}{m+1}}}{\rho} M_1(t_0, 4\rho) \leq \gamma \max \big(t_0^{\frac{1}{\mathcal{K}}} |||u_0|||_{\bar{\rho}}^{\frac{m+\alpha-1}{\mathcal{K}}} \,,\, t_0^{\frac{1}{\mathcal{H}}} |||u_0|||_{\bar{\rho}}^{\frac{m+\alpha-1}{\mathcal{H}}} \big) . \quad (4.4)$$

This implies the required minorization for t_0. Estimates (3.2) and (4.4) lead us to

$$\|u(\cdot,t)\|_{\infty,\Omega_\rho} \leq \gamma \rho^{\frac{m+1}{m+\alpha-1}} \max(t^{-\frac{N}{\mathcal{K}}} |||u_0|||_{\bar{\rho}}^{\frac{m+1}{\mathcal{K}}} \,,\, t^{-\frac{1}{\mathcal{H}}} |||u_0|||_{\bar{\rho}}^{\frac{m+1}{\mathcal{H}}}) \,,$$

for $\rho \geq \bar{\rho}$, $t < t_0$. But it is easy to see that the first term in the max function is the greater one, for $t < T_0$, provided we define $T_0 = \gamma_0 t_0$, with γ_0 suitably small.

The proof is now concluded by standard compactness arguments, and by Minty's lemma, employing the uniform L^∞ bounds just recalled, and the regularity results of [16], [20] to obtain uniform convergence of a subsequence of approximating solutions, in compact subsets of $\overline{\Omega}$.

Remark 4.1. If $|||u_0|||_{\bar{\rho}} \to 0$ as $\bar{\rho} \to \infty$, it follows immediately from the bound below for t_0 given in (4.4), that a solution u to our problem can be defined for all positive times. This is the case, for example, if $u_0 \in L^1(\Omega)$, or if u_0 is a finite measure in Ω.

5. Proof of Theorem 1.2. Let us assume that

$$\operatorname{supp} u_0 \subset \overline{\Omega_{r_0}}\,, \qquad r_0 \geq 1\,. \tag{5.1}$$

We use an energetic method, which is close to Antontsev's method [6], [9]. But, instead of reducing to a differential inequality for a suitable integral norm of the solution, we employ an iteration scheme based on the local estimates proven in the previous section.

We choose as a testing function in the weak formulation of (1.1)–(1.3), $u^\theta \zeta^{m+1}$ where $\theta > 0$ is to be selected below, and $\zeta(x)$ is a cut off function in Ω_ρ, $\rho > 4r_0$. We also assume that $0 \leq \zeta \leq 1$, $\zeta \in C^1(\overline{\Omega})$, and that $\zeta(x) \equiv 0$ in Ω_{r_0}. We get, via standard calculations,

$$\begin{aligned}\sup_{0<\tau<t} \int_{\Omega(\tau)} u^{1+\theta}\zeta^{m+1}\,\mathrm{d}x + \int_0^t\!\!\int_\Omega \zeta^{m+1} u^{\alpha+\theta-1}|Du|^{m+1}\,\mathrm{d}x\,\mathrm{d}\tau \\ \leq \gamma \|D\zeta\|^{m+1}_{\infty,\Omega} \int_0^t\!\!\int_\Omega u^{m+\alpha+\theta}\chi_{\{\zeta>0\}}\,\mathrm{d}x\,\mathrm{d}\tau\,,\end{aligned} \tag{5.2}$$

for all positive t, where of course we exploit the fact $\zeta \equiv 0$ where $u_0 \neq 0$.

Let us introduce the following sequence of cutoff functions $\{\zeta_n\}$: ζ_n is assumed to fulfill, besides the properties listed above, $\zeta_n(x) = 1$, $x \in \Omega_{\rho_n} \setminus \Omega_{\bar\rho_n}$, $\zeta_n(x) = 0$, $x \notin \Omega_{\rho_{n-1}} \setminus \Omega_{\bar\rho_{n-1}}$, $|D\zeta_n| \leq \gamma 2^n/(\sigma\rho)$, where $\rho_n = \rho + \sigma 2^{-n}\rho$, $\bar\rho_n = (\rho - \sigma 2^{-n}\rho)/2$, $n \geq 0$, $\rho > 4r_0$. Note that $\operatorname{supp}\zeta_n$ shrinks to $\overline{\Omega_\rho} \setminus \Omega_{\rho/2}$ as $n \to \infty$. Here σ is fixed arbitrarily in $(0, 1/2)$ (so that $\bar\rho_0 > r_0$). On setting $v_n = u^{(\alpha+m+\theta)/(m+1)}\zeta_n^s$, we get from an application of (5.2),

$$Y_n := \sup_{0<\tau<t} \int_{\Omega(\tau)} v_n^\varepsilon\,\mathrm{d}x + \int_0^t\!\!\int_\Omega |Dv_n|^{m+1}\,\mathrm{d}x\,\mathrm{d}\tau \leq \frac{\gamma 2^{n(m+1)}}{\rho^{m+1}\sigma^{m+1}} \int_0^t\!\!\int_\Omega v_{n-1}^{m+1}\,\mathrm{d}x\,\mathrm{d}\tau, \tag{5.3}$$

for $n \geq 1$, $\varepsilon = (m+1)(1+\theta)/(\alpha+m+\theta)$, and a large enough $s \geq 1$. To apply the embedding Theorem 2.1 to $v_{n-1}(\cdot,\tau)$, we need check that (2.2) holds in this case. We can use the same argument as in the proof of Proposition 3.1, noting that

$$|\operatorname{supp} v_n(\cdot,\tau)| \leq V(2\rho) \leq \theta_0 V(C\rho)\,, \quad (\theta_0 \text{ as in (2.2)}),$$

for all $\rho > 4r_0$, $n \geq 0$, and $C > 1$ large enough. Then, defining for $n \geq 1$

$$E_h(\tau) = \int_{\Omega(\tau)} v_{n-1}^h\,\mathrm{d}x\,, \quad h > 0\,, \qquad J(\tau) = \int_{\Omega(\tau)} |Dv_{n-1}|^{m+1}\,\mathrm{d}x\,,$$

we have from (2.3) with $q = m+1$, $p = m+1$, $\beta = \varepsilon$,

$$E_{m+1}(\tau) \le \gamma F\big(E_\varepsilon(\tau)^{\frac{\alpha+m+\theta}{m+\alpha-1}} E_{m+1}(\tau)^{-\frac{1+\theta}{m+\alpha-1}}\,,\,\rho\big) J(\tau)\,, \tag{5.4}$$

where

$$F(z,\rho) = \big(z/g(z,C\rho)\big)^{m+1} \le \gamma \max\big(z^{\frac{m+1}{N}}, z^{m+1} f(\rho)^{m+1}\big)$$

(we have used (2.13) here). We can make estimate (5.4) explicit, that is, we can write

$$\begin{aligned} E_{m+1}(\tau) \le \gamma \max\big(&J(\tau)^{\eta} E_\varepsilon(\tau)^{\frac{(\alpha+m+\theta)(1-\eta)}{1+\theta}}, \\ &J(\tau)^{\eta_1} E_\varepsilon(\tau)^{\frac{(\alpha+m+\theta)(1-\eta_1)}{1+\theta}} f(\rho)^{\eta_1(m+1)}\big)\,, \end{aligned} \tag{5.5}$$

where, recalling from Proposition 3.1 the definitions of $\mathcal{K}_q$, $\mathcal{H}_q$,

$$\eta = N(m+\alpha-1)/\mathcal{K}_{1+\theta} < 1\,, \qquad \eta_1 = (m+\alpha-1)/\mathcal{H}_{1+\theta} < 1\,.$$

Then we integrate (5.5) over $(0,t)$ to get, after applying Hölder's inequality too,

$$\begin{aligned} \int_0^t E_{m+1}(\tau)\,\mathrm{d}\tau \le \gamma \max\Big(&\Big(\int_0^t J(\tau)\,\mathrm{d}\tau\Big)^{\eta}\Big(\int_0^t E_\varepsilon(\tau)^{\frac{\alpha+m+\theta}{1+\theta}}\,\mathrm{d}\tau\Big)^{1-\eta}, \\ &\Big(\int_0^t J(\tau)\,\mathrm{d}\tau\Big)^{\eta_1}\Big(\int_0^t E_\varepsilon(\tau)^{\frac{\alpha+m+\theta}{1+\theta}}\,\mathrm{d}\tau\Big)^{1-\eta_1} f(\rho)^{\eta_1(m+1)}\Big)\,. \end{aligned} \tag{5.6}$$

Combining (5.3) with (5.6), we prove the iterative estimate

$$\begin{aligned} Y_n &\le \gamma b^n \rho^{-m-1} \max\big(Y_{n-1}^{\eta+(1-\eta)\lambda} t^{1-\eta}\,,\, Y_{n-1}^{\eta_1+(1-\eta_1)\lambda} t^{1-\eta_1} f(\rho)^{\eta_1(m+1)}\big) \\ &\le \gamma b^n \rho^{-m-1} t^{1-\eta} Y_{n-1}^{\eta+(1-\eta)\lambda} \max\big(1, I_0^{(\eta-\eta_1)(\lambda-1)} t^{\eta-\eta_1} f(\rho)^{\eta_1(m+1)}\big)\,, \end{aligned} \tag{5.7}$$

where $b > 1$, $\lambda = (\alpha+m+\theta)/(1+\theta)$, and we use $Y_n \le \gamma I_0$, $n \ge 0$, for

$$I_0 = \frac{1}{\rho^{m+1}} \int_0^t\!\!\int_{\Omega_{2\rho}} u^{\alpha+m+\theta}\,\mathrm{d}x\,\mathrm{d}\tau\,.$$

We remark that the obviously necessary assumption $\alpha+m > 1$ is exploited here in the form $\lambda > 1$, i.e., $\eta + (1-\eta)\lambda > 1$. Indeed we may then invoke Lemma 3.3 of Chapter II of [17] to infer that $Y_n \to 0$ if

$$Y_0 \le \gamma \rho^{\frac{(m+1)}{(1-\eta)(\lambda-1)}} t^{-\frac{1}{\lambda-1}} \min\big(1, I_0^{-\frac{\eta-\eta_1}{1-\eta}} t^{-\frac{\eta-\eta_1}{(1-\eta)(\lambda-1)}} f(\rho)^{-\frac{\eta_1(m+1)}{(1-\eta)(\lambda-1)}}\big)\,. \tag{5.8}$$

Clearly $u \equiv 0$ in $\Omega_\rho \setminus \Omega_{\rho/2}$ if $Y_n \to 0$. Then, the sought after bound for $Z(t)$ will be derived by imposing that ρ is selected so as to enforce (5.8). In turn, as $Y_0 \le \gamma I_0$, (5.8) is satisfied if both the following inequalities are in force

$$I_0 \le \gamma \rho^{\frac{(m+1)}{(1-\eta)(\lambda-1)}} t^{-\frac{1}{\lambda-1}} , \tag{5.9}$$

$$I_0 \le \gamma \rho^{\frac{(m+1)}{(1-\eta_1)(\lambda-1)}} t^{-\frac{1}{\lambda-1}} f(\rho)^{-\frac{(m+1)\eta_1}{(1-\eta_1)(\lambda-1)}} . \tag{5.10}$$

In fact (5.10) is the more stringent requirement, provided γ there is suitably defined, and $\rho > \tilde{\rho}(f(1))$. This follows from elementary calculations.

We need now go back to the estimation procedure carried out in Section 3. We proved there that for all $\bar{\rho} > 1$, there exists a $t_{\bar{\rho}} > 0$ (denoted by t_0 in that proof) such that

$$t\rho^{-m-1} \| u(\cdot, t) \|_{\infty, \Omega_{4\rho}} \le 1 , \qquad t \le t_{\bar{\rho}} , \ \rho \ge \bar{\rho} , \tag{5.11}$$

and that

$$t_{\bar{\rho}} \ge \gamma_0 |\!|\!| u_0 |\!|\!|_{\bar{\rho}}^{-(m+\alpha-1)} . \tag{5.12}$$

When $\|u_0\|_{1,\Omega} < \infty$, one immediately checks that

$$|\!|\!| u_0 |\!|\!|_{\bar{\rho}} \le \bar{\rho}^{-\frac{m+1}{m+\alpha-1}} V(\bar{\rho})^{-1} \| u_0 \|_{1,\Omega} \le \gamma \bar{\rho}^{-\frac{2m+\alpha}{m+\alpha-1}} f(\bar{\rho}) \| u_0 \|_{1,\Omega} . \tag{5.13}$$

Fix now $t > 0$. It follows from (5.12)–(5.13) that (5.11) is in force, for any given $\rho > 1$, if $t \le t_{\bar{\rho}}$, for $\bar{\rho} = \rho$, i.e., if $t \le \gamma \rho^{2m+\alpha} f(\rho)^{-(m+\alpha-1)} \| u_0 \|_{1,\Omega}^{-(m+\alpha-1)}$, that is

$$\rho f(\rho)^{-\frac{m+\alpha-1}{2m+\alpha}} \ge \gamma \| u_0 \|_{1,\Omega}^{\frac{m+\alpha-1}{2m+\alpha}} t^{\frac{1}{2m+\alpha}} . \tag{5.14}$$

From now on, we assume $t > 0$ is any sufficiently large positive number, in a sense to be precised, and we stipulate (5.14) as an assumption on ρ. Note that $\rho / f(\rho) \to \infty$ as $\rho \to \infty$, as a consequence of (1.5), and of (1.8), so that (5.14) amounts to a meaningful bound below for ρ. Both the restrictions $\rho > 4r_0$ and $\rho > \tilde{\rho}$ are in fact implied by (5.14), provided t is large enough, which we assume.

From Proposition 3.1, which we may apply because of (5.11), and from the obvious estimate (6.1) below, we have

$$I_0 \le \gamma \rho^{-m-1} \| u_0 \|_{1,\Omega} \max \Bigg(\int_0^t \| u_0 \|_{1,\Omega}^{\frac{(m+1)(m+\alpha+\theta-1)}{\mathcal{K}}} \tau^{-\frac{N}{\mathcal{K}}(m+\alpha+\theta-1)} \, \mathrm{d}\tau , \tag{5.15}$$
$$\int_0^t \| u_0 \|_{1,\Omega}^{\frac{(m+1)(m+\alpha+\theta-1)}{\mathcal{H}}} \tau^{-\frac{1}{\mathcal{H}}(m+\alpha+\theta-1)} f(\rho)^{\frac{(m+1)(m+\alpha+\theta-1)}{\mathcal{H}}} \, \mathrm{d}\tau \Bigg) .$$

Let us choose $0<\theta<1$ such that $N(m+\alpha+\theta-1)/\mathcal{K}<1$, $(m+\alpha+\theta-1)/\mathcal{H}<1$, i.e., $\theta<(m+1)/N$. Then both the integrals in (5.15) take a finite value, implying

$$I_0 \le \gamma\rho^{-m-1}\|u_0\|_{1,\Omega}\max\Big(t^{\frac{m+1-N\theta}{\mathcal{K}}}\|u_0\|_{1,\Omega}^{\frac{(m+1)(m+\alpha+\theta-1)}{\mathcal{K}}}, \quad t^{\frac{m+1-\theta}{\mathcal{H}}}\|u_0\|_{1,\Omega}^{\frac{(m+1)(m+\alpha+\theta-1)}{\mathcal{H}}} f(\rho)^{\frac{(m+1)(m+\alpha+\theta-1)}{\mathcal{H}}}\Big). \tag{5.16}$$

Moreover, a simple calculation shows that the second term in the $\max(\cdot,\cdot)$ function above is greater than the first one provided $1\le t^{\frac{N-1}{\mathcal{K}\mathcal{H}}}\|u_0\|_{1,\Omega}^{\frac{1}{\mathcal{H}}-\frac{1}{\mathcal{K}}} f(\rho)^{\frac{1}{\mathcal{H}}}$, which certainly holds for $\rho\ge 1$ and $t\ge\bar{t}(\|u_0\|_{1,\Omega}, f(1))$, which we assume hereafter. Finally, we satisfy (5.10) by replacing I_0 there with its just found majorization. This crucial step in the proof is completed by means of elementary algebraic manipulations, showing that (5.10) holds true if (5.14) is fulfilled (perhaps after redefining the constant γ there). Therefore $Z(t)$ is bounded above by $\tilde{z}=\min\{z : (5.14) \text{ holds for all } \rho\ge z\}$ (for $\rho=\tilde{z}$ (5.14) is fulfilled as an equality). The bound above for $Z(t)$ is finally proven by invoking Lemma 2.3.

In order to prove the bound below for $Z(t)$, we remark that for large enough t we have

$$\begin{aligned}\|u_0\|_{1,\Omega} = \int_{\Omega(t)} u\,\mathrm{d}x &\le V(Z(t))\|u(\cdot,t)\|_{\infty,\Omega_{Z(t)}} \\ &\le \gamma\frac{Z(t)}{f(Z(t))t^{\frac{1}{2m+\alpha}}} f(P(t))^{\frac{m+1}{2m+\alpha}}\|u_0\|_{1,\Omega}^{\frac{m+1}{2m+\alpha}},\end{aligned} \tag{5.17}$$

where we have exploited the already proven bound above, as well as Theorem 1.3, whose proof is independent of the current argument. If we compare (5.17) with the definition of $P(t)$ in (1.16), we readily arrive at

$$\frac{Z(t)}{f(Z(t))} \ge \gamma_0\frac{P(t)}{f(P(t))},$$

whence we derive $Z(t)\ge\gamma_0 P(t)$, with the help of Lemma 2.3. Finally, we derive the bound below for $\|u(\cdot,t)\|_{\infty,\Omega}$ from (5.17) itself, while the bound above is again a consequence of Theorem 1.3.

6. Proof of Theorem 1.3. The global existence of a solution u under the assumptions of Theorem 1.3 has been already obtained, see Remark 4.1.

From our approximation scheme in Section 4, we infer at once that

$$\int_{\Omega} u(x,t)\,\mathrm{d}x \le \int_{\Omega} \mathrm{d}u_0\,, \tag{6.1}$$

also using of course the boundary data prescribed there. We invoke again the arguments employed in Section 5, see (5.11)–(5.14), to show that we may apply the local estimates of Section 3 for $\rho \ge \gamma P(t)$. Note that those arguments make use only of the global integrability of the initial data, and not of the boundedness of its support.

In fact, we need a slightly different version of the local estimate in Proposition 3.1; this can be obtained formally substituting the "smaller" domain Ω_ρ with $\Omega_{(1+\sigma)\rho} \setminus \Omega_{(1-\sigma)\rho}$, and the "larger" domain $\Omega_{(1+\sigma)\rho}$ with $\Omega_{(1+2\sigma)\rho} \setminus \Omega_{(1-2\sigma)\rho}$. In other words, we intersect Ω with annuli rather than with balls. The proof of this estimate is just a minor variant of the one given in Section 3, and we do not reproduce it here. Then the majorisation in (1.19) follows at once.

Finally, conservation of mass (1.18) follows by multiplying the equation (1.1) (or its approximating version) by a cutoff function $\zeta(x)$, $\zeta \equiv 1$ in Ω_ρ, $\zeta \equiv 0$ in $\Omega \setminus \Omega_{2\rho}$, to get

$$\int_{\Omega} u(x,t)\zeta(x)\,\mathrm{d}x = \int_{\Omega} \zeta(x)\,\mathrm{d}u_0(x) + \int_0^t\!\!\int_{\Omega} u^{\alpha}|Du|^{m-1}Du\cdot D\zeta\,\mathrm{d}x\,\mathrm{d}\tau, \tag{6.2}$$

and then by letting $\rho \to \infty$, invoking the gradient estimates of Proposition 3.2 and (6.1) above. More exactly, we choose $\mu = m$, $\theta = \alpha$ and $q = 1$, in the Proposition, and note that, owing to the global integrability assumption stipulated presently, we have for large ρ, $G_1(t,\rho) \le \|u_0\|_{1,\Omega}$, $M_1(t,\rho) \le \gamma(t, \|u_0\|_{1,\Omega}) f(\rho)^{\frac{m+1}{2m+\alpha}}$. Therefore, (3.13) yields

$$\left| \int_0^t\!\!\int_{\Omega} u^{\alpha}|Du|^{m-1}Du\cdot D\zeta\,\mathrm{d}x\,\mathrm{d}\tau \right| \le \gamma(t, \|u_0\|_{1,\Omega}) \frac{f(\rho)^{\frac{m+\alpha-1}{2m+\alpha}}}{\rho}\,;$$

the last term vanishes as $\rho \to \infty$, as already remarked. We can then take the limit $\rho \to \infty$ in (6.2) and prove the sought after conservation property.

7. An explicit solution. We construct here an explicit solution to the Neumann problem (1.1)–(1.3), for any $\alpha \ge 0$, $m > 1$, mainly with the purpose of showing that estimate (1.19) for the growth of u as $\rho \to \infty$ is optimal. Define the domain $\Omega = \operatorname{int} S$, where $S := \bigcup_{n=1}^{\infty} \overline{C_n}$, and

$$C_n := \{x = (x', x_N) : |x_N - \xi_n| < r^n/2\,,\ |x'| \le s^{n/(N-1)}\}\,,$$

and $r > 1$, $1/r < s < 1$ are to be chosen presently. The sequence $\{\xi_n\}$ is defined by $\xi_1 = \frac{r}{2}$, $\xi_{n+1} = \xi_n + \frac{r^n}{2} + \frac{r^{n+1}}{2}$, $n \geq 1$. Therefore Ω is the union of a sequence of cylinders with height r^n and base area $\gamma(N)s^n$, and our construction is somehow similar to the example 4.3.5/4 in [19], though the geometry is different here. For example, we have obviously $|\Omega| = \infty$, because $rs > 1$. It is easy to check that, for $\Omega_\rho = \Omega \cap \{x_N < \rho\}$,

$$c_0\rho^{1-\beta} \leq V(\rho) \leq c_1\rho^{1-\beta}\,, \qquad \rho \geq r\,,$$

for $\beta = |\ln s|/\ln r < 1$, and c_0, c_1 depending on s, r, N. Then $f(\rho) = \rho^\beta$, $\rho \geq r$, in this case. From this estimate, (1.7) follows immediately, with $\nu_i(\delta) = \tilde{c}_i\delta^{1-\beta}$, $\tilde{c}_i > 0$, $i = 0, 1$.

Reasoning by induction on k one can show that

$$\forall Q \subset \Omega^{(k)} := \operatorname{int}\big(\bigcup_{n=1}^{k} \overline{C_n}\big)\,,\ \text{with } \partial Q \text{ Lipschitz continuous, one has}$$
$$|\partial Q \cap \Omega^{(k)}|_{N-1} \geq \gamma_0 \min\big(|Q|^{\frac{N-1}{N}}\,,\, |\Omega^{(k)} \setminus Q|^{\frac{N-1}{N}}\,,\, s^k\big)\,,$$

where γ_0 does not depend on k or Q. Of course this proves that (1.6) holds in this case. Next define $u(x,t) = B_n(x,t) = B(x_N - \xi_n, t; a^n)$, $x = (x', x_N) \in C_n$, $t > 0$, where the function

$$B(y,t;A) = \frac{A}{t^{\frac{1}{2m+\alpha}}}\Big(1 - \frac{C'}{A^{\frac{m+\alpha-1}{m}}}\Big(\frac{|y|}{t^{\frac{1}{2m+\alpha}}}\Big)^{\frac{m+1}{m}}\Big)_+^{\frac{m}{m+\alpha-1}}\,, \quad y \in \boldsymbol{R}, t > 0\,,$$

is a solution to the one dimensional version of (1.1), for any $A > 0$. Note that $B(y,t;A) \to CA^{\frac{2m+\alpha}{m+1}}\delta(y)$, as $t \to 0+$; here C, C' are suitable positive constants depending on m, α, and δ is Dirac's mass. The function u solves problem (1.1)–(1.3) for $0 < t < T$, provided the support of B_n is contained in C_n for all $0 < t < T$, for all $n \geq 1$. By direct inspection, this is seen to be guaranteed by $r > 2C'^{-\frac{m}{m+1}}T^{\frac{1}{2m+\alpha}}a^{\frac{m+\alpha-1}{m+1}}$ and $r > a^{(m+\alpha-1)/(m+1)}$. Finally, we select first $T > 0$ and $a > 1$, and then s, r such that all the requirements above are satisfied, as well as $s < a^{-\frac{2m+\alpha}{m+1}}$. Hence

$$\int_\Omega \mathrm{d}u(x,0) = \gamma\sum_{n=1}^{\infty}(sa^{\frac{2m+\alpha}{m+1}})^n < +\infty\,.$$

Moreover, one can check that for $0 < \sigma < \sigma_0(r)$,

$$H_n^\sigma := \Omega \cap \{(1-2\sigma)\xi_n < x_N < (1+2\sigma)\xi_n\} \subset C_n\,, \qquad \text{for all } n \geq 1.$$

Then for all such σ, $\|u(\cdot,t)\|_{1,H_n^\sigma} = \gamma(N)C(sa^{\frac{2m+\alpha}{m+1}})^n$, $0<t<T$. Therefore, for $x_n=(0,\xi_n)\in \boldsymbol{R}^N$, $0<t<T$,

$$u(x_n,t)=a^n t^{-\frac{1}{2m+\alpha}} \geq \gamma_0 t^{-\frac{1}{2m+\alpha}}\Big(\sup_{0<\tau<t}\|u(\cdot,\tau)\|_{1,H_n^\sigma}\Big)^{\frac{m+1}{2m+\alpha}} s^{-n\frac{m+1}{2m+\alpha}}$$
$$\geq \gamma_0 t^{-\frac{1}{2m+\alpha}}\Big(\sup_{0<\tau<t}\|u(\cdot,\tau)\|_{1,H_n^\sigma}\Big)^{\frac{m+1}{2m+\alpha}} f(\xi_n)^{\frac{m+1}{2m+\alpha}},$$

proving that $u(\cdot,t)$ is unbounded in Ω for any $t>0$, and, even more, that estimate (1.19) is optimal.

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