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BLOWUP BEHAVIOR OF SOLUTIONS TO THE RESCALED JÄGER-LUCKHAUS SYSTEM

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1. Introduction

In this paper, we consider the blowup behavior of solutions to the rescaled Jäger-Luckhaus system.

In [9], Jäger and Luckhaus introduced the following parabolic-elliptic system.

$$\begin{array}{lll} u_t = \nabla\cdot(\nabla u - u\nabla v) & \text{in} & \Omega\times(0,T) \\ 0 = \Delta v + u - \lambda/|\Omega| & \text{in} & \Omega\times(0,T) \\ \partial u/\partial\nu = \partial v/\partial\nu = 0 & \text{on} & \partial\Omega\times(0,T) \\ \int_\Omega v dx = 0 & \text{in} & [0,T) \\ u|_{t=0} = u_0 & \text{in} & \Omega. \end{array} \tag{1.1}$$

Here, $\Omega\subset\mathbf{R}^2$ denotes a bounded domain with smooth boundary $\partial\Omega$, ν is the outer normal unit vector, and $|\Omega|$ is the area of Ω. The initial value u_0 is smooth, nonnegative, and nontrivial. Let $\lambda = \|u_0\|_{L^1(\Omega)} > 0$. Here and henceforth, $\|\cdot\|_{L^p(\Omega)}$ denotes the standard $L^p(\Omega)$ norm for $1\leqq p\leqq\infty$.

We refer to (1.1) as the Jäger-Luckhaus system. The Jäger-Luckhaus system is a simplified system compared to the one introduced by Keller and Segel [10] or Nanjundiah [14]. Those systems describe the chemotactic feature of some organisms (cellar slime molds) sensitive to the gradient of a chemical substance secreted by themselves. We refer to the one introduced by Nanjundiah [14] as the Keller-Segel model.

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Jäger and Luckhaus [9] showed the following: if $\lambda \ll 1$, then $T_{\max} = \infty$ follows, while if $\lambda \gg 1$ then $T_{\max} < \infty$ can happen.

Here and henceforth, $T_{\max}$ denotes the maximal time for the existence of the classical solution.

If $T_{\max} < \infty$, we can show that the solution blows up at the time $T_{\max}$. Namely, $\lim_{t \to T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$.

Nagai [12] treated another simplified system of the Keller-Segel model. There, the second equation is replaced by

$$0 = \Delta v - v + u.$$

We refer to the system as the Nagai model. For radial solutions to the Nagai system, Nagai [12] showed that $\lambda = 8\pi$ is the threshold for the blowup. Namely, $\lambda < 8\pi$ implies $T_{\max} = \infty$, while $T_{\max} < \infty$ can happen if $\lambda > 8\pi$, which corresponds exactly to what Childress [4] and Childress and Percus [5] conjectured to solutions to the Keller-Segel model.

Nagai [13] and the authors [16] show the following: $T_{\max} < \infty$ occurs, if an L^1 norm of more than 8π concentrates in a sufficiently small neighborhood of a point in Ω, or if an L^1 norm of more than 4π concentrates in a sufficiently small neighborhood of a point on the boundary $\partial\Omega$.

Moreover, the authors [15] showed that if $T_{\max} < +\infty$, then the solution u to the Nagai model satisfies

$$u(\cdot,t) \to \sum_{q \in \mathcal{B}} m(q)\delta_q + f \quad \text{in } \mathcal{M}(\overline{\Omega}) \quad \text{as } t \to T_{\max}, \tag{1.2}$$

where $\mathcal{B}$ is a blowup set. Namely,

$$\mathcal{B} = \{q \in \overline{\Omega} \colon \exists t_k \uparrow T_{\max} \text{ and } \exists x_k \to q \text{ such that } u(x_k,t_k) \to +\infty \text{ as } k \to \infty\}$$

and call each $q \in \mathcal{B}$ a blowup point. Then, $T_{\max} < +\infty$ implies $\mathcal{B} \neq \emptyset$. Here, δ_q is a delta function whose support is the point q, f is a nonnegative L^1 function, and

$$m(q) \geqq m_*(q) \equiv \begin{cases} 8\pi & \text{if} \quad q \in \Omega, \\ 4\pi & \text{if} \quad q \in \partial\Omega. \end{cases}$$

By this, we obtain that $\sharp\mathcal{B} \leqq \lambda/(4\pi)$, since we can show that $\|u(\cdot,t)\|_{L^1(\Omega)} = \lambda$ in $[0,T_{\max})$.

We can show that solutions to Jäger-Luckhaus system satisfy the properties for solutions to the Nagai model mentioned above.

On the other hand, Herrero and Veázquez [7, 8] found the following blowup and radial solutions u to the Jäger-Luckhaus system and the Keller-Segel model.

$$u(\cdot, t) \to 8\pi\delta_0 + f \quad \text{in } \mathcal{M}(\overline{\Omega}) \quad \text{as } t \to T_{\max},$$

where f is a nonnegative and radial L^1 function.

The authors [17] investigated the behavior of solutions to the Nagai system in the case where infinite time blowup occurs. Here, we use the term infinite time blowup if $T_{\max} = \infty$ and $\lim_{t\to\infty} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$. There, we observe that if infinite time blowup occurs, $m(q) = m_*(q)$ for each $q \in \mathcal{B}$.

Those results suggest that $m(q) \in m_*(q)\mathcal{N}$ for each $q \in \overline{\Omega}$. Here, $\mathcal{N} = \{1, 2, 3, \dots\}$. We refer to this phenomenon as the quantumization of chemotactic collapse. Namely, we want to investigate whether the quantumization of chemotactic collapse occurs or not for any blowup solutions.

Our strategy is the following: for each blowup point $q \in \overline{\Omega}$ of (finite time) blowup solutions, we put

$$\begin{aligned} &y = \frac{x - q}{\sqrt{T_{\max} - t}}, \quad s = -\log(T_{\max} - t), \\ &z(y, s) = (T_{\max} - t)u(x, t), \quad w(y, s) = v(x, t). \end{aligned} \tag{1.3}$$

Next, we will show that the rescaled solution z blows up at infinite time. Then, by the behavior of infinite-time blowup, we will observe that

$$z(\cdot, s) \to \sum_{Q\in\mathcal{S}} m_*(Q)\delta_Q \quad \text{as } s \to \infty \tag{1.4}$$

in the sense of measures, where $\mathcal{S}$ is a blowup set of z.

For any blowup solution to the Jäger-Luckhaus system, (1.4) says that the quantumization of chemotactic collapse occurs.

Unfortunately, in this paper we will prove only Theorem 1 mentioned below.

Next, we will prove that z blows up at infinite time and that $F = 0$ in the next paper.

We introduce rescaled Jäger-Luckhaus system.

Each finite-time blowup solution to the Jäger-Luckhaus system satisfies (1.2). Then, fix each blowup point q and rescale by using (1.3). Put

$$\mathcal{O}_q(s) = e^{s/2}(\Omega - \{q\}), \quad \mathcal{E}_q = \cup_{s\geqq s_*}\mathcal{O}_q(s), \quad \Gamma_q = \cup_{s\geqq s_*}\partial\mathcal{O}_q(s),$$

where $s_* = -\log T_{\max}$. For simplicity, we assume $q = 0$ and omit q of $\mathcal{O}_q(s)$, $\mathcal{E}_q$ and Γ_q.

Then, (z,w) is a solution to the following system.

$$\begin{array}{lll} z_s = \nabla\cdot(\nabla z - z\nabla w - yz/2) & \text{in} & \mathcal{E} \\ 0 = \Delta w + z - \lambda e^{-s}/|\Omega| & \text{in} & \mathcal{E} \\ \partial z/\partial\nu = \partial w/\partial\nu = 0 & \text{on} & \Gamma \\ \int_{\mathcal{O}(s)} w\,dy = 0 & \text{in} & [s_*,\infty) \\ z|_{s=s_*} = z_0 = T_{\max}u_0(\sqrt{T_{\max}}\,\cdot) & \text{in} & \mathcal{O}(s_*). \end{array} \tag{1.5}$$

By the definition of (z,w), the solution (z,w) exists globally in time. Let

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad (0 \leqq \theta < 2\pi),$$

$H = \{y = (y_1, y_2) : y_2 > 0\}$ and

$$\mathcal{O}(\infty) = \cup_{S\geqq s_*}^{\infty} \cap_{s\geqq S} \mathcal{O}(s) = \begin{cases} R(\theta)H & \text{if} \quad 0 \in \partial\Omega, \\ \mathbf{R}^2 & \text{if} \quad 0 \in \Omega \end{cases}$$

for some $\theta \in [0, 2\pi)$. Without loss of generality, we can assume $\theta = 0$. Then we get the following theorem.

Theorem 1. *For any sequence $\{s_n\}_{n\geqq 1} \subset [0,\infty)$, there exists a subsequence $\{s_n'\}_{n\geqq 1} \subset \{s_n\}_{n\geqq 1}$ such that*

$$z(\cdot, s_n') \to \sum_{Q\in\mathcal{S}} m_*(Q)\delta_Q + F \quad \text{in } \mathcal{M}(\mathbf{R}^2) = (C_0(\mathbf{R}^2))^* \quad \text{as } n\to\infty, \tag{1.6}$$

where

$$\mathcal{S} = \{Q\in\overline{\Omega} : \exists\{s_n''\} \subset \{s_n'\} \text{ and } \exists y_n \to Q \text{ such that } z(y_n, s_n'') \to \infty \text{ as } n\to\infty\},$$

$$m_*(Q) = \begin{cases} 4\pi & \text{if} \quad Q \in \partial\mathcal{O}(\infty)\cap\mathcal{S}, \\ 8\pi & \text{if} \quad Q \in \mathcal{O}(\infty)\cap\mathcal{S}, \end{cases}$$

and F is a nonnegative $L^1(\mathbf{R}^2)$ function.

Here, $\partial\mathcal{O}(\infty)$ is empty if $0 \in \Omega$, and $\mathcal{S}$ may be empty.

2. Some fundamental properties

In this section, we prove some fundamental properties of solutions (z,w) to the rescaled Jäger-Luckhaus system.

We denote $B(Q,R) = \{y \in \mathbf{R}^2 : |Q-y| < R\}$ and $D(q,r) = \{x \in \mathbf{R}^2 : |q-x| < r\}$.

Lemma 2.1. *The solution v to* (1.1) *satisfies*

$$\|v(\cdot,t)\|_{L^2(\Omega)} \leqq C_1.$$

We denote by C_1 a positive constant which is independent of $s \geqq s_*$.

Proof. Let $G(x,y)$ be the Green's function of $-\Delta+1$ in Ω with homogeneous Neumann boundary condition and an operator T_u be

$$T_u v(\cdot,t) \equiv \int_\Omega G(\cdot,x)\Big(u(x,t) - \frac{\lambda}{|\Omega|}\Big)dx + \int_\Omega G(\cdot,x)v(x,t)dy.$$

Putting $L^2_0(\Omega) = \{f \in L^2(\Omega) : \int_\Omega f\,dx = 0\}$, the operator $-\Delta$ in Ω with homogeneous Neumann boundary condition is invertible and compact in $L^2_0(\Omega)$. Then the operator T_u is a compact and contraction map on $L^2_0(\Omega)$. This implies that T_u has a unique fixed point v in $L^2_0(\Omega)$, v has the estimate of this lemma, and v satisfies the second equation of (1.1). Then we have this lemma. □

The solution w of (1.5) satisfies

$$-\Delta w + e^{-s}w = z - \frac{\lambda}{|\Omega|}e^{-s} + e^{-s}w \quad \text{in } \mathcal{E}. \tag{2.1}$$

For any $(Q,s) \in \mathcal{E}$ and $R > 0$, we have that

$$\begin{aligned}\int_{B(Q,R)\cap\mathcal{O}(s)} |e^{-s}w(y,s)|dy &= \int_{D(e^{-s/2}Q,e^{-s/2}R)\cap\Omega} |v(x,t)|dx \\ &\leqq \;|D(e^{-s/2}Q,e^{-s/2}R)\cap\Omega|\|v(\cdot,t)\|_{L^2(\Omega)} \leqq C_1\pi R^2 e^{-s}\end{aligned} \tag{2.2}$$

by Lemma 2.1. The solutions v and w of (1.1) and (1.5) satisfy

$$-\Delta v + v = u - \frac{\lambda}{|\Omega|} + v \quad \text{in } \Omega \tag{2.3}$$

and (2.1), respectively. By Lemma 2.1 and Stampacchia's L^1 estimate [19], we observe that

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)} + \|v(\cdot,t)\|_{L^p(\Omega)} \leqq C_2 \tag{2.4}$$

for $t \in [0,T_{\max})$, $1 \leqq q < 2$, and $1 \leqq p < \infty$. C_2 is a positive constant depending only on λ, Ω, $q \in [0,1)$ and $p \in [0,\infty)$.

If $0 \in \Omega$, for any $Q \in \mathbf{R}^2$ and $R > 0$, it holds that $B(Q,R) \subset \mathcal{O}(s)$ for any sufficiently large $s \geqq s_*$. In this case, it is not necessary that we treat the boundary $\partial\mathcal{O}(s)$. Then the case where $0 \in \Omega$ is easier than the case where $0 \in \partial\Omega$.

Therefore, we treat only the case where $0 \in \partial\Omega$. Namely, we assume $H = \mathcal{O}(\infty)$. Since we assume that $\partial\Omega$ is smooth, there exists a positive constant r_1 and a conformal map $X : \overline{D(0,4r_1)\cap\Omega} \to \mathbf{R}^2$ such that

$$X(0) = 0, \quad X(D(0,4r_1)\cap\Omega) \subset \{(\xi_1,\xi_2) : \xi_2 > 0\},$$

$$X(D(0,4r_1)\cap\partial\Omega)\subset\{(\xi_1,\xi_2):\xi_2=0\},\quad \partial X/\partial x(0)=\mathrm{id},\qquad \partial X/\partial\nu=(0,-1)\quad\text{on }\partial\Omega. \tag{2.5}$$

Then the authors [15] showed that

$$\begin{aligned} G(x,x') &= \frac{1}{2\pi}\log\frac{1}{|X(x)-X(x')|}+\frac{1}{2\pi}\log\frac{1}{|X(x)-X(x')^*|} \\ &\quad +K(x,x')\quad \text{for } x,x'\in\overline{B(0,r_1)\cap\Omega} \qquad (2.6)\\ K &\in C^{1+\theta}(\overline{B(0,r_1)\cap\Omega}\times\overline{B(0,r_1)\cap\Omega})\quad\text{for some }\theta\in(0,1).\end{aligned}$$

Here and henceforth, we denote $\xi^*=(\xi_1,-\xi_2)$ for $\xi=(\xi_1,\xi_2)$.

Moreover, we have

$$|\nabla_x G(x,x')|\leqq C_3\big(\frac{1}{|x-x'|}+1\big) \tag{2.7}$$

for any x and $x'\in\overline{\Omega}$ with $x\neq x'$, where C_3 depends only on Ω. Put

$$\eta=Y(y)=Y(y,s)=e^{s/2}X(e^{-s/2}y) \tag{2.8}$$

and

$$\mathcal{G}(y,y')=G(e^{-s/2}y,e^{-s/2}y'). \tag{2.9}$$

Then we have that

$$\begin{aligned}\mathcal{G}(y,y',s)&=\frac{1}{2\pi}\log\frac{1}{|Y(y)-Y(y')|}+\frac{1}{2\pi}\log\frac{1}{|Y(y)-Y(y')^*|}\\ &\quad+\mathcal{K}(y,y',s)+s/(2\pi)\quad\text{for } y,y'\in\overline{B(0,e^{s/2}r_1)\cap\mathcal{O}(s)},\\ \mathcal{K}(y,y',s)&=K(e^{-s/2}y,e^{-s/2}y')\end{aligned} \tag{2.10}$$

and that

$$\mathcal{K}(\cdot,\cdot,s)\in C^{1+\theta}(\overline{B(0,e^{s/2}r_1)\cap\Omega}\times\overline{B(0,e^{s/2}r_1)\cap\Omega})\quad\text{for } s\geqq s_*.$$

We obtain that $\mathcal{G}$ is the Green's function $-\Delta+e^{-s}$ in $\mathcal{O}(s)$ with $\partial\cdot/\partial\nu=0$ on $\partial\mathcal{O}(s)$ and that we can get any estimates of Y and $\mathcal{G}$ by using the estimates of X and G. Then we can get the next lemma by using arguments such as one of [15, Lemma 6]. Moreover, we can define cutoff functions in $\mathcal{O}(s)$ by using an argument similar to one in [15].

Suppose we are given $Q\in H$ and $0<R'<R$ with $B(Q,2R)\subset H$. Then there exists a constant $S_1\geqq s_*$ satisfying $B(Q,3R/2)\subset\mathcal{O}(s)$ for any $s\geqq S_1$.

Here and henceforth, we denote constants S_i $(i=1,2,3,4)$ depending only on Q, R', and R.

We find cutoff functions $\Phi_{Q,R',R}$ and $\Psi_{Q,R',R}\in C^\infty(\mathbf{R}^2)$ such that

$$0\leqq\Phi_{Q,R',R}\leqq 1\quad\text{in}\quad\mathbf{R}^2,$$

$$\Phi_{Q,R',R}(y) = \begin{cases} 1 & \text{if} \quad y \in B(Q,R') \\ 0 & \text{if} \quad y \in \mathbf{R}^2 \setminus B(Q,R) \end{cases} \tag{2.11}$$
$$\Psi_{Q,R',R} = \Phi^6_{Q,R',R} \quad \text{in } \mathbf{R}^2$$

for $s \geqq S_1$. Then it holds that

$$\frac{\partial}{\partial \nu}\Phi_{Q,R',R} = 0 \quad \text{on } \partial\mathcal{O}(s) \quad \text{for } s \geqq S_1. \tag{2.12}$$

Given $Q \in \partial H$, $0 < R' < R$, we take $R''' = (2R' + R)/3$ and $R'' = (R' + 2R)/3$. Since we obtain that $Y(\cdot, s) = \mathrm{id} + O(e^{-s/2})$ as $s \to \infty$ by (2.5) and (2.8), there exists a constant $S_1 \geqq s_*$ such that

$$\begin{aligned} &B(Q,2R) \subset D(e^{s/2}r_1, 0), \\ &Y(B(Q,2R) \cap \mathcal{O}(s), s) \subset \{(\eta_1,\eta_2) : \eta_2 > 0\}, \\ &Y(B(Q,2R) \cap \partial\mathcal{O}(s)) \subset \{(\eta_1,\eta_2) : \eta_2 = 0\}, \\ &Y(B(Q,R'), s) \subset B(Q,R'''), \quad Y(B(Q,R), s) \supset B(Q,R'') \end{aligned} \tag{2.13}$$

for any $s \geqq S_1$. Let $\phi \in C^\infty(\mathbf{R}^2)$ be a cutoff function satisfying $\phi = \phi(|\eta|)$, $0 \leqq \phi \leqq 1$ in $\mathbf{R}^2$, and

$$\phi(\eta) = \begin{cases} 1 & \text{if} \quad \eta \in B(Q,R'''), \\ 0 & \text{if} \quad \eta \in \mathbf{R}^2 \setminus B(Q,R''). \end{cases} \tag{2.14}$$

Putting $\Phi_{Q,R',R}(\cdot, s) = \phi \circ Y(\cdot, s)$ and $\Psi_{Q,R',R} = \Phi^6_{Q,R',R}$ for any $s \geqq S_1$, we observe that $\Phi_{Q,R',R}$ and $\Psi_{Q,R',R} \in C^\infty(\mathbf{R}^2 \times [S_1, \infty))$ satisfy (2.11) and (2.12).

Irrespective of the case where $Q \in H$ or $Q \in \partial H$, we get the estimate of $\sup_{s \geqq S_1} \|\Phi_{Q,R',R}\|_{C^i(\mathbf{R}^2)}$ $(i = 1, 2)$. Then we can find positive constants A and B satisfying

$$|\nabla\Psi_{Q,R',R}| \leqq A\Psi^{5/6}_{Q,R',R} \quad \text{and} \quad |\nabla^2\Psi_{Q,R',R}| \leqq B\Psi^{2/3}_{Q,R',R} \tag{2.15}$$

in $\mathrm{R}^2 \times [S_1, \infty)$. Recall Y in (2.8). By $\partial X/\partial x(0) = \mathrm{id}$ and $X(0) = 0$, we observe that

$$\begin{aligned} \frac{\partial Y}{\partial s}(y,s) &= \frac{\partial}{\partial s} e^{s/2} X(e^{-s/2}y) = \frac{1}{2}e^{s/2}X(e^{-s/2}y) - \frac{1}{2}\frac{\partial X}{\partial x}(e^{-s/2}y)y \\ &= \frac{1}{2}e^{s/2}\Big(X(e^{-s/2}y) - X(0)\Big) - \frac{1}{2}\frac{\partial X}{\partial x}(e^{-s/2}y)y \\ &= \frac{1}{2}\Big(\frac{\partial X}{\partial x}(\theta e^{-s/2}y) - \frac{\partial X}{\partial x}(e^{-s/2}y)\Big)y = O(e^{-s/2})O(|y|^2), \end{aligned} \tag{2.16}$$

where $\theta \in (0,1)$. Then we have

$$\left\| \frac{\partial}{\partial s} \Phi_{Q,R',R}(\cdot, s) \right\|_{C(\mathbf{R}^2)} \leqq C_4 A \quad \text{for } s \geqq S_1. \tag{2.17}$$

Here and henceforth, we denote positive constants C_i $(i = 4, 5, 6, \dots, 12)$ depending only on λ, S_1, Q, and R.

Therefore, for $Q \in \overline{H}$, R', and R with $0 < R' < R$ we can define cutoff functions $\Phi_{Q,R',R}$ and $\Psi_{Q,R',R}$ satisfying (2.11), (2.12), (2.15), and (2.17) for $s \geqq S_1$.

Lemma 2.2. *For $s \geqq S_1$, let $\Phi = \Phi_{Q,R',R}$ and*

$$\Xi(y, y', s) = \nabla \Phi(y, s) \cdot \nabla_y \mathcal{G}(y, y', s) + \nabla \Phi(y', s) \cdot \nabla_{y'} \mathcal{G}(y, y', s).$$

Then, for any $s \geqq S_1$ it holds that $\Xi(\cdot, \cdot, s) \in L^\infty(\overline{\mathcal{O}(s)} \times \overline{\mathcal{O}(s)})$ and that

$$\|\Xi\|_{L^\infty(\overline{\mathcal{O}(s)} \times \overline{\mathcal{O}(s)})} \leqq C_5 B.$$

The next lemma is shown by using an argument similar to the proof of [15, Lemma 8].

Lemma 2.3. *Let $\Phi = \Phi_{Q,R',R}(\cdot, s)$ for $s \geqq S_1$. Then it holds that*

$$\left| \frac{d}{ds} \int_{\mathcal{O}(s)} z \Phi \, dy \right| \leqq C_6 (A + B).$$

Proof. Multiplying the first equation of (1.5) by Φ and integrating over $\mathcal{O}(s)$, we have

$$\int_{\mathcal{O}(s)} z_s \Phi \, dy = \int_{\mathcal{O}(s)} \left\{ \nabla \cdot \left(\nabla z - z \nabla w - \frac{y}{2} z \right) \right\} \Phi \, dy. \tag{2.18}$$

Let $\varphi(x,t) = \Phi(y,s)$ with (1.3). Since we obtain that

$$\frac{\partial}{\partial t}(u\varphi) = \frac{\partial}{\partial s}(z\Phi) + \nabla_y \cdot \left(\frac{y}{2} z \Phi \right),$$

we have that

$$\begin{aligned} \frac{d}{ds} \int_{\mathcal{O}(s)} z \Phi \, dy &= (T_{\max} - t) \frac{d}{dt} \int_\Omega u \varphi \, dx = \int_\Omega (T_{\max} - t) \frac{\partial}{\partial t}(u\varphi) dx \\ &= \int_{\mathcal{O}(s)} \left\{ \frac{\partial}{\partial s}(z\Phi) + \nabla_y \cdot \left(\frac{y}{2} z \Phi \right) \right\} dy. \end{aligned} \tag{2.19}$$

By (2.18) and (2.19), we observe that

$$\frac{d}{ds} \int_{\mathcal{O}(s)} z \Phi \, dy = \int_{\mathcal{O}(s)} \left\{ \frac{\partial}{\partial s}(z\Phi) + \nabla_y \cdot \left(\frac{y}{2} z \Phi \right) \right\} dy$$

$$= \int_{\mathcal{O}(s)} \Big\{ z\Big(\frac{\partial\Phi}{\partial s} + \frac{y}{2}\cdot\nabla\Phi\Big) + (\nabla\cdot(\nabla z - z\nabla w))\,\Phi \Big\} dy$$

$$= \int_{\mathcal{O}(s)} z\Big(\frac{\partial\Phi}{\partial s} + \frac{y}{2}\cdot\nabla\Phi\Big) dy + \int_{\mathcal{O}(s)} z\Delta\Phi\, dy$$

$$+ \int_{\mathcal{O}(s)} z\nabla w\cdot\nabla\Phi\, dy = I + II + III. \tag{2.20}$$

By (2.11), (2.15), and (2.17), we have that

$$|I| \leqq C_7 A \quad \text{ for } s \geqq S_1. \tag{2.21}$$

Since $\mathcal{G}$ is the Green's function of $-\Delta + e^s$ in $\mathcal{O}(s)$ with homogenous Neumann boundary condition and w satisfies (2.1), we have

$$III = \iint_{\mathcal{O}(s)\times\mathcal{O}(s)} z(y,s)\nabla\Phi(y,s)\cdot\nabla_y\mathcal{G}(y,y',s)z(y',s)dy\,dy'$$

$$+ \iint_{\mathcal{O}(s)\times\mathcal{O}(s)} z(y,s)\nabla\Phi(y,s)\cdot\nabla_y\mathcal{G}(y,y',s)\Big(e^{-s}w(y',s) - \frac{\lambda}{|\Omega|}e^{-s}\Big)dy\,dy'$$

$$= \frac{1}{2}\iint_{\mathcal{O}(s)\times\mathcal{O}(s)} \Xi(y,y',s)z(y,s)z(y',s)dy\,dy'$$

$$+ \iint_{\mathcal{O}(s)\times\mathcal{O}(s)} z(y,s)\nabla\Phi(y,s)\cdot\nabla_y\mathcal{G}(y,y',s)\Big(e^{-s}w(y',s) - \frac{\lambda}{|\Omega|}e^{-s}\Big)dy\,dy'$$

$$= IV + V. \tag{2.22}$$

By (2.6) and (2.10), we observe that

$$\nabla_y\mathcal{G}(y,y',s) = \frac{1}{2\pi}\nabla_y \log\frac{1}{|Y(y)-Y(y')|} + \frac{1}{2\pi}\nabla_y\log\frac{1}{|Y(y)-Y(y')^*|}$$

$$+ \nabla_y\mathcal{K}(y,y',s)$$

$$= -\frac{1}{2\pi|Y(y)-Y(y')|^2}\frac{\partial Y}{\partial y}(Y(y)-Y(y'))$$

$$- \frac{-1}{2\pi|Y(y)-Y(y')^*|^2}\frac{\partial Y}{\partial y}(Y(y)-Y^*(y')) + e^{-s/2}\nabla_x K(x,x'). \tag{2.23}$$

We observe that

$$\Big|\frac{\partial Y}{\partial y}(y,s)\Big| \leqq C_8 \quad \text{ for } y\in B(Q,2R) \text{ and } s \geqq S_1.$$

Then we have that for any y and $y' \in B(Q,2R)$ and $s \geqq S_1$

$$|\nabla_y \mathcal{G}(y,y',s)| \leqq C_9\Big(\frac{1}{|y-y'|}+\frac{1}{|y-y'^*|}+1\Big) \tag{2.24}$$

by (2.23).

In the case where $y \in B(Q,R)$ and $y' \notin B(Q,2R)$, it holds that $|y-y'| \geqq R$. Combining (2.7) with (2.9) implies that

$$\begin{aligned} |\nabla_y \mathcal{G}(y,y',s)| &= |e^{-s/2}\nabla_x G(e^{-s/2}x, e^{-s/2}x')| \\ &\leqq C_2\big(\tfrac{1}{|y-y'|}+1\big) = C_2\big(\tfrac{1}{R}+1\big). \end{aligned} \tag{2.25}$$

By (2.24) and (2.25), we get

$$|\nabla_y \mathcal{G}(y,y',s)| \leqq C_{10}\Big(\frac{1}{|y-y'|+1}\Big) \quad \text{for } y \in B(Q,R) \quad \text{and } y' \in \mathcal{O}(s). \tag{2.26}$$

Combining (2.25) with $e^{-s/3}\|w(\cdot,s)-\lambda/|\Omega|\|_{L^3(\mathcal{O}(s))} = \|v(\cdot,s)-\lambda/|\Omega|\|_{L^3(\Omega)}$ implies that

$$|V| \leqq C_{11}A. \tag{2.27}$$

Combining (2.20) with (2.15), (2.21), (2.28), (2.22), (2.27), and Lemma 2.2 implies that

$$|I+II+VI+V| \leqq C_{12}(A+B). \tag{2.28}$$

Thus, we have this lemma. □

The estimates (2.6) and (2.10) imply the next lemma.

Lemma 2.4. *Let $Q \in \overline{H}$, $R>0$, and $p \in [1,2)$. Then it holds that*

$$\|\nabla w\|_{L^p(B(Q,R)\cap\mathcal{O}(s))} \leqq C_{13} \qquad \textit{for } s \geqq S_1.$$

Here and henceforth, positive constants C_i ($i = 13, 14, 15$) are independent of $p \in [1,2)$.

Proof. By (2.26), we have that

$$\begin{aligned} &\Big(\int_{B(Q,R)\cap\mathcal{O}(s)} |\nabla_y w(y,s)|^p dy\Big)^{\frac{1}{p}} \\ &= \Big(\int_{B(Q,R)\cap\mathcal{O}(s)} \Big|\int_{\mathcal{O}(s)} \nabla_y \mathcal{G}(y,y',s)\Big(z(y',s)-\frac{\lambda}{|\Omega|}e^{-s}+e^{-s}w(y',s)\Big)dy'\Big|^p\Big)^{\frac{1}{p}} \\ &\leqq C_{10}\Big(\int_{\mathcal{O}(s)}\Big|z(y',s)-\frac{\lambda}{|\Omega|}e^{-s}+e^{-s}w(y',s)\Big|dy'\Big)^{\frac{p-1}{p}} \qquad (2.29) \\ &\times\Big(\int_{B(Q,R)\cap\mathcal{O}(s)}\int_{\mathcal{O}(s)}\Big|z(y',s)-\frac{\lambda}{|\Omega|}e^{-s}+e^{-s}w(y',s)\Big|\Big(\frac{1}{|y-y'|}+1\Big)^p dy'dy\Big)^{\frac{1}{p}}. \end{aligned}$$

Combining Lemma 2.1 with

$$\|w(\cdot,s)\|_{L^2(\mathcal{O}(s))} = e^{s/2}\|v(\cdot,t)\|_{L^2(\Omega)} \quad \text{and } |\mathcal{O}(s)| = e^s|\Omega|$$

implies that

$$\begin{aligned}
&\int_{\mathcal{O}(s)} \left| z(y',s) - \frac{\lambda}{|\Omega|}e^{-s} + e^{-s}w(y',s)\right| dy' \\
&\leqq \lambda + e^{-s}\frac{\lambda}{|\Omega|}|\mathcal{O}(s)| + e^{-s}\sqrt{|\Omega|}\|w(\cdot,s)\|_{L^2(\mathcal{O}(s))} \\
&\leqq 2\lambda + |\Omega|\|v(\cdot,t)\|_{L^2(\Omega)} \leqq 2\lambda + C_1|\Omega| \qquad (2.30)
\end{aligned}$$

with $e^{-s} = T_{\max} - t$. Combining (2.29) with (2.30) implies this lemma. □

In the following two lemmas, by using the reflection of w on the boundary, we assume that w is defined in $\cup_{s\geqq S_2}\cup_{Q\in\mathcal{O}(s)}B(Q,R)\times\{s\}$ for any sufficiently small $R>0$ and some $S_2 \geqq S_1$.

The next lemma is called Poincarè-Sobolev's inequality.

Lemma 2.5. *Let $p \in [1,2)$. There exists a positive constant C_{14} such that*

$$\begin{aligned}
&\left\{\frac{1}{|B(Q,R)|}\int_{B(Q,R)} \left|w(y,s) - w_{B(Q,R)}(s)\right|^{2p/(2-p)} dy\right\}^{(2-p)/(2p)} \\
&\leqq C_{14}\left\{\frac{1}{|B(Q,R)|}\int_{B(Q,R)} |\nabla w(y,s)|^p dy\right\}^{1/p}
\end{aligned}$$

for any $Q \in \mathcal{O}(s)$, $0 < R \ll 1$, and $s \geqq S_2$, where

$$w_{B(Q,R)}(s) = \frac{1}{|B(Q,R)|}\int_{B(Q,R)} w(y,s)dy.$$

The following lemma is a immediate conclusion from Lemmas 2.4 and 2.5.

Lemma 2.6. *Under the same assumption as Lemma 2.5, it holds that*

$$\left\{\frac{1}{|B(Q,R)|}\int_{B(Q,R)} \left|w(y,s) - w_{B(Q,R)}(s)\right|^{2p/(2-p)} dy\right\}^{(2-p)/(2p)} \leqq C_{15}(\lambda+1).$$

3. Compactness of solutions

In this section, for solutions to (1.5) we will show the following proposition, which is a property similar to [17, Theorem 3].

Proposition 3.1. *Let $\{s_n\}_{n\geqq 1} \subset [0,\infty)$ with $\lim_{n\to\infty} s_n = \infty$. Suppose that $Q \in \overline{\mathcal{O}(s)}$ and a positive constant R satisfy*

$$\limsup_{n\to\infty} \|z(\cdot,s_n)\|_{L^1(B(Q,R)\cap\mathcal{O}(s_n))} < m_*(Q),$$

where

$$m_*(Q) = \begin{cases} 4\pi & \text{if} \quad Q \in \partial\mathcal{O}(s), \\ 8\pi & \text{if} \quad Q \in \mathcal{O}(s). \end{cases}$$

Then there exist positive constants τ and R', and a subsequence $\{s'_n\} \subset \{s_n\}$, such that

$$\|z\|_{C^{2+\theta,1+\theta/2}(E_1)} < \infty \tag{3.1}$$

with $E_1 = \cup_{n=1}^{\infty} E_{1,n}$, $E_{1,n} = \cup_{s\in[s'_n-\tau,s'_n+\tau]}(\overline{B(Q,R') \cap \mathcal{O}(s)} \times \{s\})$ and some $\theta \in (0,1)$.

By Sobolev's inequality, we find a positive constant K satisfying

$$\|f\|^2_{L^2(\Omega)} \leqq K^2\big(\|\nabla_x f\|^2_{L^1(\Omega)} + \|f\|^2_{L^1(\Omega)}\big) \quad \text{for } f \in W^{1,1}(\Omega). \tag{3.2}$$

Then K depends only on Ω. Putting $f(x) = g(e^{\frac{s}{2}}x)$ for each $g \in W^{1,1}(\mathcal{O}(s))$, it holds that

$$\begin{aligned} \|g\|^2_{L^2(\mathcal{O}(s))} &= e^s\|f\|^2_{L^2(\Omega)} \leqq e^s K^2(\|\nabla_x f\|^2_{L^1(\Omega)} + \|f\|^2_{L^1(\mathcal{O}(s))}) \\ &= K^2(\|\nabla_y g\|^2_{L^1(\mathcal{O}(s))} + e^{-s}\|g\|^2_{L^1(\mathcal{O}(s))}). \end{aligned}$$

Putting $K_1 = K\max(e^{-s_*/2}, 1)$, we have that

$$\|g\|^2_{L^2(\mathcal{O}(s))} \leqq K_1^2(\|\nabla_y g\|^2_{L^1(\mathcal{O}(s))} + \|g\|^2_{L^1(\mathcal{O}(s))}) \ \text{ for } g \in W^{1,1}(\mathcal{O}(s)). \tag{3.3}$$

Lemma 3.1. *Let $Q \in \overline{\mathcal{O}(\infty)}$ and $\{s_n\}_{n\geqq 1} \subset [0,\infty)$ with $\lim_{n\to\infty} s_n = \infty$. Suppose that for any sufficiently small $R > 0$*

$$\|z(\cdot, s_n)\|_{L^1(B(Q,9R)\cap\mathcal{O}(s_n))} \leqq \frac{1}{128K_1^2} \quad \text{for } n \geqq 1.$$

For some constants $\tau > 0$ and $\theta \in (0,1)$ depending only on Q, λ, and R, it holds that

$$\|z\|_{C^{2+\theta,1+\theta/2}(E_2)} < \infty,$$

where $E_2 = \cup_{n=1}^{\infty} E_{2,n}$ and $E_{2,n} = \cup_{s\in[s_n-\tau,s_n+\tau]}\overline{B(Q,R) \cap \mathcal{O}(s)} \times \{s\}$.

We will prove the following lemma in the next section.

In order to prove Proposition 3.1, it is necessary that we divide the proof into the case where $Q \in \mathcal{O}(\infty)$ and the case where $0 \in \partial\Omega$ and $Q \in \partial H$.

In the case where $0 \in \partial\Omega$ and $Q \in \partial H$, it is necessary to treat the boundary $\partial\mathcal{O}(s)$ and to use the conformal map Y.

In the case where $Q \in \mathcal{O}(\infty)$, since we find a neighborhood $B(Q,R) \subset \mathcal{O}(s)$ for some $R > 0$ and any sufficiently large s, it is enough for us to treat only interior points of $\mathcal{O}(s)$. Namely, it is not necessary to treat the conformal map Y and the boundary $\partial\mathcal{O}(s)$.

Then we find that the case where $Q \in \mathcal{O}(\infty)$ is easier than the case where $0 \in \partial\Omega$ and $Q \in \partial H$. Thus, it is enough for us to treat the case where $0 \in \partial\Omega$ and $Q \in \partial H$, so we will prove Proposition 3.1 only in this case.

By the assumption of Proposition 3.1, for $Q \in \partial H$, $0 < R \ll 1$, and $\{s_n\}$ with $\lim_{n\to\infty} s_n = \infty$ there exists a subsequence $\{s'_n\} \subset \{s_n\}$ such that

$$z(\cdot, s'_n)\chi_{\overline{\mathcal{O}(s'_n)}} \to z_\infty \quad \text{as } n \to \infty \quad \text{in } \mathcal{M}(\overline{B(Q,R)}). \tag{3.4}$$

Here, for a subset $E \subset \mathbf{R}^2$ we denote

$$\chi_E(y) = \begin{cases} 1 & \text{if} \quad y \in E, \\ 0 & \text{if} \quad y \in \mathbf{R}^2 \setminus E. \end{cases}$$

Since z_∞ is a Radon measure in $\overline{B(Q,R)}$, we put

$$\mathcal{S}(Q,R) = \Big\{ P \in \overline{B(Q,R)} : z_\infty(\{P\}) \geqq \frac{1}{256K_1^2} \Big\}.$$

We observe that ${}^\sharp\mathcal{S}(Q,R) \leqq 256K_1^2\lambda$, since it holds that

$$z_\infty(\overline{B(Q,R)}) \leqq \liminf_{n\to\infty} \int_{B(Q,R)\cap\mathcal{O}(s'_n)} z(y,s'_n)dy \leqq \lambda.$$

Then we can find $R_1 \in (0,R)$ satisfying

$$\overline{B(Q,R_1)} \cap \mathcal{S}(Q,R) = \{Q\} \quad \text{or } \emptyset \tag{3.5}$$

and that

$$\varepsilon = \sup \Big\{ \Big|\frac{\partial Y}{\partial y}(y,s) - \mathrm{id}\Big| : y \in B(Q,R_1),\ s \geqq s_* \Big\} < \frac{1}{5}. \tag{3.6}$$

By the assumption of Proposition 3.1, we can find $R_2 \in (0,R_1]$, $\delta > 0$, and $N_1 \geqq 1$ such that

$$\int_{B(Q,R_2)\cap\mathcal{O}(s'_n)} z(y,s'_n)dy \leqq 4\pi - 3\delta \quad \text{for } n \geqq N_1. \tag{3.7}$$

By this, for each $P \in \overline{B(Q,R_1) \setminus \{Q\}}$ there exists $R'_2 \in (0,R_2]$ and N'_1 such that

$$\int_{B(P,R'_2)\cap\mathcal{O}(s'_n)} z(y,s'_n)dy < \frac{1}{128K_1^2} \quad \text{for } n \geqq N'_1. \tag{3.8}$$

By this and Lemma 3.1, we can find $\tau' > 0$ and $R''_2 > 0$ satisfying

$$\|z\|_{C^{2+\theta,1+(\theta/2)}(E')} < \infty$$

with $E' = \cup_{n \geqq N_1'}^{\infty} E_n'$ and $E_n' = \cup_{s \in [s_n' - \tau', s_n' + \tau']} \overline{B(P, R_2'') \cap \mathcal{O}(s)} \times \{s\}$. Then for $R_3 \in (0, R_2)$ we can find $\tau > 0$ and $N_2 \geqq 1$ satisfying

$$L_1 = \|z\|_{C^{2+\theta, 1+(\theta/2)}(E_2)} < \infty \tag{3.9}$$

with $E_2 = \cup_{n \geqq N_2}^{\infty} E_{2,n}$ and

$$E_{2,n} = \cup_{s \in [s_n' - \tau_1, s_n' - \tau_1]} \overline{(B(Q, R_2) \setminus B(Q, R_3)) \cap \mathcal{O}(s)} \times \{s\}.$$

Let f be a measurable function in $\mathbf{R}^2$,

$$\begin{aligned} \mu(\rho) &= |\{y \in \mathbf{R}^2 : f(y) > \rho\}| \quad \text{for } \rho \geqq 0, \\ f^*(\zeta) &= \inf\{\rho : \mu(\rho) \leqq \zeta\} \quad \text{for } \zeta \geqq 0. \end{aligned}$$

Then it holds that $\rho = f^*(\mu(\rho))$.

The following lemma is [17, Lemma 6].

Lemma 3.2. *Let $\mathcal{O}$ be a bounded domain in $\mathbf{R}^2$ and $p \in [1, \infty]$. Suppose that f is a nonnegative function in $W_0^{1,1}(\mathcal{O})$. Then for $\delta \in (0, |\mathcal{O}|)$ it holds that $f^* \in W^{1,p}((\delta, |\mathcal{O}|))$.*

The following lemma is an immediate conclusion from [17, Lemma 7].

Lemma 3.3. *Let Ω be a bounded domain in $\mathbf{R}^2$, $p \in [1, \infty]$, and $T > 0$. Suppose that f is a nonnegative function in $H^1(0, T; L^p(\Omega))$. Putting*

$$F(\zeta, t) = \int_{\{f > \rho\}} f(x, t) dx$$

with $\zeta = |\{x \in \mathbf{R}^2 : f(x, t) > \rho\}|$, it holds that

$$F \in L^\infty((0, |\Omega|) \times (0, T)) \cap H^1(0, T; W^{1,p}((0, |\Omega|))) \cap_{\delta > 0} L^2(0, T; W^{2,p}((\delta, |\Omega|)))$$

and that

$$\int_{\{f > \rho\}} \partial_t f(x, t) dx = \partial_t F(|\{f > \rho\}|, t) \quad \text{for a.e. } (\rho, t) \in [0, \infty) \times [0, T].$$

Let Ψ be a cutoff function defined in Section 2, and put $\varphi(x, t) = \Psi(y, s)$ with (1.3). For any $\rho > 0$, it holds that

$$\begin{aligned} &\int_{\{z\Psi > \rho\}} \left\{ (z\Psi)_s + \nabla \cdot \left(\frac{y}{2} z\Psi \right) \right\} dy = (T_{\max} - t)^2 \int_{\{z\Psi > \rho\}} (u\varphi)_t \, dy \\ &= (T_{\max} - t) \int_{\{(T_{\max} - t) u\varphi > \rho\}} (u\varphi)_t \, dx \\ &= \int_{\{(T_{\max} - t) u\varphi > \rho\}} ((T_{\max} - t) u\varphi)_t \, dx + \int_{\{(T_{\max} - t) u\varphi > \rho\}} u\varphi \, dx \end{aligned}$$

$$= (T_{\max} - t)\frac{d}{dt}\int_{\{(T_{\max}-t)u\varphi>\rho\}} u\varphi\, dx = \frac{d}{ds}\int_{\{z\Psi>\rho\}} z\Psi\, dy, \tag{3.10}$$

by using Lemma 3.3 and

$$\int_{\{(T_{\max}-t)u\varphi>\rho\}} u\varphi\, dx = \int_{\{z\Psi>\rho\}} z\Psi\, dy.$$

Let $R \in (0, R_2)$, and let $\Psi = \Psi_{Q,R/2,R}$. Putting

$$\mathcal{P}(\zeta, s) = \int_0^{\zeta} (z\Psi)^*(\zeta', s)d\zeta',$$

it holds that

$$\mathcal{P}(\mu(\rho, s), s) = \int_{\{z\Psi>\rho\}} z(y, s)\Psi(y, s)dy.$$

Then, we have the following lemma.

Lemma 3.4. *For some nonnegative constant L depending on L_1 in* (3.9), *it holds that*

$$\frac{\partial \mathcal{P}}{\partial s} - 2\pi(1-5\varepsilon)\zeta\frac{\partial^2\mathcal{P}}{\partial\zeta^2} - (\mathcal{P} + L\zeta)\frac{\partial\mathcal{P}}{\partial\zeta} - L\zeta \leqq 0$$

for almost every $\zeta \geqq 0$ and $s \in [s'_n - \tau_1, s'_n + \tau_1]$ and $n \geqq N_2$. Moreover, $\mathcal{P}$ satisfies

$$\mathcal{P}(0, s) = 0, \quad \mathcal{P}(\zeta^*, s) = \int_{\mathbf{R}^2} z(y, s)\Psi(y, s)dy,$$

with $\zeta^* = \sup_{s\geqq S_2} |\mathcal{O}(s) \cap B(Q, R)|$ and $S_2 = \max(s'_{N_2}, S_1)$.

This lemma is shown by using an argument similar to the proof of [17, Lemma 8].

Proof. For each $s \geqq S_2$, let $Y^{-1}(\cdot, s)$ be an inverse map of $Y(\cdot, s)$. Let $\widetilde{z\Psi}(\eta, s) = z(Y^{-1}(\eta, s), s) \cdot \Psi(Y^{-1}(\eta, s), s)$ and $\widetilde{w\Psi}(\eta, s) = w(Y^{-1}(\eta, s), s) \cdot \Psi(Y^{-1}(\eta, s), s)$.

The even extensions of $\widetilde{z\Psi}$ and $\widetilde{w\Psi}$ with respect to ∂H are denoted by $\overline{z\Psi}$ and $\overline{w\Psi}$, respectively. Then we see that $(z\Psi)^*$ is locally absolutely continuous on $(0, \zeta^*]$ by Lemma 3.2.

For $\rho > 0$ and $h > 0$, we put

$$T_{\rho,h}(\zeta) = \begin{cases} 0 & (\zeta \leqq \rho) \\ \zeta - \rho & (\rho < \zeta \leqq \rho + h) \\ h & (\zeta > \rho + h). \end{cases}$$

By the way of construction of cutoff functions in the previous section, we have that $T_{\rho,h}(\overline{z\Psi}(\cdot,s)) \in W^{1,\infty}(B(Q,R))$ and that $T_{\rho,h}(z(\cdot,s)\Psi(\cdot,s)) \in W^{1,\infty}(B(Q,R)\cap\mathcal{O}(s))$. By using the first equation of (1.5), we obtain

$$
\begin{aligned}
&\int_{\mathcal{O}(s)} \Big(z\Psi\Big)_s T_{\rho,h}(z\Psi) = \int_{\mathcal{O}(s)} (z_s\Psi + z\Phi_{1s}) T_{\rho,h}(z\Psi)dy \\
&= \int_{\mathcal{O}(s)} \Big[\nabla\cdot\Big(\nabla z - z\nabla w - \frac{y}{2}z\Big)\Big]\Psi T_{\rho,h}(z\Psi)dy + \int_{\mathcal{O}(s)} z\Psi_s T_{\rho,h}(z\Psi_1)dy \\
&= -\int_{\mathcal{O}(s)} \nabla(z\Psi)\cdot\nabla T_{\rho,h}(z\Psi) + \int_{\mathcal{O}(s)} (z\Psi)\nabla w\cdot\nabla T_{\rho,h}(z\Psi)dy \\
&- \int_{\mathcal{O}(s)} \Big[\nabla\cdot\Big(\frac{y}{2}z\Psi\Big)\Big] T_{\rho,h}(z\Psi)dy - \int_{\mathcal{O}(s)} gT_{\rho,h}(z\Psi)dy \\
&= -IV + V - VI - VII,
\end{aligned}
\tag{3.11}
$$

where $g = \nabla\cdot(z\nabla\Psi) + \nabla z\cdot\nabla\Psi - z\nabla w\cdot\nabla\Psi - z\Phi_{1s} - (yz/2)\cdot\nabla\Psi$. We estimate each term of the right-hand side in the following way.

For the third term, noting supp $(|\nabla\Psi|\cap B(Q,R/2)) = \emptyset$, we have

$$
\|g\|_{L^\infty(E_2)} \leqq L_2
$$

follows from (3.9). This implies

$$
\begin{aligned}
&\limsup_{h\downarrow 0}\Big|\frac{VII}{h}\Big| = \limsup_{h\downarrow 0}\Big|\frac{1}{h}\int_{\mathcal{O}(s)} g(y,s)T_{\rho,h}\big(z(y,s)\Psi(y,s)\big)dy\Big| \\
&\leqq L_2 \limsup_{h\downarrow 0}\frac{1}{h}\int_{\{z\Psi>\rho\}} h\,dy = L_2\mu(\rho,s),
\end{aligned}
$$

where $\mu(\rho,s) = \big|\big\{y\in\mathbf{R}^2 : z(y,s)\Psi(y,s) > \rho\big\}\big|$.

To estimate the first term of the right-hand side of (3.11), we recall that $Y(\cdot,s)$ is conformal. We have

$$
\int_E |\nabla f|^2 dy = \int_{Y(E,s)} |\nabla\tilde{f}|^2 d\eta = \frac{1}{2}\int_{\overline{Y}(E,s)} |\nabla\overline{f}|^2 d\eta
$$

for any $s \geqq S_2$, any $f \in C^1(\overline{B(Q,R)\cap\mathcal{O}(s)})$, and any measurable set $E \subset \overline{B(0,R)\cap\mathcal{O}(s)}$, where $\overline{Y}(E,s)$ denotes the even extension of $Y(E,s)$. Hence we obtain

$$
\begin{aligned}
\lim_{h\downarrow 0}\frac{IV}{h} &= \lim_{h\downarrow 0}\frac{1}{h}\int_{\mathcal{O}(s)} \nabla_y(z\Psi)\cdot\nabla_y T_{\rho,h}(z\Psi)dy \\
&= \lim_{h\downarrow 0}\frac{1}{h}\Big\{\int_{\{z\Psi>\rho\}} |\nabla_y(z\Psi)|^2 dy - \int_{\{z\Psi>\rho+h\}} |\nabla_y(z\Psi)|^2 dy\Big\}
\end{aligned}
$$

$$= \frac{1}{2} \lim_{h\downarrow 0} \frac{1}{h} \int_{\{\rho+h \geqq \overline{z\Psi} > \rho\}} \left|\nabla_\eta (\overline{z\Psi})\right|^2 d\eta = -\frac{1}{2} \frac{\partial}{\partial \rho} \int_{\{\overline{z\Psi} > \rho\}} \left|\nabla_\eta (\overline{z\Psi})\right|^2 d\eta.$$

Now, the standard use of the co-area formula and the isoperimetric inequality implies

$$4\pi \mathfrak{m} \Big(-\frac{\partial \mathfrak{m}}{\partial \rho} \Big)^{-1} \leqq -\frac{\partial}{\partial \rho} \int_{\{\overline{z\Psi} > \rho\}} \left|\nabla_\eta (\overline{z\Psi})\right|^2 d\eta$$

for almost every ρ, where $\mathfrak{m} = \mathfrak{m}(\rho, s) \equiv |\{\eta \in \operatorname{supp} \overline{\Psi} : \overline{z\Psi}(\eta, s) > \rho\}|$. Here we have

$$\det\Big(\frac{\partial Y}{\partial y}\Big) = \frac{\partial Y_1}{\partial y_1} \cdot \frac{\partial Y_2}{\partial y_2} - \frac{\partial Y_1}{\partial y_2} \cdot \frac{\partial Y_2}{\partial y_1} \geqq (1-\varepsilon)^2 - \varepsilon^2 = 1 - 2\varepsilon$$

and

$$\det\Big(\frac{\partial Y}{\partial y}\Big) \leq (1+\varepsilon)^2 + \varepsilon^2 \leq 1 + 3\varepsilon$$

by (3.6). Therefore,

$$\mathfrak{m}(\rho, s) = \int_{\{\overline{z\Psi} > \rho\}} d\eta = \frac{1}{2} \int_{\{z\Psi > \rho\}} \det\Big(\frac{\partial Y}{\partial y}\Big) dy \geqq \frac{1-2\varepsilon}{2} \mu(\rho, s)$$

and

$$-\frac{\partial \mathfrak{m}}{\partial \rho}(\rho, s) = \lim_{h\downarrow 0} \frac{1}{h} \Big(\mathfrak{m}(\rho, s) - \mathfrak{m}(\rho+h, s) \Big) = \lim_{h\downarrow 0} \frac{1}{h} \int_{\{\rho+h \geqq \overline{z\Psi} > \rho\}} d\eta$$
$$\leqq \frac{1+3\varepsilon}{2} \lim_{h\downarrow 0} \frac{1}{h} \int_{\{\rho+h \geqq z\Psi > \rho\}} dy = -\frac{1+3\varepsilon}{2} \frac{\partial \mu}{\partial \rho}(\rho, s)$$

follow. Those relations are summarized as

$$\lim_{h\downarrow 0} \frac{IV}{h} \geqq \frac{1}{2} \cdot 4\pi \cdot \frac{\mu}{1+3\varepsilon} \cdot (1-2\varepsilon) \cdot \Big(-\frac{\partial \mu}{\partial \rho} \Big)^{-1}$$

or equivalently,

$$\mu^{-1} \Big(-\frac{\partial \mu}{\partial \rho} \Big) \lim_{h\downarrow 0} \frac{IV}{h} \geqq \frac{1}{2} \cdot 4\pi \cdot \frac{1-2\varepsilon}{1+3\varepsilon} \geqq 2\pi \cdot (1-5\varepsilon)$$

for almost every ρ.

To handle with the second term of the right-hand side of (3.11), we put

$$S_{\rho,h}(\zeta) = \int_0^\zeta \ell \frac{d}{d\ell} T_{\rho,h}(\ell) d\ell = \begin{cases} 0 & (\zeta \leqq \rho) \\ \frac{1}{2}(\zeta^2 - \rho^2) & (\rho < \zeta \leqq \rho + h) \\ h\big(\rho + \frac{h}{2}\big) & (\zeta > \rho + h). \end{cases}$$

Then we have

$$V=\int_{\mathcal{O}(s)}\nabla w\cdot\nabla S_{\rho,h}(z\Psi)=\int_{\mathcal{O}(s)}\Big(z-\frac{\lambda}{|\Omega|}e^{-s}\Big)S_{\rho,h}(z\Psi)dy$$

by using the second equation of (1.5). This implies

$$\begin{aligned}\lim_{h\downarrow 0}\frac{V}{h}&=\lim_{h\downarrow 0}\frac{1}{2h}\int_{\{\rho+h\geqq z\Psi>\rho\}}\big(z-\frac{\lambda}{|\Omega|}e^{-s}\big)\big[(z\Psi)^2-\rho^2\big]dy\\&\quad+\lim_{h\downarrow 0}\int_{\{z\Psi>\rho+h\}}\big(z-\frac{\lambda}{|\Omega|}e^{-s}\big)\big(\rho+\frac{h}{2}\big)=\rho\int_{\{z\Psi>\rho\}}\big(z-\frac{\lambda}{|\Omega|}e^{-s}\big)dy\\&\quad\leqq\rho\int_{\{z\Psi>\rho\}}z\Psi\,dy+\rho\int_{\{z\Psi>\rho\}}z(1-\Psi)dy.\end{aligned}$$

Here, we have $\int_{\{z\Psi>\rho\}}z(1-\Psi)dy\leqq L_1\mu(\rho,s)$ by (3.9) and supp $(1-\Psi)\cap B(Q,R/2)=\emptyset$. Also, we have

$$\int_{\{z\Psi>\rho\}}z\Psi dy=\int_0^{\mu(\rho,s)}(z\Psi)^*d\zeta=\mathcal{P}(\mu(\rho,s),s)$$

and $\rho=(z\Psi)^*(\mu(\rho,s),s)=P_\zeta(\mu(\rho,s),s)$. We obtain

$$\lim_{h\downarrow 0}\frac{V}{h}\leqq\big(\mathcal{P}(\mu(\rho,s),s)+L_1\mu(\rho,s)\big)\mathcal{P}_\zeta(\mu(\rho,s),s).$$

The left-hand side of (3.11) is treated by (3.10). We have

$$\big|\{z\Psi=\rho\}\big|=0\qquad\text{for a.e. }\rho,$$

and hence

$$\lim_{h\downarrow 0}\frac{1}{h}\int_{\mathcal{O}(s)}(z\Psi)_sT_{\rho,h}(z\Psi)dy=\int_{\{z\Psi>\rho\}}(z\Psi)_s\,dy=\mathcal{P}_s(\mu(\rho,s),s)-\lim_{h\downarrow 0}\frac{IV}{h}$$

follows. Equality (3.11) implies

$$\begin{aligned}2\pi(1-5\varepsilon)\leq\mu^{-1}\big(-\frac{\partial\mu}{\partial\rho}\big)\Big\{&-\mathcal{P}_s(\mu(\rho,s),s)\\&+\Big[\mathcal{P}(\mu(\rho,s),s)+L_1\mu(\rho,s)\Big]\mathcal{P}_\zeta(\mu(\rho,s),s)+L_2\mu(\rho,t)\Big\}.\end{aligned}$$

Integrating in $\rho\in(\rho_1,\rho_2)\subset[0,|B(Q,R)|)$, we get that

$$\begin{aligned}&2\pi(1-5\varepsilon)\big(\rho_2-\rho_1\big)\\&\quad\leqq\int_{\mu(\rho_2,s)}^{\mu(\rho_1,s)}\zeta^{-1}\Big\{-\mathcal{P}_s(\zeta,s)+\Big[\mathcal{P}(\zeta,s)+L_1\zeta\Big]\mathcal{P}_\zeta(\zeta,s)+L_2\zeta\Big\}d\zeta.\end{aligned}$$

Because of $\rho_2 - \rho_1 = (z\Psi)^*(\mu(\rho_2, s), s) - (z\Psi)^*(\mu(\rho_1, s), s)$ we obtain

$$\begin{aligned} 0 &\leqq -2\pi(1-5\varepsilon)\partial_\zeta^2 \mathcal{P}(\zeta, s) = -2\pi(1-2\varepsilon)\big[\partial_\zeta (z\Psi)^*\big](\zeta, s) \\ &\leqq \zeta^{-1}\big[-\mathcal{P}_s + (\mathcal{P} + L_1\zeta)\mathcal{P}_\zeta + L_2\zeta\big]. \end{aligned}$$

We have $\mathcal{P}_\zeta \geqq 0$, and hence the inequality of this lemma holds with $L = \max(L_1, L_2)$. The latter part of the lemma is immediate, and the proof is complete. □

The following lemma is [17, Lemma 2.6]. Here, $\mathcal{L}(h) = \partial_s h - L_3\zeta\partial_\zeta^2 h - (h + L_4\zeta)\partial_\zeta h - L_4\zeta$ is a second-order parabolic operator (with the inhomogeneous term $-L_4\zeta$), where $L_3 > 0$ and $L_4 \geqq 0$ are constants.

Lemma 3.5. *Let $A = A(s) \in C^1([0, S])$ be a positive function, and $f = f(\zeta, s)$ and $g = g(\zeta, s)$ be measurable functions defined on*

$$Q_S = \Big\{(\zeta, s) : 0 < s < S,\ 0 < \zeta < A(s)\Big\},$$

satisfying the following conditions for any $\delta > 0$:
(i) $f,\ g,\ f_s,\ g_s,\ f_\zeta,\ g_\zeta \in L^\infty(Q_S)$.
(ii) $\sup_{0\leqq s\leqq S}\big\{\|f(\cdot, s)\|_{W^{2,1}(\delta, A(s))} + \|g(s)\|_{W^{2,1}(\delta, A(s))}\big\} < +\infty$.
(iii) $\mathcal{L}(f) \leqq \mathcal{L}(g)$ *almost everywhere in* Q_S.
(iv) $0 = f(0, s) \leqq g(0, s)$ *and* $f(A(s), s) \leq g(A(s), s)$ *for* $s \in [0, S]$.
(v) $f(\zeta, 0) \leqq g(\zeta, 0)$ *for* $\zeta \in [0, A(0)]$ *and* $g \geqq 0$ *in* Q_S.
Then the inequality $f \leqq g$ holds on Q_S.

As mentioned above, we prove Proposition 3.1 in only the case where $0 \in \partial\Omega$ and $Q \in \partial H$.

Proof of Proposition 3.1. By (3.7) and Lemma 2.3, we can find $\tau_2 \in (0, \tau_1)$ satisfying

$$\int_{B(Q,R_2/2)\cap\mathcal{O}(s_n')} z(y, s_n')dy \leqq 4\pi - 2\delta \tag{3.12}$$

for $s \in [s_n' - \tau_2, s_n' + \tau_2]$, $n \geqq N_2$, and $(2\tau_2)^{-2}|B(Q, R_2/2)| \geqq 1$.

For each $n \geqq N_2$, put $S \in (s_n' - \tau_2, s_n' + \tau_2)$. Let $\Psi_1 = \Psi_{Q,R_2/4,R_2/2}$ and $\mathcal{P}_1(\zeta, s) = \int_0^\zeta (z\Psi_1)^*(\zeta', s)d\zeta'$. Set $j(\kappa, s) = \mathcal{P}_1(\zeta, s)$ for $\kappa = \zeta(s - S)^{-2}$. Let

$$\begin{aligned} \mathcal{J}(j) \equiv{}& \partial_s j - 2\pi\big(1 - \frac{\delta}{16\pi}\big)\zeta(s - S)^{-2} j_{\kappa\kappa} \\ &-\big\{j + L(s - S)^2\kappa + (s - S)\kappa\big\}(s - S)^{-2} j_\kappa - L(s - S)^2\kappa. \end{aligned}$$

Then we have $\mathcal{J}(j) \leqq 0$ for almost every $s \in (S, s_n' + \tau_2))$ and $\kappa \in (0, (s - S)^{-2}|B(Q, R_2/2)|)$ by $\delta/(16\pi) > 5\varepsilon$, $j_{\kappa\kappa} \leqq 0$, and Lemma 3.4.

Let $k(\kappa) = (m\sigma_0\kappa)/(1+\sigma_0\kappa)$ with $m = 4\pi - (\delta/2)$ and $\sigma_0 = (2m/\delta) - 1$. We have

$$k(1) = \frac{m\sigma_0}{1+\sigma_0} = \frac{\sigma_0\delta}{2} = m - \frac{\delta}{2} = 4\pi - \delta.$$

Also, we have $\mathcal{P}_1(|B(Q,R_2/2)|, s) \leqq 4\pi - 2\delta$ by (3.12). Hence

$$j(1,s) = \mathcal{P}_1\big((s-S)^2, s\big) \leqq \mathcal{P}_1\big(\big|B(Q,R_2/2)\big|, s\big) \leqq 4\pi - \frac{3\delta}{2} = k(1) - \delta \quad (3.13)$$

follows for $s \in [S, s_n' + \tau_2]$. Next, we have

$$\begin{aligned}\kappa^{-1}(s-S)^2(1+\sigma_0\kappa)^3\mathcal{J}(k) &= \Big\{4\pi\big(1 - \frac{\delta}{16\pi}\big)m\sigma_0^2 - m^2\sigma_0^2\Big\} \\ &\quad - (s-S)(1+\sigma_0\kappa)\Big\{Lm\sigma_0(s-S) + m\sigma_0 + L(s-S)^3(1+\sigma_0\kappa)^2\Big\} \\ &\geqq \frac{\delta m\sigma_0^2}{4} - (s-S)(1+\sigma_0\kappa) \\ &\quad \times \Big\{Lm\sigma_0(s-S) + m\sigma_0 + L(s-S)^3(1+\sigma_0\kappa)^2\Big\}.\end{aligned}$$

Taking $\tau_3 \in (0, \tau_2]$ in

$$\frac{\delta m\sigma_0^2}{4} - 2\tau_3(1+\sigma_0)\Big\{Lm\sigma_0 2\tau_3 + m\sigma_0 + L(2\tau_3)^3(1+\sigma_0)^2\Big\} \geqq 0,$$

we get

$$\mathcal{J}(k) \geqq 0 \geqq \mathcal{J}(j) \quad (3.14)$$

for $(\kappa, s) \in (0,1] \times (S, s_n' + \tau_3]$. Finally, we have

$$j(\kappa, S) = \lim_{s\downarrow S} \mathcal{P}((s-S)^2\kappa, s) = \mathcal{P}(0,S) = 0 < J(\kappa) \quad (3.15)$$

for $\kappa \in (0,1]$. By using

$$j_\kappa(\kappa, s) = (s-S)^2(z\Phi_1)^*\big((s-S)^2\kappa, s\big) \leqq (s-S)^2\big\|z(\cdot,s)\big\|_\infty$$

and $k_\kappa(\kappa) \geqq (m\sigma_0)(1+\sigma_0)^{-2}$, we can find S' in $0 < S' - S \ll 1$ satisfying

$$j(\kappa, S') \leqq k(\kappa) \quad (3.16)$$

for $\kappa \in [0,1]$.

Now, we apply Lemma 3.5 for $f(\zeta, s) = \mathcal{P}_1(\zeta, s)$, $g(\zeta, s) = k(\zeta(s-S)^{-2}))$, $A(s) = (s-S)^2$, and $Q_{S'} = \big\{(\zeta, s) : S' < s < s_n' + \tau_3,\ \zeta \in (0, A(s))\big\}$. In fact, conditions (i) and (ii) are obvious, while conditions (iii), (iv), and (v) follow from (3.13), (3.14), (3.15), (3.16), and $\mathcal{P}(0,s) = k(0) = 0$. Thus, we get $\mathcal{P}(\zeta,s) \leqq k(\zeta(s-S)^{-2})$ for any $s \in [S', s'_n+\tau_3]$ and $\zeta \in [0,(s-S)^2]$. This means

$$\mathcal{P}(\zeta,s) \leqq \frac{m\sigma_0 s}{(s-S)^2+\sigma_0\zeta}$$

for $s \in [S', s'_n+\tau_3)$ and $\zeta \in [s'_n-\tau_3, s'_n+\tau_3]$. Finally, S' and S are also arbitrary constants satisfying $s'_n-\tau_5 < S < S' < s'_n+\tau_3$, and hence $\mathcal{P}_1(\zeta,s) \leqq (m\sigma_0\zeta)/((s-s'_n+\tau_3)^2+\sigma_0\zeta)$ holds for $s \in (s'_n-\tau_3, s'_n+\tau_3)$ and $\zeta \in (0,(s-s'_n+\tau_3)^2]$.

Combining this with $\mathcal{P}_1(0,s)=0$, we obtain

$$(z\Phi_1)^*(0,s) = P_\zeta(0,s) \leqq \partial_\zeta\Big(\frac{m\sigma_0\zeta}{(s-s'_n+\tau_3)^2+\sigma_0\zeta}\Big)\Big|_{\zeta=0} = (m\sigma_0)(s-s'_n+\tau_3)^{-2},$$

or $\|z(\cdot,s)\Phi_1(\cdot,s)\|_{L^\infty(\mathcal{O}(s))} \leqq m\sigma_0(s-s'_n+\tau_3)^{-2}$. Then the standard bootstrap argument guarantees that $\|z\|_{C^{2+\theta,1+(\theta/2)}(E_3)} < \infty$ with $E_3 = \cup_{n\geqq N_2}^{\infty} E_{3,n}$ and $E_{3,n} = \cup_{s\in[s'_n-\tau_4,s'_n+\tau_4]}\overline{B((Q,R_3)\cap\mathcal{O}(s))}\times\{s\}$ for some $\tau_4 \in (0,\tau_3)$ and $R_3 \in (0,R_2/2)$. Then we have this proposition. □

4. Proof of Lemma 3.1

In this section, we shall prove Lemma 3.1.

Proof of Lemma 3.1. Step 0. For any sufficiently small $R>0$ and $S_3 \geqq S_2$, we can define cutoff functions $\Phi_{Q,8R,9R}(\cdot,s)$ and $\Psi_{Q,8R,9R}(\cdot,s)$ for $s \geqq S_3$. Then we put $\Phi_2 = \Phi_{Q,8R,9R}(\cdot,s)$ and $\Psi_2 = \Psi_{Q,8R,9R}(\cdot,s)$ for $s \geqq S_3$. We take N_3 satisfying $s_{N_3} \geqq S_3$.

In this proof, we treat only $s \geqq S_3$ and $n \geqq N_3$.

Applying Lemma 2.3 for $\Phi=\Phi_2$, we observe that for some $\tau_5>0$

$$\|z(\cdot,s)\|_{L^1(B(Q,8R)\bigcap\mathcal{O}(s))} \leqq \frac{1}{64K_1^2} \quad \text{for } s \in [s_n-\tau_5, s_n+\tau_5]. \tag{4.1}$$

Fixing each $\tau \in [-\tau_5,\tau_5]$ and applying [17, Lemma 4.1] for $u_0^n = z(\cdot,s_n+\tau)$, we can show this lemma.

Step 1. Let Ψ be a cutoff function defined in Section 2, and let A be the constant in (2.15). Using (3.3) and an argument similar to the proof of [15, Lemma 4], we have that

$$\int_{\mathcal{O}(s)} z^2\Psi dy \leqq 2K_1^2\Big(\int_{\text{supp }\Psi\cap\mathcal{O}(s)} z\,dy\Big)\Big(\int_{\mathcal{O}(s)} z^{-1}|\nabla z|^2\Psi\,dy\Big) + K_1^2\Big(\frac{A^2}{2}+1\Big)\lambda^2. \tag{4.2}$$

Put $\Psi_3 = \Psi_{Q,7R,8R}(\cdot, s)$. Multiplying the first equation of (1.5) by $(\log z + 1)\Psi_3$ and integrating over $\mathcal{O}(s)$, we have that

$$\int_{\mathcal{O}(s)} z_s(\log z + 1)\Psi_3\, dy = \int_{\mathcal{O}(s)} \nabla\cdot(\nabla z - z\nabla w)(\log z+1)\Psi_3\, dy - \int_{\mathcal{O}(s)} \nabla\cdot\left(\frac{y}{2}z\right)(\log z+1)\Psi_3\, dy. \tag{4.3}$$

Substituting $z\log z$ and Ψ_3 for z and Φ, respectively, in (2.18), we obtain that

$$\begin{aligned}
&\frac{d}{ds}\int_{\mathcal{O}(s)} (z\log z)\Psi_3\, dy = \int_{\mathcal{O}(s)} \left\{\frac{\partial}{\partial s}[(z\log z)\Psi_3] + \nabla\cdot\left[\frac{y}{2}(z\log z)\Psi_3\right]\right\} \\
&= -\int_{\mathcal{O}(s)} \nabla z\cdot\nabla[(\log z+1)\Psi_3]dy + \int_{\mathcal{O}(s)} z\nabla w\cdot\nabla[(\log z+1)\Psi_3)]dy \\
&- \int_{\mathcal{O}(s)} z\Psi_3\, dy + \int_{\mathcal{O}(s)} (z\log z)\Big(\frac{\partial\Psi_3}{\partial s} + \frac{y}{2}\cdot\nabla\Psi_3\Big)dy \\
&= -VIII + IX - X + XI
\end{aligned} \tag{4.4}$$

by (4.3). Using the second equation of (1.5), we observe that

$$IX = \int_{\mathcal{O}(s)} z\Psi_3\Big(z - e^{-s}\frac{\lambda}{|\Omega|}\Big)dy + \int_{\mathcal{O}(s)} z\log z\nabla w\cdot\nabla\Psi_3\, dy. \tag{4.5}$$

We note that

$$VIII = \int_{\mathcal{O}(s)} z^{-1}|\nabla z|^2\Psi_3\, dy + \int_{\mathcal{O}(s)} (\log z+1)\nabla z\cdot\nabla\Psi_3\, dy. \tag{4.6}$$

Combining (4.4) with (4.5) and (4.6) implies that

$$\begin{aligned}
&\frac{d}{ds}\int_{\mathcal{O}(s)} (z\log z)\Psi_3\, dy + \int_{\mathcal{O}(s)} z^{-1}|\nabla z|^2\Psi_3\, dy + \Big(\frac{\lambda}{|\Omega|}e^{-s}+1\Big)\int_{\mathcal{O}(s)} z\Psi_3\, dy \\
&= \int_{\mathcal{O}(s)} z^2\Psi_3\, dy - \int_{\mathcal{O}(s)} (\log z+1)\nabla z\cdot\nabla\Psi_3\, dy \\
&+ \int_{\mathcal{O}(s)} (z\log z)\nabla(w - w_{B(Q,8R)})\cdot\nabla\Psi_3\, dy \\
&+ \int_{\mathcal{O}(s)} (z\log z)\Big(\frac{\partial\Psi_3}{\partial s} + \frac{y}{2}\cdot\nabla\Psi_3\Big)dy = XII - XIII + XIV + XV.
\end{aligned} \tag{4.7}$$

Since it holds that $(|\log z|+1)^\alpha z^\beta \leqq z^2 + C_{\alpha,\beta}$ $(z>0)$ for any $\alpha>0$ and $\beta\in(0,2)$, we get the following inequity.

$$|VIII| \leqq A\int_{\mathcal{O}(s)} (|\log z|+1) z^{1/2}\Psi_3^{1/3}\cdot z^{-1/2}|\nabla z|\Psi_3^{1/2}dy$$

$$\leqq \frac{1}{8}\int_{\mathcal{O}(s)} z^{-1}|\nabla z|^2\Psi_3\,dy + \frac{1}{3}\int_{\mathcal{O}(s)} z^2\Psi_3\,dy + C_{16}. \tag{4.8}$$

Here and henceforth, A and B are positive constants satisfying (2.15) with $\Psi=\Psi_{Q,iR,(i+1)R}$ $(i=2,3,4,\dots,7)$, and we denote by C_i $(i=16,17,18,\dots,30)$ positive constants depending only on S_3, N_3, λ, A, B, R, and Q. However, we find that A and B depend only on R and Q.

We have that

$$XIV = -\int_{\mathcal{O}(s)} (w - w_{B(Q,8R)})(\log z+1)\nabla z\cdot\nabla\Psi_3\,dy$$

$$-\int_{\mathcal{O}(s)} (w - w_{B(Q,8R)})(z\log z)\Delta\Psi_3\,dy. \tag{4.9}$$

Since we can estimate each term of the right-hand side of (4.9) by Lemma 2.6, then we get the following estimate.

$$|XIV| \leqq \frac{1}{8}\int_{\mathcal{O}(s)} z^{-1}|\nabla z|^2\Psi_3\,dy + \frac{2}{3}\int_{\mathcal{O}(s)} z^2\Psi_3\,dy + C_{17}. \tag{4.10}$$

Noting (2.15) and (2.17), we get

$$\left|\frac{\partial\Psi}{\partial s} + \frac{y}{2}\cdot\nabla\Psi\right| \leqq C_{18}\Psi^{5/6} \tag{4.11}$$

for $\Psi=\Psi_{Q,iR,(i+1)R}$ $(i=1,2,3,\dots,7)$. Then we obtain that

$$|XV| \leqq \frac{1}{3}\int_{\mathcal{O}(s)} z^2\Psi_1\,dy + C_{19}. \tag{4.12}$$

Combining (4.7) with (4.8), (4.10), and (4.12) implies that

$$\frac{d}{ds}\int_{\mathcal{O}(s)} (z\log z)\Psi_3\,dy + \frac{3}{4}\int_{\mathcal{O}(s)} z^{-1}|\nabla z|^2\Psi_3\,dy \leqq \frac{7}{3}\int_{\mathcal{O}(s)} z^2\Psi_3\,dy + C_{20}. \tag{4.13}$$

By this and (4.2) with $\Psi=\Psi_3$, we obtain that

$$\frac{d}{ds}\int_{\mathcal{O}(s)} (z\log z)\Psi_3\,dy + \left(\frac{3}{4} - \frac{14}{3}K_1^2\int_{B(Q,8R)\cap\mathcal{O}(s)} z\,dy\right)\int_{\mathcal{O}(s)} z^{-1}|\nabla z|^2\Psi_3\,dy$$

$$+\int_{\mathcal{O}(s)} z^2\Psi_3\,dy \leqq C_{20} + \frac{7}{3}K_1^2\Big(\frac{A^2}{2}+1\Big)\lambda^2. \tag{4.14}$$

Combining (4.1) with (4.14) implies

$$\frac{d}{ds}\int_{\mathcal{O}(s)}(z\log z)\Psi_3\,dy+\int_{\mathcal{O}(s)} z^2\Psi_3\,dy+\frac{1}{2}\int_{\mathcal{O}(s)} z^{-1}|\nabla z|^2\Psi_3\,dy \leqq C_{21}. \tag{4.15}$$

Combining

$$\begin{aligned}
&3\Big\{\int_{\mathcal{O}(s)}(z\log z+e^{-1})\Psi_3\,dy\Big\}^{3/2}\\
&\leqq 3\Big(\int_{\mathcal{O}(s)}\Psi_3\,dy\Big)^{1/2}\int_{\mathcal{O}(s)}(z\log z+e^{-1})^{3/2}\Psi_3\,dy\\
&\leqq 3|B(Q,8R)|^{1/2}\int_{\mathcal{O}(s)}(z^{7/4}+C_{22})\Psi_3\,dy \leqq \int_{\mathcal{O}(s)} z^2\Psi_3\,dy + C_{23}
\end{aligned}$$

with (4.15) implies that

$$\begin{aligned}
\frac{d}{ds}\int_{\mathcal{O}(s)}(z\log z+e^{-1})\Psi_3\,dy + 3\Big(\int_{\mathcal{O}(s)}(z\log z+e^{-1})\Psi_3\,dy\Big)^{3/2}&\\
\leqq C_{21}+C_{23}+\frac{d}{ds}\int_{\mathcal{O}(s)} e^{-1}\Psi_3\,dy& \qquad (4.16)
\end{aligned}$$

for $s \in [s_n-\tau_5, s_n+\tau_5]$. For each $\tau \in [0, 2\tau_5]$, we get that for each $s \in [s_n-\tau_5+\tau, s_n+\tau_5+\tau]$

$$\begin{aligned}
&\frac{d}{ds}\Big\{(s-s_n+\tau_5-\tau)^{-2}+\int_{\mathcal{O}(s)} e^{-1}\Psi_5\,dy\Big\}\\
&+3\Big\{(s-s_n+\tau_5-\tau)^{-2}+\int_{\mathcal{O}(s)} e^{-1}\Psi_5\,dy\Big\}^{3/2}\\
&\qquad\geqq (s-s_n+\tau_5-\tau)^{-3}+\frac{d}{ds}\int_{\mathcal{O}(s)} e^{-1}\Psi_5\,dy
\end{aligned}$$

and find $\tau_6 \geqq \tau_5$ such that $C_{21}+C_{23} \leqq \tau_6^3$. Then we get that

$$\int_{\mathcal{O}(s)} z\log z\Psi_3\,dy \leqq \min(s-s_n+\tau_5, \tau_6)^{-2} \quad \text{for } s \in [s_n-\tau_7, s_n+\tau_7].$$

Then putting $\tau_7 = \min(\tau_5/2, \tau_6)$, we have that

$$\int_{\mathcal{O}(s)} z\log z\Psi_3\,dy \leqq \tau_7^{-2} \quad \text{for } s \in [s_n-\tau_7, s_n+\tau_7]. \tag{4.17}$$

Combining (4.17) with (4.15) implies that

$$\int_{s_n-\tau_7}^{s_n+\tau_7}\int_{\mathcal{O}(s')} z^{-1}|\nabla z|^2\Psi_3\,dy\,ds' \leqq 4C_{21}\tau_7 + 2\tau_7^{-2}. \tag{4.18}$$

Step 2. Put $\Phi_4 = \Phi_{Q,6R,7R}$ and $\Psi_4 = \Psi_{Q,6R,7R}$. Multiplying the first equation of (1.5) by $z\Phi_4$ and integrating over $\mathcal{O}(s)$, we have that

$$\begin{aligned}\frac{d}{ds}\int_{\mathcal{O}(s)} z^2\Psi_4\,dy &= \int_{\mathcal{O}(s)}\Big[\frac{\partial}{\partial s}(z^2\Psi_4) + \nabla\cdot\Big(\frac{y}{2}z^2\Psi_4\Big)\Big]dy \\ &= -2\int_{\mathcal{O}(s)}|\nabla z|^2\Psi_4\,dy - 2\int_{\mathcal{O}(s)} z\nabla z\cdot\nabla\Psi_4\,dy + 2\int_{\mathcal{O}(s)} z(\nabla w\cdot\nabla z)\Psi_4\,dy \\ &\quad + 2\int_{\mathcal{O}(s)} z^2\nabla w\cdot\nabla\Psi_4\,dy - \int_{\mathcal{O}(s)} z^2\Psi_4\,dy + \int_{\mathcal{O}(s)} z^2\Big(\frac{\partial\Psi_4}{\partial s} + \frac{y}{2}\cdot\nabla\Psi_4\Big)dy\end{aligned} \tag{4.19}$$

by using a calculation similar to that in (4.4). Since

$$\begin{aligned}2\int_{\mathcal{O}(s)} z(\nabla w\cdot\nabla z)\Psi_4\,dy &= \int_{\mathcal{O}(s)} z^3\Psi_4\,dy + \int_{\mathcal{O}(s)}(w - w_{B(Q,7R)})\nabla z^2\cdot\nabla\Psi_4\,dy \\ &\quad + \int_{\mathcal{O}(s)} z^2(w - w_{B(Q,7R)})\Delta\Psi_4\,dy\end{aligned}$$

and

$$\begin{aligned}2\int_{\mathcal{O}(s)} z^2\nabla w\cdot\nabla\Psi_4\,dy &= 2\int_{\mathcal{O}(s)} z^2\nabla(w - w_{B(Q,7R)})\cdot\nabla\Psi_4\,dy \\ &= -2\int_{\mathcal{O}(s)}(w - w_{B(Q,7R)})\nabla z^2\cdot\nabla\Psi_4\,dy - 2\int_{\mathcal{O}(s)} z^2\nabla(w - w_{B(Q,7R)})\Delta\Psi_4\,dy,\end{aligned}$$

then we have

$$\begin{aligned}\frac{d}{ds}\int_{\mathcal{O}(s)} z^2\Psi_4\,dy &= -2\int_{\mathcal{O}(s)}|\nabla z|^2\Psi_4\,dy - 2\int_{\mathcal{O}(s)} z\nabla z\cdot\nabla\Psi_4\,dy \\ &\quad + \int_{\mathcal{O}(s)} z^3\Psi_4\,dy - \int_{\mathcal{O}(s)}(w - w_{B(Q,7R)})\nabla z^2\cdot\nabla\Psi_4\,dy \\ &\quad - \int_{\mathcal{O}(s)} z^2(w - w_{B(Q,7R)})\Delta\Psi_4\,dy \\ &\quad - \int_{\mathcal{O}(s)} z^2\Psi_4\,dy + \int_{\mathcal{O}(s)} z^2\Big(\frac{\partial\Psi_4}{\partial s} + \frac{y}{2}\cdot\nabla\Psi_4\Big)dy \\ &= -XVI - XVII + XVIII - XIX - XX - XXI + XXII.\end{aligned}$$

We get the following inequalities by using Lemma 2.6, (4.11), and a calculation similar to the one in the previous step:

$$|XVII| \leqq \frac{1}{4}\int_{\mathcal{O}(s)} |\nabla z|^2 \Psi_4 \, dy + \frac{1}{3}\int_{\mathcal{O}(s)} z^3 \Psi_4 \, dy + C_{24},$$
$$|XIX| \leqq \frac{1}{4}\int_{\mathcal{O}(s)} |\nabla z|^2 \Psi_4 \, dy + \frac{1}{3}\int_{\mathcal{O}(s)} z^3 \Psi_4 \, dy + C_{25},$$
$$|XX| \leqq \frac{1}{3}\int_{\mathcal{O}(s)} z^3 \Psi_4 \, dy + C_{26},$$
$$|XXII| \leqq \frac{1}{3}\int_{\mathcal{O}(s)} z^3 \Psi_4 \, dy + C_{27}.$$

Then we observe that

$$\frac{d}{ds}\int_{\mathcal{O}(s)} z^2 \Psi_4 \, dy + \frac{3}{2}\int_{\mathcal{O}(s)} |\nabla z|^2 \Psi_4 \, dy \leqq \frac{7}{3}\int_{\mathcal{O}(s)} z^3 \Psi_4 \, dy + C_{28}. \tag{4.20}$$

By using (3.3) and an argument similar to the one in the proof of [15, Lemma 4], we have that

$$\int_{\mathcal{O}(s)} z^3 \Psi_4 \, dy \leqq \frac{72 K_1^2}{\log \ell}\Big\{ \int_{B(Q,7R)\cap\mathcal{O}(s)} (z \log z + e^{-1}) dy \Big\}\Big\{ \int_{\mathcal{O}(s)} |\nabla z|^2 \Psi_4 \, dy \Big\} + 10|B(Q,7R)|\ell^3 + C_{29} \quad \text{for } \ell > 1. \tag{4.21}$$

In (4.21), putting $\ell = \exp\Big(2^4 \cdot 3 \cdot 7 K_1^2 (\tau_7^{-2} + e^{-1}|B(Q,7R)|)\Big)$, we have that

$$\frac{7}{3}\int_{\mathcal{O}(s)} z^3 \Psi_4 \, dy \leqq \frac{1}{2}\int_{\mathcal{O}(s)} |\nabla z|^2 \Psi_4 \, dy + 10|B(Q,7R)| \exp\Big(2^4 \cdot 3 \cdot 7 K_1^2 (\tau_7^{-2} + e^{-1}|B(Q,7R)|)\Big) + C_{29}$$

for $s \in [s_n - \tau_7, s_n + \tau_7]$, by (4.17). Combining this with (4.20) implies that

$$\frac{d}{ds}\int_{\mathcal{O}(s)} z^2 \Psi_4 \, dy + \int_{\mathcal{O}(s)} |\nabla z|^2 \Psi_4 \, dy \leqq C_{30} \tag{4.22}$$

for $s \in [s_n - \tau_7, s_n + \tau_7]$.

By (4.18), for each $n \geqq N_3$ there exists $s_n' \in (s_n - \tau_7, s_n - \tau_7/2)$ satisfying

$$\int_{\mathcal{O}(s_n')} z^{-1}(y, s_n') |\nabla z(y, s_n')|^2 \Psi_3 \, dy \leqq 8 C_{21} + 4\tau_7^{-3}. \tag{4.23}$$

By this and (4.2) with $\Psi = \Psi_4$, we get

$$\int_{\mathcal{O}(s'_n)} z^2(y, s'_n)\Psi_4(y, s'_n)\, dy \leqq C_{31}.$$

Here and henceforth, we denote by C_i $(i = 31, 32, 33, \dots, 44)$ positive constants depending only on S_3, λ, N_4, A, B, R, Q, and τ_i $(i = 7, 8, 9, 10)$.

Combining this with (4.22) implies that

$$\int_{\mathcal{O}(s)} z^2\Psi_4\, dy + \int_{s_n-\tau_8}^{s}\int_{\mathcal{O}(s')} |\nabla z|^2\Psi_4\, dy\, ds' \quad\leqq\quad C_{32} \tag{4.24}$$

for $s \in [s_n - \tau_8, s_n + \tau_8]$, where $\tau_8 = \tau_7/2$.

Step 3. Put $\Phi_5 = \Phi_{Q,5R,6R}$ and $\Psi_5 = \Psi_{Q,5R,6R}$. Multiplying the first equation of (1.5) by $z^2\Psi_5$ and integrating over $\mathcal{O}(s)$, we observe that

$$\begin{aligned}
&\frac{d}{ds}\int_{\mathcal{O}(s)} z^3\Psi_5\, dy = \int_{\mathcal{O}(s)} \Big[\frac{\partial}{\partial s}(z^3\Psi_5) + \nabla\cdot\Big(\frac{y}{2}z^3\Psi_5\Big)\Big]dy\\
&= -6\int_{\mathcal{O}(s)} z|\nabla z|^2\Psi_5\, dy - 3\int_{\mathcal{O}(s)} z^2\nabla z\cdot\nabla\Psi_5\, dy + 6\int_{\mathcal{O}(s)} z^2(\nabla w\cdot\nabla z)\Psi_5\, dy\\
&+ 3\int_{\mathcal{O}(s)} z^3\nabla w\cdot\nabla\Psi_5\, dy - \int_{\mathcal{O}(s)} 2z^3\Psi_5\, dy + \int_{\mathcal{O}(s)} z^3\Big(\frac{\partial\Psi_5}{\partial s} + \frac{y}{2}\cdot\nabla\Psi_5\Big)dy.
\end{aligned}$$

Putting $\tilde{z} = z^{3/2}$, we have that

$$\begin{aligned}
&\frac{d}{ds}\int_{\mathcal{O}(s)} \tilde{z}^2\Psi_5\, dy = -\frac{8}{3}\int_{\mathcal{O}(s)} |\nabla\tilde{z}|^2\Psi_5\, dy - 2\int_{\mathcal{O}(s)} \tilde{z}\nabla\tilde{z}\cdot\nabla\Psi_5\, dy\\
&+ 4\int_{\mathcal{O}(s)} \tilde{z}(\nabla w\cdot\nabla\tilde{z})\Psi_5\, dy + 3\int_{\mathcal{O}(s)} \tilde{z}^2\nabla w\cdot\nabla\Psi_5\, dy\\
&- \int_{\mathcal{O}(s)} 2\tilde{z}^2\Psi_5\, dy + \int_{\mathcal{O}(s)} \tilde{z}^2\Big(\frac{\partial\Psi_5}{\partial s} + \frac{y}{2}\cdot\nabla\Psi_5\Big)\, dy.
\end{aligned} \tag{4.25}$$

Since (4.25) is a similar form of (4.19) and inequality (4.24) implies the boundedness of $\|\tilde{z}(\cdot, s)\|_{L^1(B(Q,6R)\cap\mathcal{O}(s))}$, we get

$$\int_{\mathcal{O}(s)} \tilde{z}\log\tilde{z}\Psi_5\, dy \leqq C_{33} \quad \text{for } s \in [s_n - \tau_8, s_n + \tau_8],$$

by an argument similar to the one in Step 1. Then, an argument similar to the one in Step 2 implies that

$$\frac{d}{ds}\int_{\mathcal{O}(s)} \tilde{z}^2\Psi_5\, dy + \int_{\mathcal{O}(s)} |\nabla\tilde{z}|^2\Psi_5\, dy \leqq C_{34} \ \text{ for } s \in [s_n - \tau_8, s_n + \tau_8]. \tag{4.26}$$

(4.24) says

$$\frac{2}{\tau_8}\int_{s_n-\tau_8}^{s_n-\tau_8/2}\int_{\mathcal{O}(s)} z^{-2}|\nabla z^2|^2\Psi_5\,dy \leqq C_{35}$$

for $s\in[s_n-\tau_8, s_n+\tau_8]$. Using this and applying (4.2) with $\Psi=\Psi_5$ for z^2, we have the boundedness of $\|z(\cdot,s_n')\|_{L^4(B(Q,5R)\cap\mathcal{O}(s))}$ for some $s_n'\in[s_n-\tau_8, s_n-\tau_8/2]$. Then, we observe that for some $s_n'\in[s_n-\tau_8, s_n-\tau_8/2]$

$$\int_{\mathcal{O}(s_n')} z^3(y,s_n')\Psi_5(y,s_n')dy \leqq C_{36}.$$

Combining this with (4.26) implies that

$$\int_{\mathcal{O}(s)} z^3\Psi_5\,dy + \int_{s_n-\tau_9}^{s}\int_{\mathcal{O}(s')} z|\nabla z|^2\Psi_5\,dy\,ds' \leqq C_{37} \tag{4.27}$$

for $s\in[s_n-\tau_9, s_n+\tau_9]$ with $\tau_9=\tau_8/2$, by an argument similar to the one in Step 2. Combining (2.25) with (4.27), (2.4), and $e^{-s/3}\|w(\cdot,s)\|_{L^3(\mathcal{O}(s))}=\|v(\cdot,t)\|_{L^3(\Omega)}$ implies that

$$\|\nabla w(\cdot,s)\|_{L^\infty(B(Q,4R)\cap\mathcal{O}(s))}\leqq C_{38} \quad \text{for } s\in[s_n-\tau_9, s_n-\tau_9]. \tag{4.28}$$

Step 4. Put $\Phi_6=\Phi_{Q,3R,4R}$ and $\Psi_6=\Psi_{Q,3R,4R}$. Since we observe that

$$\begin{aligned}
&\frac{d}{ds}\int_{\mathcal{O}(s)}|\nabla z|^2\Psi_6\,dy = \int_{\mathcal{O}(s)}\Big[\frac{\partial}{\partial s}(|\nabla z|^2\Psi_6)+\nabla\cdot\Big(\frac{y}{2}|\nabla z|^2\Psi_6\Big)\Big]dy\\
&= 2\int_{\mathcal{O}(s)} z_s(-\nabla\cdot(\Psi_6\nabla z))dy + \int_{\mathcal{O}(s)}\nabla\cdot\Big(\frac{y}{2}|\nabla z|^2\Big)\Psi_6\,dy\\
&+\int_{\mathcal{O}(s)}|\nabla z|^2\Big(\frac{\partial\Psi_6}{\partial s}+\frac{y}{2}\cdot\nabla\Psi_6\Big)dy
\end{aligned}$$

and that

$$\begin{aligned}
&\int_{\mathcal{O}(s)}\nabla\cdot\Big(\frac{y}{2}|\nabla z|^2\Big)\Psi_6\,dy = \int_{\mathcal{O}(s)}\Big\{\sum_{i,j=1}^{2}\frac{\partial z}{\partial y_i}\frac{\partial}{\partial y_i}\Big(y_j\frac{\partial z}{\partial y_j}\Big)\Big\}\Psi_6\,dy\\
&= 2\int_{\mathcal{O}(s)}\Big(\nabla\cdot\Big(\frac{y}{2}z\Big)\Big)(-\nabla\cdot(\Psi_6\nabla z))dy - \int_{\mathcal{O}(s)}|\nabla z|^2\Psi_6\,dy,
\end{aligned}$$

we get that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\int_{\mathcal{O}(s)}|\nabla z|^2\Psi_6\,dy = -\int_{\mathcal{O}(s)}|\Delta z|^2\Psi_6\,dy + \int_{\mathcal{O}(s)}[\nabla\cdot(z\nabla w)]\nabla\cdot(\Psi_6\nabla z)dy\\
&-\int_{\mathcal{O}(s)}(\Delta z)\nabla\Psi_6\cdot\nabla z\,dy - \frac{1}{2}\int_{\mathcal{O}(s)}|\nabla z|^2\Psi_6\,dy + \frac{1}{2}\int_{\mathcal{O}(s)}|\nabla z|^2\Big(\frac{\partial\Psi_6}{\partial s}+\frac{y}{2}\cdot\nabla\Psi_6\Big)dy
\end{aligned}$$

$$= -XXIII + XXIV - XXV - XXVI + XXVII,$$

by using the first equation of (1.5). By using the second equation of (1.5), we have the following equation:

$$\begin{aligned} XXIV - XXV &= \int_{\mathcal{O}(s)} \Big(\nabla z \cdot \nabla w + e^{-s}\frac{\lambda}{|\Omega|}z - z^2\Big)(\Delta z)\Psi_6\,dy \\ &+ 2\int_{\mathcal{O}(s)} z|\nabla z|^2\Psi_6\,dy - \int_{\mathcal{O}(s)}(\Delta z)\nabla z\cdot\nabla\Psi_6\,dy \\ &+ \int_{\mathcal{O}(s)}\Big(\nabla z\cdot\nabla w + e^{-s}\frac{\lambda}{|\Omega|}z - z^2\Big)(\nabla z\cdot\nabla\Psi_6)dy. \end{aligned} \tag{4.29}$$

Note (4.11) and $\Psi_6 \leqq \Psi_6^{5/6} \leqq \Psi_6^{2/3} \leqq \Psi_5 \leqq \Psi_4$. Combining (4.29) with (4.24) and (4.28) implies that

$$\begin{aligned} &\frac{1}{2}\frac{d}{ds}\int_{\mathcal{O}(s)}|\nabla z|^2\Psi_6\,dy + \frac{1}{2}\int_{\mathcal{O}(s)}|\Delta z|^2\Psi_6\,dy \leqq 2\int_{\mathcal{O}(s)} z|\nabla z|^2\Psi_5\,dy \\ &+ C_{39}\Big(\int_{\mathcal{O}(s)}|\nabla z|^2\Psi_4\,dy + \int_{\mathcal{O}(s)} z^4\Psi_4\,dy + 1\Big). \end{aligned} \tag{4.30}$$

Combining (4.24) with $H^1(B(Q,6R)\cap\mathcal{O}(s)) \subset L^4(B(Q,6R)\cap\mathcal{O}(s))$ implies that

$$\int_{s_n-\tau_8}^{s_n+\tau_8}\int_{B(Q,6R)\cap\mathcal{O}(s)} z^4\,dy\,ds \leqq C_{40}. \tag{4.31}$$

By (4.24), we find some $s_n' \in [s_n - \tau_8, s_n - \tau_8/2]$ such that

$$\int_{\mathcal{O}(s_n')}|\nabla z(y,s_n')|^2\Psi_4(y,s_n')\,dy = \frac{2}{\tau_8}\int_{s_n-\tau_8}^{s_n-\tau_8/2}\int_{\mathcal{O}(s')}|\nabla z|^2\Psi_4\,dy\,ds' \leqq C_{41}\frac{2}{\tau_8}.$$

Combining this with (4.30), (4.31), (4.24), and (4.27) implies that

$$\int_{\mathcal{O}(s)}|\nabla z|^2\Psi_6\,dy + \int_{s_n-\tau_9}^{s}\int_{\mathcal{O}(s')}|\Delta z|^2\Psi_6\,dy\,ds' \leqq C_{42} \tag{4.32}$$

for $s \in [s_n - \tau_9, s_n + \tau_9]$ with $\tau_9 = \tau_8/2$. By (4.32), we find $s_n' \in [s_n - \tau_9, s_n - \tau_9/2]$ satisfying

$$\int_{\mathcal{O}(s_n')}|\Delta z|^2\Psi_6\,dy = \frac{2}{\tau_9}\int_{s_n-\tau_9}^{s_n-\tau_9/2}\int_{\mathcal{O}(s')}|\Delta z|^2\Psi_6\,dy\,ds' \leqq C_{42}\frac{2}{\tau_9}.$$

By this and (4.24), we can find $s_n' \in [s_n - \tau_9, s_n - \tau_9/2]$ such that

$$\|z(\cdot,s_n')\|_{L^\infty(B(Q,3R)\cap\mathcal{O}(s_n'))} \leqq C_{43} \tag{4.33}$$

by the Sobolev imbedding theorem (see [1]).

Step 5. Put $\Phi_7 = \Phi_{Q,2R,3R}$ and $\Psi_7 = \Phi_{Q,2R,3R}$. Let $p \geqq 1$. Using the first equation of (1.5), we observe that

$$\begin{aligned}&\frac{1}{p+1}\frac{d}{ds}\int_{\mathcal{O}(s)} z^{p+1}\Psi_7^{p+1}dy\\&= \frac{1}{p+1}\int_{\mathcal{O}(s)}\Big\{\frac{\partial}{\partial s}(z^{p+1}\Psi_7^{p+1}) + \nabla\cdot\Big(\frac{y}{2}z^{p+1}\Psi_7^{p+1}\Big)\Big\}dy\\&= -\int_{\mathcal{O}(s)}\nabla(z^p\Psi_7^{p+1})\cdot\nabla z\,dy + \int_{\mathcal{O}(s)} z\nabla(z^p\Psi_7^{p+1})\cdot\nabla w\,dy\\&\quad - \frac{p}{p+1}\int_{\mathcal{O}(s)} z^{p+1}\Psi_7^{p+1}dy + \int_{\mathcal{O}(s)} z^p\Psi_7^p\Big(\frac{\partial\Psi_7}{\partial s} + \frac{y}{2}\cdot\nabla\Psi_3\Big)dy.\end{aligned}$$

By using Gagliardo-Nirenberg's inequality in Ω and $y = e^{s/2}x$, we have that

$$\Big\|f\Big\|_{L^q(\mathcal{O}(s))} \leqq K_2\Big(\Big\|\nabla f\Big\|^2_{L^2(\mathcal{O}(s))} + \Big\|f\Big\|^2_{L^2(\mathcal{O}(s))}\Big)^{\frac{1-(1/q)}{2}}\Big\|f\Big\|^{1/q}_{L^1(\mathcal{O}(s))}, \quad (4.34)$$

where $K_2 > 0$ is independent of $s \geqq s_*$ and $q \in [1, q_0]$ for given $q_0 > 1$. Put $z_1 = z\Psi_7$. Noting (4.11), (4.34), and (4.28), and using the arguments in Step 3 of the proof of [15, Lemma 5], we have the following inequality.

$$\frac{d}{ds}\int_{\mathcal{O}(s)} z_1^{p+1}dy + \int_{\mathcal{O}(s)} z_1^{p+1}dy \leqq C_{44}(p+1)^6\Big\{\int_{\mathcal{O}(s)} z_1^{(p+1)/2}dy + 1\Big\}^2.$$

Here and henceforth, we denote by C_i ($i = 44, 45, 46, \ldots, 49$) positive constants which are independent of $p \geqq 1$. Moreover, we obtain that C_{43} is independent of $p \geqq 1$.

Combining this with (4.33) gives that

$$\begin{aligned}&\sup_{s\in[s_n', s_n+\tau_9]}\Big\{\int_{\mathcal{O}(s)} z_1^{p+1}dy + 1\Big\}\\&\leqq \max\Big\{C_{44}(p+1)^6\sup_{s\in[s_n', s_n+\tau_9]}\Big(\int_{\mathcal{O}(s)} z_1^{(p+1)/2}dy + 1\Big)^2, C_{41}^{p+1}|B(Q,3R)| + 1\Big\},\end{aligned}$$

where s_n' is the constant in (4.33). Therefore,

$$a_k = \sup\Big\{\int_{\mathcal{O}(s)} z_1^{2^k}dy + 1 : s \in [s_n', s_n + \tau_9] \text{ and } n \geqq N_4\Big\}$$

satisfies

$$a_{k+1} \leqq C_{45}\max\Big\{2^{6(k+1)}a_k^2,\ \Big(|B(Q,2R)| + 1\Big)\Big(C_{41} + 1\Big)^{2^{k+1}}\Big\}$$

$$\leqq C_{45} 2^{6(k+1)} \max \left\{ a_k^2,\ \left(C_{41}+1\right)^{2^{k+1}} \right\} \tag{4.35}$$

for $k = 1, 2, 3, \ldots$. Let $d = C_{43} + 1$. Then, (4.35) is reduced to

$$a_{k+1} \leqq C_{45}^{2^k - 1} \cdot 2^{\sum_{i=1}^{k} 6(i+1)2^{k-i}} \cdot \max \left\{ a_2^{2^k},\ d^{2^{k+1}} \right\}$$

for $k = 1, 2, 3, \ldots$. Letting $\tau_{10} = \tau_9/2$ and $k \to \infty$,

$$\sup_{s \in [s_n - \tau_{10}, s_n + \tau_{10}]} \|z_1(\cdot, s)\|_{L^\infty(\mathcal{O}(s))}$$
$$\leqq C_{46} \max \left\{ \left(\sup_{s \in [s'_n, s_n - \tau_{10}]} \|z_1(\cdot, s)\|^2_{L^2(\mathcal{O}(s))} + 1 \right)^{1/2},\ d \right\}$$

follows. Noting this and (4.24), we observe

$$\|z(\cdot, s)\|_{L^\infty(B(Q,2R) \cap \mathcal{O}(s))} \leqq C_{47} \quad \text{for } s \in [s_n - \tau_{10}, s_n - \tau_{10}] \text{ and } n \geqq N_4.$$

Combining this with

$$\|z(\cdot, s)\|_{L^\infty(\mathcal{O}(s))} \leqq C_{48} \quad \text{for } s \in [s_*, S_3]$$

implies that

$$\|z(\cdot, s)\|_{L^\infty(B(Q,2R) \cap \mathcal{O}(s))} \leqq C_{49} \quad \text{for } s \in [s_n - \tau_{10}, s_n - \tau_{10}] \text{ and } n \geqq 1.$$

By using this and standard arguments for parabolic regularities, we have this lemma. □

5. Proof of theorem

Since $\|z(\cdot, s)\|_{L^1(\mathcal{O}(s))} = \lambda$ for $s \geqq s_*$, there exists a subsequence $\{s'_n\}$ such that

$$z(\cdot, s'_n)\chi_{\mathcal{O}(s)} \to z_\infty \quad \text{as } n \to \infty \quad \text{in } \mathcal{M}(\mathbf{R}^2).$$

By Proposition 3.1, for $Q \in \overline{H}$ it holds that $z_\infty(\{Q\}) = 0$ or $\geqq m_*(Q)$. Then, putting $\mathcal{S}' = \{Q \in \overline{\mathcal{O}(\infty)} : z_\infty(\{Q\}) \geqq m_*(Q)\}$, we have $\sharp \mathcal{S}' \leqq \lambda/(4\pi) < \infty$. By this and Proposition 3.1, there exists a subsequence $\{s'_n\} \subset \{s_n\}$ and a nonnegative function F such that

$$z(\cdot, s'_n)\chi_{\overline{\mathcal{O}(s'_n)}} \to \sum_{Q \in \mathcal{S}'} m(Q)\delta_Q + F \quad \text{as } n \to \infty \quad \text{in } \mathcal{M}(\mathbf{R}^2), \tag{5.1}$$

where $m(Q) \geqq m_*(Q)$, supp$F \subset \overline{\mathcal{O}(\infty)}$ and $F \in L^1(\mathbf{R}^2) \cap C^1_{loc}(\overline{\mathcal{O}(\infty)} \setminus \mathcal{S}')$.

Then it is enough for us to proof $m(Q) \leqq m_*(Q)$. In this section, we consider only the case where $0 \in \partial\Omega$ and $Q \in \partial H$. Then $m_*(Q) = 4\pi$.

Put $\Psi_{i+7} = \Psi_{Q, 4^i R_7, 2 \cdot 4^i R_7}$ $(i = 1, 2)$ and $m(y, s) = |Y(y, s) - Y(Q, s)|^2/2$.

Lemma 5.1.

$$\begin{aligned}&\left|\Xi_m(y,y',s)+\frac{1}{\pi}\Psi_8(y)\Psi_9(y')\right|\\&\leqq C_{40}R^{-1}(|y|+|y'|)\Psi_8^{1/2}(y)\Psi_9(y')+C_{50}R^{-1}|y'|\Psi_9(y'),\end{aligned}$$

where

$$\begin{aligned}\Xi_m(y,y',s)&=[\nabla(m\Psi_8)(y)\cdot\nabla_y\mathcal{G}(y,y',s)]\Psi_9(y')\\&\quad+[\nabla(m\Psi_8)(y')\cdot\nabla_{y'}\mathcal{G}(y,y',s)]\Psi_9(y).\end{aligned}$$

Here and henceforth, C_i $(i=50,51,52,\dots,58)$ is independent of R. We can proof this lemma the same way as [16, Lemma 3.2] was proved, by replacing the conformal map X, weight m, Green's function G and cutoff function ψ_i in [16, Lemma 3.2] as Y, G, m, and Ψ_{i+6} defined above. Thus we omit the proof.

Lemma 5.2. *Let $Q\in\partial H\cap\mathcal{S}'$ and S_4 be sufficiently large. Suppose that*

$$\int_{\mathcal{O}(s)\cap B(Q,R)} z(y,s)dy>4\pi\quad\text{for some } s\geqq S_4 \text{ and } R>0.$$

Then there exists $\varepsilon>0$ depending on $\int_{\mathcal{O}(s)\cap B(Q,R)} z(y,s)dy-4\pi$ satisfying the following: if

$$\frac{1}{R^2}\int_{\mathcal{O}(s)\cap B(Q,R)}|y-Q|^2 z(y,s)dy<\varepsilon,$$

then z blows up in a finite time.

Proof. We observe that

$$\begin{aligned}&\frac{d}{ds}\int_{\mathcal{O}(s)}(zm\Psi_8)dy=\int_{\mathcal{O}(s)}\nabla\cdot(\nabla z-z\nabla w)(zm\Psi_8)dy\\&+\int_{\mathcal{O}(s)}\left\{(m\Psi_8)_s+\frac{1}{2}\nabla(ym\Psi_8)\right\}z\,dy=XXVIII+XXIX.\end{aligned}\tag{5.2}$$

We observe that $|m_s|+|\nabla m|\leqq C_{51}m^{1/2}$ and that $|\Psi_{8s}|+|\nabla\Psi_8|\leqq C_{52}A\Psi_8^{5/6}$. Then we have that

$$|XXIX|\leqq C_{53}\lambda^{1/2}\Big(\int_{\mathcal{O}(s)} mz\Psi_8\,dy\Big)^{1/2}.\tag{5.3}$$

We get the following estimate of $XXIX$, by using Lemma 5.1 and an argument similar to the proof of [16, Theorem 3].

$$XXVIII\leqq 2\lambda_{\Psi_8}-\frac{1}{2\pi}\lambda_{\Psi_8}^2+C_{54}R^{-1}\lambda^{3/2}I_{\Psi_8}^{1/2}+C_{54}\lambda(\lambda_{\Psi_9}-\lambda_{\Psi_8}),\tag{5.4}$$

where

$$\lambda_{\Psi_i} = \int_{\mathcal{O}(s)} z\Psi_i\,dy \quad \text{and} \quad \lambda_{\Psi_i} = \int_{\mathcal{O}(s)} mz\Psi_i\,dy$$

$(i = 7, 8)$. Combining (5.4) with (5.2) and (5.3) implies that

$$\frac{d}{ds}\lambda_{\Psi_8} \leqq 2\lambda_{\Psi_8} - \frac{1}{2\pi}\lambda_{\Psi_8}^2 + C_{55}(\lambda^{1/2} + \lambda^{3/2})R^{-1}I_{\Psi_8}^{1/2} + C_{56}\lambda(\lambda_{\Psi_9} - \lambda_{\Psi_8}). \tag{5.5}$$

Note that $0 < C_{57}|Y(y,s) - Y(Q,s)| \leqq |y - Q| \leqq C_{58}|Y(y,s) - Y(Q,s)|$ if $|y-Q| \ll 1$. Applying the argument of the proof of [16, Theorem 3] to (5.5), we get this lemma. □

We shall continue to prove Theorem 1.

If there exists $Q \in \mathcal{S}' \cap \partial H$ such that $z_\infty(Q) > 4\pi$, then there exists $\delta > 0$ such that $z_\infty(\{Q\}) \geqq 4\pi + \delta$. By (5.1), for any sufficiently small $R > 0$ and any sufficiently large n it holds that

$$\int_{\mathcal{O}(s'_n)\cap B(Q,R)} z(y, s'_n)dy > 4\pi$$

$$\frac{1}{R^2}\int_{\mathcal{O}(s'_n)\cap B(Q,R)} |y - Q|^2 z(y, s'_n)dy < \varepsilon,$$

where ε is the constant in Lemma 5.2.

Combining this with Lemma 5.2 implies that z blows up in a finite time. This is a contradiction. Then we have this theorem with $\mathcal{S} = \mathcal{S}'$. □

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Note added to the proof. If the blowup point is "type II," then (1.4) is improved as $z(\cdot, s) \to m_*(x_0)\delta_0$. Also, it always holds that $m(q) = m_*(q)$ in (1.2). These results will be published in a monograph by the second author. A special case is also proven in a forthcoming paper.